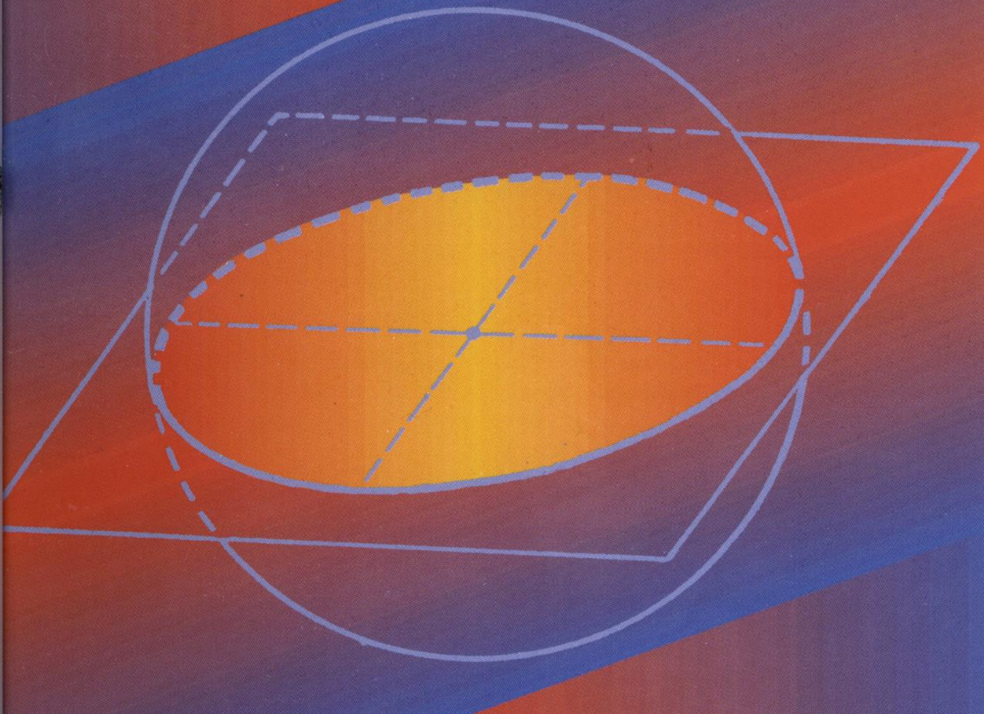
Mathematics and the Imagination



Edward Kasner
and James Newman

Mathematics and the Imagination

Edward Kasner
and
James Newman

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Introduction

The fashion in books in the last decade or so has turned increasingly to popular science. Even newspapers, Sunday supplements and magazines have given space to relativity, atomic physics, and the newest marvels of astronomy and chemistry. Symptomatic as this is of the increasing desire to know what happens in laboratories and observatories, as well as in the awe-inspiring conclaves of scientists and mathematicians, a large part of modern science remains obscured by an apparently impenetrable veil of mystery. The feeling is widely prevalent that science, like magic and alchemy in the Middle Ages, is practiced and can be understood only by a small esoteric group. The mathematician is still regarded as the hermit who knows little of the ways of life outside his cell, who spends his time compounding incredible and incomprehensible theories in a strange, clipped, unintelligible jargon.

Nevertheless, intelligent people, weary of the nervous pace of their own existence—the sharp impact of the happenings of the day—are hungry to learn of the accomplishments of more leisurely, contemplative lives, timed by a slower, more deliberate clock than their own. Science, particularly mathematics, though it seems less practical and less real than the news contained in the latest radio dispatches, appears to be building the one permanent and stable edifice in an age where all others are either crumbling or being blown to bits. This is not to say that science has not also undergone revolutionary changes. But it has happened quietly and honorably. That which is no longer useful has been rejected only after mature deliberation, and the building has been reared steadily on the creative achievements of the past.

Thus, in a certain sense, the popularization of science is a duty to be performed, a duty to give courage and comfort to the men and women of good will everywhere who are gradually losing their faith in the life of reason. For most of the sciences the veil of mystery is gradually being torn asunder. Mathematics, in large measure, remains unrevealed. What most popular books on mathematics have tried to do is either to discuss it philosophically, or to make clear the stuff once learned and already forgotten. In this respect our purpose in writing has been somewhat different. "Haute vulgarisation" is the term applied by the French to that happy result which neither offends by its condescension nor leaves obscure in a mass of technical verbiage. It has been our aim to extend the process of "haute vulgarisation" to those outposts of mathematics which are mentioned, if at all, only in a whisper; which are referred to, if at all, only by name; to show by its very diversity something of the character of mathematics, of its bold, untrammeled spirit, of how, as both an art and a science, it has continued to lead the creative faculties beyond even imagination and intuition. In the compass of so brief a volume there can only be snapshots, not portraits. Yet, it is hoped that even in this kaleidoscope there may be a stimulus to further interest in and greater recognition of the proudest queen of the intellectual world.

MATHEMATICS AND THE IMAGINATION

I will not go so far as to say that to construct a history of thought without profound study of the mathematical ideas of successive epochs is like omitting Hamlet from the play which is named after him. That would be claiming too much. But it is certainly analogous to cutting out the part of Ophelia. This simile is singularly exact. For Ophelia is quite essential to the play, she is very charming,—and a little mad. Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.

—ALFRED NORTH WHITEHEAD,
Science and the Modern World.

New Names for Old

For out of olde feldes, as men seith,
Cometh al this newe corn fro yeer to yere;
And out of olde bokes, in good feith,
Cometh al this newe science that men lere.
—CHAUCER

EVERY ONCE in a while there is house cleaning in mathematics. Some old names are discarded, some dusted off and refurbished; new theories, new additions to the household are assigned a place and name. So what our title really means is new *words* in mathematics; not new names, but new words, new terms which have in part come to represent new concepts and a reappraisal of old ones in more or less recent mathematics. There are surely plenty of words already in mathematics as well as in other subjects. Indeed, there are so many words that it is even easier than it used to be to speak a great deal and say nothing. It is mostly through words strung together like beads in a necklace that half the population of the world has been induced to believe mad things and to sanctify mad deeds. Frank Vizetelly, the great lexicographer, estimated that there are 800,000 words in use in the English language. But mathematicians, generally quite modest, are not satisfied with these 800,000; let us give them a few more.

We can get along without new names until, as we advance in science, we acquire new ideas and new forms.

A peculiar thing about mathematics is that it does not use so many long and hard names as the other sciences. Besides, it is more conservative than the other sciences in that it clings tenaciously to old words. The terms used by Euclid in his *Elements* are current in geometry today. But an Ionian physicist would find the terminology of modern physics, to put it colloquially, pure Greek. In chemistry, substances no more complicated than sugar, starch, or alcohol have names like these: Methylpropenylenedihydroxycinnamenylacrylic acid, or, 0-anhydrosulfaminobenzoine, or, protocatechuicaldehydemethylene. It would be inconvenient if we had to use such terms in everyday conversation. Who could imagine even the aristocrat of science at the breakfast table asking, "Please pass the 0-anhydrosulfaminobenzoic acid," when all he wanted was sugar for his coffee? Biology also has some tantalizing tongue twisters. The purpose of these long words is not to frighten the exoteric, but to describe with scientific curtness what the literary man would take half a page to express.

In mathematics there are many easy words like "group," "family," "ring," "simple curve," "limit," etc. But these ordinary words are sometimes given a very peculiar and technical meaning. In fact, here is a booby-prize definition of mathematics: *Mathematics is the science which uses easy words for hard ideas.* In this it differs from any other science. There are 500,000 known species of insects and every one has a long Latin name. In mathematics we are more modest. We talk about "fields," "groups," "families," "spaces," although much more meaning is attached to these words than ordinary conversation implies. As its use becomes more and more technical, nobody can guess the mathematical meaning of a word any more than one could guess that a "drug store" is a place where they sell ice-cream sodas and umbrellas. No one could guess the meaning of the word "group" as it is used in mathematics. Yet it is so important that whole courses are given on the theory of "groups," and hundreds of books are written about it.

Because mathematicians get along with common words, many amusing ambiguities arise. For instance, the word "function" probably expresses the most important idea in the whole history of mathematics. Yet, most people hearing it would think of a "function" as meaning an evening social affair, while others, less socially minded, would think of their livers. The word "function" has at least a dozen meanings, but few people suspect the mathematical one. The mathematical meaning (which we shall elaborate upon later) is expressed most simply by a *table*. Such a table gives the relation between two variable quantities when the value of one variable quantity is determined by the value of the other. Thus, one variable quantity may express the years from 1800 to 1938, and the other, the number of men in the United States wearing handle-bar mustaches; or one variable may express in decibels the amount of noise made by a political speaker, and the other, the blood pressure units of his listeners. You could probably never guess the meaning of the word "ring" as it has been used in mathematics. It was introduced into the newer algebra within the last twenty years. The theory of rings is much more recent than the theory of groups. It is now found in most of the new books on algebra, and has nothing to do with either matrimony or bells.

Other ordinary words used in mathematics in a peculiar sense are "domain," "integration," "differentiation." The uninitiated would not be able to guess what they represent; only mathematicians would know about them. The word "transcendental" in mathematics has not the meaning it has in philosophy. A mathematician would say: The number π, equal to 3.14159 . . . , is transcendental, because it is not the root of any algebraic equation with integer coefficients.

Transcendental is a very exalted name for a small number, but it was coined when it was thought that transcendental numbers were as rare as quintuplets. The work of Georg Cantor in the realm of the infinite has since proved that of all the numbers in mathematics, the transcendental ones are the most common, or, to use the word in a slightly different sense, the least transcendental. We shall talk of this later when we speak of another famous transcendental number, *e*, the base of the natural logarithms. Immanuel Kant's "transcendental epistemology" is what most educated people might think of when the word transcendental is used, but in that sense it has nothing to do with mathematics. Again, take the word "evolution," used in mathematics to denote the process most of us learned in elementary school, and promptly forgot, of extracting square roots, cube roots, etc. Spencer, in his philosophy, defines evolution as "an integration of matter, and a dissipation of motion from an indefinite, incoherent homogeneity to a definite, coherent heterogeneity," etc. But that, fortunately, has nothing to do with mathematical evolution either. Even in Tennessee, one may extract square roots without running afoul of the law.

As we see, mathematics uses simple words for complicated ideas. An example of a simple word used in a complicated way is the word "simple." "Simple curve" and "simple group" represent important ideas in higher mathematics.

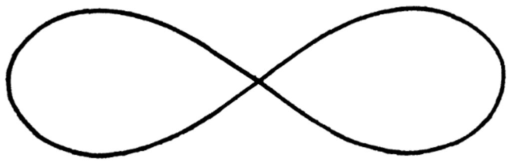

FIG. 1

The above is not a simple curve. A simple curve is a closed curve which does not cross itself and may look like Fig. 2. There are many important theorems about such figures that make the word worth while. Later, we are

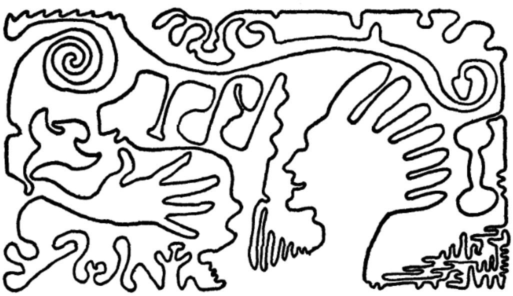

FIG. 2

going to talk about a queer kind of mathematics called "rubber-sheet geometry," and will have much more to say about simple curves and nonsimple ones. A French mathematician, Jordan, gave the fundamental theorem: every simple curve has one inside and one outside. That is, every simple curve divides the plane into two regions, one inside the curve, and one outside.

There are some groups in mathematics that are "simple" groups. The definition of "simple group" is really so hard that it cannot be given here. If we wanted to get a clear idea of what a simple group was, we should probably have to spend a long time looking into a great many books, and then, without an extensive mathematical background, we should probably miss the point. First of all, we should have to define the concept "group." Then we should have to give a definition of subgroups, and then of self-conjugate subgroups, and then we should be able to tell what a simple group is. A simple group is simply a group without any self-conjugate subgroups—simple, is it not?

Mathematics is often erroneously referred to as the science of common sense. Actually, it may transcend common sense and go beyond either imagination or intuition. It has become a very strange and perhaps frightening subject from the ordinary point of view, but anyone who penetrates into it will find a veritable fairyland, a fairyland which is strange, but makes sense, if not common sense. From the ordinary point of view mathematics deals with strange things. We shall show you that occasionally it does deal with strange things, but mostly it deals with familiar things in a strange way. If you look at yourself in an ordinary mirror, regardless of your physical attributes, you may find yourself amusing, but not strange; a subway ride to Coney Island, and a glance at yourself in one of the distorting mirrors will convince you that from another point of view you may be strange as well as amusing. It is largely a matter of what you are accustomed to. A Russian peasant came to Moscow for the first time and went to see the sights. He went to the zoo and saw the giraffes. You may find a moral in his reaction as plainly as in the fables of La Fontaine. "Look," he said, "at what the Bolsheviks have done to our horses." That is what modern mathematics has done to simple geometry and to simple arithmetic.

There are other words and expressions, not so familiar, which have been invented even more recently. Take, for instance, the word "turbine." Of course, that is already used in engineering, but it is an entirely new word in geometry. The mathematical name applies to a certain diagram. (Geometry, whatever others may think, is the study of different shapes, many of them very beautiful, having harmony, grace and symmetry. Of course, there are also fat books written on abstract geometry, and abstract space in which neither a diagram nor a shape appears. This is a very important branch of mathematics, but it is not the geometry studied by the Egyptians and the Greeks. Most of us, if we can play chess at all, are content to play it on a board with wooden

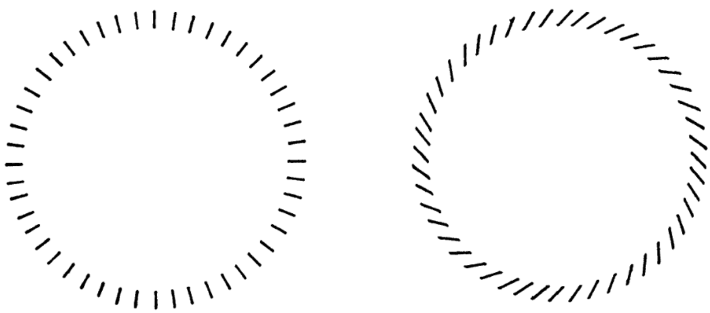

FIG. 3.—Turbines.

chess pieces; but there are some who play the game blindfolded and without touching the board. It might be a fair analogy to say that abstract geometry is like blindfold chess—it is a game played without concrete objects.) Above you see a picture of a turbine, in fact, two of them.

A turbine consists of an infinite number of "elements" filled in continuously. An element is not merely a point;

it is a point with an associated direction—like an iron filing. A turbine is composed of an infinite number of these elements, arranged in a peculiar way: the points must be arranged on a perfect circle, and the inclination of the iron filings must be at the same angle to the circle throughout. There are thus an infinite number of elements of equal inclination to the various tangents of the circle. In the special case where the angle between the direction of the element and the direction of the tangent is zero, what would happen? The turbine would be a circle. In other words, the theory of turbines is a generalization of the theory of the circle. If the angle is ninety degrees, the elements point toward the center of the circle. In that special case we have a normal turbine (see left-hand diagram).

There is a geometry of turbines, instead of a geometry of circles. It is a rather technical branch of mathematics which concerns itself with working out continuous groups of transformations connected with differential equations and differential geometry. The geometry connected with the turbine bears the rather odd name of "turns and slides."

*

The circle is one of the oldest figures in mathematics. The straight line is the simplest line, but the circle is the simplest nonstraight curve. It is often regarded as the limit of a polygon with an infinite number of sides. You can see for yourself that as a series of polygons is inscribed in a circle with each polygon having more sides than its predecessor, each polygon gets to look more and more like a circle.[1]

The Greeks were already familiar with the idea that as a regular polygon increases in the number of its sides, it differs less and less from the circle in which it is inscribed. Indeed, it may well be that in the eyes of an omniscient creature, the circle would look like a polygon with an infinite number of straight sides.[2] However, in the absence of complete omniscience, we shall continue

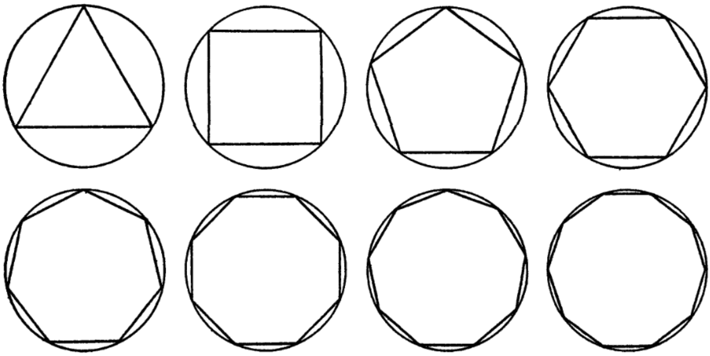

FIG. 4.—The circle as the limit of inscribed polygons.

to regard a circle as being a nonstraight curve. There are some interesting generalizations of the circle when it is viewed in this way. There is, for example, the concept denoted by the word "cycle," which was introduced by a French mathematician, Laguerre. A cycle is a circle with an arrow on it, like this:

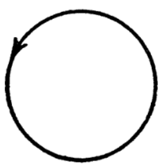

FIG. 5.

If you took the same circle and put an arrow on it in the opposite direction, it would become a different cycle.

The Greeks were specialists in the art of posing problems which neither they nor succeeding generations of mathematicians have ever been able to solve. The three most famous of these problems—the squaring of the circle, the duplication of the cube, and the trisection of an angle—we shall discuss later. Many well-meaning, self-appointed, and self-anointed mathematicians, and a motley assortment of lunatics and cranks, knowing neither history nor mathematics, supply an abundant crop of "solutions" of these insoluble problems each year. However, some of the classical problems of antiquity have been solved. For example, the theory of cycles was used by Laguerre in solving the problem of Apollonius: given three fixed circles, to find a circle that touches them all. It turns out to be a matter of elementary high

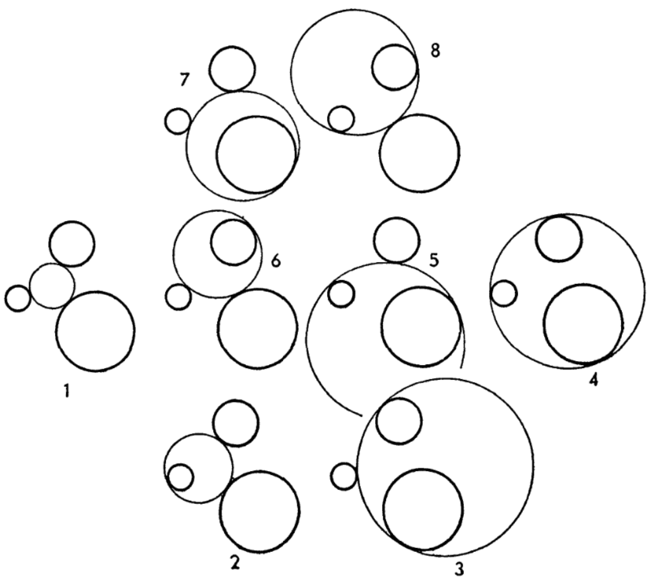

FIG. 6(a).—The eight solutions of the problem of Apollonius. Each lightly drawn circle is in contact with 3 heavily drawn ones.

school geometry, although it involves ingenuity, and any brilliant high school student could work it out. It has eight answers, as shown in Fig. 6(a).

They can all be constructed with ruler and compass, and many methods of solution have been found. Given three *circles*, there will be eight circles touching all of them. Given three *cycles*, however, there will be only one clockwise cycle that touches them all. (Two cycles are said to touch each other only if their arrows agree in direction at the point of contact.) Thus, by using the idea of cycles, we have one definite answer instead of eight. Laguerre made the idea of cycles the basis of an elegant theory.

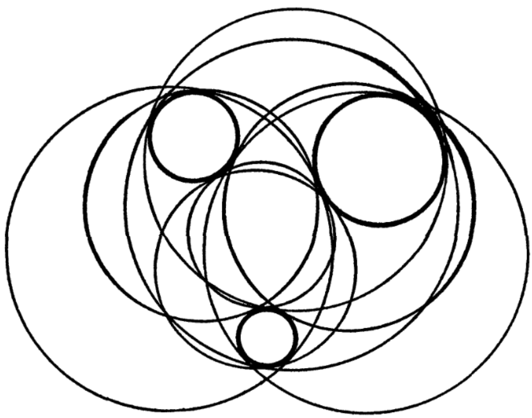

FIG. 6(b).—The eight solutions of Appolonius merged into one diagram.

Another variation of the circle introduced by the eminent American mathematician, C. J. Keyser, is obtained by taking a circle and removing one point.[3] This creates a serious change in conception. Keyser calls it "a pathocircle," (from pathological circle). He has used it in discussing the logic of axioms.

We have made yet another change in the concept of circle, which introduces another word and a new diagram. Take a circle and instead of leaving one point out, simply emphasize one point as the initial point. This is to be called a "clock." It has been used in the theory of polygenic functions. "Polygenic" is a word recently introduced into the theory of complex functions —about 1927. There was an important word, "monogenic," introduced in the nineteenth century by the famous French mathematician, Augustin Cauchy, and used in the classical theory of functions. It is used to denote functions that have a single derivative at a point, as in the differential calculus. But most functions, in the complex domain, have an infinite number of derivatives at a point. If a function is not monogenic, it can never be bigenic, or trigenic. Either the derivative has one value or an infinite number of values—either monogenic or polygenic, nothing intermediate. Monogenic means one rate of growth. Polygenic means many rates of growth. The complete derivative of a polygenic function is represented by a congruence (a double infinity) of clocks, all with different starting points, but with the

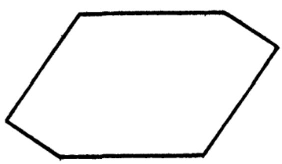

FIG. 7.—The parhexagon.

same uniform rate of rotation. It would be useless to attempt to give a simplified explanation of these concepts. (The neophyte will have to bear with us over a few intervals like this for the sake of the more experienced mathematical reader.)

The going has been rather hard in the last paragraph, and if a few of the polygenic seas have swept you overboard, we shall throw you a hexagonal life preserver. We may consider a very simple word that has been introduced in elementary geometry to indicate a certain kind of hexagon. The word on which to fix your attention is "parhexagon." An ordinary hexagon has six arbitrary sides. A parhexagon is that kind of hexagon in which any side is both equal and parallel to the side opposite to it (as in Fig. 7).

If the opposite sides of a quadrilateral are equal and parallel, it is called a parallelogram. By the same reasoning that we use for the word parhexagon, a parallelogram might have been called a parquadrilateral.

Here is an example of a theorem about the parhexagon: take any irregular hexagon, not necessarily a parhexagon, ABCDEF. Draw the diagonals AC, BD, CE, DF, EA, and FB, forming the six triangles, ABC, BCD, CDE, DEF, EFA, and FAB. Find the six centers of gravity, A′, B′, C′, D′, E′, and F′ of these triangles. (The center of gravity of a triangle is the point at which the triangle would balance if it were cut out of cardboard and supported only at that point; it coincides with the

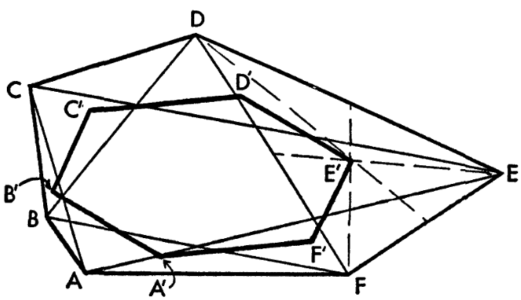

FIG. 8.—*ABCDEF* is an irregular hexagon. *A′B′ C′D′E′F′* is a parhexagon.

point of intersection of the medians.) Draw A′B′, B′C′, C′D′, D′E′, E′F′, and F′A′. Then the new inner hexagon A′B′C′D′E′F′ will always be a parhexagon.

The word radical, favorite call to arms among Republicans, Democrats, Communists, Socialists, Nazis, Fascists, Trotskyites, etc., has a less hortatory and bellicose character in mathematics. For one thing, everybody knows its meaning: i.e., square root, cube root, fourth root, fifth root, etc. Combining a word previously defined with this one, we might say that the extraction of a root is the evolution of a radical. The square root of 9 is 3; the square root of 10 is greater than 3, and the most famous and the simplest of all square roots, the first incommensurable number discovered by the Greeks, the square root of 2, is 1.414. . . . There are also composite radicals—expressions like $\sqrt{7} + \sqrt[5]{10}$. The symbol for a radical is not the hammer and sickle, but a sign three or four centuries old, and the idea of the mathematical radical is even older than that. The concept of the "hyperradical," or "ultraradical," which means something higher than a radical, but lower than a transcendental, is of recent origin. It has a symbol which we shall see in a moment. First, we must say a few words about radicals in general. There are certain numbers and functions in mathematics which are not expressible in the language of radicals and which are generally not well understood. Many ideas for which there are no concrete or diagrammatic representations are difficult to explain. Most people find it impossible to think without words; it is necessary to give them a word and a symbol to pin their attention. Hyperradical or ultraradical, for which hitherto there have been neither words, nor symbols, fall into this category.

We first meet these ultraradicals, not in Mexico City, but in trying to solve equations of the fifth degree. The Egyptians solved equations of the first degree perhaps 4000 years ago. That is, they found that the solution of the equation $ax + b = 0$, which is represented in geometry by a straight line, is $x = \frac{-b}{a}$. The quadratic equation $ax^2 + bx + c = 0$ was solved by the Hindus and the Arabs with the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The various conic sections, the circle, the ellipse, the parabola, and the hyperbola, are the geometric pictures of quadratic equations in two variables.

Then in the sixteenth century the Italians solved the equations of third and fourth degree, obtaining long formulas involving cube roots and square roots. So that by the year 1550, a few years before Shakespeare was born, the equation of the first, second, third, and fourth degrees had been solved. Then there was a delay of 250 years, because mathematicians were struggling with the equation of the fifth degree—the general quintic. Finally, at the beginning of the nineteenth century, Ruffini and Abel showed that equations of the fifth degree could not be solved with radicals. The general quintic is thus not like the general quadratic, cubic or biquadratic. Nevertheless, it presents a problem in algebra which theoretically can be solved by algebraic operations. Only, these operations are so hard that they cannot be expressed by the symbols for radicals. These new higher things are

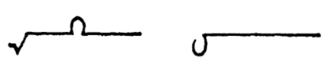

FIG. 9.—A portrait of two ultra-radicals.

named "ultraradicals," and they too have their special symbols (shown in Fig. 9).

With such symbols combined with radicals, we can solve equations of the fifth degree. For example, the solution of $x^5 + x = a$ may be written $x = \sqrt{\hat{a}}$ or $x = \int\overline{a}$. The usefulness of the special symbol and name is apparent. Without them the solution of the quintic equation could not be compactly expressed.

*

We may now give a few ideas somewhat easier than those with which we have thus far occupied ourselves. These ideas were presented some time ago to a number of children in kindergarten. It was amazing how well they understood everything that was said to them. Indeed, it is a fair inference that kindergarten children can enjoy lectures on graduate mathematics as long as the mathematical concepts are clearly presented.

It was raining and the children were asked how many raindrops would fall on New York. The highest answer was 100. They had never counted higher than 100 and what they meant to imply when they used that number was merely something very, very big—as big as they could imagine. They were asked how many raindrops hit the roof, and how many hit New York, and how many single raindrops hit all of New York in 24 hours. They soon got a notion of the bigness of these numbers even though they did not know the symbols for them. They were certain in a little while that the number of raindrops was a great deal bigger than a hundred. They were asked to think of the number of grains of sand on the beach at Coney Island and decided that the number of grains of sand and the number of raindrops were about the same. But the important thing is that they realized that the number was *finite, not infinite.* In this respect they showed their distinct superiority over many scientists who to this day use the word infinite when they mean some big number, like a billion billion.

Counting, something such scientists evidently do not realize, is a precise operation.* It may be wonderful but there is nothing vague or mysterious about it. If you count something, the answer you get is either perfect or all wrong; there is no half way. It is very much like catching a train. You either catch it or you miss it, and if you miss it by a split second you might as well have come a week late. There is a famous quotation which illustrates this:

> "Oh, the little more, and how much it is!
> And the little less, and what worlds away!"

A big number is big, but it is definite and it is finite. Of course in poetry, the finite ends with about three thousand; any greater number is infinite. In many poems, the poet will talk to you about the infinite number of stars. But, if ever there was a hyperbole, this is it, for nobody, not even the poet, has ever seen more than three thousand stars on a clear night, without the aid of a telescope.

With the Hottentots, infinity begins at three.† Ask a Hottentot how many cows he owns, and if he has more than three he'll say "many." The number of raindrops

* No one would say that 1 + 1 is "about equal to 2." It is just as silly to say that a billion billion is not a finite number, simply because it is big. Any number which may be named, or conceived of in terms of the integers is finite. *Infinite means something quite different*, as we shall see in the chapter on the googol.

† Although, in all fairness, it must be pointed out that some of the tribes of the Belgian Congo can count to a million and beyond.

falling on New York is also "many." It is a large finite number, but nowhere near infinity.

Now here is the name of a very large number: "Googol."* Most people would say, "A googol is so large that you cannot name it or talk about it; it is so large that it is infinite." Therefore, we shall talk about it, explain exactly what it is, and show that it belongs to the very same family as the number 1.

A googol is this number which one of the children in the kindergarten wrote on the blackboard:

100000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000
00000

The definition of a googol is: 1 followed by a hundred zeros. It was decided, after careful mathematical researches in the kindergarten, that the number of raindrops falling on New York in 24 hours, or even in a year or in a century, is much less than a googol. Indeed, the googol is a number just larger than the largest numbers that are used in physics or astronomy. All those numbers require less than a hundred zeros. This information is, of course, available to everyone, but seems to be a great secret in many scientific quarters.

A very distinguished scientific publication recently came forth with the revelation that the number of snow crystals necessary to form the ice age was a billion to the billionth power. This is very startling and also very silly. A billion to the billionth power looks like this:

$$1000000000^{1000000000}.$$

A more reasonable estimate and a somewhat smaller number would be 10^{30}. As a matter of fact, it has been estimated that if the entire universe, which you will con-

* Not even approximately a Russian author.

cede is a trifle larger than the earth, were filled with protons and electrons, so that no vacant space remained, the total number of protons and electrons would be 10^{110} (i.e., 1 with 110 zeros after it). Unfortunately, as soon as people talk about large numbers, they run amuck. They seem to be under the impression that since zero equals nothing, they can add as many zeros to a number as they please with practically no serious consequences. We shall have to be a little more careful than that in talking about big numbers.

To return to Coney Island, the number of grains of sand on the beach is about 10^{20}, or more descriptively, 100000000000000000000. That is a large number, but not as large as the number mentioned by the divorcee in a recent divorce suit who had telephoned that she loved the man "a million billion billion times and eight times around the world." It was the largest number that she could conceive of, and shows the kind of thing that may be hatched in a love nest.

Though people do a great deal of talking, the total output since the beginning of gabble to the present day, including all baby talk, love songs, and Congressional debates, totals about 10^{16}. This is ten million billion. Contrary to popular belief, this is a larger number of words than is spoken at the average afternoon bridge.

A great deal of the veneration for the authority of the printed word would vanish if one were to calculate the number of words which have been printed since the Gutenberg Bible appeared. It is a number somewhat larger than 10^{16}. A recent popular historical novel alone accounts for the printing of several hundred billion words.

The largest number seen in finance (though new records are in the making) represents the amount of money in circulation in Germany at the peak of the inflation. It was less than a googol—merely

496,585,346,000,000,000,000.

A distinguished economist vouches for the accuracy of this figure. The number of marks in circulation was very nearly equal to the number of grains of sand on Coney Island beach.

The number of atoms of oxygen in the average thimble is a good deal larger. It would be represented by perhaps 1000000000000000000000000000. The number of electrons, in size exceedingly smaller than the atoms, is much more enormous. The number of electrons which pass through the filament of an ordinary fifty-watt electric lamp in a minute equals the number of drops of water that flow over Niagara Falls in a century.

One may also calculate the number of electrons, not only in the average room, but over the whole earth, and out through the stars, the Milky Way, and all the nebulae. The reason for giving all these examples of very large numbers is to emphasize the fact that no matter how large the collection to be counted, a finite number will do the trick. We will have occasion later on to speak of infinite collections, but those encountered in nature, though sometimes very large, are all definitely finite. A celebrated scientist recently stated in all seriousness that he believed that the number of pores (through which leaves breathe) of all the leaves, of all the trees in all the world, would certainly be infinite. Needless to say, he was not a mathematician. The number of electrons in a single leaf is much bigger than the number of pores of all the leaves of all the trees of all the world. And still the number of all the electrons in the entire universe can be found by means of the physics of Einstein. It is a good deal less than a googol—perhaps one with seventy-nine zeros, 10^{79}, as estimated by Eddington.

Words of wisdom are spoken by children at least as often as by scientists. The name "googol" was invented by a child (Dr. Kasner's nine-year-old nephew) who was asked to think up a name for a very big number, namely, 1 with a hundred zeros after it. He was very certain that this number was not infinite, and therefore equally certain that it had to have a name. At the same time that he suggested "googol" he gave a name for a still larger number: "Googolplex." A googolplex is much larger than a googol, but is still finite, as the inventor of the name was quick to point out. It was first suggested that a googolplex should be 1, followed by writing zeros until you got tired. This is a description of what would happen if one actually tried to write a googolplex, but different people get tired at different times and it would never do to have Carnera a better mathematician than Dr. Einstein, simply because he had more endurance. The googolplex then, is a specific finite number, with so many zeros after the 1 that the number of zeros is a googol. A googolplex is much bigger than a googol, much bigger even than a googol times a googol. A googol times a googol would be 1 with 200 zeros, whereas a googolplex is 1 with a googol of zeros. You will get some idea of the size of this very large but finite number from the fact that there would not be enough room to write it, if you went to the farthest star, touring all the nebulae and putting down zeros every inch of the way.

One might not believe that such a large number would ever really have any application; but one who felt that way would not be a mathematician. A number as large as the googolplex might be of real use in problems of combination. This would be the type of problem in which it might come up scientifically:

Consider this book which is made up of carbon and nitrogen and of other elements. The answer to the question, "How many atoms are there in this book?" would certainly be a finite number, even less than a googol. Now imagine that the book is held suspended by a string, the end of which you are holding. How long will it be necessary to wait before the book will jump up into your hand? Could it conceivably ever happen? One answer might be "No, it will never happen without some external force causing it to do so." But that is not correct. The right answer is that it will almost *certainly* happen *sometime* in less than a googolplex of years—perhaps tomorrow.

The explanation of this answer can be found in physical chemistry, statistical mechanics, the kinetic theory of gases, and the theory of probability. We cannot dispose of all these subjects in a few lines, but we will try. Molecules are always moving. Absolute rest of molecules would mean absolute zero degrees of temperature, and absolute zero degrees of temperature is not only nonexistent, but impossible to obtain. All the molecules of the surrounding air bombard the book. At present the bombardment from above and below is nearly the same and gravity keeps the book down. It is necessary to wait for the favorable moment when there happens to be an enormous number of molecules bombarding the book from below and very few from above. Then gravity will be overcome and the book will rise. It would be somewhat like the effect known in physics as the Brownian movement, which describes the behavior of small particles in a liquid as they dance about under the impact of molecules. It would be analogous to the Brownian movement on a vast scale.

But the probability that this will happen in the near future or, for that matter, on any specific occasion that we might mention, is between $\frac{1}{\text{googol}}$ and $\frac{1}{\text{googolplex}}$. To be reasonably sure that the book will rise, we should have to wait between a googol and a googolplex of years.

When working with electrons or with problems of combination like the one of the book, we need larger numbers than are usually talked about. It is for that reason that names like googol and googolplex, though they may appear to be mere jokes, have a real value. The names help to fix in our minds the fact that we are still dealing with finite numbers. To repeat, a googol is 10^{100}; a googolplex is 10 to the googol power, which may be written $10^{10^{100}} = 10^{\text{googol}}$.

We have seen that the number of years that one would have to wait to see the miracle of the rising book would be less than a googolplex. In that number of years the earth may well have become a frozen planet as dead as the moon, or perhaps splintered to a number of meteors and comets. The real miracle is not that the book will rise, but that with the aid of mathematics, we can project ourselves into the future and predict with accuracy *when* it will probably rise, i.e., some time between today and the year googolplex.

*

We have mentioned quite a few new names in mathematics—new names for old and new ideas. There is one more new name which it is proper to mention in conclusion. Watson Davis, the popular science reporter, has given us the name "mathescope." With the aid of the magnificent new microscopes and telescopes, man, midway between the stars and the atoms, has come a little closer to both. The mathescope is not a physical instrument; it is a purely intellectual instrument, the ever-increasing insight which mathematics gives into the fairyland which lies beyond intuition and beyond imagination. Mathematicians, unlike philosophers, say nothing about ultimate truth, but patiently, like the makers of the great microscopes, and the great telescopes, they grind their lenses. In this book, we shall let you see through the newer and greater lenses which the mathematicians have ground. Be prepared for strange sights through the mathescope!

FOOTNOTES

1. See the Chapter on PIE.—P. 10.
2. See the Chapter on Change and Changeability—Section on Pathological Curves.—P.11.
3. N.B. This is a diagram which the reader will have to imagine, for it is beyond the capacity of any printer to make a circle with one point omitted. A point, having no dimensions, will, like many of the persons on the Lord High Executioner's list, never be missed. So the circle with one point missing is purely conceptual, not an idea which can be pictured.—P.13.

Beyond the Googol

If you do not expect the unexpected, you will not find it;
for it is hard to be sought out, and difficult.

—HERACLITUS

MATHEMATICS MAY well be a science of austere logical propositions in precise canonical form, but in its countless applications it serves as a tool and a language, the language of description, of number and size. It describes with economy and elegance the elliptic orbits of the planets as readily as the shape and dimensions of this page or a corn field. The whirling dance of the electron can be seen by no one; the most powerful telescopes can reveal only a meager bit of the distant stars and nebulae and the cold far corners of space. But with the aid of mathematics and the imagination the very small, the very large—all things may be brought within man's domain.

To count is to talk the language of number. To count to a googol, or to count to ten is part of the same process; the googol is simply harder to pronounce. The essential thing to realize is that the googol and ten are kin, like the giant stars and the electron. Arithmetic—this counting language—makes the whole world kin, both in space and in time.

To grasp the meaning and importance of mathematics, to appreciate its beauty and its value, arithmetic must first be understood, for mostly, since its beginning, mathematics has been arithmetic in simple or elaborate attire. Arithmetic has been the queen and the handmaiden of the sciences from the days of the astrologers of Chaldea and the high priests of Egypt to the present days of relativity, quanta, and the adding machine. Historians may dispute the meaning of ancient papyri, theologians may wrangle over the exegesis of Scripture, philosophers may debate over Pythagorean doctrine, but all will concede that the numbers in the papyri, in the Scriptures and in the writings of Pythagoras are the same as the numbers of today. As arithmetic, mathematics has helped man to cast horoscopes, to make calendars, to predict the risings of the Nile, to measure fields and the height of the Pyramids, to measure the speed of a stone as it fell from a tower in Pisa, the speed of an apple as it fell from a tree in Woolsthorpe, to weigh the stars and the atoms, to mark the passage of time, to find the curvature of space. And although mathematics is also the calculus, the theory of probability, the matrix algebra, the science of the infinite, it is still the art of counting.

*

Everyone who will read this book can count, and yet, what is counting? The dictionary definitions are about as helpful as Johnson's definition of a net: "A series of reticulated interstices." *Learning to compare is learning to count.* Numbers come much later; they are an artificiality, an abstraction. Counting, matching, comparing are almost as indigenous to man as his fingers. Without the faculty of comparing, and without his fingers, it is unlikely that he would have arrived at numbers.

One who knows nothing of the formal processes of counting is still able to compare two classes of objects, to determine which is the greater, which the less. Without knowing anything about numbers, one may ascertain whether two classes have the same number of elements; for example, barring prior mishaps, it is easy to show that we have the same number of fingers on both hands by simply matching finger with finger on each hand.

To describe the process of matching, which underlies counting, mathematicians use a picturesque name. They call it putting classes into a "one-to-one reciprocal correspondence" with each other. Indeed, that is all there is to the art of counting as practiced by primitive peoples, by us, or by Einstein. A few examples may serve to make this clear.

In a monogamous country it is unnecessary to count both the husbands and the wives in order to ascertain the number of married people. If allowances are made for the few gay Lotharios who do not conform to either custom or statute, it is sufficient to count either the husbands or the wives. There are just as many in one class as in the other. The correspondence between the two classes is one-to-one.

There are more useful illustrations. Many people are gathered in a large hall where seats are to be provided. The question is, are there enough chairs to go around? It would be quite a job to count both the people and the chairs, and in this case unnecessary. In kindergarten children play a game called "Going to Jerusalem"; in a room full of children and chairs there is always one less chair than the number of children. At a signal, each child runs for a chair. The child left standing is "out." A chair is removed and the game continues. Here is the solution to our problem. It is only necessary to ask everyone in the hall to be seated. If everyone sits down and no chairs are left vacant, it is evident that there are as many chairs as people. In other words, without actually knowing the number of chairs or people, one does know that the number is the same. The two classes—chairs and people—have been shown to be equal in number by a one-to-one correspondence. To each person corresponds a chair, to each chair, a person.

In counting any class of objects, it is this method alone which is employed. One class contains the things to be counted; the other class is always at hand. It is the class of integers, or "natural numbers," which for convenience we regard as being given in serial order: 1, 2, 3, 4, 5, 6, 7 . . . Matching in one-to-one correspondence the elements of the first class with the integers, we experience a common, but none the less wonderful phenomenon—the last integer necessary to complete the pairings denotes *how many* elements there are.

*

In clarifying the idea of counting, we made the unwarranted assumption that the concept of number was understood by everyone. The number concept may seem intuitively clear, but a precise definition is required. While the definition may seem worse than the disease, it is not as difficult as appears at first glance. Read it carefully and you will find that it is both explicit and economical.

Given a class C containing certain elements, it is possible to find other classes, such that the elements of each may be matched one to one with the elements of C. (Each of these classes is thus called "equivalent to C.") All such classes, including C, whatever the character of their elements, share one property in common: all of them have the same *cardinal number*, which is called the *cardinal number* of the class C.[1]

The cardinal number of the class *C* is thus seen to be the *symbol* representing the set of all classes that can be put into one-to-one correspondence with *C*. For example, the number 5 is simply the name, or symbol, attached to the set of all the classes, each of which can be put into one-to-one correspondence with the fingers of one hand.

Hereafter we may refer without ambiguity to the number of elements in a class as the cardinal number of that class or, briefly, as "its cardinality." The question, "How many letters are there in the word *mathematics?*" is the same as the question, "What is the cardinality of the class whose elements are the letters in the word *mathematics?*" Employing the method of one-to-one correspondence, the following graphic device answers the question, and illustrates the method:

M	A	T	H	E	M	A	T	I	C	S
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
1	2	3	4	5	6	7	8	9	10	11

It must now be evident that this method is neither strange nor esoteric; it was not invented by mathematicians to make something natural and easy seem unnatural and hard. It is the method employed when we count our change or our chickens; it is the proper method for counting any class, no matter how large, from ten to a googolplex—and beyond.

Soon we shall speak of the "beyond" when we turn to classes which are not finite. Indeed, we shall try *to measure our measuring class*—the integers. One-to-one correspondence should, therefore, be thoroughly understood, for an amazing revelation awaits us: Infinite classes can also be counted, and by the very same means. But before we try to count them, let us practice on some very big numbers—big, but not infinite.

*

"Googol" is already in our vocabulary: It is a big number—one, with a hundred zeros after it. Even bigger is the googolplex: 1 with a googol zeros after it. Most numbers encountered in the description of nature are much smaller, though a few are larger.

Enormous numbers occur frequently in modern science. Sir Arthur Eddington claims that there are, not approximately, but exactly $136 \cdot 2^{256}$ protons,* and an equal number of electrons, in the universe. Though not easy to visualize, this number, as a symbol on paper, takes up little room. Not quite as large as the googol, it is completely dwarfed by the googolplex. None the less, Eddington's number, the googol, and the googolplex are finite.

A veritable giant is Skewes' number, even bigger than a googolplex. It gives information about the distribution of primes[2] and looks like this:

$$10^{10^{10^{34}}}$$

Or, for example, the total possible number of moves in a game of chess is:

$$10^{10^{50}}$$

And speaking of chess, as the eminent English mathematician, G. H. Hardy, pointed out—if we imagine the

* Let no one suppose that Sir Arthur has counted them. But he does have a theory to justify his claim. Anyone with a better theory may challenge Sir Arthur, for who can be referee? Here is his number written out: 15,747,724,136,275,002,577,605,653,961,181,555,468,-044,717,914,527,116,709,336,231,425,076,185,631,031,276—accurate, he says, to the last digit.

entire universe as a chessboard, and the protons in it as chessmen, and if we agree to call any interchange in the position of two protons a "move" in this cosmic game, then the total number of possible moves, of all odd coincidences, would be Skewes' number:

$$10^{10^{10^{34}}}$$

No doubt most people believe that such numbers are part of the marvelous advance of science, and that a few generations ago, to say nothing of centuries back, no one in dream or fancy could have conceived of them.

There is some truth in that idea. For one thing, the ancient cumbersome methods of mathematical notation made the writing of big numbers difficult, if not actually impossible. For another, the average citizen of today encounters such huge sums, representing armament expenditures and stellar distances, that he is quite conversant with, and immune to, big numbers.

But there were clever people in ancient times. Poets in every age may have sung of the stars as infinite in number, when all they saw was, perhaps, three thousand. But to Archimedes, a number as large as a googol, or even larger, was not disconcerting. He says as much in an introductory passage in *The Sand Reckoner*, realizing that a number is not infinite merely because it is enormous.

There are some, King Gelon, who think that the number of the sand is infinite in multitude; and I mean by the sand, not only that which exists about Syracuse and the rest of Sicily, but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed its multitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the earth, including in it all the seas and the hollows of the earth filled up to a height equal to that of the highest of the mountains, would be many times further still from recognizing that any number could be expressed which exceeded the multitude of the sand so taken. But I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described, but also that of a mass equal in magnitude to the universe.

The Greeks had very definite ideas about the infinite. Just as we are indebted to them for much of our wit and our learning, so are we indebted to them for much of our sophistication about the infinite. Indeed, had we always retained their clear-sightedness, many of the problems and paradoxes connected with the infinite would never have arisen.

Above everything, we must realize that "very big" and "infinite" are entirely different.* By using the method of one-to-one correspondence, the protons and electrons in the universe may theoretically be counted as easily as the buttons on a vest. Sufficient and more than sufficient for that task, or for the task of counting any finite collection, are the integers. But measuring the

* There is no point where the very big starts to merge into the infinite. You may write a number as big as you please; it will be no nearer the infinite than the number 1 or the number 7. Make sure that you keep this distinction very clear and you will have mastered many of the subtleties of the transfinite.

totality of integers is another problem. To measure such a class demands a lofty viewpoint. Besides being, as the German mathematician Kronecker thought, the work of God, which requires courage to appraise, the class of integers is infinite—which is a great deal more inconvenient. It is worse than heresy to measure our own endless measuring rod!

*

The problems of the infinite have challenged man's mind and have fired his imagination as no other single problem in the history of thought. The infinite appears both strange and familiar, at times beyond our grasp, at times natural and easy to understand. In conquering it, man broke the fetters that bound him to earth. All his faculties were required for this conquest—his reasoning powers, his poetic fancy, his desire to know.

To establish the science of the infinite involves the principle of *mathematical induction.* This principle affirms the power of reasoning by recurrence. It typifies almost all mathematical thinking, all that we do when we construct complex aggregates out of simple elements. It is, as Poincaré remarked, "at once necessary to the mathematician and irreducible to logic." His statement of the principle is: "If a property be true of the number one, and if we establish that it is true of $n + 1$,* provided it be of n, it will be true of all the whole numbers." Mathematical induction is not derived from experience, rather is it an inherent, intuitive, almost instinctive property of the mind. "*What we have once done we can do again.*"

If we can construct numbers to ten, to a million, to a googol, we are led to believe that there is no stopping,

* Where n is any integer.

no end. Convinced of this, we need not go on forever; the mind grasps that which it has never experienced—the infinite itself. Without any sense of discontinuity, without transgressing the canons of logic, the mathematician and philosopher have bridged in one stroke the gulf between the finite and the infinite. The mathematics of the infinite is a sheer affirmation of the inherent power of reasoning by recurrence.

In the sense that "infinite" means "without end, without bound," simply "not finite," probably everyone understands its meaning. No difficulty arises where no precise definition is required. Nevertheless, in spite of the famous epigram that mathematics is the science in which we do not know what we are talking about, at least we shall have to agree to talk about the same thing. Apparently, even those of scientific temper can argue bitterly to the point of mutual vilification on subjects ranging from Marxism and dialectical materialism to group theory and the uncertainty principle, only to find, on the verge of exhaustion and collapse, that they are on the same side of the fence. Such arguments are generally the results of vague terminology; to assume that everyone is familiar with the precise mathematical definition of "infinite" is to build a new Tower of Babel.

Before undertaking a definition, we might do well to glance backwards to see how mathematicians and philosophers of other times dealt with the problem.

The infinite has a double aspect—the infinitely large, and the infinitely small. Repeated arguments and demonstrations, of apparently apodictic force, were advanced, overwhelmed, and once more resuscitated to prove or disprove its existence. Few of the arguments were ever refuted—each was buried under an avalanche of others. The happy result was that the problem never became any clearer.*

*

The warfare began in antiquity with the paradoxes of Zeno; it has never ceased. Fine points were debated with a fervor worthy of the earliest Christian martyrs, but without a tenth part of the acumen of medieval theologians. Today, some mathematicians think the infinite has been reduced to a state of vassalage. Others are still wondering what it is.

Zeno's puzzles may help to bring the problem into sharper focus. Zeno of Elea, it will be recalled, said some disquieting things about motion, with reference to an arrow, Achilles, and a tortoise. This strange company was employed on behalf of the tenet of Eleatic philosophy —that all motion is an illusion. It has been suggested, probably by "baffled critics," that "Zeno had his tongue in cheek when he made his puzzles." Regardless of motive, they are immeasurably subtle, and perhaps still defy solution.†

One paradox—the Dichotomy—states that it is impossible to cover any given distance. The argument: First, half the distance must be traversed, then half of the remaining distance, then again half of what remains,

* No one has written more brilliantly or more wittily on this subject than Bertrand Russell. See particularly his essays in the volume *Mysticism and Logic.*

† To be sure, a variety of explanations have been given for the paradoxes. In the last analysis, the explanations for the riddles rest upon the interpretation of the foundations of mathematics. Mathematicians like Brouwer, who reject the infinite, would probably not accept any of the solutions given.

and so on. It follows that some portion of the distance to be covered always remains, and therefore motion is impossible! A solution of this paradox reads:

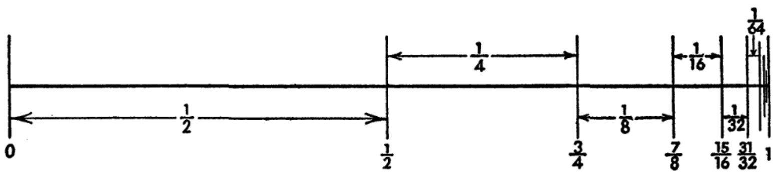

FIG. 10.

The successive distances to be covered form an infinite geometric series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$$ [3]

each term of which is half of the one before. Although this series has an infinite number of terms, its sum is *finite* and equals 1. Herein, it is said, lies the flaw of the Dichotomy. Zeno assumed that any totality composed of an infinite number of parts must, itself, be infinite, whereas we have just seen an infinite number of elements which make up the finite totality—1.

The paradox of the tortoise states that Achilles, running to overtake the tortoise, must first reach the place where it started:—but the tortoise has already departed. This comedy, however, is repeated indefinitely. As Achilles arrives at each new point in the race, the tortoise having been there, has already left. Achilles is as unlikely to catch him as a rider on a carrousel the rider ahead.

Finally: the arrow in flight must be moving every instant of time. But at every instant it must be *somewhere* in space. However, if the arrow must always be in some one place, it cannot at every instant also be in transit, for to be in transit is to be *nowhere*.

Aristotle and lesser saints in almost every age tried to demolish these paradoxes, but not very creditably. Three German professors succeeded where the saints had failed. At the end of the nineteenth century, it seemed that Bolzano, Weierstrass and Cantor had laid the infinite to rest, and Zeno's paradoxes as well.

The modern method of disposing of the paradoxes is not to dismiss them as mere sophisms unworthy of serious attention. The history of mathematics, in fact, recounts a poetic vindication of Zeno's stand. Zeno was, at one time, as Bertrand Russell has said, "A notable victim of posterity's lack of judgement." That wrong has been righted. In disposing of the infinitely small, Weierstrass showed that the moving arrow *is* really always at rest, and that we live in Zeno's changeless world. The work of Georg Cantor, which we shall soon encounter, showed that if we are to believe that Achilles *can* catch the tortoise, we shall have to be prepared to swallow a bigger paradox than any Zeno ever conceived of: THE WHOLE IS NO GREATER THAN MANY OF ITS PARTS!

The infinitely small had been a nuisance for more than two thousand years. At best, the innumerable opinions it evoked deserved the laconic verdict of Scotch juries: "Not proven." Until Weierstrass appeared, the total advance was a confirmation of Zeno's argument against motion. Even the jokes were better. Leibniz, according to Carlyle, made the mistake of trying to explain the infinitesimal to a Queen—Sophie Charlotte of Prussia. She informed him that the behavior of her courtiers made her so familiar with the infinitely small, that she needed no mathematical tutor to explain it. But philosophers and mathematicians, according to Russell, "having less acquaintance with the courts, continued to discuss this topic, though without making any advance."

Berkeley, with the subtlety and humor necessary for an Irish bishop, made some pointed attacks on the infinitesimal, during the adolescent period of the calculus, that had the very best, sharp-witted, scholastic sting. One could perhaps speak, if only with poetic fervor, of the infinitely large, but what, pray, was the infinitely small? The Greeks, with less than their customary sagacity, introduced it in regarding a circle as differing infinitesimally from a polygon with a large number of equal sides. Leibniz used it as the bricks for the infinitesimal calculus. Still, no one knew what it was. The infinitesimal had wondrous properties. It was not zero, yet smaller than any quantity. It could be assigned no quantity or size, yet a sizable number of infinitesimals made a very definite quantity. Unable to discover its nature, happily able to dispense with it, Weierstrass interred it alongside of the phlogiston and other once-cherished errors.

*

The infinitely large offered more stubborn resistance. Whatever it is, it is a doughty weed. The subject of reams of nonsense, sacred and profane, it was first discussed fully, logically, and without benefit of clergy-like prejudices by Bernhard Bolzano. *Die Paradoxien des Unendlichen*, a remarkable little volume, appeared posthumously in 1851. Like the work of another Austrian priest, Gregor Mendel, whose distinguished treatise on the principles of heredity escaped oblivion only by chance, this important book, charmingly written, made no great impression on Bolzano's contemporaries. It is the creation of a clear, forceful, penetrating intelligence. For the first time in twenty centuries the infinite was treated as a problem in science, and not as a problem in theology.

Both Cantor and Dedekind are indebted to Bolzano for the foundations of the mathematical treatment of the infinite. Among the many paradoxes he gathered and explained, one, dating from Galileo, illustrates a typical source of confusion:

Construct a square—*ABCD*. About the point *A* as center, with one side as radius, describe a quarter-circle, intersecting the square at *B* and *D*. Draw *PR* parallel to *AD*, cutting *AB* at *P*, *CD* at *R*, the diagonal *AC* at *N*, and the quarter-circle at *M*.

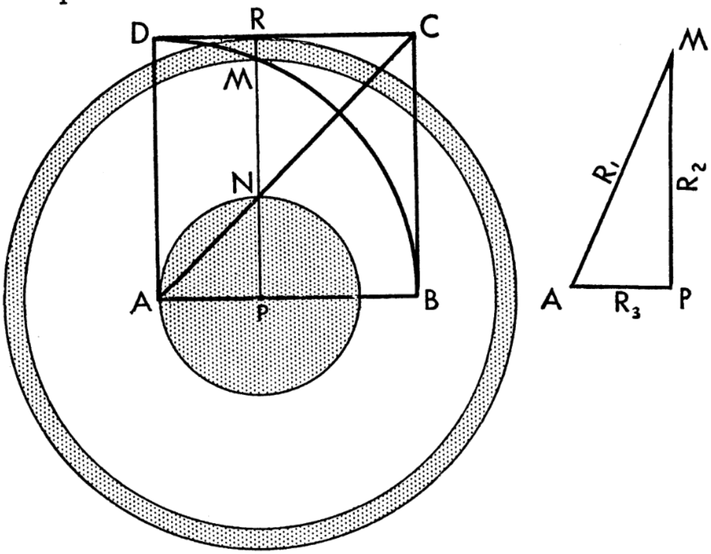

FIG. 11.—Extract triangle *APM* from the figure. It is not hard to see that its three sides equal respectively the radii of the three circles.
Thus

$$R_1^2 - R_2^2 = R_3^2$$

or,

$$\pi R_1^2 - \pi R_2^2 = \pi R_3^2$$

or, the two shaded areas are equal.

By a well-known geometrical theorem, it can be shown that if *PN*, *PM* and *PR* are radii, the following relationship exists:

$$\pi\overline{PN}^2 = \pi\overline{PR}^2 - \pi\overline{PM}^2 \qquad (1)$$

Permit *PR* to approach *AD*. Then the circle with *PN* as radius becomes smaller, and the ring between the circles with *PM* and *PR* as radii becomes correspondingly smaller. Finally, when *PR* becomes identical with *AD*, the radius *PN* vanishes, leaving the point *A*, while the ring between the two circles *PM* and *PR* contracts into one periphery with *AD* as radius. From equation (1) it may be concluded that the *point A* takes up as much area as the *circumference* of the circle with *AD* as radius.

Bolzano realized that there is only an *appearance* of a paradox. The two classes of points, one composed of a single member, the point *A*, the other of the points in the circumference of the circle with *AB* as radius, take up exactly the same amount of area. The area of both is zero! The paradox springs from the erroneous conception that the number of points in a given configuration is an indication of the area which it occupies. Points, finite or infinite in number, have no dimensions and can therefore occupy no area.

Through the centuries such paradoxes had piled up. Born of the union of vague ideas and vague philosophical reflections, they were nurtured on sloppy thinking. Bolzano cleared away most of the muddle, preparing the way for Cantor. It is to Cantor that the mathematics of the infinitely large owes its coming of age.

*

Georg Cantor was born in St. Petersburg in 1845, six years before Bolzano's book appeared. Though born in Russia, he lived the greater part of his life in Germany, where he taught at the University of Halle. While Weierstrass was busy disposing of the infinitesimal, Cantor set himself the apparently more formidable task at the other pole. The infinitely small might be laughed out of existence, but who dared laugh at the infinitely large? Certainly not Cantor! Theological curiosity prompted his task, but the mathematical interest came to subsume every other.

In dealing with the science of the infinite, Cantor realized that the first requisite was to define terms. His definition of "infinite class" which we shall paraphrase, rests upon a paradox. AN INFINITE CLASS HAS THE UNIQUE PROPERTY THAT THE WHOLE IS NO GREATER THAN SOME OF ITS PARTS. That statement is as essential for the mathematics of the infinite as THE WHOLE IS GREATER THAN ANY OF ITS PARTS is for finite arithmetic. When we recall that two classes are equal if their elements can be put into one-to-one correspondence, the latter statement becomes obvious. Zeno would not have challenged it, in spite of his scepticism about the obvious. But what is obvious for the finite is false for the infinite; our extensive experience with finite classes is misleading. Since, for example, the class of men and the class of mathematicians are both finite, anyone realizing that some men are not mathematicians would correctly conclude that the class of men is the larger of the two. He might also conclude that the number of integers, even and odd, is greater than the number of even integers. But we see from the following pairing that he would be mistaken:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2 & 4 & 6 & 8 & 10 & 12 & 14\ldots \end{array}$$

Under every integer, odd or even, we may write its double—an even integer. That is, we place each of the elements of the class of all the integers, odd and even, into a one-to-one correspondence with the elements of the class composed solely of even integers. This process may be continued to the googolplex and beyond.

Now, the class of integers is infinite. No integer, no matter how great, can describe its cardinality (or numerosity). Yet, since it is possible to establish a one-to-one correspondence between the class of even numbers and the class of integers, we have succeeded in counting the class of even numbers just as we count a finite collection. The two classes being perfectly matched, we must conclude that they have the same cardinality. That their cardinality is the same we *know*, just as we knew that the chairs and the people in the hall were equal in number when every chair was occupied and no one was left standing. Thus, we arrive at the fundamental paradox of all infinite classes:—There exist component parts of an infinite class which are just as great as the class itself. THE WHOLE IS NO GREATER THAN SOME OF ITS PARTS!

The class composed of the even integers *is thinned out* as compared with the class of all integers, but evidently "thinning out" has not the slightest effect on its cardinality. Moreover, there is almost no limit to the number of times this process can be repeated. For instance, there are as many square numbers and cube numbers as there are integers. The appropriate pairings are:

1	2	3	4	5	6 . . .	1	2	3	4	5	6 . . .
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕
1	4	9	16	25	36 . . .	1	8	27	64	125	216 . . .
1^2	2^2	3^2	4^2	5^2	6^2	1^3	2^3	3^3	4^3	5^3	6^3

Indeed, from any denumerable class there can always be removed a denumerably infinite number of denumerably infinite classes without affecting the cardinality of the original class.

*

Infinite classes which can be put into one-to-one correspondence with the integers, and thus "counted," Cantor called *countable*, or *denumerably infinite*. Since all finite sets are countable, and we can assign to each one a number, it is natural to try to extend this notion and assign to the class of all integers a number representing its cardinality. Yet, it is obvious from our description of "infinite class" that no ordinary integer would be adequate to describe the cardinality of the whole class of integers. In effect, it would be asking a snake to swallow itself entirely. Thus, the first of the transfinite numbers was created to describe the cardinality of countable infinite classes. Etymologically old, mathematically new, $\aleph$ (aleph), the first letter of the Hebrew alphabet, was suggested. However, Cantor finally decided to use the compound symbol $\aleph_0$ (Aleph-Null). If asked, "How many integers are there?" it would be correct to reply, "There are $\aleph_0$ integers."

Because he suspected that there were other transfinite numbers, in fact an infinite number of transfinites, and the cardinality of the integers the smallest, Cantor affixed to the first $\aleph$ a small zero as subscript. The cardinality of a denumerably infinite class is therefore referred to as $\aleph_0$ (Aleph-Null). The anticipated transfinite numbers form a hierarchy of alephs: $\aleph_0, \aleph_1, \aleph_2, \aleph_3 \ldots$

All this may seem very strange, and it is quite excusable for the reader by now to be thoroughly bewildered. Yet, if you have followed the previous reasoning step by step, and will go to the trouble of rereading, you will see that nothing which has been said is repugnant to straight thinking. Having established what is meant by counting in the finite domain, and what is meant by number, we decided to extend the counting process to infinite classes. As for our right to follow such a procedure, we have the same right, for example, as those who decided that man had crawled on the surface of the earth long enough and that it was about time for him to fly. It is our right to venture forth in the world of ideas as it is our right to extend our horizons in the physical universe. One restraint alone is laid upon us in these adventures of ideas: that we abide by the rules of logic.

Upon extending the counting process it was evident at once that no finite number could adequately describe an infinite class. If any number of ordinary arithmetic describes the cardinality of a class, that class must be finite, even though there were not enough ink or enough space or enough time to write the number out. We shall then require an entirely new kind of number, nowhere to be found in finite arithmetic, to describe the cardinality of an infinite class. Accordingly, the totality of integers was assigned the cardinality "aleph." Suspecting that there were *other* infinite classes with a cardinality *greater* than that of the totality of integers, we supposed a whole hierarchy of alephs, of which the cardinal number of the totality of integers was named Aleph-Null to indicate it was the smallest of the transfinites.

Having had an interlude in the form of a summary, let us turn once more to scrutinize the alephs, to find if, upon closer acquaintance, they may not become easier to understand.

The arithmetic of the alephs bears little resemblance to that of the finite integers. The immodest behavior of $\aleph_0$ is typical.

A simple problem in addition looks like this:

$$\begin{aligned} \aleph_0 + 1 &= \aleph_0 \\ \aleph_0 + \text{googol} &= \aleph_0 \\ \aleph_0 + \aleph_0 &= \aleph_0 \end{aligned}$$

The multiplication table would be easy to teach, easier to learn:

$$\begin{aligned} 1 \times \aleph_0 &= \aleph_0 \\ 2 \times \aleph_0 &= \aleph_0 \\ 3 \times \aleph_0 &= \aleph_0 \\ n \times \aleph_0 &= \aleph_0 \end{aligned}$$

where n represents any finite number.

Also,

$$\begin{aligned} (\aleph_0)^2 &= \aleph_0 \times \aleph_0 \\ &= \aleph_0 \end{aligned}$$

And thus,

$$(\aleph_0)^n = \aleph_0$$

when n is a finite integer.

There seems to be no variation of the theme; the monotony appears inescapable. But it is all very deceptive and treacherous. We go along obtaining the same result, no matter what we do to $\aleph_0$, when suddenly we try:

$$(\aleph_0)^{\aleph_0}$$

This operation, at last, creates a new transfinite. But before considering it, there is more to be said about countable classes.

*

Common sense says that there are many more fractions than integers, for between any two integers there is an infinite number of fractions. Alas—common sense is amidst alien corn in the land of the infinite. Cantor discovered a simple but elegant proof that the rational fractions form a denumerably infinite sequence equivalent to the class of integers. Whence, this sequence must have the same cardinality.*

The set of all rational fractions is arranged, not in order of increasing magnitude, but in order of ascending numerators and denominators in an array:

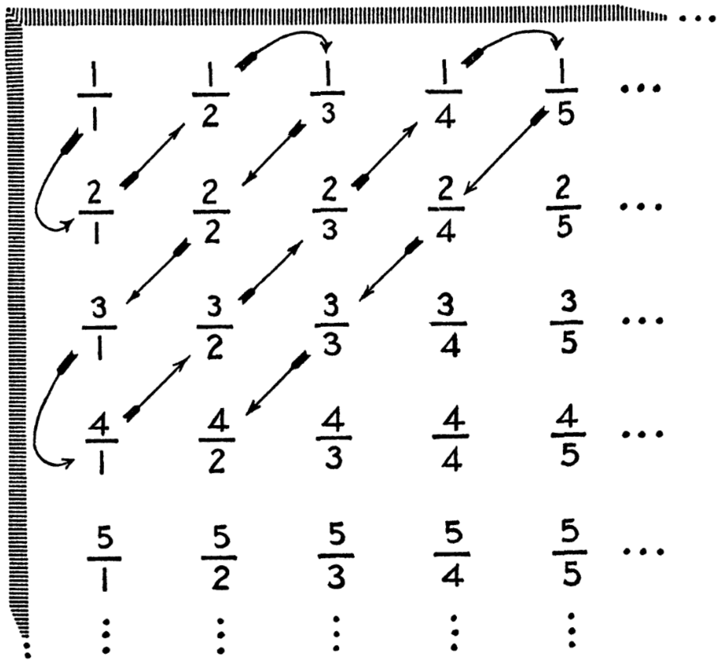

FIG. 12.—Cantor's array.

Since each fraction may be written as a pair of integers, i.e., $\frac{3}{4}$ as (3,4), the familiar one-to-one correspondence

* It has been suggested that at this point the tired reader puts the book down with a sigh—and goes to the movies. We can only offer

with the integers may be effected. This is illustrated in the above array by the arrows.

1	2	3	4	5	6	7	8	9	
↕	↕	↕	↕	↕	↕	↕	↕	↕	...
(1,1)	(2,1)	(1,2)	(1,3)	(2,2)	(3,1)	(4,1)	(3,2)	(2,3)	

Cantor also found, by means of a proof (too technical to concern us here) based on the "height" of algebraic equations, that the class of all algebraic numbers, numbers which are the solutions of algebraic equations with integer coefficients, of the form:

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$$

is denumerably infinite.

But Cantor felt that there were other transfinites, that there were classes which were not countable, which could not be put into one-to-one correspondence with the integers. And one of his greatest triumphs came when he succeeded in showing that there are classes with a cardinality greater than $\aleph_0$.

The class of real numbers composed of the rational and irrational numbers† is such a class. It contains those irrationals which are algebraic as well as those which are not. The latter are called *transcendental numbers.*[4]

in mitigation that this proof, like the one which follows on the non-countability of the real numbers, is tough and no bones about it. You may grit your teeth and try to get what you can out of them, or conveniently omit them. The essential thing to come away with is that Cantor found that the rational fractions are countable but that the set of real numbers is not. Thus, in spite of what common sense tells you, there are no more fractions than there are integers and there are more real numbers between 0 and 1 than there are elements in the whole class of integers.

† Irrational numbers are numbers which *cannot* be expressed as rational fractions. For example, $\sqrt{2}$, $\sqrt{3}$, e, π. The class of real numbers is made up of rationals like 1, 2, 3, $\frac{1}{4}$, $\frac{17}{32}$, and irrationals as above.

Two important transcendental numbers were known to exist in Cantor's time: π, the ratio of the circumference of a circle to its diameter, and e, the base of the natural logarithms. Little more was known about the class of transcendentals: it was an enigma. What Cantor had to prove, in order to show that the class of real numbers was nondenumerable (i.e., too big to be counted by the class of integers), was the unlikely fact that the class of transcendentals was nondenumerable. Since the rational and the algebraic numbers were known to be denumerable, and the sum of any denumerable number of denumerable classes is also a denumerable class, the sole remaining class which could make the totality of real numbers nondenumerable was the class of transcendentals.

He was able to devise such a proof. If it can be shown that the class of real numbers between 0 and 1 is nondenumerable, it will follow *a fortiori* that all the real numbers are nondenumerable. Employing a device often used in advanced mathematics, the *reductio ad absurdum*, Cantor assumed that to be true which he suspected was false, and then showed that this assumption led to a contradiction. He assumed that the real numbers between 0 and 1 were countable and could, therefore, be paired with the integers. Having proved that this assumption led to a contradiction, it followed that its opposite, namely, that the real numbers could *not* be paired with the integers (and were therefore not countable), was true.

To count the real numbers between 0 and 1, it is required that they all be expressed in a uniform way and a method of writing them down in order be devised so that they can be paired one to one with the integers. The first requirement can be fulfilled, for it is possible to express every real number as a nonterminating decimal. Thus, for example: [5]

$$\frac{1}{3} = .3333\ldots \qquad \frac{3}{14} = .214285712142857 1\ldots$$

$$\frac{1}{9} = .1111111\ldots \qquad \frac{\sqrt{2}}{2} = \frac{1.414\ldots}{2} = .707\ldots$$

Now, the second requirement confronts us. *How shall we make the pairings?* What system may be devised to ensure the appearance of *every* decimal? We did find a method for ensuring the appearance of every rational fraction. Of course, we could not actually write them all, any more than we could actually write all the integers; but the method of increasing numerators and denominators was so explicit that, if we had had an infinite time in which to do it, we could actually have set down all the fractions and have been certain that we had not omitted any. Or, to put it another way: It was always certain and determinate after a fraction had been paired with an integer, what the next fraction would be, and the next, and the next, and so on.

On the other hand, when a real number, expressed as a nonterminating decimal, is paired with an integer, what method is there for determining what the next decimal in order should be? You have only to ask yourself, which shall be the *first* of the nonterminating decimals to pair with the integer 1, and you have an inkling of the difficulty of the problem. Cantor however *assumed* that such a pairing does exist, without attempting to give its explicit form. His scheme was: With the integer 1 pair the decimal $.a_1a_2a_3\ldots$, with the integer 2, $.b_1b_2b_3\ldots$, etc. Each of the letters represents a digit of the nonterminating decimal in which it appears. The determinate array of pairing between the decimals and the integers would then be:

$$1 \longleftrightarrow 0.\ a_1\ a_2\ a_3\ a_4\ a_5 \ldots$$
$$2 \longleftrightarrow 0.\ b_1\ b_2\ b_3\ b_4\ b_5 \ldots$$
$$3 \longleftrightarrow 0.\ c_1\ c_2\ c_3\ c_4\ c_5 \ldots$$
$$4 \longleftrightarrow 0.\ d_1\ d_2\ d_3\ d_4\ d_5 \ldots$$

. . . .

. . . .

. . . .

That was Cantor's array. But at once it was evident that it glaringly exhibited the very contradiction for which he had been seeking. And in this defeat lay his triumph. For no matter *how* the decimals are arranged, by whatever system, by whatever scheme, it is always possible to construct an infinity of others which are not present in the array. The point is worth repeating: having contrived a general form for an array which we believed would include *every* decimal, we find, in spite of all our efforts, that *some* decimals are bound to be omitted. This, Cantor showed by his famous "diagonal proof." The conditions for determining a decimal omitted from the array are simple. It must differ from the first decimal in the array in its first place, from the second decimal in the array in its second place, from the third decimal in its third place, and so on. But then, *it must differ from every decimal in the entire array in* at least one place. If (as illustrated in the figure) we draw a diagonal line through our model array and write a new decimal, each digit of which shall differ from every digit intercepted by the diagonal, this new decimal cannot be found in the array.

$$\begin{array}{lcl}
1 & \longleftrightarrow & 0.\ a_1\ a_2\ a_3\ a_4\ a_5\ \ldots \\
2 & \longleftrightarrow & 0.\ b_1\ b_2\ b_3\ b_4\ b_5\ \ldots \\
3 & \longleftrightarrow & 0.\ c_1\ c_2\ c_3\ c_4\ c_5\ \ldots \\
4 & \longleftrightarrow & 0.\ d_1\ d_2\ d_3\ d_4\ d_5\ \ldots \\
5 & \longleftrightarrow & 0.\ e_1\ e_2\ e_3\ e_4\ e_5\ \ldots \\
\cdot & & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot & & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot & & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}$$

The new decimal may be written:—

$$0.\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5 \ldots;$$

where α_1 differs from a_1, α_2 differs from b_2, α_3 from c_3 α_4 from d_4, α_5 from e_5, etc. Accordingly, it will differ from each decimal in at least one place, from the *n*th decimal in at least its *n*th place. This proves conclusively that there is no way of including all the decimals in any possible array, no way of pairing them off with the integers. Therefore, as Cantor set out to prove:

1. The class of transcendental numbers is not only infinite, but also not countable, i.e., nondenumerably infinite.
2. The real numbers between 0 and 1 are infinite and not countable.
3. *A fortiori*, the class of all real numbers is nondenumerable.

*

To the noncountable class of real numbers, Cantor assigned a new transfinite cardinal. It was one of the alephs, but which one remains unsolved to this day. It is suspected that this transfinite, called the "cardinal of the continuum," which is represented by *c* or *C*, is identical with $\aleph_1$. But a proof acceptable to most mathematicians has yet to be devised.

The arithmetic of C is much the same as that of $\aleph_0$. The multiplication table has the same dependable monotone quality. But when C is combined with $\aleph_0$, it swallows it completely. Thus:

$$C + \aleph_0 = C \qquad C - \aleph_0 = C$$
$$C \times \aleph_0 = C \text{ and even } C \times C = C$$

Again, we hope for a variation of the theme when we come to the process of involution. Yet, for the moment, we are disappointed, for $C^{\aleph_0} = C$. But just as $(\aleph_0)^{\aleph_0}$ does not equal $\aleph_0$, so C^C does not equal C.

We are now in a position to solve our earlier problem in involution, for actually Cantor found that $(\aleph_0)^{\aleph_0} = C$. Likewise C^C gives rise to a new transfinite, greater than C. This transfinite represents the cardinality of the class of all one-valued functions. It is also one of the $\aleph$'s, but again, which one is unknown. It is often designated by the letter F.[6] In general, the process of involution, when repeated, continues to generate higher transfinites.

Just as the integers served as a measuring rod for classes with the cardinality $\aleph_0$, the class of real numbers serves as a measuring rod for classes with the cardinality C. Indeed, there are classes of geometric elements which can be measured in no other way except by the class of real numbers.

From the geometric notion of a point, the idea is evolved that on any given line segment there are an infinite number of points. The points on a line segment are also, as mathematicians say, "everywhere dense." This means that between any two points there is an infinitude of others. The concept of two immediately adjoining points is, therefore, meaningless. This property of being "everywhere dense," constitutes one of the essential characteristics of a *continuum.* Cantor, in referring to the "cardinality of the continuum," recognized that it applies alike to the class of real numbers and the class of points on a line segment. Both are everywhere dense, and both have the same cardinality, *C.* In other words, it is possible to pair the points on a line segment with the real numbers.

Classes with the cardinality *C* possess a property similar to classes with the cardinality $\aleph_0$: they may be thinned out without in any way affecting their cardinality. In this connection, we see in very striking fashion another illustration of the principle of transfinite arithmetic, that the whole is no greater than many of its parts. For instance, it can be proved that there are as many points on a line one foot long as there are on a line one yard long. The line segment *AB* in Fig. 13 is three times as long as the line *A'B'*. Nevertheless, it is possible to put the class of all points on the segment *AB* into a one-to-one correspondence with the class of points on the segment *A'B'*.

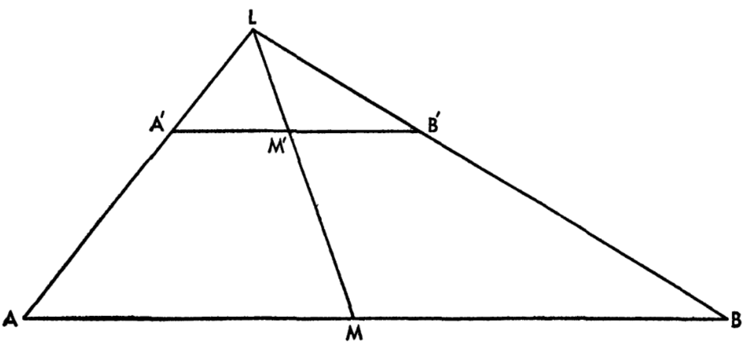

FIG. 13.

Let *L* be the intersection of the lines *AA'* and *BB'*. If then to any point *M* of *AB*, there corresponds a point

M' of $A'B'$, which is on the line LM, we have established the desired correspondence between the class of points on $A'B'$ and those on AB. It is easy to see intuitively and to prove geometrically that this is always possible, and that, therefore, the cardinality of the two classes of points is the same. Thus, since $A'B'$ is smaller than AB, it may be considered a proper part of AB, and we have again established that an infinite class may contain as proper parts, subclasses equivalent to it.

There are more startling examples in geometry which illustrate the power of the continuum. Although the statement that a line one inch in length contains as many points as a line stretching around the equator, or as a line stretching from the earth to the most distant stars, is startling enough, it is fantastic to think that a line segment one-millionth of an inch long has as many points as there are in all three-dimensional space in the entire universe. Nevertheless, this is true. Once the principles of Cantor's theory of transfinites is understood, such statements cease to sound like the extravagances of a mathematical madman. The oddities, as Russell has said, "then become no odder than the people at the antipodes who used to be thought impossible because they would find it so inconvenient to stand on their heads." Even conceding that the treatment of the infinite is a form of mathematical madness, one is forced to admit, as does the Duke in *Measure for Measure:*

> "If she be mad,—as I believe no other,—
> Her madness hath the oddest frame of sense,
> Such a dependency of thing on thing,
> As e'er I heard in madness."

*

Until now we have deliberately avoided a definition of "infinite class." But at last our equipment makes it possible to do so. We have seen that an infinite class, whether its cardinality is $\aleph_0$, C, or greater, may be thinned out in a countless variety of ways, without affecting its cardinality. In short, the whole is no greater than many of its parts. Now, this property does not belong to finite classes at all; it belongs only to infinite classes. Hence, it is a unique method of determining whether a class is finite or infinite. Thus, our definition reads: *An infinite class is one which can be put into one-to-one reciprocal correspondence with a proper subset of itself.*

Equipped with this definition and the few ideas we have gleaned we may re-examine some of the paradoxes of Zeno. That of Achilles and the tortoise may be expressed as follows: Achilles and the tortoise, running the same course, must each occupy the same number of distinct positions during their race. However, if Achilles is to catch his more leisurely and determined opponent, he will have to occupy *more* positions than the tortoise, in the same elapsed period of time. Since this is manifestly impossible, you may put your money on the tortoise.

But don't be too hasty. There are better ways of saving money than merely counting change. In fact, you had best bet on Achilles after all, for he is likely to win the race. Even though we may not have realized it, we have just finished proving that he could overtake the tortoise by showing that a line a millionth of an inch long has just as many points as a line stretching from the earth to the furthest star. In other words, the points on the tiny line segment can be placed into one-to-one correspondence with the points on the great line, for there is no relation between the number of points on a line and its length. But this reveals the error in thinking that Achilles cannot catch the tortoise. The statement that Achilles must occupy as many distinct positions as the tortoise is correct. So is the statement that he must travel a greater distance than the tortoise in the same time. The only incorrect statement is the inference that since he must occupy the same number of positions as the tortoise he cannot travel further while doing so. Even though the classes of points on each line, which correspond to the several positions of both Achilles and the tortoise are equivalent, the line representing the path of Achilles is much longer than that representing the path of the tortoise. Achilles may travel much further than the tortoise without successively touching more points.

The solution of the paradox involving the arrow in flight requires a word about another type of continuum. It is convenient and certainly familiar to regard time as a continuum. The time continuum has the same properties as the space continuum: the successive instants in any elapsed portion of time, just as the points on a line, may be put into one-to-one correspondence with the class of real numbers; between any two instants of time an infinity of others may be interpolated; time also has the mathematical property mentioned before—it is everywhere dense.

Zeno's argument stated that at every instant of time the arrow was somewhere, in some place or position, and therefore, could not at any instant be in motion. Although the statement that the arrow had at every moment to be in some place is true, the conclusion that, therefore, it could not be moving is absurd. Our natural tendency to accept this absurdity as true springs from our firm conviction that motion is entirely different from rest.

We are not confused about the position of a body when it is at rest—we feel there is no mystery about the state of rest. We should feel the same when we consider a body in motion.

When a body is at rest, it is in one position at one instant of time and at a later instant it is still in the same position. When a body is in motion, there is a one-to-one correspondence between every instant of time and every new position. To make this clear we may construct two tables: One will describe a body at rest, the other, a body in motion. The "rest" table will tell the life history

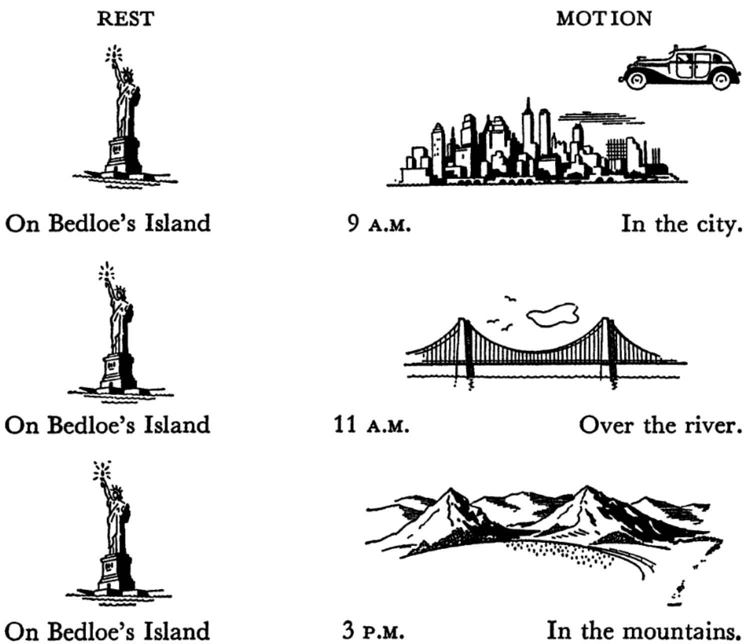

FIG. 14.—At the times shown, the Statue of Liberty is at the point shown, while the taxi's passengers see the different scenes shown at the right.

and the life geography of the Statue of Liberty, while the "motion" table will describe the Odyssey of an automobile.

The tables indicate that to every instant of time there corresponds a position of the Statue of Liberty and of the taxi. There is a one-to-one space-time correspondence for rest as well as for motion.

No paradox is concealed in the puzzle of the arrow when we look at our table. Indeed, it would be strange if there were gaps in the table; if it were impossible, at any instant, to determine exactly what the position of the arrow is.

Most of us would swear by the existence of motion, but we are not accustomed to think of it as something which makes an object occupy different positions at different instants of time. We are apt to think that motion endows an object with the strange property of being continually nowhere. Impeded by the limitations of our senses which prevent us from perceiving that an object in motion simply occupies one position after another and does so rather quickly, we foster an illusion about the nature of motion and weave it into a fairy tale. Mathematics helps us to analyze and clarify what we perceive, to a point where we are forced to acknowledge, if we no longer wish to be guided by fairy tales, that we live either in Mr. Russell's changeless world or in a world where motion is but a form of rest. The story of motion is the same as the story of rest. It is the same story told at a quicker tempo. The story of rest is: "It is here." The story of motion is: "It is here, it is there." Because, in this respect, it resembles Hamlet's father's ghost is no reason to doubt its existence. Most of our beliefs are chained to less substantial phantoms. Motion is perhaps not easy for our senses to grasp, but with the aid of mathematics, its essence may first be properly understood.

*

At the beginning of the twentieth century it was generally conceded that Cantor's work had clarified the concept of the infinite so that it could be talked of and treated like any other respectable mathematical concept. The controversy which arises wherever mathematical philosophers meet, on paper, or in person, shows that this was a mistaken view. In its simplest terms this controversy, so far as it concerns the infinite, centers about the questions: Does the infinite exist? Is there such a thing as an infinite class? Such questions can have little meaning unless the term mathematical "existence" is first explained.

In his famous "Agony in Eight Fits," Lewis Carroll hunted the snark. Nobody was acquainted with the snark or knew much about it except that it existed and that it was best to keep away from a boojum. The infinite may be a boojum, too, but its existence in any form is a matter of considerable doubt. Boojum or garden variety, the infinite certainly does not exist in the same sense that we say, "There are fish in the sea." For that matter, the statement "There is a number called 7" refers to something which has a different existence from the fish in the sea. "Existence" in the mathematical sense is wholly different from the existence of objects in the physical world. A billiard ball may have as one of its properties, in addition to whiteness, roundness, hardness, etc., a relation of circumference to diameter involving the number π. We may agree that the billiard ball and π both exist; we must also agree that the billiard ball and π lead different kinds of lives.

There have been as many views on the problem of existence since Euclid and Aristotle as there have been philosophers. In modern times, the various schools of mathematical philosophy, the Logistic school, Formalists, and Intuitionists, have all disputed the somewhat less than glassy essence of mathematical being. All these disputes are beyond our ken, our scope, or our intention. A stranger company even than the tortoise, Achilles, and the arrow, have defended the existence of infinite classes —defended it in the same sense that they would defend the existence of the number 7. The Formalists, who think mathematics is a meaningless game, but play it with no less gusto, and the Logistic school, which considers that mathematics is a branch of logic—both have taken Cantor's part and have defended the alephs. The defense rests on the notion of self-consistency. "Existence" is a metaphysical expression tied up with notions of being and other bugaboos worse even than boojums. But the expression, "self-consistent proposition" sounds like the language of logic and has its odor of sanctity. A proposition which is not self-contradictory is, according to the Logistic school, a true existence statement. From this standpoint the greater part of Cantor's mathematics of the infinite is unassailable.

New problems and new paradoxes, however, have been discovered arising out of parts of Cantor's structure because of certain difficulties already inherent in classical logic. They center about the use of the word "all." The paradoxes encountered in ordinary parlance, such as "All generalities are false including this one," constitute a real problem in the foundations of logic, just as did the Epimenides paradox whence they sprang. In the Epimenides, a Cretan is made to say that all Cretans are liars, which, if true, makes the speaker a liar for telling the truth. To dispose of this type of paradox the Logistic school invented a "Theory of Types." The theory of types and the axiom of reducibility on which it is based must be accepted as axioms to avoid paradoxes of this kind. In order to accomplish this a reform of classical logic is required which has already been undertaken. Like most reforms it is not wholly satisfactory—even to the reformers—but by means of their theory of types the last vestige of inconsistency has been driven out of the house that Cantor built. The theory of transfinites may still be so much nonsense to many mathematicians, but it is certainly consistent. The serious charge Henri Poincaré expressed in his aphorism, "La logistique n'est plus stérile: elle engendre la contradiction," has been successfully rebutted by the logistic doctrine so far as the infinite is concerned.

To Cantor's alephs then, we may ascribe the same existence as to the number 7. An existence statement free from self-contradiction may be made relative to either. For that matter, there is no valid reason to trust in the finite any more than in the infinite. It is as permissible to discard the infinite as it is to reject the impressions of one's senses. It is neither more, nor less scientific to do so. In the final analysis, this is a matter of faith and taste, but *not* on a par with rejecting the belief in Santa Claus. Infinite classes, judged by finite standards, generate paradoxes much more absurd and a great deal less pleasing than the belief in Santa Claus; but when they are judged by the appropriate standards, they lose their odd appearance, behave as predictably as any finite integer.

At last in its proper setting, the infinite has assumed a respectable place next to the finite, just as real and just as dependable, even though wholly different in character. Whatever the infinite may be, it is no longer a purple cow.

FOOTNOTES

1. We distinguish cardinal from *ordinal numbers*, which denote the relation of an element in a class to the others, with reference to some system of order. Thus, we speak of the *first* Pharaoh of Egypt, or of the *fourth* integer, in their customary order, or of the *third* day of the week, etc. These are examples of ordinals.—P. 30.
2. For the definition of primes, see the Chapter on PIE.—P. 32.
3. This series is said to CONVERGE TO A LIMIT—1. Discussion of this concept must be postponed to the chapters on PIE and the calculus.—P. 38.
4. A transcendental number is one which is not the root of an algebraic equation with integer coefficients. See PIE.—P. 49.
5. Any terminating decimal, such as .4, has a nonterminating form .3999. . . —P. 51.
6. A simple geometric interpretation of the class of all one-valued functions F is the following: With each point of a line segment, associate a color of the spectrum. The class F is then composed of all possible combinations of colors and points that can be conceived.—P. 54.

PIE (π, i, e)
Transcendental and Imaginary

In order to reach the Truth, it is necessary, once in one's life, to put everything in doubt—so far as possible.
—DESCARTES

PERHAPS PURE science begins where common sense ends; perhaps, as Bergson says, "Intelligence is characterized by a natural lack of comprehension of life." [1] But we have no paradoxes to preach, no epigrams to sell. It is only that the study of science, particularly mathematics, often leads to the conclusion that one need only say that a thing is unbelievable, impossible, and science will prove him wrong. Good common sense makes it plain that the earth is flat and stands still, that the Chinese and the Antipodeans walk about suspended by their feet like chandeliers, that parallel lines never meet, that space is infinite, that negative numbers are as real as negative cows, that -1 has no square root, that an infinite series must have an infinite sum, or that it must be possible with ruler and compass alone to construct a square exactly equal in area to a given circle.

Just how far have we been carried by common sense in arriving at these conclusions? Not very far! Yet some of the statements seem quite plausible, even inescapable. It would be wrong to say that science has proved that all are false. We may still cling to the Euclidean hypothesis that parallel lines never meet and remain always equidistant, as long as we remember it is merely a hypothesis, but the statements about the squaring of the circle, the square root of -1, and about infinite series belong in a different category.

The circle *can not* be squared with ruler and compass. -1 *has* a square root. An infinite series *can have* a finite sum. Three symbols, π, i, e, have enabled mathematicians to prove these statements, three symbols which represent the fruits of centuries of mathematical research. How do they stand up to common sense?

*

The most famous problem in the entire history of mathematics is the "squaring of the circle." Two other problems which challenged Greek geometers, the "duplication of the cube" and the "trisection of an angle," may, as a matter of interest, be briefly considered with the first, even though squaring the circle alone involves π.

In the infancy of geometry, it was discovered that it was possible to measure the area of a figure bounded by straight lines. Indeed, geometry was devised for that very purpose—to measure the fields in the valley of the Nile, where each year the floods from the rising river obliterated every mark made by the farmer to indicate which fields were his and which his neighbor's. Measuring areas bounded by curved lines presented greater difficulties, and an effort was made to reduce every problem of this type to one of measuring areas with straight boundaries. Clearly, if a square can be constructed with the area of a given circle, by measuring the area of the square, that of the circle is determined. The expression "squaring the circle" derives its name from this approach.

The number π is the ratio of the circumference of a circle to its diameter. The area of a circle of radius r is given by the formula πr^2. Now the area of a square with side of length A is A^2. Thus, the algebraic statement: $A^2 = \pi r^2$ expresses the equivalence in area between a given square and a circle. Taking square roots of both sides of this equation yields $A = r\sqrt{\pi}$. As r is a known quantity, the problem of squaring the circle is, in effect, the computation [2] of the value of π.

Since mathematicians have succeeded in computing π with extraordinary exactitude, what then is meant by the statement, "It is impossible to square the circle"? Unfortunately, this question is still shrouded in many misapprehensions. But these would vanish if the problem were understood.

*

Squaring the circle is proclaimed *impossible*, but what does "impossible" mean in mathematics? The first steam vessel to cross the Atlantic carried, as part of its cargo, a book that "proved" it was impossible for a steam vessel to cross anything, much less the Atlantic. Most of the savants of two generations ago "proved" that it would be forever impossible to invent a practical heavier-than-air flying machine. The French philosopher, Auguste Comte, demonstrated that it would always be impossible for the human mind to discover the chemical constitution of the stars. Yet, not long after this statement was made the spectroscope was applied to the light of the stars, and we now know more about their chemical constitution, including those of the distant nebulae, than we know about the contents of our medicine chest. As just one illustration, helium was discovered in the sun before it was found in the earth.

Museums and patent offices are filled with cannons, clocks, and cotton gins, already obsolete, each of which confounded predictions that their invention would be impossible. A scientist who says that a machine or a project is impossible only reveals the limitations of his day. Whatever the intentions of the prophet, the prediction has none of the qualities of prophecy. "It is impossible to fly to the moon" is meaningless, whereas "We have not yet devised a means of flying to the moon" is not.

Statements about impossibility in mathematics are of a wholly different character. A problem in mathematics which may not be solved for centuries to come is not always impossible. "Impossible" in mathematics means *theoretically* impossible, and has nothing to do with the present state of our knowledge. "Impossible" in mathematics does *not* characterize the process of making a silk purse out of a sow's ear, or a sow's ear out of a silk purse; it *does* characterize an attempt to prove that 7 times 6 is 43 (in spite of the fact that people not good at arithmetic often achieve the impossible). By the rules of arithmetic 7 times 6 is 42, just as by the rules of chess, a pawn must make at least 5 moves before it can be queened.

Where theoretical proof that a problem cannot be solved is lacking, it is legitimate to attempt a solution, no matter how improbable the hope of success. For centuries the construction of a regular polygon of 17 sides was rightly considered difficult, but falsely considered impossible, for the nineteen-year-old Gauss in 1796 succeeded in finding an elementary construction.[3] On the other hand, many famous problems, such as Fermat's Last Theorem,[4] have defied solution to this day in spite of heroic researches. To determine whether we have the right to say that squaring the circle, trisecting the angle, or duplicating the cube is *impossible*, we must find logical proofs, involving purely mathematical reasoning. Once such proofs have been adduced, to continue the search for a solution is to hunt for a three-legged biped.[5]

*

Having determined what mathematicians mean by impossible, the bare statement, "It is impossible to square the circle" still remains meaningless. To give it meaning we must specify *how* the circle is to be squared. When Archimedes said, "Give me a place to stand and I will move the earth," he was not boasting of his physical powers but was extolling the principle of the lever. When it is said that the circle cannot be squared, all that is meant is that this *cannot be done with ruler and compass alone,* although with the aid of the integraph or higher curves the operation does become possible.

Let us repeat the problem: It is required to construct a square equal in area to a given circle, by means of an exact theoretical plan, using only two instruments: the ruler and compass. By a ruler is meant a straightedge, that is, an instrument for drawing a straight line, not for measuring lengths. By a compass is meant an instrument with which a circle with any center and any radius can be drawn. These instruments are to be used a finite number of times, so that limits or converging processes with an infinite number of steps may not be employed.[6] The construction, by purely logical reasoning, depending only on Euclid's axioms and theorems, is to be absolutely exact.

The concepts of "limit" and "convergence" are more fully explained elsewhere,[7] but a word about them here is in place.

Consider the familiar series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$ The sum of the first 5 terms of this series is 1.9375; the sum of the first 10 terms is 1.9980 . . . ; the sum of the first 15 is 1.999781 . . . What is readily apparent is that this series tends to choke off, i.e., the additional terms which are added become so small that even a vast number will not cause the series to grow beyond a finite bound. In this instance the bound, or limit, is 2. Such a series which chokes off is said to "*converge*"[8] to a "*limit.*"

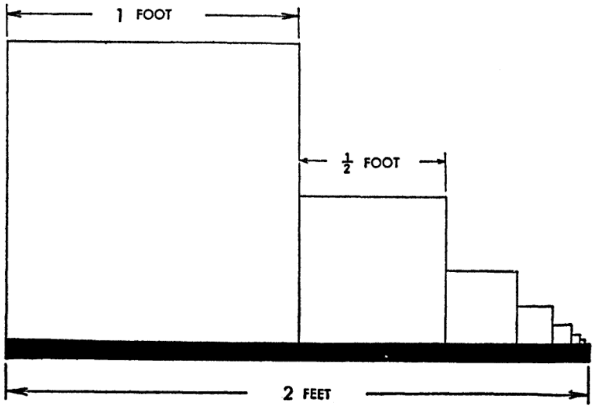

FIG. 15.—An infinite number of terms with a finite sum. If the width of the first block is one foot, the width of the second $\frac{1}{2}$ foot, of the third $\frac{1}{4}$ foot, of the fourth $\frac{1}{8}$ foot, and so one, then an infinite number of blocks rests on the 2-foot bar, that is:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 2.$$

The geometric analogues of the concepts of limit and convergence are equally fruitful. A circle may be regarded as the limit of the polygons with increasing number of sides which may be successively inscribed in it, or circumscribed about it, and its area as the common limit of both of these sets of polygons.

This is not a rigorous definition of limit and convergence, but too often mathematical rigor serves only to bring about another kind of rigor—*rigor mortis* of mathematical creativeness.

To return to squaring the circle: the Greeks, and later mathematicians, sought an exact construction with ruler and compass, but always failed. As we shall see later, all ruler and compass constructions are geometric equivalents of first- and second-degree *algebraic* equations and combinations of such equations. But the German mathematician Lindemann, in 1882, published a proof that π is a *transcendental* number and thus any equation which satisfies it cannot be algebraic and surely not algebraic of first or second degree. It follows that the statement, "The squaring of the circle is impossible with ruler and compass alone," *is* meaningful.

So far as the other two problems are concerned, thanks in part to the work of "the marvelous boy . . . who perished in his prime," the sixteen-year-old Galois, it was established about one hundred years ago that the duplication of the cube and the trisection of an angle are also impossible with ruler and compass. We may allude to them briefly.

There is a story among the Greeks that the problem of duplicating the cube originated in a visit to the Delphic oracle. There was an epidemic raging at the time, and the oracle said the epidemic would cease only if a cubical altar to Apollo were doubled in size. The masons and architects made the mistake of *doubling* the side of the cube, but that made the volume *eight* times as great. Of course the oracle was not satisfied, and the Greek mathematicians, on re-examining the problem began to see that the right answer involved, not doubling the side, but multiplying it by the cube root of 2. This could not be done geometrically with ruler and compass. They finally succeeded by using other instruments and higher curves. The oracle was appeased and the epidemic ceased. You may believe the story or not, much as you choose, but you cannot "duplicate the cube."[9]

The trisection of an angle has received a good deal of attention in the newspapers during the past few years because monographs continue to crop up which claim to solve the problem completely. The fallacies contained in these "solutions" are of four kinds: they are sometimes merely approximate and not exact; instruments other than the ruler and compass are occasionally used, either wittingly or unwittingly; at times there is a logical fallacy in the intended proof; and often only special and not general angles are considered. An angle can be bisected but not trisected by elementary geometry, since the first problem involves merely square roots, while the second involves cube roots, which, as we have stated, cannot be constructed with ruler and compass.

*

The difficulty in squaring the circle, as stated at the outset, lies in the nature of the number π. This remarkable number, as Lindemann proved, cannot be the root of an algebraic equation with integer coefficients.[10] It is therefore not expressible by rational operations, or by the extraction of square roots, and as only such operations can be translated into an equivalent ruler and compass construction, it is impossible to square the circle. The parabola is a more complicated curve than a circle, but nevertheless, as Archimedes knew, any area bounded by a parabola and a straight line can be determined by rational operations, and hence the "parabola can be squared."

Lindemann's proof is too technical to concern us here. If, however, we consider the history and development of π, we shall be in a better position to understand its purpose without being compelled to master its difficulties.

If a triangle is *inscribed* in a circle (Fig. 16), the area of the inscribed triangle will be less than the area of the circle:

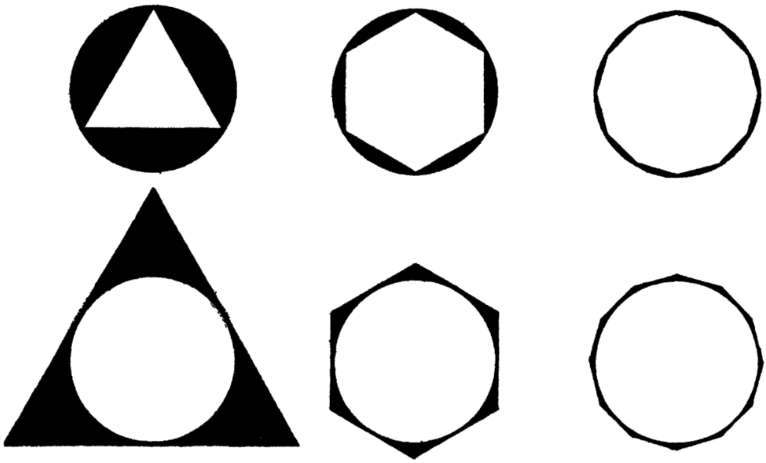

FIG. 16.—The circle as the limit of inscribed and circumscribed polygons.

The difference between the area of the circle and the triangle are the three shaded portions of the circle. Now consider the same circle with a triangle *circumscribed* about it (Fig. 16). The area of the circumscribed triangle will be greater than the area of the circle. The three shaded portions of the triangle again represent the difference in area. It may readily be seen that if the number of sides of the inscribed figure is doubled, the area of the resulting hexagon will be less than the area of the circle, but closer to it than the area of the inscribed triangle. Similarly, if the number of sides of the circumscribed triangle is doubled, the area of the circumscribed hexagon will still be greater than the area of the circle but, again, closer to it than the area of the circumscribed triangle. By well-known, simple, geometric methods, employing only ruler and compass, the number of sides of the inscribed and circumscribed polygons may be doubled as many times as desired. The area of the successively inscribed polygons will approach that of the circle, but will always remain *slightly less;* the area of the circumscribed polygons will also approach that of the circle but their area will always remain *slightly greater*. The common value approached by both is the area of the circle. In other words, the circle is the *limit* of these two series of polygons. If the radius of the circle is equal to 1, its area, which equals πr^2, is simply π.

This method of increasing and decreasing polygons for computing the value of π was known to Archimedes, who, employing polygons of 96 sides, showed that π is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$. Somewhere in between lies the area of the circle.

Archimedes' approximation for π is considerably closer than that given in the Bible. In the Book of Kings, and in Chronicles, π is given as 3. Egyptian mathematicians gave a somewhat more accurate value—3.16. The familiar decimal—3.1416, used in our schoolbooks, was already known at the time of Ptolemy in 150 A.D.

Theoretically, Archimedes' method for computing π by increasing the number of sides of the polygons may be extended indefinitely, but the requisite calculations soon become very cumbersome. None the less, during the Middle Ages such calculations were zealously carried out.

Francisco Vieta, the most eminent mathematician of the sixteenth century, though not a professional, made a great advance in the calculation of π in determining its