*G. Polya*

# MATHEMATICS AND PLAUSIBLE REASONING

## VOLUME II
## PATTERNS OF PLAUSIBLE INFERENCE

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*VOLUME II*
*OF MATHEMATICS*
*AND PLAUSIBLE*
*REASONING*

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Volume II

# Patterns of Plausible Inference

# XII

# SOME CONSPICUOUS PATTERNS

*I do not wish, at this stage, to examine the logical justification of this form of argumentation; for the present, I am considering it as a* practice, *which we can observe in the habits of men and animals.*—BERTRAND RUSSELL[1]

**1. Verification of a consequence.** In the first volume of this work on *Induction and Analogy in Mathematics* we found some opportunity to familiarize ourselves with the practice of plausible reasoning. In the present second volume we undertake to describe this practice in general terms. The examples of the first part have already indicated certain forms or *patterns* of plausible reasoning. In the present chapter we undertake to formulate some such patterns explicitly.[2]

We begin with a pattern of plausible inference which is of so general use that we could extract it from almost any example. Yet let us take an example which we have not yet discussed before.

The following conjecture is due to Euler:[3] *Any integer of the form* $8n + 3$ *is the sum of a square and of the double of a prime.* Euler could not prove this conjecture, and the difficulty of a proof appears perhaps even greater today than in Euler's time. Yet Euler verified his statement for all integers of the form $8n + 3$ under 200; for $n = 1, 2, \ldots 10$ see Table I.

Table I

$$11 = 1 + 2 \times 5$$
$$19 = 9 + 2 \times 5$$
$$27 = 1 + 2 \times 13$$
$$35 = 1 + 2 \times 17 = 9 + 2 \times 13 = 25 + 2 \times 5$$
$$43 = 9 + 2 \times 17$$
$$51 = 25 + 2 \times 13$$
$$59 = 1 + 2 \times 29 = 25 + 2 \times 17 = 49 + 2 \times 5$$
$$67 = 9 + 2 \times 29$$
$$75 = 1 + 2 \times 37 = 49 + 2 \times 13$$
$$83 = 1 + 2 \times 41 = 9 + 2 \times 37 = 25 + 2 \times 29 = 49 + 2 \times 17$$

[1] *Philosophy*, W. W. Norton & Co., 1927, p. 80.

[2] Parts of this chapter were used in my address "On plausible reasoning" printed in the *Proceedings of the International Congress of Mathematicians* 1950, vol. 1, p. 739–747.

[3] *Opera Omnia*, ser. 1, vol. 4, p. 120–124. In this context, Euler regards 1 as a prime; this is needed to account for the case $3 = 1 + 2 \times 1$.

Such empirical work can be easily carried further; no exception has been found in numbers under 1000.[4] Does this prove Euler's conjecture? By no means; even verification up to 1,000,000 would prove nothing. Yet each verification renders the conjecture somewhat more credible, and we can see herein a general pattern.

Let $A$ denote some clearly formulated conjecture which is, at present, neither proved, nor disproved. (For instance, $A$ may be Euler's conjecture that, for $n = 1, 2, 3, \ldots$ ,

$$8n + 3 = x^2 + 2p$$

where $x$ is an integer and $p$ a prime.) Let $B$ denote some consequence of $A$; also $B$ should be clearly stated and neither proved, nor disproved. (For instance, $B$ may be the first particular case of Euler's conjecture not listed in Table I which asserts that $91 = x^2 + 2p$.) For the moment we do not know whether $A$ or $B$ is true. We do know, however, that

$$A \text{ implies } B.$$

Now, we undertake to check $B$. (A few trials suffice to find out whether the assertion about 91 is true or not.) If it turned out that $B$ is false, we could conclude that $A$ also is false. This is completely clear. We have here a classical elementary pattern of reasoning, the "modus tollens" of the so-called hypothetical syllogism:

$$\begin{array}{c} A \text{ implies } B \\ B \text{ false} \\ \hline A \text{ false} \end{array}$$

The horizontal line separating the two premises from the conclusion stands as usual for the word "therefore." We have here *demonstrative inference* of a well-known type.

What happens if $B$ turns out to be true? (Actually, $91 = 9 + 2 \times 41 = 81 + 2 \times 5$.) There is no demonstrative conclusion: the verification of its consequence $B$ does not prove the conjecture $A$. Yet such verification renders $A$ more credible. (Euler's conjecture, verified in one more case, becomes somewhat more credible.) We have here a pattern of *plausible inference*:

$$\begin{array}{c} A \text{ implies } B \\ B \text{ true} \\ \hline A \text{ more credible} \end{array}$$

The horizontal line again stands for "therefore." We shall call this pattern the *fundamental inductive pattern* or, somewhat shorter, the "inductive pattern."

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[4] Communication of Professor D. H. Lehmer.

This inductive pattern says nothing surprising. On the contrary, it expresses a belief which no reasonable person seems to doubt: *The verification of a consequence renders a conjecture more credible.* With a little attention, we can observe countless reasonings in everyday life, in the law courts, in science, etc., which appear to conform to our pattern.

**2. Successive verification of several consequences.** In the present section, I use the phrase "discussion of a theorem" in the specific meaning "discussion, or survey, of some particular cases and some more immediate consequences of the theorem." I think that the discussion of the theorems presented is useful both in advanced and in elementary classes. Let us consider a very elementary example. Let us assume that you teach a class in solid geometry and that you have to derive the formula for the area of the lateral surface of the frustum of a cone. Of course, the cone is a right circular cone, and you are given the radius of the base $R$, the radius of the top $r$, and the altitude $h$. You go through the usual derivation and you arrive at the result:

*A. The area of the lateral surface of the frustum is*

$$\pi(R + r)\sqrt{(R - r)^2 + h^2}.$$

We call this theorem $A$ for future reference.

Now comes the discussion of the theorem $A$. You ask the class: *Can you check the result?* If there is no response, you give more explicit hints: Can you check the result by *applying* it? Can you check it by applying it to some *particular case you already know*? Eventually, with more or less collaboration from the part of your class, you get down to various known cases. If $R = r$, you obtain a first noteworthy particular case:

$B_1$. *The area of the lateral surface of a cylinder is* $2\pi rh$.

Of course, $h$ stands for the altitude of the cylinder and $r$ for the radius of its base. We call $B_1$ this consequence of $A$ for future reference. The consequence $B_1$ has been treated already in your class and so it serves as a confirmation of $A$.

You obtain another particular case of $A$ in setting $r = 0$ which yields:

$B_2$. *The area of the lateral surface of a cone is* $\pi R\sqrt{R^2 + h^2}$.

Here $h$ denotes the altitude of the cone and $R$ the radius of its base. Also this consequence $B_2$ of $A$ was known before and serves as a further confirmation of $A$.

There is a less obvious but interesting particular case corresponding to $h = 0$:

$B_3$. *The area of the annulus between two concentric circles with radii $R$ and $r$ is* $\pi R^2 - \pi r^2$.

This consequence $B_3$ of $A$ is clear from plane geometry and yields still another confirmation of $A$.

different from the pattern that we have just stated, but rather a complementary form of it:

$$\begin{array}{c} A \text{ implies } B_{n+1} \\ B_{n+1} \text{ is very similar to the formerly verified} \\ \text{consequences } B_1, B_2, \ldots B_n \text{ of } A \\ B_{n+1} \text{ true} \\ \hline A \text{ just a little more credible} \end{array}$$

*The verification of a new consequence counts more or less according as the new consequence differs more or less from the formerly verified consequences.*

**3. Verification of an improbable consequence.** In a little known short note[5] Euler considers, for positive values of the parameter $n$, the series

$$(1) \quad 1 - \frac{x^2}{n(n+1)} + \frac{x^4}{n(n+1)(n+2)(n+3)} - \frac{x^6}{n\ldots(n+5)} + \cdots$$

which converges for all values of $x$. He observes the sum of the series and its zeros for $n = 1, 2, 3, 4$.

| | |
|---|---|
| $n = 1$: sum $\cos x$, | zeros $\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$ |
| $n = 2$: sum $(\sin x)/x$, | zeros $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ |
| $n = 3$: sum $2(1 - \cos x)/x^2$, | zeros $\pm 2\pi, \pm 4\pi, \pm 6\pi, \ldots$ |
| $n = 4$: sum $6(x - \sin x)/x^3$, | no real zeros. |

Euler observes a difference: in the first three cases all the zeros are real, in the last case none of the zeros is real. Euler notices a more subtle difference between the first two cases and the third case: for $n = 1$ and $n = 2$, the distance between two consecutive zeros is $\pi$ (provided that we disregard the zeros next to the origin in the case $n = 2$) but for $n = 3$ the distance between consecutive zeros is $2\pi$ (with a similar proviso). This leads him to a striking observation: in the case $n = 3$ all the zeros are double zeros. "Yet we know from Analysis," says Euler, "that two roots of an equation always coincide in the transition from real to imaginary roots. Thus we may understand why all the zeros suddenly become imaginary when we take for $n$ a value exceeding 3." On the basis of these observations he states a surprising conjecture: the function defined by the *series* (1) *has only real zeros, and an infinity of them, when* $0 < n \leq 3$, *but has no real zero at all when* $n > 3$. In this statement he regards $n$ as a continuously varying parameter.

In Euler's time questions about the reality of the zeros of transcendental equations were absolutely new, and we must confess that even today we possess no systematic method to decide such questions. (For instance, we

[5] *Opera Omnia*, ser. 1, vol. 16, sect. 1, p. 241–265.

cannot prove or disprove Riemann's famous hypothesis.) Therefore, Euler's conjecture appears extremely bold. I think that the courage and clearness with which he states his conjecture are admirable.

Yet Euler's admirable performance is understandable to a certain extent. Other experts perform similar feats in dealing with other subjects, and each of us performs something similar in everyday life. In fact, Euler *guessed the whole from a few scattered details.* Quite similarly, an archaeologist may reconstitute with reasonable certainty a whole inscription from a few scattered letters on a worn-out stone. A paleontologist may describe reliably the whole animal after having examined a few of its petrified bones. When a person whom you know very well starts talking in a certain way, you may predict after a few words the whole story he is going to tell you. Quite similarly, Euler guessed the whole story, the whole mathematical situation, from a few clearly recognized points.

It is still remarkable that he guessed it from so few points, by considering just four cases, $n = 1, 2, 3, 4$. We should not forget, however, that circumstantial evidence may be very strong. A defendant is accused of having blown up the yacht of his girl friend's father, and the prosecution produces a receipt signed by the defendant acknowledging the purchase of such and such an amount of dynamite. Such evidence strengthens the prosecution's case immensely. Why? Because the purchase of dynamite by an ordinary citizen is a very unusual event in itself, but such a purchase is completely understandable if the purchaser intends to blow up something or somebody. Please observe that this court case is very similar to the case $n = 3$ of Euler's series. That all roots of an equation written at random turn out to be double roots is a very unusual event in itself. Yet it is completely understandable that in the transition from two real roots to two imaginary roots a double root appears. The case $n = 3$ is the strongest piece of circumstantial evidence produced by Euler and we can perceive herein a general pattern of plausible inference:

$$\frac{\begin{array}{c} A \text{ implies } B \\ B \text{ very improbable in itself} \\ B \text{ true} \end{array}}{A \text{ very much more credible}}$$

Also this pattern appears as a modification or a sophistication of the fundamental inductive pattern (sect. 1). Let us add, without specific illustration for the moment, the complementary pattern which explains the same idea from the reverse side:

$$\frac{\begin{array}{c} A \text{ implies } B \\ B \text{ quite probable in itself} \\ B \text{ true} \end{array}}{A \text{ just a little more credible}}$$

The first statement is the isoperimetric theorem, the second a celebrated conjecture of Lord Rayleigh. Our tables yield sound inductive evidence for both statements but, of course, no proof.

The situation has changed since we considered these tables in sect. 10.1 and 10.4. In the meantime we have seen a proof for the isoperimetric theorem (sect. 10.6–10.8, ex. 10.1–10.15). The geometrical minimum property of the circle, inductively supported by Table II, has been proved. It is natural to expect that the analogous physical minimum property of the circle, inductively supported by Table III, will also turn out to be true. In expecting this we follow an important pattern of plausible inference:

$$\frac{\begin{array}{c} A \text{ analogous to } B \\ B \text{ true} \end{array}}{A \text{ more credible}}$$

*A conjecture becomes more credible when an analogous conjecture turns out to be true.*

The application of this pattern to the situation discussed seems sensible. Yet there are further promising indications in this situation.

**5. Deepening the analogy.** The Tables II and III, side by side, seem to offer further suggestions. The ten figures considered do not appear in exactly the same sequence in both tables. There is something peculiar about this sequence. The arrangement in Table II appears not very different from that in Table III, but this is not the main point. The tables contain various kinds of figures: rectangles, triangles, sectors. How are the *figures of the same kind* arranged? How would a shorter table look listing only figures of one kind? The tables contain a few regular figures: the equilateral triangle, the square, and, let us not forget it, the circle. How are the regular figures arranged? Could we compare somehow figures of different kinds, for instance, triangles and sectors? Could we broaden the inductive basis by adding further figures to our tables? (In this we are much restricted. It is not difficult to compute areas and perimeters, but the principal frequency is difficult to handle and its explicit expression is known in very few cases only.) Eventually we obtain Table IV.

Table IV exhibits a remarkable parallelism between these two quantities depending on the shape of a variable plane figure: the perimeter and the principal frequency. (We should not forget that the area of the variable figure is fixed, = 1.) If we know the perimeter, we are by no means able to compute the principal frequency or vice versa. Yet, judging from Table IV, we should think that, in many simple cases, these two quantities *vary in the same direction.* Consider the two columns of numerical data in this table and pass from any row to the next row: if there is an increase in one of the columns, there is a corresponding increase in the other, and if there is a decrease in one of the columns, there is a corresponding decrease in the other.

Table IV

Perimeters and principal frequencies of figures of equal area

| Figure | Perimeter | Pr. frequency |
|---|---|---|
| Rectangles: | | |
| 1 : 1 (square) | 4.00 | 4.443 |
| 3 : 2 | 4.08 | 4.624 |
| 2 : 1 | 4.24 | 4.967 |
| 3 : 1 | 4.64 | 5.736 |
| Triangles: | | |
| 60° 60° 60° | 4.56 | 4.774 |
| 45° 45° 90° | 4.84 | 4.967 |
| 30° 60° 90° | 5.08 | 5.157 |
| Sectors: | | |
| 180° (semicircle) | 4.10 | 4.803 |
| 90° (quadrant) | 4.03 | 4.551 |
| 60° (sextant) | 4.21 | 4.616 |
| 45° | 4.44 | 4.755 |
| 36° | 4.68 | 4.916 |
| 30° | 4.93 | 5.084 |
| Regular figures: | | |
| circle | 3.55 | 4.261 |
| square | 4.00 | 4.443 |
| equilateral triangle | 4.56 | 4.774 |
| Triangles versus sectors: | | |
| tr. 60° 60° 60° | 4.56 | 4.774 |
| sector 60° | 4.21 | 4.616 |
| tr. 45° 45° 90° | 4.84 | 4.967 |
| sector 45° | 4.44 | 4.755 |
| tr. 30° 60° 90° | 5.08 | 5.157 |
| sector 30° | 4.93 | 5.084 |

Let us focus our attention on the rectangles. If the ratio of the length to the width increases from 1 to $\infty$, so that the shape varies from a square to an infinitely elongated rectangle, both the perimeter and the principal frequency seem to increase steadily. The square which, being a regular figure, is "nearest" to the circle among all quadrilaterals, has the minimum perimeter and also the minimum principal frequency. Of the three triangles listed, the equilateral triangle which, being a regular figure, is "nearest" to the circle among all triangles has the minimum perimeter and also the minimum principal frequency. The behavior of the sectors is more complex. As the angle of the sector varies from 180° to 0°, the perimeter first decreases, attains a minimum, and then increases; and the principal frequency varies in the same manner. Let us now look at the regular figures. The equilateral triangle has 3 axes of symmetry, the square has 4 such axes, and the circle an infinity.

As far as we can see from Table IV, both the perimeter and the principal frequency seem to decrease as the number of the axes of symmetry increases. In the last section of Table IV we matched each triangle against the sector whose angle is equal to the least angle of the triangle. In all three cases, the sector turned out to be "more circular," having the shorter perimeter and the lower principal frequency.

What we definitely know about these regularities goes, of course, only as far as Table IV goes. That these regularities hold beyond the limits of the experimental material collected is suggested and rendered plausible by Table IV, but is by no means proved. And so Table IV led us to several new conjectures which are similar to Rayleigh's conjecture although, of course, of much more limited scope.

How does Table IV influence our confidence in Rayleigh's conjecture? Can we find in Table IV any reasonable ground for it that we did not notice before in discussing the Tables II and III?

We certainly can. First of all, the Table IV contains a few more particular cases in which Rayleigh's conjecture is verified (the 30° 60° 90° triangle, the sectors with opening 45°, 36°, and 30°). Yet there is more than that. The analogy between the isoperimetric theorem and Rayleigh's conjecture has been considerably deepened; the facts listed in Table IV add several new aspects to this analogy. Now it seems to be reasonable to consider a conclusion from analogy as becoming stronger if the analogy itself, on which the conclusion is based, becomes stronger. And so Table IV considerably strengthens Rayleigh's case.

**6. Shaded analogical inference.** Yet there is still something more. As we have observed, Table IV suggests several conjectures which are analogous to (but of more limited scope than) Rayleigh's conjecture. Table IV suggests these conjectures and lends them some plausibility too. Yet this circumstance quite reasonably raises somewhat the plausibility of Rayleigh's original conjecture. If you think so too, you think according to the following pattern:

$$\begin{array}{c} A \text{ analogous to } B \\ B \text{ more credible} \\ \hline A \text{ somewhat more credible} \end{array}$$

*A conjecture becomes somewhat more credible when an analogous conjecture becomes more credible.* This is a weakened or *shaded* form of the pattern formulated in sect. 4.

## EXAMPLES AND COMMENTS ON CHAPTER XII

**1.** Table I, exhibiting some inductive evidence for Euler's conjecture mentioned in sect. 1, is very similar to the table in sect. 1.3, or to Tables I,

**6.** Set

$$a^n + b^n + c^n = s_n$$

for $n = 1, 2, 3, \ldots$ and define $p$, $q$, and $r$ by the identity in $x$

$$(x - a)(x - b)(x - c) = x^3 - px^2 + qx - r$$

so that

$$p = a + b + c,$$

$$q = ab + ac + bc,$$

$$r = abc.$$

(In the usual terminology, $p$, $q$, and $r$ are the "elementary symmetric functions" of $a$, $b$, and $c$, and $s_n$ a "sum of like powers.") Obviously, $p = s_1$. It is asserted that, for arbitrary values of $a$, $b$, and $c$,

$$q = \frac{2s_1^5 - 5s_1^2s_3 + 3s_5}{5(s_1^3 - s_3)},$$

$$r = \frac{s_1^6 - 5s_1^3s_3 - 5s_3^2 + 9s_1s_5}{15(s_1^3 - s_3)}$$

provided that the denominator does not vanish. Check these formulas in the particular case $a = 1$, $b = 2$, $c = 3$ and in three more cases displayed in the table:

| Case | $a$ | $b$ | $c$ |
|---|---|---|---|
| (1) | 1 | 2 | 3 |
| (2) | 1 | 2 | −3 |
| (3) | 1 | 2 | 0 |
| (4) | 1 | 2 | −2 |

Devise further checks. Especially, try to generalize the cases (2), (3), and (4).

**7.** Let $A$, $B_1$, $B_2$, $B_3$, and $B_4$ have the meaning given them in sect. 2. Does the verification of $B_4$, coming after that of $B_1$, $B_2$, and $B_3$, supply additional inductive evidence for $A$?

**8.** Let us recall Euler's "Most Extraordinary Law" and the meaning of the abbreviations $T$, $C_1$, $C_2$, $C_3$, $\ldots$ $C_1^*$, $C_2^*$, $C_3^*$, $\ldots$ explained in sect. 6.3. Euler supported the theorem $T$, when he was not yet able to prove it, inductively, by verifying its consequences $C_1, C_2, C_3, \ldots C_{20}$. (He went even further, perhaps.) Then he discovered that also $C_1^*$, $C_2^*$, $C_3^*$, $\ldots$ are consequences of $T$, and verified $C_1^*, C_2^*, \ldots C_{20}^*, C_{101}^*, C_{301}^*$. Thanks to these new verifications Euler's confidence was, presumably, much strengthened: but was it justifiably strengthened? [Closer attention to detail is needed here than in ex. 2.]

**9.** We return to Euler's conjecture discussed in sect. 1; for the sake of brevity, we call it the "conjecture $E$." Let us note concisely the meaning of this abbreviation,

$$E: \qquad 8n + 3 = x^2 + 2p.$$

The idea that led Euler to his conjecture $E$ deserves mention. Euler devoted much of his work to those celebrated propositions of Number Theory that Fermat has stated without proof. One of these (we call it the "conjecture $F$") says that any integer is the sum of three trigonal numbers. Let us note concisely the meaning of this abbreviation,

$$F: \qquad n = \frac{x(x-1)}{2} + \frac{y(y-1)}{2} + \frac{z(z-1)}{2}.$$

Euler observed that *if* his conjecture $E$ were true, Fermat's conjecture $F$ would easily follow. That is, Euler satisfied himself that $E$ implies $F$. (For details, see the next ex. 10.) Bent on proving Fermat's conjecture $F$, Euler naturally desired that his conjecture $E$ should be true. Is this mere wishful thinking? I do not think so; the relations considered yield some weak but not unreasonable ground for believing Euler's conjecture $E$ according to the following scheme:

$$\begin{array}{c} E \text{ implies } F \\ F \text{ credible} \\ \hline E \text{ somewhat more credible} \end{array}$$

Here is another pattern of plausible inference. The reader should compare it with the fundamental inductive pattern.

**10.** In proving that $E$ implies $F$ (in the notation of the foregoing ex. 9), Euler used a deeper result which he proved previously: a prime number of the form $4n + 1$ is a sum of two squares. (This was discussed inductively in ex. 4.4.) Taking this for granted, prove that actually $E$ implies $F$.

**11.** After having conceived his conjecture discussed in sect. 3, Euler tested it by computing the first zeros of his series for a few values of $n$. (By a "first zero" we mean a zero the absolute value of which is a minimum. If $x$ is a first zero of the series in question, also $-x$ is a zero, and $x$ and $-x$ are "first zeros." Therefore, $x$ is real if, and only if, $x^2$ is positive.) Of course, Euler had to compute these zeros approximately. A method (Daniel Bernoulli's method) which he frequently used for such a purpose yielded the following sequences of approximate values for the first zero $x$ in the cases $n = 1/2, 1/3, 1/4$.

| $n = 1/2$ | $n = 1/3$ | $n = 1/4$ |
|---|---|---|
| $4x^2 \sim 3.000$ | $9x^2 \sim 4.0000$ | $16x^2 \sim 5.0000$ |
| 3.281 | 4.2424 | 5.2232 |
| 3.291 | 4.2528 | 5.2302 |
| 3.304 | 4.2532 | 5.2304 |

(2) The numerical data quoted in ex. 11 led the author to suspect the general theorem proved l.c. This is a small but concrete example of the use of the inductive method in mathematical research.

**13.** In ch. IV we investigated inductively the sum of four odd squares; see sect. 4.3–4.6, Table I. Later we tackled the analogous problems involving four arbitrary squares and eight squares; see ex. 4.10–4.23 and Tables II and III. The former investigation certainly helped us to recognize the law in the latter cases. Should our confidence in the result of the latter investigation also be enhanced by the result of the former?

**14.** *Inductive conclusion from fruitless efforts.* Construct, by ruler and compasses, the side of a square equal in area to a circle of given radius. This is the strict formulation of the famous problem of the quadrature of the circle, conceived by the Greeks. It was not forgotten in the Middle Ages, although we cannot believe that many people then understood its strict formulation; Dante refers to it at the theological culmination of the *Divina Commedia*, toward the end of the concluding Canto. The problem was about two thousand years old as the French Academy resolved that manuscripts purporting to square the circle will not be examined. Was the Academy narrow-minded? I do not think so; after the fruitless efforts of thousands of people in thousands of years there was some ground to suspect that the problem is insoluble.

You are moved to give up a task that withstands your repeated efforts. You desist only after many and great efforts if you are stubborn or deeply concerned. You desist after a few cursory trials if you are easy going or not seriously concerned. Yet in any case there is a sort of inductive conclusion. The conjecture under consideration is:

*A*. It is impossible to do this task.

You observe:

*B*. Even I cannot do this task.

This, in itself, is *very* unlikely indeed. Yet certainly

$$A \text{ implies } B$$

and so your observation of *B* renders *A* more credible, by the fundamental inductive pattern.

The impossibility of squaring the circle, strictly formulated, was proved in 1882, by Lindemann, after the basic work of Hermite. There are other similar problems dating from the Greeks (the Trisection of an Angle and the Duplication of the Cube) that, after the accumulated evidence of fruitless efforts, have been ultimately proved insoluble. After fruitless efforts to construct a "perpetuum mobile" the physicists formulated the "principle of the impossibility of a perpetual motion," and this principle turned out remarkably fruitful.

# XIII

## FURTHER PATTERNS AND FIRST LINKS

> *When we have intuitively understood some simple propositions . . . it is useful to go through them with a continuous, uninterrupted motion of thought, to meditate upon their mutual relations, and to conceive distinctly several of them, as many as possible, simultaneously. In this manner our knowledge will grow more certain, and the capacity of the mind will notably increase.*—DESCARTES[1]

**1. Examining a consequence.** We consider a situation which frequently occurs in mathematical research. We wish to decide whether a clearly formulated mathematical proposition $A$ is true or not. We have, perhaps, some intuitive confidence in the truth of $A$, but that is not enough: we wish to prove $A$ or disprove it. We work at this problem, but without decisive success. After a while we notice a consequence $B$ of $A$. This $B$ is a clearly formulated mathematical proposition of which we know that it follows from $A$:

$$A \text{ implies } B.$$

Yet we do not know whether $B$ is true or not. Now it seems that $B$ is more accessible than $A$; for some reason or other we have the impression that we shall have better success with $B$ than we had with $A$. Therefore, we switch to examining $B$. We work to answer the question: is $B$ true or false? Finally we succeed in answering it. *How does this answer influence our confidence in $A$?*

That depends on the answer. If we find that $B$, this consequence of $A$, is false, we can infer with certainty that $A$ must also be false. Yet if we find that $B$ is true, there is no demonstrative inference: although its consequence $B$ turned out to be true, $A$ itself could be false. Yet there is a heuristic inference: since its consequence $B$ turned out to be true, $A$ itself seems to

[1] The eleventh of his Rules for the Direction of the Mind. See *Oeuvres*, edited by Adam and Tannery, vol. 10, 1908, p. 407.

deserve more confidence. According to the nature of our result concerning $B$, we follow a demonstrative or a heuristic pattern:

| *Demonstrative* | *Heuristic* |
|---|---|
| $A$ implies $B$ | $A$ implies $B$ |
| $B$ false | $B$ true |
| $A$ false | $A$ more credible |

We met these patterns already in sect. 12.1 where we called the heuristic pattern the fundamental inductive pattern. We shall meet with similar but different patterns in the following sections.

**2. Examining a possible ground.** We consider another situation that frequently occurs in mathematical research. We wish to decide whether the clearly formulated proposition $A$ is true or not, we wish to prove $A$ or disprove it. After some indecisive work we hit upon another proposition $B$ from which $A$ would follow. We do not know whether $B$ is true or not, but we have satisfied ourselves that

$$A \text{ is implied by } B.$$

Thus, if we could prove $B$, the desired $A$ would follow; $B$ is a possible ground for $A$. We may be tired of $A$, or $B$ may appear to us more promising than $A$; for some reason or other we switch to examining $B$. Our aim is now to prove or disprove $B$. Finally we succeed. How will our result concerning $B$ influence our confidence in $A$?

That depends on the nature of our result. If we find that $B$ is true, we can conclude that $A$ which is implied by $B$ (follows from $B$, is a consequence of $B$) is also true. Yet if we find that $B$ is false, there is no demonstrative conclusion: $A$ could still be true. But we have been obliged to discard a possible ground for $A$, we have one chance less to prove $A$, our hope to prove $A$ from $B$ has been wrecked: if there is any change at all in our confidence in $A$ in consequence of the disproof of $B$, it can only be a change for the worse. In short, according to the nature of our result concerning $B$, we follow a demonstrative or a heuristic pattern:

| *Demonstrative* | *Heuristic* |
|---|---|
| $A$ implied by $B$ | $A$ implied by $B$ |
| $B$ true | $B$ false |
| $A$ true | $A$ less credible |

Observe that the first premise is the same in both patterns. The second premises are diametrically opposite, and the conclusions are also opposite, although not quite as far apart.

(1) The term *proposition* may be taken in a more general meaning, but most of the time it will be sufficient and even advantageous to think of some clearly formulated mathematical proposition of which *for the moment we do not know whether it is true or not.* (A good example of a proposition for a more advanced reader is the celebrated "Riemann hypothesis": Riemann's $\xi$-function has only real zeros. We do not know, in spite of the efforts of many excellent mathematicians, whether this proposition is true or false.) We shall use capitals $A$, $B$, $C$, . . . to denote propositions.

(2) The *negation* of the proposition $A$ is a proposition that is true if, and only if, $A$ is false. We let non-$A$ stand for the negation of $A$.

(3) The two statements "$A$ is false" and "non-$A$ is true" amount exactly to the same. We can substitute one for the other in any context without changing the import, the truth, or the falsity, of the whole text. Two statements which can be so substituted for each other are termed *equivalent.* Thus, the statement "$A$ is false" is equivalent to the statement "non-$A$ is true." It will be convenient to write this in the abbreviated form:

"$A$ false" eq. "non-$A$ true."

It is also correct to say that

"$A$ true" eq. "non-$A$ false."

"non-$A$ true" eq. "$A$ false,"

"non-$A$ false" eq. "$A$ true."

(4) We say that the two propositions $A$ and $B$ are *incompatible* with each other, if both cannot be true. The proposition $A$ can be true or false, $B$ can be true or false; if we consider $A$ and $B$ jointly, four different cases can arise:

| | |
|---|---|
| $A$ true, $B$ true | $A$ true, $B$ false |
| $A$ false, $B$ true | $A$ false, $B$ false. |

If we say that $A$ is incompatible with $B$, we mean that the first of these four cases (in the north-west corner) is excluded. Incompatibility is always mutual. Therefore,

"$A$ incompatible with $B$" eq. "$B$ incompatible with $A$."

(5) We say that $A$ implies $B$ (or $B$ is implied by $A$, or $B$ follows from $A$, or $B$ is a consequence of $A$, etc.) if $A$ and non-$B$ are incompatible. Thus the concept of implication is characterized by the following equivalence:

"$A$ implies $B$" eq. "$A$ incompatible with non-$B$."

To know that $A$ implies $B$ is important. For the moment we do not know whether $A$ is true or not and we are in the same state of ignorance concerning $B$. If, however, it should turn out some day that $A$ is true, we shall know right away that non-$B$ must be false and so $B$ must be true.

We know that

"*A* incompatible with non-*B*" eq. "non-*B* incompatible with *A*."

We know also that

"non-*B* incompatible with *A*" eq. "non-*B* implies non-*A*."

From the chain of the last three equivalences we conclude:

"*A* implies *B*" eq. "non-*B* implies non-*A*."

This last equivalence is quite important in itself and it will be essential in the following consideration.

(6) The few points of formal logic discussed in this section enable us already to clarify the relation between the demonstrative patterns encountered in the three foregoing sections.

Let us start from the demonstrative pattern formulated in sect. 1 (the "modus tollens"):

*A* implies *B*
*B* false
―――――
*A* false

It is understood that this pattern is generally applicable. Let us apply it in substituting non-*A* for *A* and non-*B* for *B*. We obtain

non-*A* implies non-*B*
non-*B* false
―――――
non-*A* false

We have seen, however, in the foregoing that

"non-*A* implies non-*B*" eq. "*B* implies *A*"

"non-*B* false" eq. "*B* true"

"non-*A* false" eq. "*A* true"

Le. us substitute for the premises and the conclusion of the last considered pattern the three corresponding equivalent statements here displayed. Then we obtain:

*B* implies *A*
*B* true
―――――
*A* true

which is the demonstrative pattern of sect. 2, the "modus ponens."

We leave to the reader to derive similarly the demonstrative pattern of sect. 3 from that of sect. 1.

can be interpreted as confirmation or refutation. Such interpretation depends on some kind of plausible reasoning the difficulties of which in "physical situations" begin a step earlier than in "mathematical situations." We shall try to reduce this distinction to its simplest expression.

Suppose that we are investigating a mathematical conjecture by examining its consequences. Let *A* stand for the conjecture and *B* for one of its consequences, so that *A* implies *B*. We arrive at a final decision concerning *B*: we disprove *B* or we prove it and, accordingly, we face one or the other of the following two situations:

| | |
|---|---|
| *A* implies *B* | *A* implies *B* |
| *B* false | *B* true |
| ————— | ————— |

We shall call these "mathematical situations." We have considered them repeatedly and we know what reasonable inference we can draw from each.

Suppose now that we are investigating a physical conjecture *A* and that we test experimentally one of its consequences *B*. We cannot arrive at an absolute decision concerning *B*; our experiments may show, however, that *B*, or its contrary, is very hard to believe. Accordingly, we face one or the other of the following two situations:

| | |
|---|---|
| *A* implies *B* | *A* implies *B* |
| *B* scarcely credible | *B* almost certain |
| ————— | ————— |

We call these "physical situations." What inference is reasonable in these situations? (The empty space under the horizontal line that suggests the word "therefore" symbolizes the open question.)

In each of the four situations considered we have two data or *premises*. The first premise is the same in all four situations; all the difference between them hinges on the second premise. This second premise is on the level of pure formal logic in the "mathematical" situations, but on a much vaguer level in the "physical" situations. This difference seems to me essential; it may account for the additional difficulties of the physical situations.

Let us survey the four situations "with a continuous uninterrupted motion of thought," as Descartes liked to say (see the motto at the beginning of this chapter). Let us imagine that our confidence in *B* changes gradually, varies "continuously." We imagine that *B* becomes less credible, then still less credible, scarcely believable, and finally false. On the other hand, we imagine that *B* becomes more credible, then still more credible, practically certain, and finally true. If the *strength of our conclusion varies continually in the same direction as the strength of our confidence in B*, there is little doubt what our conclusion should be since the two extreme cases ($B$ false, $B$ true) are clear. In this manner we arrive at the following patterns:

$$\frac{A \text{ implies } B \qquad B \text{ less credible}}{A \text{ less credible}} \qquad\qquad \frac{A \text{ implies } B \qquad B \text{ more credible}}{A \text{ somewhat more credible}}$$

The word "somewhat" in the second pattern is inserted to remind us that the conclusion is, of course, weaker than in the fundamental inductive pattern. *Our confidence in a conjecture is influenced by our confidence in one of its consequences and varies in the same direction.* We shall call these patterns *shaded*; the first is a shaded demonstrative pattern, the second is the shaded version of the fundamental inductive pattern. The term "shaded" intends to indicate the weakening of the second premise: "less credible" instead of "false"; "more credible" instead of "true." We have already used this term in this meaning in sect. 12.6.

We obtained the shaded patterns just introduced from their extreme cases, the "modus tollens" and the fundamental inductive pattern discussed in sect. 1, by weakening the second premise. In the same manner we can obtain other shaded patterns from the patterns formulated in sects. 2 and 3. We state just one here (all are listed in the next section). The heuristic pattern of sect. 3 yields the following shaded pattern:

$$\frac{A \text{ incompatible with } B \qquad B \text{ less credible}}{A \text{ somewhat more credible}}$$

**7. A table.** In order to list concisely the patterns discussed in this chapter, it will be convenient to use some abbreviations. We write

$A \rightarrow B$ for $A$ implies $B$,

$A \leftarrow B$ for $A$ is implied by $B$,

$A \mid B$ for $A$ incompatible with $B$.

The symbols introduced are used by some authors writing on symbolic logic.[3] In this notation, the two formulas

$$A \rightarrow B, \qquad B \leftarrow A$$

are exactly equivalent and so are the formulas

$$A \mid B, \qquad B \mid A.$$

[3] D. Hilbert and W. Ackerman, *Grundzüge der theoretischen Logik.*

We shall also abbreviate "credible" as "cr." and "somewhat" as "s." See Table I.

Table I

| | (1) Demonstrative | (2) Shaded Demonstrative | (3) Shaded Inductive | (4) Inductive |
|---|---|---|---|---|
| 1. Examining a consequence | $A \to B$<br>$B$ false | $A \to B$<br>$B$ less cr. | $A \to B$<br>$B$ more cr. | $A \to B$<br>$B$ true |
| | $A$ false | $A$ less cr. | $A$ s. more cr. | $A$ more cr. |
| 2. Examining a possible ground | $A \leftarrow B$<br>$B$ true | $A \leftarrow B$<br>$B$ more cr. | $A \leftarrow B$<br>$B$ less cr. | $A \leftarrow B$<br>$B$ false |
| | $A$ true | $A$ more cr. | $A$ s. less cr. | $A$ less cr. |
| 3. Examining a conflicting conjecture | $A \mid B$<br>$B$ true | $A \mid B$<br>$B$ more cr. | $A \mid B$<br>$B$ less cr. | $A \mid B$<br>$B$ false |
| | $A$ false | $A$ less cr. | $A$ s. more cr. | $A$ more cr. |

**8. Combination of simple patterns.** The following situation can easily arise in mathematical research. We investigate a theorem $A$. This theorem $A$ is clearly formulated, but we do not know and we wish to find out whether it is true or false. After a while we hit upon a possible ground: we see that $A$ can be derived from another theorem $H$

$$A \text{ is implied by } H$$

and so we try to prove $H$. We do not succeed in proving $H$, but we notice that one of its consequences $B$ is true. The situation is:

$$\begin{array}{c} A \text{ implied by } H \\ B \text{ implied by } H \\ B \text{ true} \\ \hline \end{array}$$

Is there a reasonable inference concerning $A$ from these premises?

There is one, I think, and we can even obtain it by combining two of the patterns surveyed in the foregoing section. In fact, the fundamental inductive pattern yields:

$$\begin{array}{c} H \text{ implies } B \\ B \text{ true} \\ \hline H \text{ more credible} \end{array}$$

In obtaining this conclusion, we have used only two of our three premises. Let us combine the unused third premise with the conclusion just obtained;

There is, of course, the important difference that now we do not have $H$, we just hope for $H$. With this proviso, however, we can regard the two premises

$A$ implied by $H$

$B$ implied by $H$

as equivalent to one:

$A$ analogous to $B$.

In substituting this one premise for those two premises in the above compound pattern we arrive at the fundamental pattern of plausible inference first exhibited in sect. 12.4:

$$\frac{\begin{array}{c}A \text{ analogous to } B\\ B \text{ true}\end{array}}{A \text{ more credible}}$$

**10. Qualified inference.** We come back again to the fundamental inductive pattern. It is the first pattern that we introduced and it is the most conspicuous form of plausible reasoning. It is concerned with the verification of a consequence of a conjecture and the resulting change in our opinion. It says something about the direction of this change; such a verification can only increase our confidence in the conjecture. It says nothing about the amount of the change; the increase of confidence can be great or small. Indeed, it can be tremendously great or ridiculously small.

The aim of the present section is to clarify the circumstances on which such an important difference depends. We begin by recalling one of our examples (sect. 12.3).

A defendant is accused of having blown up the yacht of his girl friend's father and the prosecution produces a receipt signed by the defendant acknowledging the purchase of such and such an amount of dynamite.

The evidence against the defendant appears very strong. Why does it appear so? Let us insist on the general features of the case. Two statements play an essential role.

$A$: the defendant blew up that yacht.

$B$: the defendant acquired explosives.

At the beginning of the proceedings, the court has to consider $A$ as a conjecture. The prosecution works to render $A$ more credible to the jurors, the defense works to render it less credible.

At the beginning of the proceedings also $B$ has to be considered as a conjecture. Later, after the introduction of that receipt in court (the authenticity of the signature was not challenged by the defense) $B$ has to be considered as a proven fact.

Certain relations between $A$ and $B$, however, should be clear from the beginning.

$A$ without $B$ is impossible. If the defendant blew up the yacht, he had some explosives. He had to acquire these explosives somehow: by purchase, stealing, gift, inheritance or otherwise. That is

$A$ implies $B$.

$B$ without $A$ is not impossible, but must appear extremely unlikely from the outset. To buy dynamite is very unusual anyhow for an ordinary citizen. To buy dynamite without the intention of blowing up something or somebody would be nonsense. It was easy to suspect that the defendant had quite strong emotional and financial grounds for blowing up that yacht. It was difficult to suspect any purpose for the purchase of dynamite, except blowing up the yacht. And so, as we said, $B$ without $A$ appears extremely unlikely.

Let us nail down this important constituent of the situation: *the credibility of B, before the event, viewed under the assumption that A is not true.* We shall abbreviate this precise but long description as "the credibility of $B$ without $A$." Thus, we can say:

$B$ without $A$ is hardly credible.

Now, we can see the essential premises and the whole pattern of the plausible inference that impressed us with its cogency:

$A$ implies $B$
$B$ without $A$ hardly credible
$B$ true
---
$A$ very much more credible

For better understanding let us imagine that that important constituent of the situation, the credibility of $B$ without $A$, changes gradually, varies continuously between its extreme cases.

$A$ implies $B$. If, conversely, also $B$ implies $A$, so that $A$ and $B$ imply each other mutually, the credibility of $B$ without $A$ attains its minimum, is nil. In this case, if $B$ is true, also $A$ is true, so that the credibility of $A$ attains its maximum.

$A$ implies $B$. That is, $B$ is certain when $A$ is true. If the credibility of $B$ without $A$ approaches its maximum, $B$ is almost certain when $A$ is false. Therefore, $B$ is almost certain anyway. When an event happens that looks almost certain in advance, we do not get much new information and so we cannot draw surprising consequences. (The purchase of a loaf of bread, for instance, can hardly ever yield such a strong circumstantial evidence, as the purchase of dynamite.)

This expresses essentially the same thing as the complementary patterns formulated in sect. 12.2, but perhaps a little better. In fact, we may regard the explicit mention of analogy as an advantage.

**12. On rival conjectures.** If there are two different conjectures, *A* and *B*, aimed at explaining the same phenomenon, we regard them as opposed to each other even if they are not proved to be logically incompatible. These conjectures *A* and *B* may or may not be incompatible, but one of them tends to render the other superfluous. This is enough opposition, and we regard *A* and *B* as *rival* conjectures.

There are cases in which we treat rival conjectures almost *as if* they were incompatible. For example, we have two rival conjectures *A* and *B* but, in spite of some effort, we cannot think of a third conjecture explaining the same phenomenon; then each of the two conjectures *A* and *B* is the "unique obvious rival" of the other. A short schematic illustration may clarify the meaning of the term.

Let us say that *A* is the emission theory of light that goes back to Newton and that *B* is the wave theory of light that originated with Huyghens. Let us also imagine that we discuss these matters in the time after Newton and Huyghens, but before Young and Fresnel when, in fact, much inconclusive discussion of these theories took place. Nobody showed, or pretended to show, that these two theories are logically incompatible, and still less that they are the only logically possible alternatives; but there were no other theories of light prominently in view, although the physicists had ample opportunity to invent such theories: each theory was the unique obvious rival of the other. And so any argument that seemed to speak against one of the two rival theories was readily interpreted as speaking for the other.

In general, the relation between rival conjectures is similar to the relation between rivals in any other kind of competition. If you compete for a prize, the weakening of the position of any of your rivals means some strengthening of your position. You do not gain much by a slight setback to one of your many obscure rivals. You gain more if such setback occurs to a dangerous rival. You gain still more if your most dangerous rival drops out of the race. If you have a unique obvious rival, any weakening or strengthening of his position influences your position appreciably. And something similar happens between competing conjectures. There is a pattern of plausible reasoning which we attempt to make somewhat more explicit in Table II.

Table II

| | |
|---|---|
| *A* incompatible with *B*<br>*B* false | *A* incompatible with *B*<br>*B* less credible |
| *A* more credible | *A* somewhat more credible |
| *A* rival of *B*<br>*B* false | *A* rival of *B*<br>*B* less credible |
| *A* a little more credible | *A* very little more credible |

The disposition of Table II is almost self-explanatory. This Table contains four patterns arranged in two rows and two columns. The first row contains two patterns already considered; see sect. 3, the end of sect. 6, and the last row of Table I. In passing from the first row to the second row, we weaken the first premise; in fact we substitute for a clear relation of formal logic between *A* and *B* a somewhat diffuse relation which, however, makes some sense in practice. This weakening of the first premise renders the conclusion correspondingly weaker, as the verbal expression attempts to convey. In passing from the first column to the second column we weaken the second premise, which renders the conclusion correspondingly weaker. The pattern in the southeast corner has no premise that would make sense in demonstrative logic and its conclusion is the weakest.

It is important to emphasize that the verbal expressions used are slightly misleading. In fact, the specifications added to "credible" ("somewhat," "a little," "very little") should not convey any *absolute*, only a *relative*, degree of credibility. They indicate only the change in strength as we pass from one row to the other, or from one column to the other. Even the weakest of the four patterns may yield a weighty conclusion if the conviction that the conjecture *A* has no other dangerous rival than *B* is strong enough. In fact, this pattern will play some role in the next chapter.

**13. On judicial proof.** The reasoning by which a tribunal arrives at its decisions may be compared with the inductive reasoning by which the naturalist supports his generalizations. Such comparisons have been already offered and debated by authorities on legal procedure.[5] Let us begin the discussion of this interesting point by considering an example.

(1) The manager of a popular restaurant that is kept open to late hours returned to his suburban home, as usual, well after midnight. As he left his car to open the door of his garage, he was held up and robbed by two masked individuals. The police, searching the premises, found a dark grey rag in the front yard of the victim; the rag might have been used by one of the holdup men for covering his face. The police questioned several persons in the nearby town. One of the men questioned had an overcoat with a big hole in the lining, but otherwise in good condition. The rag found in the front yard of the victim of the holdup was of the same material as the lining and fitted into the hole exactly. The man with the overcoat was arrested and charged with participation in the holdup.

(2) Many of us may feel that such a charge was amply justified by the related circumstances. But why? What is the underlying idea?

The charge is not a statement of facts, but the expression of a suspicion, of a *conjecture*:

*A*. The man with the overcoat participated in the holdup.

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[5] J. H. Wigmore, *The principles of judicial proof*, Boston, 1913; cf. p. 9–12, 15–17.

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