MATHEMATICS FOR HUMAN FLOURISHING

FRANCIS SU

WITH REFLECTIONS BY CHRISTOPHER JACKSON

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mathematics for human flourishing

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with reflections by

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preface

This book is not about how great mathematics is, though it is, indeed, a glorious endeavor. Nor does it focus on what math can do, though it undeniably can do many things. Rather, this is a book that grounds mathematics in what it means to be a human being and to live a more fully human life.

This book grew out of a speech I gave in January 2017 at the end of my term as the president of the Mathematical Association of America. Although I was addressing a conference of mathematicians, the underlying themes were universal and the message resonated in ways I could not have anticipated. The tearful response of the audience showed me that there truly is a need, even among those who do math for a living, to talk about our longings for the common good, and the need for us to be better human beings to one another. After the speech was reported in *Quanta Magazine* and *Wired,* I received numerous letters from people whose experiences of math matched my own: hurtful ones when it is not practiced well, and joyful ones when we see how different it can be.

To welcome everyone to this conversation, I've aimed this book at a wide audience—especially those of you who don't see yourselves as "math people." Maybe the way for you to see yourself in mathematics is not for me to convince you that math is great or that math does lots of wonderful things, but for me to

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show you that math is intimately tied to being human. For then your deepest human desires reveal your mathematical nature—and you need only to awaken it.

I won't assume much background; I know we've all had different experiences in math, and it's totally fine to come to this book as you are. I'll refer to some mathematical ideas here and there and try to connect them to things you may know, in much the same way that you might have casual conversations about philosophy or music or sports. You may be reading this book on behalf of someone you know who is learning mathematics, and for that reason I will sometimes give advice to those who teach. No matter who you are, I hope you'll read this book as an invitation and think of the ideas as conversation starters—in the home, in the classroom, or among friends—for how to imagine mathematics in a new way.

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1
flourishing

Every being cries out silently to be read differently.
Simone Weil

Christopher Jackson is an inmate in a high-security federal prison. He's been in trouble with the law since he was fourteen. He didn't finish high school, he had an addiction to hard drugs, and at age nineteen he was involved in a string of armed robberies that landed him in prison with a thirty-two-year sentence.

By now, you've probably formed a mental image of who Christopher is, and you might be wondering why I'm opening

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with his story. When you think about who does mathematics, would you think of Christopher?

Yet he wrote me a letter after seven years in prison. He said:

> I've always had a proclivity for mathematics, but being in a very early stage of youth and also living in some adverse circumstances, I never came to understand the true meaning and benefit of pursuing an education. . . .
>
> Over the last 3 years I have purchased and studied a multitude of books to give me a profound and concrete understanding of Algebra I, Algebra II, College Algebra, Geometry, Trigonometry, Calculus I and Calculus II.

When you think about who does mathematics, would you think of Christopher?

Every being cries out silently to be read differently.

Simone Weil (1909–1943) was a well-known French religious mystic and a widely revered philosopher. She is probably less well known as the younger sister of André Weil, one of history's most famous number theorists.

For Simone, to *read* someone means to interpret or make a judgment about them. She's saying, "Every being cries out silently to be judged differently." I wonder if Simone was crying out about herself. For she, too, loved and participated in mathematics, but she often compared herself unfavorably to her brother. In a letter to a mentor, she wrote:

> At fourteen I fell into one of those fits of bottomless despair that come with adolescence, and I seriously thought of dying because of the mediocrity of my natural faculties. The exceptional gifts of my brother, who had a childhood

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Simone Weil, around 1937.
Photo courtesy of Sylvie Weil.

> and youth comparable to those of Pascal, brought my own inferiority home to me. I did not mind having no visible successes, but what did grieve me was the idea of being excluded from that transcendent kingdom to which only the truly great have access and wherein truth abides. I preferred to die rather than live without that truth.[1]

We know Simone loved mathematics because she used mathematical examples throughout her philosophical writing.[2] And you'll find her with André in photos of Bourbaki, a group of reformist French mathematicians, as a conspicuously lone woman. Their prank-filled meetings were perhaps not the most inviting places to be a woman.[3]

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A meeting of Bourbaki, around 1938. Simone Weil is seen at left, leaning over her notes. André Weil is waving the bell. Photo courtesy of Sylvie Weil.

I often wonder what her relationship to mathematics would have been like if she hadn't always been in André's shadow.[4]

Every being cries out silently to be read differently.

I am a joyful enthusiast of mathematics, a teacher of mathematics, a researcher in mathematics, and former president of the Mathematical Association of America. So you might think that my relationship to mathematics has always been solid. I don't like the word *success,* but people think of me as successful, as if the true measure of my mathematical achievement were the awards I've received or the papers I've published. Even though

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I've had advantages, including a middle-class background and parents who pushed me to excel, my pursuit of math, even for nobler reasons than achievement, has had its obstacles.

As a child I was attracted to the beautiful ideas of mathematics, and I longed to learn more. But I grew up in a small rural town in south Texas, with limited opportunities. There were few advanced math or science courses available in my high school, since college wasn't a standard option for its students. I didn't have a large network of friends excited about math. My parents, as motivated as they were to help me learn, didn't know where to look to nourish my interest in mathematics; finding such resources was even harder in the era before the internet existed. I mainly relied on older books from the public library. My love for math deepened when I was an undergraduate at the University of Texas, and I was admitted to Harvard for my PhD. But I felt out of place there, since my college degree was not from an Ivy League school and, unlike many of my peers, I did not have a full slate of graduate courses when I entered. I felt like Simone Weil, standing next to future André Weils, thinking I would never be able to flourish in mathematics if I was not like them.

I was told by one professor, *You don't have what it takes to be a successful mathematician.* That unkind remark forced me to consider, among other things, why I wanted to do mathematics. To do mathematics means more than just learning the facts of mathematics—it means seeing oneself as a capable mathematical learner who has the confidence and the habits of mind to tackle new problems. Unexpectedly, I joined the company of the multitude of people who have been wounded by harsh judgments and have questioned their capacity for math. There are many others who question the point of learning math, and still others who don't have access to a quality mathematical educa-

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tion. In the face of so many obstacles, it is a fair question for all of us to consider:

Why do mathematics?

Why was Christopher sitting in a prison cell studying calculus, even though he wouldn't be using it as a free man for another twenty-five years? What benefit does mathematics have for him? Why was Simone so captivated by transcendent mathematical truths? What do they offer, that she so desperately yearned to know more? Why should you persist in learning math or in seeing yourself as a mathematical explorer when others are telling you in subtle and not-so-subtle ways that you don't belong?

In this present moment, society is also asking what its relationship with mathematics should be. Is mathematics only a tool to make you "college- and career-ready" so you can achieve your real aims in life? Or is mathematics unnecessary for most of us and relevant only to an elite few? What value is there in studying math if you'll never use what you're learning? Tomorrow's jobs may not even use the math you learn today.

Amid the great societal shifts wrought by the digital revolution and the transition to an information economy, we are witnessing the rapid transformation of the ways we work and live. Mathematical tools are now prominent in every sector of the workforce, including the most dominant ones; presently, technology companies are the four most valuable companies in the world.[5] This means that power is now even more vested in those with mathematical skills.[6] In the span of a young person's lifetime, the tools of our daily lives have become mathematical as well. Search engines now satisfy our every investigative whim, with algorithms powered by linear algebra and advertising powered by game theory. Smartphones have become our digital

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butlers, storing our data in algebraically locked closets, interpreting our voice commands with statistical sensibilities, and pleasing us with a selection of analytically decompressed music.

Yet society has not taken seriously its obligation to provide a vibrant mathematics education for everyone. In many schools, teachers lack sufficient support. Outdated curricula and pedagogies prevent many students from experiencing math as a fascinating area of exploration, culturally relevant and important in all spheres of life. We hear voices in the public square saying that high school students don't need algebra, or that few people need to be good at math—implying that math is best left to the mathematicians.[7] Some college mathematics faculty effectively declare the same thing by abdicating the teaching of introductory classes, or by viewing the undergraduate math degree as only a pipeline for the production of math PhDs. Over many decades and at all levels—elementary school through college—there have been calls to change the way math is taught;[8] nevertheless, change has been slow, in part because the math curriculum has often served as a backdrop to political quarrels over the nature of education itself.[9]

We are not educating ourselves as well as we should, and like most injustices, this especially harms the most vulnerable. Lack of access to mathematics and lack of welcome in mathematics have had devastating consequences for the poor and other disadvantaged groups.[10] Not tapping everyone's potential is a loss for all of us and will limit the ability of future generations to solve the problems they will face.

Our failure to invest in people is already affecting us now. We are easily manipulated when we don't understand how new technologies work but expect them to make decisions on our behalf. We've been unaware of the ways that algorithms are

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used to sort us and track us and divide us—showing us different news, selling us different loans, and stirring different emotions in us than in our neighbors.[11] We witness entrepreneurs unwilling to critique the technologies they are inventing, politicians unable to hold them accountable because of a lack of mathematical sophistication, and a general public unprepared to contemplate its relationship to these technologies.

We all know there's math under the hood, but otherwise math seems cold, logical, and lifeless. No wonder we don't feel a personal connection to it. No wonder we don't feel a responsibility for how it's used.

You and I can change that. All of us have the capacity to embrace the wonder, power, and responsibility of mathematics by nourishing our affection for it. The need to do so in today's world cannot be overstated, and the stakes are high.

A society without mathematical affection is like a city without concerts, parks, or museums. To miss out on mathematics is to live without an opportunity to play with beautiful ideas and see the world in a new light. To grasp mathematical beauty is a unique and sublime experience that everyone should demand.

All of us—no matter who you are or where you're from—can cultivate mathematical affection. All of us can have a different relationship to mathematics than we imagined. All of us can read ourselves and one another differently.

I'm speaking to the demoralized, who've been injured by words someone said about their math abilities. I'm speaking to the disenchanted, for whom math has become boring. I'm speaking to those who haven't had the resources or the confidence to get a mathematical education but have always been curious about how things work. I'm speaking to the artist who never thought math was beautiful, the social worker who never

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thought math was relational, and the mathematician who never thought math was accessible to anyone else.

And I'm speaking to both those who teach math and those who think they'll never teach math—because *every single one of us is, whether we realize it or not, a teacher of math.* We all communicate attitudes about mathematics through what we say to others, and our words have indelible effects. You can communicate negativity: "I was never any good at math." "That subject is for boys." "Don't hang out with her—she's a nerd." "Son, I'm not a math person, and you probably have my genes." "Why would you take another math class?" Or you can communicate positivity: "Math is an exploratory adventure." "You *can* improve your math skills, just like I can improve my free throws." "Math is power to see hidden patterns." "Everyone has promise in mathematics."

You may be a parent someday, or an aunt or uncle, a youth group leader, a community volunteer, or in another position where you influence others—if so, you will be a teacher of math. If you help kids with their homework, you are a teacher of math. If you are afraid to help kids with their homework, you are teaching an attitude about math. Studies show that parents with math anxiety pass on that anxiety to their children. In fact, math-anxious parents are more likely to pass on math anxiety if they try to help their kids with their math homework than if they don't.[12] So your disposition toward math matters as much for a child's sake as for your own.

Reading ourselves differently will require all of us—those who've failed at math and those who've been successful—to change our view of what mathematics is and who should be learning it. This will require teachers to change their view of how they should be teaching it. We'll need to speak about it in a different way—and if we do, more of us will be drawn to mathe-

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matics when we see how mathematics connects to our deepest human desires.

So if you ask me, “Why do mathematics?” I will say this: “Mathematics helps people flourish.”

Mathematics is for human flourishing.

Human flourishing refers to a wholeness—of being and doing, of realizing one’s potential and helping others do the same, of acting with honor and treating others with dignity, of living with integrity even in challenging circumstances. It is not the same as happiness, and it is not just a state of mind. The well-lived life is a life of human flourishing. The ancient Greeks had a word for human flourishing—*eudaimonia*—which they viewed as the highest good: “the good composed of all goods; an ability which suffices for living well.”[13] There is a similar word in Hebrew—*shalom*—which is used as a greeting. *Shalom* is sometimes translated as “peace,” but the word has a far richer context. To say *shalom* to someone is to wish that they will flourish and live well. And Arabic has a related word: *salaam*.

A basic question, taken up by human beings throughout the ages, is: How do you achieve human flourishing? What is the well-lived life? The philosopher Aristotle said that flourishing comes through the exercise of virtue. The Greek concept of virtue is excellence of character that leads to excellence of conduct. So it includes more than just moral virtue; for instance, traits like courage and wisdom and patience are also virtues.

I claim that the proper practice of mathematics cultivates virtues that help people flourish. These virtues serve you well no matter what profession you choose or where your life takes you. And the movement toward virtue is aroused by basic human desires—the universal longings that we all have—which fundamentally motivate everything we do. These desires can be chan-

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neled into the pursuit of mathematics; the resulting virtues can enable you to flourish.

Consider this analogy: if doing mathematics is like navigating a sailboat, then human desires are the winds that power the sails, and virtues are the qualities of character that sailing builds—mindfulness, attention, and harmony with the wind. Of course, sailing is useful in getting us from point A to point B, but that is not the only reason to sail. And there are technical skills that must be mastered to sail well, but we don't learn sailing to become better at tying knots. Similarly, math skills are valuable, but they cannot serve as goals. The skills society needs from math may change, but the virtues needed from math will not.

In promoting the human side of mathematics, I join a growing chorus of those who have called for a humanization of mathematics and math education, often in service of addressing long-standing inequities, by shifting away from contextless portrayals of mathematics to reveal its social and cultural dimensions.[14] That laudable goal will not be possible—and will often be resisted—unless we name a purpose for learning mathematics that is more than just memorizing procedures for an eventual career.

When some people ask, "When am I ever going to use this?" what they are really asking is "When am I ever going to value this?"[15] They're equating math's value with utility because they haven't seen that they can value anything more. A grander, more purposeful vision of mathematics would tap into the desires that can entice us to do mathematics as well as the virtues that mathematics can build.

Therefore, each of the following chapters is devoted to one basic human desire whose fulfillment is a sign of human flourishing. In each, I illustrate how the pursuit of mathematics can meet this desire, and I illuminate the virtues that are cultivated

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by engaging in math in this way. Changing the practices of mathematics so that they do, in fact, meet these desires is our common responsibility if everyone is to flourish in mathematics.

I know some may hear me broach virtue and think I'm saying, Math makes you better than other people. No—I'm not saying that math gives you greater claim to worthiness or human dignity. I'm saying that the pursuit of math can, if grounded in human desires, build aspects of character and habits of mind that will allow you to live a more fully human life and experience the best of what life has to offer. None of us is wholly virtuous; we all are works in progress with room to grow. And there are many ways to grow in virtue, not just in mathematics. But does the proper practice of mathematics build *particular* virtues, like the ability to think clearly and to reason well? Unequivocally yes, and it may do so in a distinctive way.

Because I speak so highly of mathematics, you may think I idolize mathematics as the ultimate pursuit, to be prized above all other pursuits in life. That is not the case either—we must each discover what gives our soul its greatest purpose. Still, mathematics is a marvelous human endeavor—worth the effort to explore and participate in, and worth the effort to help others do the same—because it meets basic human desires and contributes in unique ways to a life lived well.

I hope that by seeing yourself in these desires, you can see yourself as a *mathematical explorer,* who can think in mathematical ways and who is welcome in mathematical spaces. And where the practice of mathematics is not yet grounded in these desires, I hope you will join me in changing it. In doing so, you will have new ways to experience mathematics, not just as a toolbox of facts and skills, but as a force for the flourishing of all.

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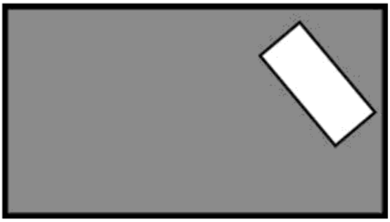
Mathematical exploration begins with questions. So I'm scattering a few puzzles throughout this book. No pressure—you can skip them if you wish, or think about just the ones that seem enticing. Hints and solutions can be found in the back, but before you look there, I recommend playing around with each problem.

DIVIDING BROWNIES

A father bakes brownies in a rectangular pan as an after-school snack for his two daughters. Before his daughters get home, his wife comes along and removes a rectangle from somewhere in the middle, with the sides of the rectangle not necessarily parallel to the sides of the pan.

How can he make one straight cut and divide the remainder of the brownies evenly between his two daughters so that they get the same area?

A version of this puzzle was featured on the NPR show *Car Talk.*[a]



TOGGLING LIGHT SWITCHES

Imagine 100 lightbulbs, each with a switch numbered 1 through 100, all in a row, and all lights off. Suppose you do the following: toggle all switches that are multiples of 1, then toggle all switches that are multiples of 2, then toggle all switches that

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are multiples of 3, etc., all the way to multiples of 100. (To toggle a switch means to flip it on if it's off and off if it's on.)

When you are done, which lightbulbs are on and which are off? Do you see a pattern? Can you explain it?

a. See https://www.cartalk.com/puzzler/cutting-holey-brownies.

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Nov. 26, 2013

Hello, Mr. Su. My name is Christopher Jackson, an inmate at United States Penitentiary McCreary in Pine Knot, Kentucky. I am 27 years old and I've been in prison a little over 7 years. I have a 32 year sentence for a string of armed robberies I committed in which no one was hurt, at the age of 19, while suffering from a severe addiction to a multitude of hard drugs.

I've always had a proclivity for mathematics, but being in a very early stage of youth and also living in some adverse circumstances, I never came to understand the true meaning and benefit of pursuing an education. At the age of 14, I became more involved with the juvenile justice system, because I left the house of my guardians, soon after I left off from attending school, and began becoming more involved in a criminal lifestyle. At the age of 17, at the prodding of my case manager and others supporting me, I got my G.E.D. and enrolled in Atlanta Technical College, but didn't attend but for a few days, and returned back deeper into my life of crime. Over the next couple of years, I struggled with addiction while going back and forth to jail until I committed the crimes I am now serving this sentence for. After pleading guilty to two charges in my indictment at the age of 21, I was remanded to federal custody and sent here to this facility, where I have spent the last 4 years of my life.

Over the last 7 years, I have developed an acute interest in studies and books concerning philosophy, mathematics, finance, economics, business, and politics. And over the last 3 years I have purchased and studied a multitude of books to give me a profound and concrete understanding of Algebra I, Algebra II, College Algebra, Geometry, Trigonometry, Calculus I and Calculus II.

The vast majority of problems in my life have stemmed from my hardheartedness and my unwillingness to listen to people

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who actually did know better than me or had positions of authority. Because even with a missing and deceased father, I had a mother, aunties, and a grandmother that really and actually attempted to raise me right. Every day that I wake up and go about my life I try not to allow the present circumstances that I've created for myself to determine the passions and the future I want to see myself in.

I became aware of your institution from a book I read by a Professor who teaches there, and by your institution being mentioned a couple of times on a television show I watch regularly.

I am a person of limited means, but I would like to know if there is a program that you have so I can by way of correspondence earn a degree in the art of mathematics from your school. I know you are a very busy person so I wouldn't want to take up any more of your time. But thank you for your time and consideration.

Christopher Jackson

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2
exploration

It is like being lost in a jungle and trying to use all the knowledge that you can gather to come up with some new tricks, and with some luck you might find a way out.
Maryam Mirzakhani

The world of math is more weird and wonderful than some people want to tell you.
Eugenia Cheng

My friend Christopher Jackson is a mathematical explorer. He's limited by circumstance, but not by imagination. He's curi-

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ous and inventive; he's fearless and persistent. Chris likes the challenge a good problem can bring.

For the past several years, Chris has been on a journey. He's exploring mathematics with fresh eyes, beginning to see that it's different from the dry and uninspired form of math he'd been taught before. He's growing in his knowledge and love of the subject, in spite of his isolation in prison and the difficulties that brings. I've had the privilege of observing, at a distance, his transformation.

Life hasn't been easy for Chris. He grew up in a working-class neighborhood of Augusta, Georgia, raised by his mother, with help from his aunts and his grandmother. Chris never knew his father, who succumbed to a crack cocaine addiction and died tragically in a car accident before Chris was two, hit by an eighteen-wheeler on a highway. Chris had some good influences—his mother instilled a love for books in him by reading to him frequently—but he had some negative influences as well. That led, in his teenage years, to drug addiction and the series of crimes that he described in his first letter to me.

I was cautious of, but also captivated by, that letter from the Pine Knot penitentiary in November 2013 (see facing page). It was handwritten, with orderly penmanship and earnest lines. I imagined a young man who had labored with care over every word. I couldn't see him—I could know him only through his writing—but perhaps the content of his character was more visible that way. I was touched by Chris's reflections about his troubled past, the future he would like to see himself in, and the way he had pursued his interest in math by teaching himself from books. I regretted that my school didn't have a distance program for him.

Chris and I have corresponded by mail, off and on, for six years now. We've talked about our mutual interest in mathemat-

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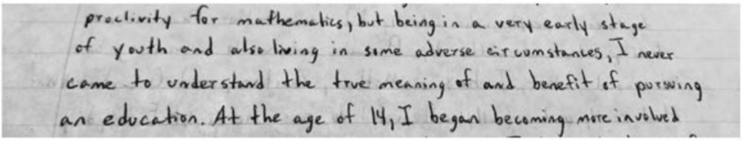
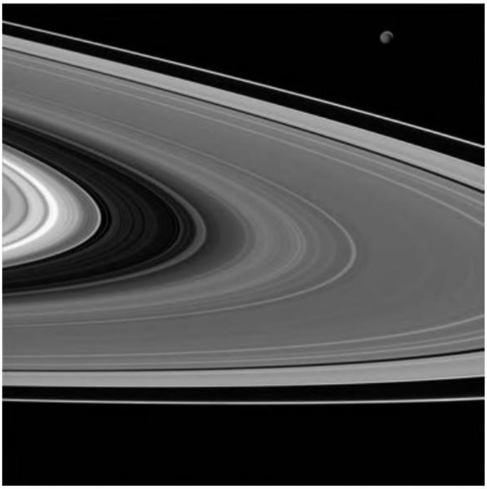

proclivity for mathematics, but being in a very early stage of youth and also living in some adverse circumstances, I never came to understand the true meaning of and benefit of pursuing an education. At the age of 14, I began becoming more involved

ics and about life. With Chris's permission, I'm sharing excerpts from our correspondence, because his insights and experience amplify everything I am saying in this book. This is not a story about how I helped Chris do math in prison. Rather, this is a story about how Chris is coming to see himself, and mathematics, in a new way. His insights and his journey have inspired *me,* in writing a book about flourishing, to believe more fully that mathematics has something to offer everyone.

I am a mathematical explorer. My journey has been different than Chris's, but we've both been enticed by the power of mathematical exploration to awaken the imagination. As a child, I loved the stars, and in the rural Texas town where I lived, far from any big city, I could see even the dimmest ones. I begged my parents for a telescope, but we didn't have the money. So I devoured books on astronomy and dreamed about space. I wanted to be an astronaut, to visit other worlds and encounter strange new life-forms. That seemed exciting, until I realized how long it would take to travel to even the nearest star and thought about all the people I'd have to leave behind. But that didn't stop me from fantasizing. I fed myself a steady diet of science fiction, captivated by stories like Isaac Asimov's *Nightfall,* a novella about what happens when night finally comes to civilization on a planet with six suns. I could visit this world in my mind.

My childhood imagination was further stoked by the journeys of the *Pioneer* and *Voyager* probes through the solar sys-



Mimas by Saturnshine. The moon Mimas is being illuminated by sunlight reflected off Saturn. The Cassini division is the largest gap in the rings, visible on the left side of the photo. Image courtesy NASA/JPL-Caltech/Space Science Institute. Taken by the *Cassini* spacecraft on February 16, 2015.

tem in the late 1970s and early 1980s. For the first time, scientists captured up-close photographs of the moons of Jupiter and the rings of Saturn. Reaching these worlds required years of creative planning for all scenarios, good and bad, that these crewless probes might encounter. Just as the scientists themselves were making their discoveries from a distance, I could vicariously explore these worlds from a little town in south Texas. I loved poring over the *Voyager* images printed in the newspaper.

Mathematics can literally be seen in these worlds. The rings of Saturn encircle the planet in its equatorial plane. From a distance they look like stationary annular bands, but the rings are basically made of a multitude of boulder-size rocks (moonlets) that contain mostly ice and orbit the planet because of the force

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of gravity. The astronomer Galileo Galilei first observed the rings through a telescope in 1610. Unsure of what they were, he playfully referred to them as ears.[1] Later astronomers identified these structures as rings, with gaps between them. The *Voyager* probes enabled us to see the structure of the rings in finer detail, such as their pattern of high- and low-density ripples, much like the grooves on an old vinyl record.

Some of the structure of the rings, as I learned, can be explained with mathematical insight. All of the icy rocks at the same distance from Saturn take the same time to do one orbit—that's called their orbital period. The rocks that are farther away from Saturn have longer orbital periods and slower speeds than the ones that are closer, because they are less influenced by the planet's gravity. Think of the rings like a racetrack around Saturn, in which runners on the inner lanes go faster and also have less distance to travel than those on the outer lanes.

Something special happens when the orbital period of a rock is in one of certain exact whole-number ratios with the period of a moon of Saturn. For instance, suppose you have a rock and a moon revolving around Saturn such that in the time it takes the moon to do one orbit on an outer lane, the rock does precisely two on an inner lane. Every two orbits, the rock will pass the moon in exactly the same location in the rock's orbit.

The moon's strongest gravitational tug on the rock happens at these times of closest approach. Because these repeated tugs happen in the same location, they tend to reinforce each other and perturb the rock's orbit, much as pushing a child on a swing in sync with its motion causes the child to go higher. Thus, all the rocks at that distance from Saturn, with the same orbital period, tend to swing out of that orbit. This effect is called *resonance,* and when it is very strong it can create a gap in the rings.

The largest gap, some 3,000 miles wide, is called the Cassini

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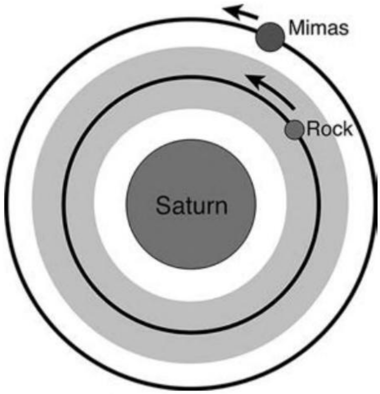

Icy rock in an inner orbit around Saturn catching up with Mimas. The effects of Mimas's gravity may accumulate and perturb the rock's orbit if the rock always passes the moon in the same location.

division and is the result of a two-to-one resonance between orbiting rocks and Saturn's moon Mimas. You can get resonance effects for other small-whole-number ratios of moon-to-rock periods (like three-to-two or four-to-three), though they are less pronounced, often more like ripples than gaps. Resonance effects between moons and rocks can explain many features in Saturn's rings.[2] In effect, we are seeing numerical ratios—simple fractions—whipping up visible patterns in the delicate orbital dances of these icy rocks! For a kid like me, it was enchanting that a little exploration, using mathematics and the imagination, could shed insight on objects 900 million miles away.

Mathematical exploration is very much like space exploration, but of a different kind of space—a space of ideas. You don't know what you'll find when you start. You send out probes to test theories. You are captivated by mystery, motivated by

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questions, undeterred by setbacks. You make discoveries from a distance: because the ideas themselves are not physical, you access this space through reason. Exploration and understanding are at the heart of what it means to do mathematics.

Unfortunately, *exploration* is not a word one associates with mathematics if one thinks it is just arithmetic—or something advanced and *even more* dreary that was discovered and settled long ago.

School mathematics sets you up for future exploration, but imagine how different our experiences would be if we could explore math *now,* as we learn it. Picture what it would be like to learn the rules of basketball and practice only free throws but never see a game and never play—until you're ready to go professional.[3] Learning wouldn't have been joyful, and you wouldn't now be prepared.

Exploration is a deep human desire and a mark of human flourishing. You don't need a lot of resources, except your mind, to be a math explorer, and as a result, you can embark on an adventure from anywhere—a prison, a small rural town, a far-flung corner of the earth. It is no surprise, then, that math explorers can be found in every society throughout history. This is most readily apparent in the games that people play, especially games of strategy, which generate interesting mathematical questions.

Achi is a game played by the Ashanti people of Ghana in West Africa. This two-player game is played on a three-by-three diagram of horizontal and vertical lines and two diagonals. Achi is like tic-tac-toe, but with a twist. The players each have just four pieces, which they take turns placing on the board's nine positions. The object is to get three in a row along one of the straight lines. If neither player has gotten three in a row by the time all

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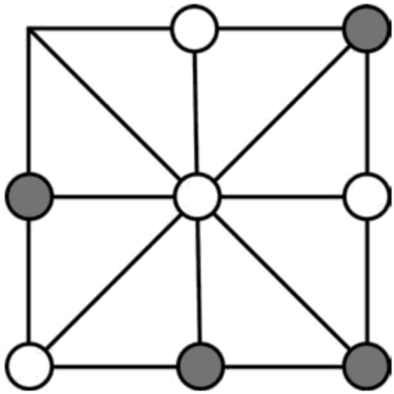

All eight pieces have been placed on this Achi board and there is no winner yet, so the players will take turns pushing one of their pieces onto the empty position until one gets three in a row.

the pieces have been placed, then there is a single empty space on the board. At that point the game moves into a second phase, in which the players take turns pushing one of their pieces along a line onto whatever space is empty. No jumping is allowed. The player who first gets three in a row wins.[4]

This is the standard description of Achi, but the rules leave some ambiguity.[5] For instance, in the second phase, what happens if a player is stuck, with no move to make? Is it impossible for a player to get stuck if both players play smartly (i.e., don't pass up any obvious opportunities to win)? You'll also need to decide whether players should be forced to make a move if they have moves to make. Mathematical reasoning can help you answer these questions and decide which options generate a more interesting game. In these variants, can the game of Achi go on forever, or can one player have a winning strategy (a plan of moves to guarantee a win no matter what the other

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player does)? What if each player has only three pieces instead of four? Can you create an interesting variant of Achi with different lines?

If you ask questions like these, you are a mathematical explorer. You are exploring the space of all possible ways that the game play could proceed. You send out "probes" by trying things. You don't know what you'll find when you start. When you discover answers using mathematical reasoning, you are doing it from a distance, because you can know things about how the game will play out *without* actually playing out every possible scenario.

For instance, in the game of tic-tac-toe there's a clever argument, called a *strategy-stealing argument,* to show that the second player cannot have a winning strategy: If she did have such a strategy, the first player could, by ignoring his own first move, pretend that he was the second player and use her strategy in responding to her! If that strategy ever suggested playing a move that he'd already made, he could just place his next move anywhere else—that's an extra move for him, and in tic-tac-toe any extra move can only help a player to win. The fact that both players can force a win is a contradiction and means that our assumption that the second player had a winning strategy was false. So the first player must be able to force a win or a draw. It's amazing that we can deduce this through mathematical reasoning—without playing a single game of tic-tac-toe.

Every culture plays games of strategy.[6] Every culture has math explorers, because strategic thinking is mathematical thinking. One of the best ways to claim your heritage in mathematics is to find a game of strategy from your own cultural history and embrace the kind of thinking the game requires. Probe it with exploratory questions.

Mathematical exploration begins with questions. The only re-

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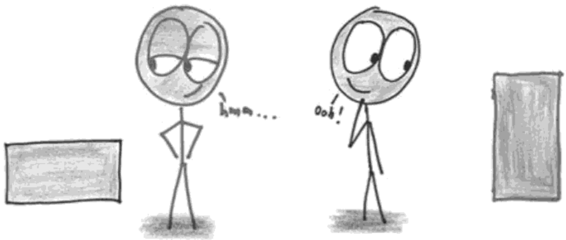
Create two rectangles so that the first has a bigger perimeter, and the second a bigger area.



A Bad Drawing, courtesy of Ben Orlin, from his book *Math with Bad Drawings.*

book *Math with Bad Drawings,* the math teacher Ben Orlin discusses the difference between a dull problem and an exploratory problem.[10] He gives this example:

> Find the area and perimeter of a rectangle with height 3 and width 11.

This problem is dull because it reduces area and perimeter to simple formulas that never force you to grapple with their original meaning. He notes that here, "'area' doesn't refer to the number of one-by-one squares it takes to cover the rectangle; it's just 'the two numbers, multiplied.'" If you do twenty of these problems, you'll never learn anything about geometry. A more interesting, exploratory problem is at the top of this page.

Hmmm . . . much better. This version requires deeper insight about the nature of rectangles and is way more interesting. Orlin notes that you can take it up another notch with this version: "Create two rectangles so that the first has exactly twice the perimeter of the second, and the second has exactly twice

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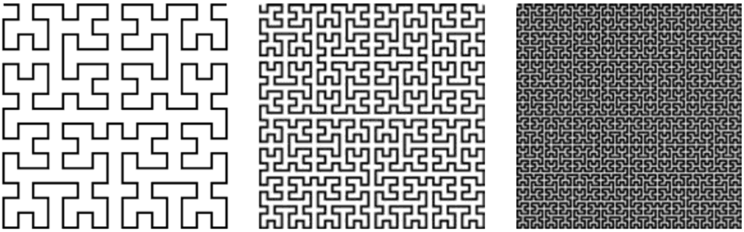

A space-filling curve is constructed as a limit of curves like these, which wind their way through a given region of space in a denser way each time. The mathematical work here is to show that such a limit exists.

the area of the first." Wrestling with good problems like these, you'll invent your own ways of thinking about things and develop your own ways of solving them. This is the best kind of learning.

Exploration cultivates an *expectation of enchantment.* Explorers are excited by the thrill of finding the unexpected, especially things weird and wonderful. It's why hikes through unfamiliar terrain entice us, why unexplored caves beckon to us, why the strange creatures of the deep-sea ocean floor fascinate us—what else may be lurking down there? There's similar enchantment to be found in the zoo of strange discoveries in math. One such curious creature is a *space-filling curve:* a single curve that touches every point inside a square. Although it can't be drawn, only approximated, math tells us this creature exists. As bizarre as they are, space-filling curves now have applications in computer science and image processing.

Another strange animal is the *Banach-Tarski paradox:* the surprising result that a solid ball can be cut into five pieces and reassembled to form *two* solid balls each the same size as the original. You might wonder why this can't be done with balls of gold(!), and the answer—that real matter is not infinitely divis-

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ible like idealized space is—helps us grasp the difference between the nature of real things and mathematical models of them. If you go through life with the eyes of exploration, every new landscape is an opportunity to imagine fanciful things, exercise your creative skills, and discover hidden treasure.

Linda Furuto is a mathematics explorer, and she helps others see themselves as explorers too. She grew up on the North Shore of Oʻahu in Hawaiʻi, spearfishing, diving, swimming, and surfing. Although she struggled with math as a child because she didn't see its relevance, she can now see math embedded everywhere around her, from the dynamics of the oceans to the optimization involved in maximizing her time underwater. Now, as a math education professor at the University of Hawaiʻi at Mānoa, Linda helps students see how mathematics is connected to their cultural histories. She shows students how seeing the world as a math explorer can inform their understanding of marine biology and conservation—the linear functions modeling how coral reefs are being cleared of invasive algae, the matrices describing ocean debris collection, and the quadratic equations involved in sustaining limited island resources. She takes students sailing on *Hōkūleʻa* (Star of gladness), a double-hulled canoe of the Polynesian Voyaging Society, on which they learn about the traditional practice of wayfinding among the Indigenous peoples of Hawaiʻi and the Pacific.[11] Such techniques rely solely on observing clues from nature and the heavens to navigate without instruments. Over the past four decades, *Hōkūleʻa* has sailed more than 160,000 nautical miles, including the Mālama Honua Worldwide Voyage (begun in 2013), dispelling any doubts about the reliability of this ancient practice.[12] Linda's role is apprentice navigator and education specialist on land and sea. She helps students explore the trigonometry and calculus em-

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bedded in knowing wind dynamics and sail mechanics, and why these are significant beyond memorizing formulas:

> I believe it is important that students know what is written in our textbooks, because they contain important information. However, equally critical is that our students understand and realize that their ancestors sailed thousands of miles across the Pacific Ocean without any kind of modern navigational tool—by the sun, the moon, the stars, the winds, the tides, migratory bird patterns, and more. They traversed oceanic highways in the past, and our students are capable of doing the same things inside and outside of our classrooms today.[13]

Indeed, the wayfinders were mathematical explorers of their society, using attentive study, logical reasoning, and spatial intuition to solve the problems they encountered in their cultural moment. Mathematical explorers have been part of every civilization in every corner of the earth, and Linda sees the importance of drawing the straight line from the mathematical explorers in her students' cultural history to the mathematical identity she'd like them to embrace.

Do you have problems you want to solve? Oceans you want to navigate? Patterns in the starry spheres of your life that you wish to understand? Then you can be a math explorer, since you were born with the human capacities to inquire and to reason. Dream of the sun, the moon, the stars, and the world that you will discover. Imaginative, creative, and unexpected enchantments await.

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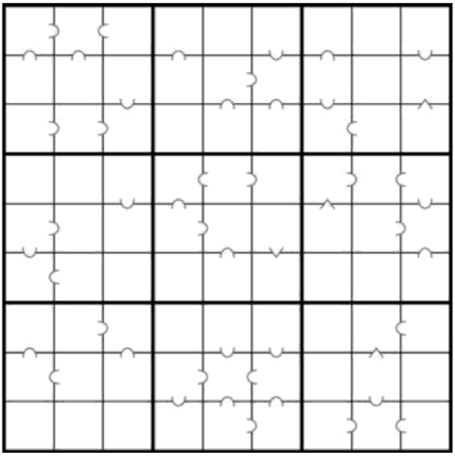
"DIVIDES" SUDOKU

Sudoku is a puzzle that you solve by exploration. This unusual version is courtesy of Philip Riley and Laura Taalman of Brainfreeze Puzzles, from their book *Naked Sudoku*.[a] There are no numerical clues (it's "naked"), yet it has a unique solution.

Rules: Fill in the cells of the grid so that the numbers 1 through 9 appear exactly once in each row, column, and three-by-three block (the usual sudoku rules). In addition, each time a cell's value exactly divides that of one of its neighbors within a three-by-three block, their common border is marked with the symbol ⊂, whose orientation tells us something: cell A ⊂ cell B indicates that the value of cell A exactly divides the value of cell B. A few "greater than" (>) symbols are also provided.



Getting started: You might first think about which numbers divide other ones. For instance, 4 divides 8, so 4 ⊂ 8. Also, 1 divides 3 and 3 divides 9, so 1 ⊂ 3 ⊂ 9. The 1s are usually easy to place, because 1 evenly divides all numbers 1 to 9.

a. Philip Riley and Laura Taalman, Brainfreeze Puzzles, *Naked Sudoku* (New York: Puzzlewright, 2009), 125.

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us *plus* all our luggage. I grew more skeptical on the four-hour journey as we wound our way along bumpy dirt paths populated only by goats. Was this a shortcut? Were there really no paved roads to this village?

Then, as we drove over a particularly uneven path, the car's front tires rolled over a bump and the body got wedged against the bump and would not budge. Our front and back tires spun helplessly against the soft dirt. We were stuck.[1]

This wasn't looking good. We were isolated, miles from any real civilization, on a path that was unlikely to receive any traffic. The day was fading fast, and we could not walk dozens of miles before nightfall.

Our predicament didn't seem like a math problem—it had no numbers, no symbols, no formulas—but I couldn't escape the feeling that my mathematical training might help us. I recalled having seen something like this problem before—in a book by the popular mathematics writer Martin Gardner. A truck is stuck beneath an overpass, the problem goes, unable to back up because of traffic, but too tall to proceed forward. What to do? I remembered the answer: let some air out of the tires. This would lower the truck enough to let it roll under the overpass (see next page).

That puzzle seemed somehow similar to our predicament, yet

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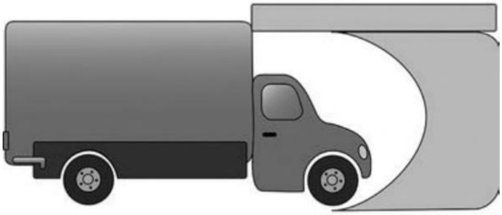

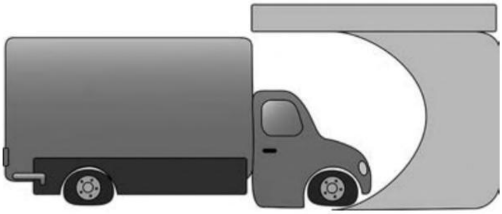

somehow different—we were stuck not under a pass over us, but over a bump under us. Perhaps we could inflate the tires . . . but sadly, we had no pump. What could we do?

There's a moment in problem solving, as you start to brainstorm and think of potential strategies, when you must make sense of the problem as it really is—when you must strip away its nonessential elements so you can classify it and make connections between this problem and the catalog of problems you've tackled in the past. When you do that, you are wrestling with its underlying meaning.

Indeed, when you want to grasp the meaning of something, you are always asking about its relationship to other things. If you muse about the meaning of life, you are contemplating your place in this world. Or, if you ponder the meaning of a strange event, you have made a choice not to view the event in isolation,

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but to think about its causes or its implications for other events. And if you look up the meaning of a word, you'll get a definition that places this word in relation to other words.

When the writer Jorge Luis Borges said that "every word is a dead metaphor," quoting the poet Leopoldo Lugones, he meant that every word has a meaning that comes from a certain history—a context in which that word originates. For instance, *calculus* used to mean "small stone," like the kind that you would find in an abacus, to do arithmetic with. Today the word refers to a much more complicated kind of addition. The word *geometry* used to mean "land measurement." Today it refers to the mathematical insights that inform the measuring of almost anything. Words don't exist in isolation. Each one carries with it metaphors from an ancient but ongoing conversation.

Likewise, mathematical ideas, too, are metaphors. Think about the number 7. To say anything interesting about 7, you have to place it in conversation with other things. To say that 7 is a prime is to talk about its relationship with its factors: those numbers that divide evenly into 7. To say that 7 is 111 in binary notation is to have it dialogue with the number 2. To say that 7 is the number of days in a week is to make it converse with the calendar. Thus, the number 7 is both an abstract idea and several concrete metaphors: a prime, a binary number, and days in a week. Similarly, the Pythagorean theorem is a statement relating the three sides of a right triangle, but it is also, metaphorically, every proof you learn that illuminates why it is true and every application you see that shows you why it is useful. So the theorem grows in meaning for you each time you see a new proof or see it used in a new way. Every mathematical idea carries with it metaphors that shape its meaning. No idea can survive in isolation—it will die.

This is why mathematics, like poetry, can be so satisfying.

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teners to itself and to one another. It is no different with mathematics. Connecting ideas is essential for building meaning in mathematics, and those who do it become natural story builders and storytellers.

Too often in my math education I've been given a concept and asked to do exercises with it but wasn't taught its significance. I would struggle with the concept, because even though it had a definition, it had no meaning—no connection to a larger narrative. However, there were many instances when a story, as captured by a pithy phrase, helped me see the bigger picture. In calculus, when someone said, "Integration by parts is the inverse of the product rule," both concepts became clearer. In statistics, I heard this story: "Learning statistics is learning to be a good detective with data." And in all of mathematics there is this lesson: "Objects are not as important as the functions between objects." This maxim recapitulates what I am saying about mathematical meaning: objects don't have meaning independent of their relationships to other objects. Functions are relationships; functions tell a story.

Now, there are many ways to build a story. Think again of the Pythagorean theorem, which says that the side lengths a, b, and c of a right triangle (a triangle with one ninety-degree angle) satisfy the relationship

$$a^2 + b^2 = c^2$$

where c is the side length of the hypotenuse (the longest side). This is a fact without context and easily forgotten, until you build a story.

You might create a *geometric story,* by drawing the right triangle with squares on each side and realizing that the theorem says the areas of the two smaller squares must add up to that of the largest square (see next page).

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