G. Polya

MATHEMATICS AND PLAUSIBLE REASONING

VOLUME I

INDUCTION AND ANALOGY IN MATHEMATICS

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OF MATHEMATICS AND PLAUSIBLE REASONING

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Induction and Analogy in Mathematics

I

INDUCTION

It will seem not a little paradoxical to ascribe a great importance to observations even in that part of the mathematical sciences which is usually called Pure Mathematics, since the current opinion is that observations are restricted to physical objects that make impression on the senses. As we must refer the numbers to the pure intellect alone, we can hardly understand how observations and quasi-experiments can be of use in investigating the nature of the numbers. Yet, in fact, as I shall show here with very good reasons, the properties of the numbers known today have been mostly discovered by observation, and discovered long before their truth has been confirmed by rigid demonstrations. There are even many properties of the numbers with which we are well acquainted, but which we are not yet able to prove; only observations have led us to their knowledge. Hence we see that in the theory of numbers, which is still very imperfect, we can place our highest hopes in observations; they will lead us continually to new properties which we shall endeavor to prove afterwards. The kind of knowledge which is supported only by observations and is not yet proved must be carefully distinguished from the truth; it is gained by induction, as we usually say. Yet we have seen cases in which mere induction led to error. Therefore, we should take great care not to accept as true such properties of the numbers which we have discovered by observation and which are supported by induction alone. Indeed, we should use such a discovery as an opportunity to investigate more exactly the properties discovered and to prove or disprove them; in both cases we may learn something useful.—EULER[1]

1. Experience and belief. Experience modifies human beliefs. We learn from experience or, rather, we ought to learn from experience. To make the best possible use of experience is one of the great human tasks and to work for this task is the proper vocation of scientists.

A scientist deserving this name endeavors to extract the most correct belief from a given experience and to gather the most appropriate experience in order to establish the correct belief regarding a given question. The

[1] Euler, *Opera Omnia*, ser. 1, vol. 2, p. 459, Specimen de usu observationum in mathesi pura.

scientist's procedure to deal with experience is usually called *induction.* Particularly clear examples of the inductive procedure can be found in mathematical research. We start discussing a simple example in the next section.

2. Suggestive contacts. Induction often begins with observation. A naturalist may observe bird life, a crystallographer the shapes of crystals. A mathematician, interested in the Theory of Numbers, observes the properties of the integers 1, 2, 3, 4, 5, ...

If you wish to observe bird life with some chance of obtaining interesting results, you should be somewhat familiar with birds, interested in birds, perhaps you should even like birds. Similarly, if you wish to observe the numbers, you should be interested in, and somewhat familiar with, them. You should distinguish even and odd numbers, you should know the squares 1, 4, 9, 16, 25, . . . and the primes 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, ... (It is better to keep 1 apart as "unity" and not to classify it as a prime.) Even with so modest a knowledge you may be able to observe something interesting.

By some chance, you come across the relations

$$3 + 7 = 10, \quad 3 + 17 = 20, \quad 13 + 17 = 30$$

and notice some resemblance between them. It strikes you that the numbers 3, 7, 13, and 17 are odd primes. The sum of two odd primes is necessarily an even number; in fact, 10, 20, and 30 are even. What about the *other* even numbers? Do they behave similarly? The first even number which is a sum of two odd primes is, of course,

$$6 = 3 + 3.$$

Looking beyond 6, we find that

$$\begin{aligned} 8 &= 3 + 5 \\ 10 &= 3 + 7 = 5 + 5 \\ 12 &= 5 + 7 \\ 14 &= 3 + 11 = 7 + 7 \\ 16 &= 3 + 13 = 5 + 11. \end{aligned}$$

Will it go on like this forever? At any rate, the particular cases observed suggest a general statement: *Any even number greater than* 4 *is the sum of two odd primes.* Reflecting upon the exceptional cases, 2 and 4, which cannot be split into a sum of two odd primes, we may prefer the following more sophisticated statement: *Any even number that is neither a prime nor the square of a prime, is the sum of two odd primes.*

We arrived so at formulating a *conjecture.* We found this conjecture by *induction.* That is, it was suggested by observation, indicated by particular instances.

These indications are rather flimsy; we have only very weak grounds to believe in our conjecture. We may find, however, some consolation in the fact that the mathematician who discovered this conjecture a little more than two hundred years ago, Goldbach, did not possess much stronger grounds for it.

Is Goldbach's conjecture true? Nobody can answer this question today. In spite of the great effort spent on it by some great mathematicians, Goldbach's conjecture is today, as it was in the days of Euler, one of those "many properties of the numbers with which we are well acquainted, but which we are not yet able to prove" or disprove.

Now, let us look back and try to perceive such steps in the foregoing reasoning as might be typical of the inductive procedure.

First, we *noticed some similarity.* We recognized that 3, 7, 13, and 17 are primes, 10, 20, and 30 even numbers, and that the three equations $3 + 7 = 10$, $3 + 17 = 20$, $13 + 17 = 30$ are *analogous* to each other.

Then there was a step of *generalization.* From the examples 3, 7, 13, and 17 we passed to all odd primes, from 10, 20, and 30 to all even numbers, and then on to a possibly general relation

$$\text{even number} = \text{prime} + \text{prime}.$$

We arrived so at a clearly formulated general statement, which, however, is merely a conjecture, merely *tentative.* That is, the statement is by no means proved, it cannot have any pretension to be true, it is merely an attempt to get at the truth.

This conjecture has, however, some *suggestive points of contact* with experience, with "the facts," with "reality." It is true for the particular even numbers 10, 20, 30, also for 6, 8, 12, 14, 16.

With these remarks, we outlined roughly a first stage of the inductive process.

3. Supporting contacts. You should not put too much trust in any unproved conjecture, even if it has been propounded by a great authority, even if it has been propounded by yourself. You should try to prove it or to disprove it; you should *test* it.

We test Goldbach's conjecture if we examine some new even number and decide whether it is or is not a sum of two odd primes. Let us examine, for instance, the number 60. Let us perform a "quasi-experiment," as Euler expressed himself. The number 60 is even, but is it the sum of two primes? Is it true that

$$60 = 3 + \text{prime?}$$

No, 57 is not a prime. Is

$$60 = 5 + \text{prime?}$$

The answer is again "No": 55 is not a prime. If it goes on in this way, the conjecture will be exploded. Yet the next trial yields

$$60 = 7 + 53$$

and 53 is a prime. The conjecture has been verified in one more case.

The contrary outcome would have settled the fate of Goldbach's conjecture once and for all. If, trying all primes under a given even number, such as 60, you never arrive at a decomposition into a sum of two primes, you thereby explode the conjecture irrevocably. Having verified the conjecture in the case of the even number 60, you cannot reach such a definite conclusion. You certainly do not prove the theorem by a single verification. It is natural, however, to interpret such a verification as a *favorable sign*, speaking for the conjecture, rendering it *more credible*, although, of course, it is left to your personal judgement how much weight you attach to this favorable sign.

Let us return, for a moment, to the number 60. After having tried the primes 3, 5, and 7, we can try the remaining primes under 30. (Obviously, it is unnecessary to go further than 30 which equals 60/2, since one of the two primes, the sum of which should be 60, must be less than 30.) We obtain so all the decompositions of 60 into a sum of two primes:

$$60 = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 29 + 31.$$

We can proceed systematically and examine the even numbers one after the other, as we have just examined the even number 60. We can *tabulate* the results as follows:

$$6 = 3 + 3$$
$$8 = 3 + 5$$
$$10 = 3 + 7 = 5 + 5$$
$$12 = 5 + 7$$
$$14 = 3 + 11 = 7 + 7$$
$$16 = 3 + 13 = 5 + 11$$
$$18 = 5 + 13 = 7 + 11$$
$$20 = 3 + 17 = 7 + 13$$
$$22 = 3 + 19 = 5 + 17 = 11 + 11$$
$$24 = 5 + 19 = 7 + 17 = 11 + 13$$
$$26 = 3 + 23 = 7 + 19 = 13 + 13$$
$$28 = 5 + 23 = 11 + 17$$
$$30 = 7 + 23 = 11 + 19 = 13 + 17.$$

4. Observe the values of the consecutive sums

$$1, \quad 1+8, \quad 1+8+27, \quad 1+8+27+64, \quad \ldots .$$

Is there a simple rule?

5. The three sides of a triangle are of lengths l, m, and n, respectively. The numbers l, m, and n are positive integers, $l \leqq m \leqq n$. Find the number of different triangles of the described kind for a given n. [Take $n = 1, 2, 3, 4, 5, \ldots$.] Find a general law governing the dependence of the number of triangles on n.

6. The first three terms of the sequence 5, 15, 25, . . . (numbers ending in 5) are divisible by 5. Are also the following terms divisible by 5?

The first three terms of the sequence 3, 13, 23, . . . (numbers ending in 3) are prime numbers. Are also the following terms prime numbers?

7. By formal computation we find

$$(1 + 1!x + 2!x^2 + 3!x^3 + 4!x^4 + 5!x^5 + 6!x^6 + \ldots)^{-1}$$
$$= 1 - x - x^2 - 3x^3 - 13x^4 - 71x^5 - 461x^6 \ldots .$$

This suggests two conjectures about the following coefficients of the right hand power series: (1) they are all negative; (2) they are all primes. Are these two conjectures equally trustworthy?

8. Set

$$\left(1 - \frac{x}{1} + \frac{x^2}{2} - \frac{x^3}{3} + \ldots\right)^{-1} = A_0 + \frac{A_1 x}{1!} + \frac{A_2 x^2}{2!} + \ldots .$$

We find that for

$n =$	0	1	2	3	4	5	6	7	8	9
$A_n =$	1	1	1	2	4	14	38	216	600	6240.

State a conjecture.

9. The great French mathematician Fermat considered the sequence

$$5, 17, 257, 65537, \ldots,$$

the general term of which is $2^{2^n} + 1$. He observed that the first four terms (here given), corresponding to $n = 1, 2, 3$, and 4, are primes. He conjectured that the following terms are also primes. Although he did not prove it, he felt so sure of his conjecture that he challenged Wallis and other English mathematicians to prove it. Yet Euler found that the very next term, $2^{32} + 1$, corresponding to $n = 5$, is not a prime: it is divisible by 641.[2] See the passage of Euler at the head of this chapter: "Yet we have seen cases in which mere induction led to error."

10. In verifying Goldbach's conjecture for $2n = 60$ we tried successively the primes p under $n = 30$. We could have also tried, however, the primes p' between $n = 30$ and $2n = 60$. Which procedure is likely to be more advantageous for greater n?

11. In a dictionary, you will find among the explanations for the words "induction," "experiment," and "observation" sentences like the following.

"Induction is inferring a general law from particular instances, or a production of facts to prove a general statement."

"Experiment is a procedure for testing hypotheses."

"Observation is an accurate watching and noting of phenomena as they occur in nature with regard to cause and effect or mutual relations."

Do these descriptions apply to our example discussed in sect. 2 and 3?

12. *Yes and No.* The mathematician as the naturalist, in testing some consequence of a conjectural general law by a new observation, addresses a question to Nature: "I suspect that this law is true. Is it true?" If the consequence is clearly refuted, the law cannot be true. If the consequence is clearly verified, there is some indication that the law may be true. Nature may answer Yes or No, but it whispers one answer and thunders the other, its Yes is provisional, its No is definitive.

13. *Experience and behavior.* Experience modifies human behavior. And experience modifies human beliefs. These two things are not independent of each other. Behavior often results from beliefs, beliefs are potential behavior. Yet you can see the other fellow's behavior, you cannot see his beliefs. Behavior is more easily observed than belief. Everybody knows that "a burnt child dreads the fire," which expresses just what we said: experience modifies human behavior.

Yes, and it modifies animal behavior, too.

In my neighborhood there is a mean dog that barks and jumps at people without provocation. But I have found that I can protect myself rather easily. If I stoop and pretend to pick up a stone, the dog runs away howling. All dogs do not behave so, and it is easy to guess what kind of experience gave this dog this behavior.

The bear in the zoo "begs for food." That is, when there is an onlooker around, it strikes a ridiculous posture which quite frequently prompts the onlooker to throw a lump of sugar into the cage. Bears not in captivity probably never assume such a preposterous posture and it is easy to imagine what kind of experience led to the zoo bear's begging.

A thorough investigation of induction should include, perhaps, the study of animal behavior.

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[2] Euler, *Opera Omnia*, ser. 1, vol. 2, p. 1–5. Hardy and Wright, *The Theory of Numbers*, p. 14–15.

14. *The logician, the mathematician, the physicist, and the engineer.* "Look at this mathematician," said the logician. "He observes that the first ninety-nine numbers are less than hundred and infers hence, by what he calls induction, that all numbers are less than a hundred."

"A physicist believes," said the mathematician, "that 60 is divisible by all numbers. He observes that 60 is divisible by 1, 2, 3, 4, 5, and 6. He examines a few more cases, as 10, 20, and 30, taken at random as he says. Since 60 is divisible also by these, he considers the experimental evidence sufficient."

"Yes, but look at the engineers," said the physicist. "An engineer suspected that all odd numbers are prime numbers. At any rate, 1 can be considered as a prime number, he argued. Then there come 3, 5, and 7, all indubitably primes. Then there comes 9; an awkward case, it does not seem to be a prime number. Yet 11 and 13 are certainly primes. 'Coming back to 9,' he said, 'I conclude that 9 must be an experimental error.' "

It is only too obvious that induction can lead to error. Yet it is remarkable that induction sometimes leads to truth, since the chances of error appear so overwhelming. Should we begin with the study of the obvious cases in which induction fails, or with the study of those remarkable cases in which induction succeeds? The study of precious stones is understandably more attractive than that of ordinary pebbles and, moreover, it was much more the precious stones than the pebbles that led the mineralogists to the wonderful science of crystallography.

II

GENERALIZATION, SPECIALIZATION, ANALOGY

> *And I cherish more than anything else the Analogies, my most trustworthy masters. They know all the secrets of Nature, and they ought to be least neglected in Geometry.*—KEPLER

1. Generalization, Specialization, Analogy, and Induction. Let us look again at the example of inductive reasoning that we have discussed in some detail (sect. 1.2, 1.3). We started from observing the *analogy* of the three relations

$$3 + 7 = 10, \quad 3 + 17 = 20, \quad 13 + 17 = 30,$$

we *generalized* in ascending from 3, 7, 13, and 17 to all primes, from 10, 20, and 30 to all even numbers, and then we *specialized* again, came down to test particular even numbers such as 6 or 8 or 60.

This first example is extremely simple. It illustrates quite correctly the role of generalization, specialization, and analogy in inductive reasoning. Yet we should examine less meager, more colorful illustrations and, before that, we should discuss generalization, specialization, and analogy, these great sources of discovery, for their own sake.

2. Generalization is passing from the consideration of a given set of objects to that of a larger set, containing the given one. For example, we generalize when we pass from the consideration of triangles to that of polygons with an arbitrary number of sides. We generalize also when we pass from the study of the trigonometric functions of an acute angle to the trigonometric functions of an unrestricted angle.

It may be observed that in these two examples the generalization was effected in two characteristically different ways. In the first example, in passing from triangles to polygons with n sides, we replace a constant by a variable, the fixed integer 3 by the arbitrary integer n (restricted only by the inequality $n \geqq 3$). In the second example, in passing from acute angles to arbitrary angles α, we remove a restriction, namely the restriction that $0° < \alpha < 90°$.

We often generalize in passing from just one object to a whole class containing that object.

3. Specialization is passing from the consideration of a given set of objects to that of a smaller set, contained in the given one. For example, we specialize when we pass from the consideration of polygons to that of regular polygons, and we specialize still further when we pass from regular polygons with n sides to the regular, that is, equilateral, triangle.

These two subsequent passages were effected in two characteristically different ways. In the first passage, from polygons to regular polygons, we introduced a restriction, namely that all sides and all angles of the polygon be equal. In the second passage we substituted a special object for a variable, we put 3 for the variable integer n.

Very often we specialize in passing from a whole class of objects to just one object contained in the class. For example, when we wish to check some general assertion about prime numbers we pick out some prime number, say 17, and we examine whether that general assertion is true or not for just this prime 17.

4. Analogy. There is nothing vague or questionable in the concepts of generalization and specialization. Yet as we start discussing analogy we tread on a less solid ground.

Analogy is a sort of similarity. It is, we could say, similarity on a more definite and more conceptual level. Yet we can express ourselves a little more accurately. The essential difference between analogy and other kinds of similarity lies, it seems to me, in the intentions of the thinker. Similar objects agree with each other in some aspect. If you intend to reduce the aspect in which they agree to definite concepts, you regard those similar objects as *analogous*. If you succeed in getting down to clear concepts, you have *clarified* the analogy.

Comparing a young woman to a flower, poets feel some similarity, I hope, but usually they do not contemplate analogy. In fact, they scarcely intend to leave the emotional level or reduce that comparison to something measurable or conceptually definable.

Looking in a natural history museum at the skeletons of various mammals, you may find them all frightening. If this is all the similarity you can find between them, you do not see much analogy. Yet you may perceive a wonderfully suggestive analogy if you consider the hand of a man, the paw of a cat, the foreleg of a horse, the fin of a whale, and the wing of a bat, these organs so differently used, as composed of similar parts similarly related to each other.

The last example illustrates the most typical case of clarified analogy; two *systems* are analogous, if they *agree in clearly definable relations of their respective parts.*

This aim suggests that we describe squares on the three sides of our right triangle. And so we arrive at the not unfamiliar part I of our compound figure, fig. 2.3. (The reader should draw the parts of this figure as they arise, in order to see it in the making.)

(2) Discoveries, even very modest discoveries, need some remark, the recognition of some relation. We can discover the following proof by observing the *analogy* between the familiar part I of our compound figure

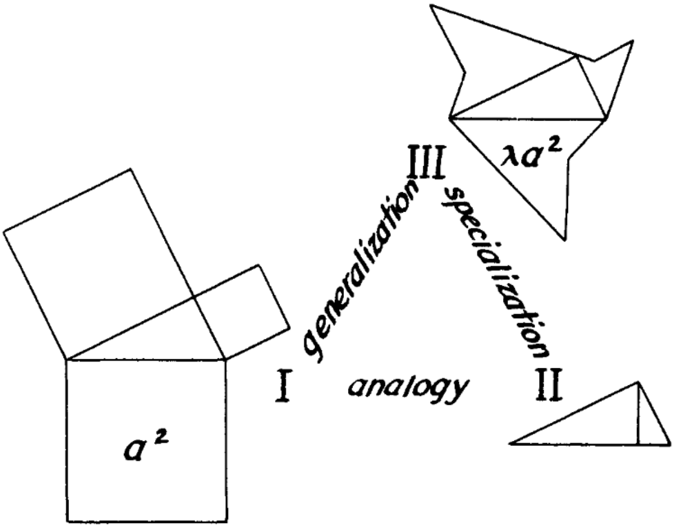

Fig. 2.3.

and the scarcely less familiar part II: the same right triangle that arises in I is divided in II into two parts by the altitude perpendicular to the hypotenuse.

(3) Perhaps, you fail to perceive the analogy between I and II. This analogy, however, can be made explicit by a common *generalization* of I and II which is expressed in III. There we find again the same right triangle, and on its three sides three polygons are described which are similar to each other but arbitrary otherwise.

(4) The area of the square described on the hypotenuse in I is a^2. The area of the irregular polygon described on the hypotenuse in III can be put equal to λa^2; the factor λ is determined as the ratio of two given areas. Yet then, it follows from the similarity of the three polygons described on the sides a, b, and c of the triangle in III that their areas are equal to λa^2, λb^2, and λc^2, respectively.

Now, if the equation (A) should be true (as stated by the theorem that we wish to prove), then also the following would be true:

(B) $$\lambda a^2 = \lambda b^2 + \lambda c^2.$$

In fact, very little algebra is needed to derive (B) from (A). Now, (B) represents a *generalization* of the original theorem of Pythagoras: *If three similar polygons are described on three sides of a right triangle, the one described on the hypotenuse is equal in area to the sum of the two others.*

It is instructive to observe that this generalization is *equivalent* to the special case from which we started. In fact, we can derive the equations (A) and (B) from each other, by multiplying or dividing by λ (which is, as the ratio of two areas, different from 0).

(5) The general theorem expressed by (B) is equivalent not only to the special case (A), but to any other special case. Therefore, if any such special case should turn out to be obvious, the general case would be demonstrated.

Now, trying to *specialize* usefully, we look around for a suitable special case. Indeed II represents such a case. In fact, the right triangle described on its own hypotenuse is similar to the two other triangles described on the two legs, as is well known and easy to see. And, obviously, the area of the whole triangle is equal to the sum of its two parts. And so, the theorem of Pythagoras has been proved.

The foregoing reasoning is eminently instructive. A case is instructive if we can learn from it something applicable to other cases, and the more instructive the wider the range of possible applications. Now, from the foregoing example we can learn the use of such fundamental mental operations as generalization, specialization, and the perception of analogies. There is perhaps no discovery either in elementary or in advanced mathematics or, for that matter, in any other subject that could do without these operations, especially without analogy.

The foregoing example shows how we can ascend by generalization from a special case, as from the one represented by I, to a more general situation as to that of III, and redescend hence by specialization to an analogous case, as to that of II. It shows also the fact, so usual in mathematics and still so surprising to the beginner, or to the philosopher who takes himself for advanced, that the general case can be logically equivalent to a special case. Our example shows, naïvely and suggestively, how generalization, specialization, and analogy are naturally combined in the effort to attain the desired solution. Observe that only a minimum of preliminary knowledge is needed to understand fully the foregoing reasoning.

6. Discovery by analogy. Analogy seems to have a share in all discoveries, but in some it has the lion's share. I wish to illustrate this by an example which is not quite elementary, but is of historic interest and far more impressive than any quite elementary example of which I can think.

Jacques Bernoulli, a Swiss mathematician (1654–1705), a contemporary of Newton and Leibnitz, discovered the sum of several infinite series, but did not succeed in finding the sum of the reciprocals of the squares,

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \dots .$$

"If somebody should succeed," wrote Bernoulli, "in finding what till now withstood our efforts and communicate it to us, we shall be much obliged to him."

The problem came to the attention of another Swiss mathematician, Leonhard Euler (1707–1783), who was born at Basle as was Jacques Bernoulli and was a pupil of Jacques' brother, Jean Bernoulli (1667–1748). He found various expressions for the desired sum (definite integrals, other series), none of which satisfied him. He used one of these expressions to compute the sum numerically to seven places (1.644934). Yet this is only an approximate value and his goal was to find the exact value. He discovered it, eventually. Analogy led him to an extremely daring conjecture.

(1) We begin by reviewing a few elementary algebraic facts essential to Euler's discovery. If the equation of degree n

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0$$

has n different roots

$$\alpha_1, \alpha_2, \dots \alpha_n$$

the polynomial on its left hand side can be represented as a product of n linear factors,

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n =$$
$$a_n(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n).$$

By comparing the terms with the same power of x on both sides of this identity, we derive the well known relations between the roots and the coefficients of an equation, the simplest of which is

$$a_{n-1} = -a_n(\alpha_1 + \alpha_2 + \dots + \alpha_n);$$

we find this by comparing the terms with x^{n-1}.

There is another way of presenting the decomposition in linear factors. If none of the roots $\alpha_1, \alpha_2, \dots \alpha_n$ is equal to 0, or (which is the same) if a_0 is different from 0, we have also

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$
$$= a_0\left(1 - \frac{x}{\alpha_1}\right)\left(1 - \frac{x}{\alpha_2}\right) \dots \left(1 - \frac{x}{\alpha_n}\right)$$

and

$$a_1 = -a_0\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \dots + \frac{1}{\alpha_n}\right).$$

There is still another variant. Suppose that the equation is of degree $2n$, has the form

$$b_0 - b_1x^2 + b_2x^4 - \ldots + (-1)^n b_n x^{2n} = 0$$

and $2n$ different roots

$$\beta_1, \quad -\beta_1, \quad \beta_2, \quad -\beta_2, \quad \ldots \quad \beta_n, \quad -\beta_n.$$

Then

$$b_0 - b_1x^2 + b_2x^4 - \ldots + (-1)^n b_n x^{2n}$$
$$= b_0\left(1 - \frac{x^2}{\beta_1^2}\right)\left(1 - \frac{x^2}{\beta_2^2}\right)\ldots\left(1 - \frac{x^2}{\beta_n^2}\right)$$

and

$$b_1 = b_0\left(\frac{1}{\beta_1^2} + \frac{1}{\beta_2^2} + \ldots + \frac{1}{\beta_n^2}\right).$$

(2) Euler considers the equation

$$\sin x = 0$$

or

$$\frac{x}{1} - \frac{x^3}{1\cdot2\cdot3} + \frac{x^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{x^7}{1\cdot2\cdot3\cdots7} + \ldots = 0.$$

The left hand side has an infinity of terms, is of "infinite degree." Therefore, it is no wonder, says Euler, that there is an infinity of roots

$$0, \quad \pi, \quad -\pi, \quad 2\pi, \quad -2\pi, \quad 3\pi, \quad -3\pi, \quad \ldots .$$

Euler discards the root 0. He divides the left hand side of the equation by x, the linear factor corresponding to the root 0, and obtains so the equation

$$1 - \frac{x^2}{2\cdot3} + \frac{x^4}{2\cdot3\cdot4\cdot5} - \frac{x^6}{2\cdot3\cdot4\cdot5\cdot6\cdot7} + \ldots = 0$$

with the roots

$$\pi, \quad -\pi, \quad 2\pi, \quad -2\pi, \quad 3\pi, \quad -3\pi, \quad \ldots .$$

We have seen an analogous situation before, under (1), as we discussed the last variant of the decomposition in linear factors. Euler concludes, by analogy, that

$$\frac{\sin x}{x} = 1 - \frac{x^2}{2\cdot3} + \frac{x^4}{2\cdot3\cdot4\cdot5} - \frac{x^6}{2\cdot3\cdots7} + \ldots$$
$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\ldots,$$

$$\frac{1}{2\cdot3} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots,$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}.$$

This is the series that withstood the efforts of Jacques Bernoulli—but it was a daring conclusion.

(3) Euler knew very well that his conclusion was daring. "The method was new and never used yet for such a purpose," he wrote ten years later. He saw some objections himself and many objections were raised by his mathematical friends when they recovered from their first admiring surprise.

Yet Euler had his reasons to trust his discovery. First of all, the numerical value for the sum of the series which he has computed before, agreed to the last place with $\pi^2/6$. Comparing further coefficients in his expression of $\sin x$ as a product, he found the sum of other remarkable series, as that of the reciprocals of the fourth powers,

$$1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625} + \dots = \frac{\pi^4}{90}.$$

Again, he examined the numerical value and again he found agreement.

(4) Euler also tested his method on other examples. Doing so he succeeded in rederiving the sum $\pi^2/6$ for Jacques Bernoulli's series by various modifications of his first approach. He succeeded also in rediscovering by his method the sum of an important series due to Leibnitz.

Let us discuss the last point. Let us consider, following Euler, the equation

$$1 - \sin x = 0.$$

It has the roots

$$\frac{\pi}{2}, \quad -\frac{3\pi}{2}, \quad \frac{5\pi}{2}, \quad -\frac{7\pi}{2}, \quad \frac{9\pi}{2}, \quad -\frac{11\pi}{2}, \quad \dots .$$

Each of these roots is, however, a double root. (The curve $y = \sin x$ does not intersect the line $y = 1$ at these abscissas, but is tangent to it. The derivative of the left hand side vanishes for the same values of x, but not the second derivative.) Therefore, the equation

$$1 - \frac{x}{1} + \frac{x^3}{1 \cdot 2 \cdot 3} - \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots = 0$$

has the roots

$$\frac{\pi}{2}, \quad \frac{\pi}{2}, \quad -\frac{3\pi}{2}, \quad -\frac{3\pi}{2}, \quad \frac{5\pi}{2}, \quad \frac{5\pi}{2}, \quad -\frac{7\pi}{2}, \quad -\frac{7\pi}{2}, \quad \dots$$

and Euler's analogical conclusion leads to the decomposition in linear factors

$$1 - \sin x = 1 - \frac{x}{1} + \frac{x^3}{1 \cdot 2 \cdot 3} - \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$$

$$= \left(1 - \frac{2x}{\pi}\right)^2 \left(1 + \frac{2x}{3\pi}\right)^2 \left(1 - \frac{2x}{5\pi}\right)^2 \left(1 + \frac{2x}{7\pi}\right)^2 \dots .$$