Texts for Quantitative Critical Thinking

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Mathematics for Sustainability

Foreword by
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Springer

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Contents

Foreword

I'm excited to introduce this book to you, because it may be different from any math text you've read before. It will change the way you look at the world and enlarge the way you think about mathematics. No longer will you be just a spectator when people give you quantitative information—you will become an active participant who can engage and contribute new insights to any discussion. Just look at the verbs that underlie the chapter titles: measure, flow, connect, change, risk, decide!

Here's what stands out to me when I read this book: there are many math books that will feed you knowledge, but it is rare to see a book like this one that will help you cultivate wisdom.

There is a deep difference between knowledge and wisdom. A knowledgeable person may be armed with facts, but a wise person considers how to act in light of those facts. A knowledgeable person may think an answer is the end of an investigation, whereas a wise person considers the new questions that result. And a knowledgeable person might ignore the human element of a problem that a wise person deems essential to understand. As the authors illustrate, mathematics that pays attention to human considerations can help you look at the world with a new lens, help you frame important questions, and help you make wise decisions.

Sustainability asks: how can we be wise stewards of Earth's resources? One way or another this question will impinge on some aspect of your life, if it hasn't already. Sustainability is an economic concern because resources are limited. Sustainability is a moral concern, because any scarcity of Earth's resources will harm the weak and vulnerable first. And sustainability is a scientific concern, because we may have the power to improve the lives of those who will be affected.

I know that each of the authors shares a deep vocational commitment in bringing this book to you, and as evidence, I'll speak personally of the author I have the privilege to know as a friend: John Roe, a man of deep grace and humility who made this book his highest professional priority while battling a difficult illness. For him, this project grew out of a conviction and prayerful reflection that his knowledge as a mathematician and an educator could be channeled into wise action on matters that will impact us all.

The authors have poured their hearts into this remarkably important and timely book, and I hope you will engage it with the same fervor. Because it concerns the world you live in, how you will need to live in it, and the problems that you—yes YOU—can solve so that all of us can live in it well.

Francis Edward Su
Benediktsson-Karwa Professor of Mathematics, Harvey Mudd College
Past President, Mathematical Association of America

Before We Begin. . .

0.1 To the Student

A Letter from the Authors

Dear Student,

The world that you are inheriting is full of bright possibilities—and also of big problems. Many of the problems center on *sustainability questions* like "can this (key part of our social or economic system) last?" or to put it in a way that has a little more math in it, "*how long* can this last?" For example, modern society is a profligate consumer of energy, most of which is supplied by fossil fuels: coal, oil, and natural gas. Fossil fuel supplies, though of vast and unknown size, are limited. *How long can they last?* What's more, there is a strong scientific consensus that the carbon dioxide gas (also known as CO_2) released by burning fossil fuels is affecting Earth's climate, making it more unstable. *How much more CO_2 can we emit before climate instability becomes dangerous?* These are big problems. Bright possibilities for addressing them include renewable energy sources like wind and solar. No doubt you have heard all of these things before—as well as many other news stories about "sustainability," both positive and negative.

We started developing the "Mathematics for Sustainability" course, and writing this book, because of three convictions:

- Many of the key choices that humans will have to make in the twenty-first century are rooted in sustainability questions. These include choices that we must make together, as citizens, as well as choices related to individual lifestyles.
- In a democracy, as many people as possible need to participate in well-informed discussion of these sustainability questions. They are too important to be left to "experts."
- We may engage with sustainability questions from a wide variety of perspectives, including scientific, technological, political, ethical, and religious. For many of these discussions, we need some knowledge of mathematics in order to participate in a well-informed way.

The aim of this book is to help you, the student, gain that mathematical knowledge and the ability to apply it to sustainability questions.

You may not consider yourself a "math person." Your studies may center on music or English or art or education or architecture or agriculture.[1] But if you want to find out *for yourself* what "the numbers say"—not just to choose which "expert" you prefer to listen to—then this book is for you. Together, we will find out how to model sustainability on local, regional, and global scales. We will learn about

[1] Students in all these majors have succeeded in the course on which this book is based.

measurement, flow, connectivity, change, risk, and decision-making. Some of the topics we discuss will probably be challenging, perhaps even unsettling. Whatever conclusions you reach, this book will prepare you to think critically about your own and other people's arguments and to support them with careful mathematical reasoning.

As citizens in a democracy, you will ultimately be the ones whose decisions will guide your world toward a sustainable future. We wish you the very best in your studies, and as you participate in building the sustainable society of the future.

John Roe, Russ deForest, Sara Jamshidi
August 2017

Sustainability—The Key Idea

In spring 2017, our Earth's human population surpassed $7\frac{1}{2}$ billion. Here's a question. What do you imagine was the population of Earth two thousand years ago, at the beginning of the Common Era?

Demographers (scientists and historians who study population) obviously don't know an exact answer to this question. But they are able to make some good estimates, which are in the range of 200 to 300 million people (Section 8.2). That is to say, the number of people on the whole planet twenty centuries ago was roughly the same as the number in the United States (U.S.) today. Or, to put it differently, the population of our Earth has increased by twenty-five times over that period.

That population increase has not been a steady one. Most of the growth has occurred in the last century. And many other measures of human activity follow a similar pattern. Take a look at the graphs in Figure 1, which are taken from a book by Will Steffen [308]. These graphs show a pattern of accelerating increase that mathematicians call *exponential growth*.[2] This is important news: some good (most societies have regarded large families and increased wealth as good, for example) and some less so (Steffen's book includes similar curves about pollution and overuse of resources, which most would regard as bad). Both "goods" and "bads" have been growing exponentially, especially over the past two hundred years, since the Industrial Revolution got into gear. *Can this pattern continue?*

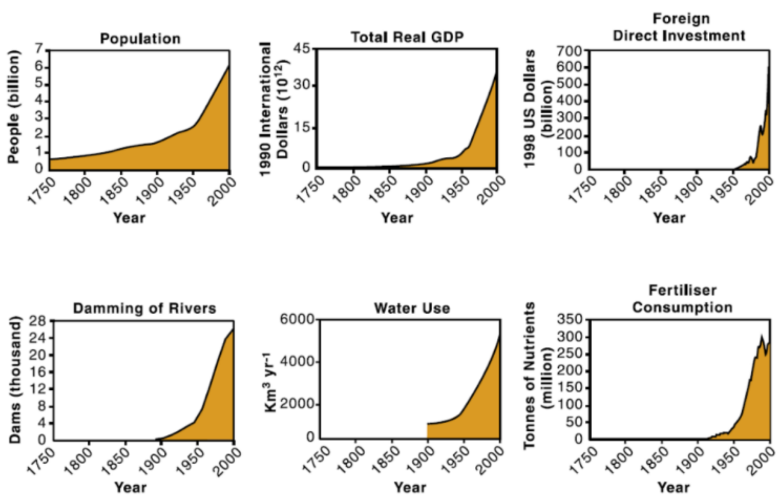


Figure 1: Some measures of the "size" of humanity (from [308]).

Some think so. In July 2015, one presidential candidate declared that his objective for the United States was "4 percent annual growth as far as the eye can see." That is about the growth rate for the

[2] See Section 3.4 for more about this concept.

curves in Figure 1. Others, though, look at similar data and see it differently. "[These] remarkable charts. . . ," writes Gus Speth, former dean of the Yale School of Forestry and Environmental Studies, "reveal the story of humanity's impact on the natural earth. The pattern is clear: if we could speed up time, it would seem as if the global economy is crashing against the earth—the Great Collision. And like the crash of an asteroid, the damage is enormous." [305]. "Societies are now traveling together in the midst of unfolding calamity down a path that links two worlds," he continues. "Behind is the world we have lost, ahead is the world we are making. . . . The old world, nature's world, continues, of course; but we are steadily closing it down, roping it off. It flourishes in our art and literature and in our imaginations. But it is disappearing."

The "old world" that Speth describes is a world in which Earth appears to be huge, teeming with life, abundant, exhilarating and dangerous. Humanity exists on the margin. In the "old world," to ask about humanity's impact on nature might seem absurd: much more important to worry about nature's impact on human beings (diseases? predators? food shortages?) By contrast, the iconic image of the "new world" is the Apollo astronaut's view of Earth: the "blue marble" (Figure 2), floating in space, gemlike and beautiful, yet cut down to finite size by human achievement. In this finite world it makes sense to ask: How long can we keep growing? Have we already become too big? Can our complex society remain diverse, active, and productive for an extended period of time? Or could we overshoot the limits of our resources and then decline, as many earlier civilizations have done [96]?

Figure 2: The "blue marble."

This is **the** sustainability question from which all other such questions derive. We can put it another way by thinking of the successive generations of humanity on this planet. Each generation inherits Earth's resources from its predecessors and passes them on to its successors. In a lasting or sustainable society, each generation would leave the world system in as good a shape as it found it; my generation's enjoyment of Earth's bounty would not deprive the next generation (yours) of the opportunity for similar enjoyment. Thus we arrive at the famous definition[3] given in the Brundtland Report [255] as early as 1987:

Definition 1

Sustainability is the ability of a social, economic, or ecological system to meet the needs of the present generation without compromising the ability of future generations to meet their own needs. A process or practice is **sustainable** to the extent that it contributes to the sustainability of the social, economic, or ecological systems in that it is embedded.

It is important to recognize that working for sustainability *does not* mean just trying to keep things as they are. "Things as they are" include patterns of privilege and inequality that deprive many members of the *present* generation of their basic needs. A sustainable world system must continue to extend the provision of these needs to a growing share of its population—ultimately, to all—even

[3]The Bruntland Report in fact defines "sustainable development"; we have slightly modified its language to arrive at a definition of "sustainability" itself.

as it also works to ensure that these needs can be provided in a way that doesn't ask future generations to pay the bill. This double challenge, we believe, will define the century that you live in.

We should also recognize that questions about sustainability and about the needs of the present and future generations are inherently value-laden. Particularly when we approach these questions on a global scale, we should expect to encounter different value judgments concerning how human well-being is defined and what are the basic needs whose provision should be sustained.

Think about it...

The Brundtland definition of sustainability, Definition 1, was written in 1987. Yet such ideas appear many years earlier in the thought of indigenous peoples around the world. For instance, the Iroquois Confederacy's "Great Law of Peace," which is older than the U.S. Constitution, contains a clause that is often paraphrased as, "In every deliberation, we must consider the impact on the seventh generation... even if it requires having skin as thick as the bark of a pine." Thus, Westerners' recent "discovery" of the notion of sustainability might be more properly described as a "rediscovery" of ideas that are rooted in many traditions (including some of our own). How do you feel about this? Why might Western society have lost touch with the sustainability idea for part of its history?

Sustainability and Resilience

Figure 3: Marine rescue operation in Galveston, Texas, following Hurricane Harvey, August 31, 2017.

The time scale on which we ask sustainability questions is a long one—many generations. Some changes to our world are not perceptible unless we take such a long view. The rate at which Earth's climate is changing, measured by global average surface temperature, is no more than a couple of hundredths of a degree per year: of little consequence from one year to the next but (as we will see) hugely significant on longer time scales. But change does not always come in tiny steps: our world also experiences sudden, extreme events. (Think of Hurricane Harvey, which flooded Houston in August 2017, doing damage initially estimated at $60 billion.) Extreme events are unavoidable: the question is not how to avoid them, but how quickly we can bounce back from them. This "self-healing" capacity is **resilience**.

Definition 2

Resilience is the ability of a social or economic system to absorb and respond to a sudden shock without damage to its core functioning.

Sustainability and resilience are not the same, but they are closely related: a system that is already near the limits of sustainability may be pushed to collapse by a shock that a more resilient system would easily have survived. In the same way, an infection that a healthy person would shrug off may be fatal for someone whose immune system is already compromised by malnutrition. Many historical examples can be found in [96].

One way to think about the relationship between sustainability and resilience is through the idea of *control*. It is tempting to believe that the more elements of our socio-ecological system we can bring under human control, the better we will be able to steer it on a safe path and so the more sustainable it will be. For example, wildfire management in the Western U.S., and elsewhere, focused for many years on fire suppression: bringing fires under human control. This seemingly allowed development in wildfire-threatened areas to be lasting, or "sustainable." Only recently has it become clear that these measures, which indeed halted many local fires, also set the stage for huge buildups of fuel that make the forests less *resilient* to the possibility of a catastrophic burn. Similar ideas apply to flood control measures like overflow dams and levees. In Section 4.4 we will develop a mathematical perspective on *tipping points*—situations in which a system lacks resilience because a small shock can produce an overwhelming response.

Ecosystem Services

In Speth's "new world" there is a tight connection between the economy, environmental quality, human well-being, and the well-being of the entire community of living and nonliving things that we call Earth's *ecosystem*. This requires a significant change in our thinking. From the perspective of the "old world," Earth was simply a source of abundance: nature provided. Our "new world" perspective requires a deeper understanding of what it is that nature provides, and indeed of the fact that human activity can no longer be so neatly distinguished from "nature" at all. Economic activity and human prosperity are embedded in the natural world, and they depend on and are ultimately constrained by the productive capacity of the whole Earth ecosystem (Figure 4)—which itself is constrained by the rate at which energy arrives from the Sun.

In Figure 4, we should therefore think of the outermost oval, which represents the entire ecosystem of the Earth, as of more or less unchangeable size. That means the inner ones cannot grow too much. In other words, the definition of a sustainable society implicitly involves the uncomfortable idea of *limits*: limits on the "goods" that are available to each generation and also limits on the capacity of Earth's ecosystem to absorb the "bads" generated by human activities. The recognition that such limits exist sets the stage for the mathematical tools we develop throughout this book. We need to be able to interpret and use quantitative information to assess the size and nature of such limits and so to reach well-justified decisions about sustainability questions. If we lack such tools, we'll be tempted to believe either that there "are no limits" or that if limits exist, they are so far off that they need be of no concern to us. That may be true in some cases, but in others we may find that the limits are startlingly close.

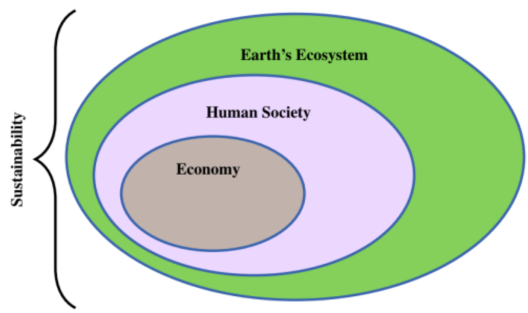


Figure 4: Sustainability and limits. Redrawn from [65].

The notion of **ecosystem services** [342] provides one way to conceptualize this dependence of human activity on the planet's ecosystems.

Definition 3

Ecosystem services are benefits that human society obtains from Earth's ecosystems. These services range from pollination of food plants and provisioning of freshwater and other resources to climate regulation, soil production, and recreational opportunities.

Specific ecosystem services are not fixed once and for all: human activity may enhance or degrade the capacity of Earth's ecosystems to provide these services. Moreover, many ecosystem services (such as clean air, clean water, climate regulation) are *public goods*—they cannot be "owned" in any practical sense. While some ecosystem services are local or regional (think of the services provided by a community park or a state forest), many are global in nature (like the pollution-absorbing capacity of the atmosphere): effective management and stewardship therefore require international cooperation. In Chapter 6 we develop mathematical tools to study such cooperation. This will allow us to understand some of the pitfalls and difficulties in reaching mutually beneficial agreements.

Think about it…

Gaylord Nelson, former U.S. senator for Wisconsin and founder of Earth Day, wrote in 2002:

> *The economy is a wholly owned subsidiary of the environment, not the other way around.*

What do you think he means? Do you agree?

About This Book

A few words about the *structure* of this book. **Part I** is the longest part of the text. In it, we develop mathematical tools and apply them to short examples. It is organized into six chapters corresponding to key concepts that arise in the mathematical study of sustainability: measuring, flowing, connecting, changing, risking, and deciding. Here is an overview of those concepts:

Chapter 1: Measuring. In this chapter we discuss how to measure, and how to express *how big* some quantity is. From the point of view of sustainability, the kind of answer that matters is often not some absolute number, but a *comparison* or a *level of importance*, so we'll talk about how we might judge whether some quantity represents something important, and how we might make a decision on how to respond.

Chapter 2: Flowing. It's common to talk about the *balance of nature*. But this image of a "balance" can suggest something that is static, unchanging. That is not the way natural ecosystems operate. Individual components of the system are constantly changing, even as the system as a whole maintains its equilibrium. For example, consider a mountain lake. The water level in the lake may stay the same, but the actual water in the lake today is not the same as the water that was there yesterday: new water has arrived through rain and snowmelt, old water has left through runoff and evaporation. It's the balance between these various *flow* processes that keeps the water level—the *stock*—constant. This chapter is devoted to exploring these concepts of flow and stock in detail.

Chapter 3: Connecting. Our lives are more interconnected now than at any time in history, and not just through social media. Rather than most of our food being grown close to where we live, for example, we have gotten used to obtaining food from supermarkets that are supplied by a production and transportation network that reaches all over the globe. Energy supply, too, is a vast and complex network of tanker routes, pipelines, electrical grid connections, and truck deliveries, to mention only a few. Human-made networks like these are not the only ones: we have also grown much more aware of the complexity of the *natural* networks that connect the web of life on earth. In this chapter we will study the mathematical language that is used to understand these various kinds of networks.

Chapter 4: Changing. In this chapter we will look at examples in which stock-flow systems are out of equilibrium—that is, how they respond to *change*. Although we start by studying a simple model of continuous growth or decay, from the point of view of sustainability the important questions arise when growth is limited by some external factor (such as the fact that we live on a finite planet). We'll study how the *strength of feedbacks* governs a system's response to change, and how this leads to the key idea of a *tipping point*—a moment when the system "flips" suddenly to a new state. Some scientists are concerned that Earth's climate system might be approaching one or more tipping points. We'll ask how it might be possible to tell.

Chapter 5: Risking. Any kind of realistic thinking about sustainability must consider likelihoods or *risks*. Nuclear power generation does not produce any greenhouse gas emissions, and it is constant and reliable, but what about the possibility of a catastrophic accident? This question asks us to balance the near-certainty of a steady benefit against a small risk of disaster. It is hard to do so without numbers. How likely are you to die from a nuclear reactor meltdown? The math involved in working with these sorts of questions is called *probability and statistics*. In this chapter, we are going to learn about these techniques and how they can help us make good decisions when faced with limited knowledge and uncertain outcomes.

Finally, **Chapter 6: Deciding** is where the rubber meets the road. In the end, you and your generation are going to have to make some sustainability decisions. These will range from the personal, through the local, to the national and global. Human behavior does not always follow the "rational" principles discussed in Chapter 5. Even if mine always did, I am not the only person involved. Other people's decisions interact with mine in a complicated way—we are in fact a *network* of decision-makers, with no final authority. This decision-making interaction can be studied mathematically, which we will do. The chapter concludes, however, with an extended reflection on how our sustainability decisions can never be *purely* mathematical, but must also engage our fundamental personal and ethical commitments.

In **Part II (Chapter 7)** of the book we provide a collection of Case Studies in which we apply the mathematical tools developed in Part I to answer particular questions related to sustainability and to explore extended examples. We believe that it is important that you, the student, learn to write extended pieces of this sort, and a student writing requirement has been a major part of the course on which this book is based. Why? This is how you build the ability to assess, develop, and present quantitative evidence in support of your own ideas. These skills are vital to you as a future leader, as an engaged citizen, and as an effective advocate for the things you care about. We don't want you just to learn mathematical techniques but also to be able to incorporate them in extended and persuasive written pieces—pieces that might be published in a course blog, in a student or hometown newspaper, or even on a national platform.

Finally, **Part III (Chapter 8)** of the book contains reference material: suggestions for further reading, tables of useful data, and the list of figures, bibliography, and index. We suggest checking Part III regularly, especially if you need some numerical information (such as the heat capacity of water, or the amount of sunlight that falls on Earth, or the planet's estimated coal reserves) to help you answer one of the exercises or formulate an argument in one of your more extended written pieces.

Online Resources

This book has a *website*, http://math-for-sustainability.com, as well as an email address for comments and suggestions, comments@math-for-sustainability.com. On the website you will find many different kinds of supporting materials:

- Hints and solutions to selected exercises.
- Online calculators and examples. Especially later in the book we will describe many models for sustainability-related processes that *change over time* (such models are called *dynamic*). The best way to visualize such a model is also dynamic, like a video clip rather than a collection of snapshots. That can't be done on paper, but it can be done online, and the website uses several different modeling tools to help you see the way our dynamic models evolve over time.
- Extended discussions of particular topics (refreshing the case studies in part II of the book, some of which may become outdated as technology advances or the environment changes). If *you* publish a written piece in a newspaper somewhere, please write to us (using the email address above) and tell us! We'll be happy to link to your writing from the book's website.
- Corrections or updates. We've tried hard to make this the best book possible. But there are bound to be some mistakes left. What's more, some information will simply become outdated. We'll keep a list of corrections or updates on the website. Again, if *you* notice an apparent mistake, please email us. You'll be helping many future students by doing so.

Conclusion

Figure 5: Pope Francis has said: "These ancient stories...bear witness to a conviction which we today share: that everything is interconnected, and that genuine care for our own lives and our relationships to nature is inseparable from . . . justice and faithfulness to others" [123].

We hope that through this book you will gain a clearer understanding of the sustainability issues that we humans face and of some choices that we need to make. But let's be clear: mathematics cannot make these choices *for* us. As remarked above, the choices human beings make on such fundamental questions reflect their deepest ethical and personal commitments (compare Figure 5). What mathematics can do, though, is to *inform* our choices by making their likely consequences clearer. It can help us *prioritize* issues by ranking them in terms of the relative size of the risk they pose and the relative severity of their potential outcomes. Understanding mathematics can help us *avoid* falling for some plausible-looking "solutions" that really don't achieve much or are even harmful. Finally, mathematics carries its *own values* also, like communicating clearly, reasoning logically, and considering all possibilities. These values, as well as the specific content of mathematics, can help us all in the decisions that we will all have to make together.

Summary of Ideas: To the Student

- **Sustainability** refers to the ability of a social or economic system to keep functioning without degrading its environment—to provide for its own needs and also preserve the ability of future generations to provide for their needs.
- **Resilience** refers to the ability of a social or economic system to "self-heal"—to recover from a disruptive event.
- **Ecosystem services** refer to the benefits that society receives from Earth's ecosystems.
- Many questions about sustainability and resilience involve *measurement*, *change*, *connection*, and *risk*, all of which can be expressed in the language of mathematics.
- **This book** introduces some of the mathematical ideas that are helpful in making decisions that involve sustainability.

0.2 To the Instructor

> *If you want to make a course interesting, then you should study something of interest [356].*

This text supports a course that is aimed at college students—many thousands of them, in our large universities—who would not describe themselves as "mathematicians" or "scientists" but who need to take at least one course that supports quantitative literacy as part of their degree requirements. Often such students have found themselves steered into courses in the precalculus sequence: courses that may be excellent preparation for future scientists and engineers, but that fail to catch a student's attention as their last experience of mathematics. One of us sometimes asks such students, "Would you rather learn the quadratic formula or would you rather save the world?" This book is for those students who would like to save the world, or at least take a step in that direction. Similarly, it is for those instructors who would like to "teach as if life matters" [328], or at least take a step toward using the mathematics classroom to help students think more clearly about some of the issues that are only going to become of increasing importance over the twenty-first century.

Each instructor will, of course, use this book in the way that they see fit. However, one of our primary goals is to advance student skill in quantitative literacy. A **required** student writing component has played an essential role in accomplishing this goal and has accounted for over one-third of the total grade in the course as we have taught it. The Association of American Colleges and Universities provides the following definition for quantitative literacy [249]:

> Quantitative Literacy (QL) is a "habit of mind," competency, and comfort working with numerical data. Individuals with strong QL-skills possess the ability to reason and solve quantitative problems from a wide array of authentic contexts and everyday life situations. They understand and can create sophisticated arguments supported by quantitative evidence and they can clearly communicate those arguments in a variety of formats.

Because of its intended audience, the book does not require any mathematics beyond high school algebra; the most complicated idea that appears is a fourth root, which shows up in a few places in Section 2.4. In particular, *no calculus* is required to read and study this book. If the instructor does know calculus, though, they will find that it provides background to the chapters on "Flowing" and "Changing," and may even find it helpful—for themselves, not for the students—to investigate how our presentation can be translated into the classical language of differential equations.

Nor does the book require (from you, the instructor) a great deal of sustainability-specific mathematical background. Naturally, the more you know, the more you will be able to help your students, but in the end what we're presenting in Part I of the book is a self-contained set of mathematical techniques, and if you learn about them from the text, that will be fine. Should you wish to pursue the material further, the reading suggested in Section 8.1 gives some possible starting points.

The book's website is at `http://math-for-sustainability.com`. As well as the student-oriented material described in the previous section, this website contains additional resources and suggestions specifically for instructors. These include a quite specific description of the writing component of the course as we have taught it. You're welcome to use this model exactly as it is, to adapt it, or to do something entirely different. We do believe, however, that this book will be most effective if it is used in conjunction with a requirement for some student response in the form of extended writing. Our experience suggests that most students are glad to have the opportunity to integrate their mathematical learning with social and environmental concerns and to express themselves in this way.

0.3 Acknowledgments

A book like this does not come into being without the support and critique of many people. We are grateful to all the many students who have worked through different versions of the book in their Math 33 class at Penn State, and for the comments and suggestions they have provided. We're also grateful for the advice of those who have read all or part of the manuscript, including Peter Buckland, David Hunter, Marta Mendoza, Liane Roe, and the anonymous readers who reviewed the manuscript on behalf of Springer. Numerous colleagues from Penn State and elsewhere have given generously of their time to make special presentations in one or more Math 33 classes; we are deeply grateful to all of you. Finally, we especially want to thank Francis Su for the generous and impassioned foreword that he has contributed to the book.

- *John Roe writes:* I want to express my thanks to Karl "Baba" Bralich, to Fletcher Harper, to Byron Smith, to Christopher Uhl, and to Marty Walter, for their various challenges to embrace this work with my whole self. Marty's book *Mathematics for the Environment* [340] helped me see that something like this volume would be possible. I am deeply grateful to my undergraduate research assistant for the start of the project, Kaley Weinstein, whose flow of ideas was an inspiration. And I want to humbly acknowledge the support and love of my whole family, who now and ever have meant more than I can possibly say.

- *Russ deForest writes:* Foremost, I would like to express my gratitude to John Roe for inviting me to be a part of this project, for his kind leadership, and for his deep and personal commitment to the ideals of general education. I would like to thank Rachel Hoellman for her essential role as a peer assistant and tutor through multiple iterations and refinements of the accompanying course, and to Matt Lupold for his current work in this position. Finally, I would like to thank the many engaged students who make teaching a joyous effort.

- *Sara Jamshidi writes:* I am inexpressibly grateful to John Roe for giving me an opportunity to be part of this important project; he has given me an expansive understanding of all that a mathematician can be and I will carry that with me for the rest of my career and life. I am thankful for my friends Duane Graysay, Shiv Karunakaran, and Monica Smith Karunakaran, who have taught me so much about mathematics education. I am also indebted to my partner, Aleksey Zelenberg, who provided invaluable support and feedback for my work here, and to my father, Morid Jamshidi, who was the first person to show me that mathematics can be used in civics and ethics. Finally, I would like to thank you, the reader, for picking up this book and endeavoring to better understand sustainability through a mathematical lens; I hope this becomes an empowering tool in your life.

Part I

Fundamental Concepts

CHAPTER 1

Measuring

Think about this quotation from a BBC news broadcast:

> Archaeologists announced Friday that they have discovered human footprints in England that are between 800,000 and 1 million years old—the most ancient found outside Africa, and the earliest evidence of human life in northern Europe [188].

We know 1 million years is a long time, but "a long time" is subjective. Your friend might say that she sat in traffic for "a long time" because she drove 20 miles in one hour, but one hour does not compare to 1 million years. Now suppose your friend instead said it took her one hour to travel 3 million centimeters. How do we interpret that? Was that a large or small distance to traverse in that amount of time?

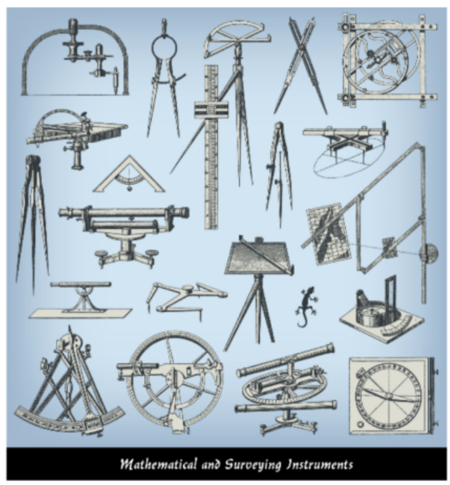


Figure 1: Measuring instruments.

The most basic mathematical question we can ask is: how big? But there is (or ought to be) an immediate comeback in any real-world situation: compared to what? Is a million a big number? A million dollars is a lot of dollars for an ordinary person. But a million molecules of carbon dioxide is a tiny amount for most purposes, and even a million dollars is a rounding error if you are thinking about the total of the U.S. federal budget. What about a million centimeters? Could you walk that far in a day? In a week? In a month? Not just the numbers but also the *units* matter: a million centimeters might be a manageable distance to walk (we'll see!), but we can be quite sure that a million miles is not.

In this chapter we will review basic information about numbers, units, measurements, and comparisons. We will study *scientific notation*, which is a convenient way to work with very large and very small numbers, and we will look at some of the most effective ways to communicate numerical information in *human terms*. We'll also learn some of the skills of *estimation*: how to get a useful rough idea of the size of some quantity, even when an exact answer is not available.

J. Roe et al., *Mathematics for Sustainability*, Texts for Quantitative Critical Thinking,
https://doi.org/10.1007/978-3-319-76660-7_1

1.1 Units and Measurement

Objectives

- ☐ I can identify the **number part** and the **unit part** of a physical measurement.
- ☐ I can keep track of **units** throughout a calculation.
- ☐ I can work with **pure numbers** and their **percentage** equivalents.
- ☐ I am familiar with standard units of time, distance, weight (or mass), and temperature.
- ☐ I can **convert** the same measurement into different units.
- ☐ I can communicate the meaning of a measurement by relating it to everyday human experience.

1.1.1 Number Part and Unit Part

Let's consider some simple examples of measurements:

(a) The fuel tank of my car holds 13 gallons.

(b) We need 5 pounds of potatoes.

(c) Today's high temperature is predicted to be 91 degrees Fahrenheit.

(d) The distance by road from New York City to San Francisco is about 2900 miles.

(e) Abraham Lincoln took approximately 2 minutes to deliver the Gettysburg Address.

(f) The Hoover Dam is 726 feet high.

(g) The area of Lake Mead (the lake impounded by the Hoover Dam) is 640 square kilometers.

As you can see from these examples, a physical measurement is made up of two parts:

- the **number part**—like "13" or "5" or "726";
- the **unit part**—"gallons" or "pounds" or "feet."

It is only the number and the unit part *together* that make a complete measurement. If my gas tank has a 13-gallon capacity, and I for some reason decide to measure in teaspoons instead, I could say that my tank holds about 10,000 teaspoons. Both "13 gallons" and "10,000 teaspoons" are complete measurements, though the first version is much more useful for most purposes. But it would make no sense just to say "The capacity of my gas tank is 13." Without the *unit part* (gallons), the *number part* (13) does not tell us anything.

Critical Thinking

When you read an article that gives numerical information, always ask, "What are the units?" If no units are given, the information is meaningless.

Problem 1: Identify the number part and the unit part in the following measurements:

(i) The radius of the Earth is approximately 4,000 miles.

(ii) At full takeoff power a Boeing 747 burns over 3 gallons of jet fuel per second.

(iii) The current U.S. population is about 320 million.

Figure 2: Boeing 747 at takeoff.

Solution: In example (i) it is clear that the number part is 4,000 and the unit part is "miles."

In example (ii), the number part is 3, but what are the units? The "3" refers to a *rate of fuel consumption*, and the units in which this is measured are "gallons per second," which we may abbreviate as "gal/ sec." Notice that the units are *not* gallons alone, or seconds alone, but the combination "gallons per second." [1]

Example (iii) looks at first as though it does not have any units. To see what the units are, try to expand the statement a bit: ask yourself, "the current U.S. population *of what* are we describing?" Clearly the answer is "human beings," and that tells us the unit part of our measurement: "human beings" or "people."

Remark 1: When describing the unit part of a measurement, we don't distinguish between singular and plural. Thus, in both "1 inch" and "2 inches," the units are "inches"; in "1 person" and "300 people" the units are "people" (or "persons" if you prefer). Some units have standard abbreviations: for example, "2 inches" can also be written "2 in."

1.1.2 The Unit-Factor Method

Suppose that my car's gas mileage is 33 miles per gallon. And suppose also that, as in the first example above, my car's tank holds 13 gallons. How far can I travel on a full tank?

You probably know that to obtain the answer, we need to *multiply* 13 by 33. But did you know that the multiplication can be done with the units as well? Like this:

$$13\,\cancel{\text{gal}} \times \frac{33\,\text{mi}}{1\,\cancel{\text{gal}}} = 13 \times 33\,\text{mi} = 429\,\text{mi}.$$

Using the abbreviations "mi" for miles and "gal" for gallons, the tank capacity is 13 gal, and the fuel consumption is 33 mi/ gal, which we write as a fraction $\dfrac{33\,\text{mi}}{1\,\text{gal}}$. When we multiply these two together the gallons cancel, leaving the answer (with the correct units) as 429 mi, that is, 429 miles.

This is an example of the **unit-factor method**.

Definition 1

Using the **unit-factor method**, whenever we multiply or divide physical quantities, we must multiply or divide their number parts *and* their unit parts.

The unit-factor method is valuable in many kinds of problems. It's especially helpful when we deal with problems involving **unit conversions**, like the one below.

[1] You may also see this abbreviated as "gal sec^{-1}" in some books; this notation is not wrong, but we'll try to avoid it.

Problem 2: There are 1,760 yards in a mile, and there are 36 inches in a yard. Using this information, express one million inches in miles.

Solution: The key idea is that conversion problems can be expressed in terms of “multiplying by 1” in different ways—and that multiplying a quantity by 1 does not change that quantity. For example, the statement “there are 36 inches in a yard,” part of the information given in the problem, can be re-expressed as

$$\frac{1\,\text{yd}}{36\,\text{in}} = 1 \quad \text{or} \quad \frac{36\,\text{in}}{1\,\text{yd}} = 1. \tag{1}$$

Both ways of writing the fraction are correct; we have to figure out which way will be more helpful to us. Now a million inches multiplied by 1 is still a million inches; but if we multiply a million inches by the left-hand expression for 1 in the display (labeled (1)) above, we get the useful fact that

$$1{,}000{,}000\,\text{in} = 1{,}000{,}000\,\cancel{\text{in}} \times \frac{1\,\text{yd}}{36\,\cancel{\text{in}}} = \frac{1{,}000{,}000\,\text{yd}}{36} \approx 27{,}780\,\text{yd}.$$

(The symbol “$\approx$” means “is approximately equal to.”) We can use the same idea a second time to convert from yards to miles, using the fraction $\dfrac{1\,\text{mi}}{1{,}760\,\text{yd}} = 1$:

$$27{,}780\,\text{yd} = 27{,}780\,\cancel{\text{yd}} \times \frac{1\,\text{mi}}{1{,}760\,\cancel{\text{yd}}} = \frac{27{,}780\,\text{mi}}{1{,}760} \approx 15.8\,\text{mi}.$$

The answer to our problem, therefore, is approximately 15.8 miles. If we wanted to shorten our work, we could combine two steps into one and write

$$1{,}000{,}000\,\text{in} = 1{,}000{,}000\,\cancel{\text{in}} \times \frac{1\,\cancel{\text{yd}}}{36\,\cancel{\text{in}}} \times \frac{1\,\text{mi}}{1{,}760\,\cancel{\text{yd}}} = \frac{1{,}000{,}000}{36 \times 1{,}760}\,\text{mi} \approx 15.8\,\text{mi}.$$

This is also a correct solution.

The unit-factor method is a great help in conversion problems because keeping track of the units automatically lets us know when to multiply and when to divide. For instance, in the first line of our solution we multiplied by $1\,\text{yd}/36\,\text{in} = 1$ to convert our measurement from inches to yards. Only this form of the conversion factor allows the “inches” to cancel, and that cancellation is the signal that tells us we are heading in the right direction. Watch how this works in our next example:

Problem 3: *Beefy Acres* is an imaginary cow/calf farming operation located in the Southeast, used as an example in a pamphlet published by the Natural Resources Conservation Service (a division of the U.S. Department of Agriculture) [235]. Beefy Acres has 20 acres of pastureland, and each acre produces, on average, 9,500 pounds of forage per year. To thrive, each cow requires approximately 17,500 pounds of forage per year. What is the maximum number of cows that can be supported sustainably on Beefy Acres?

Solution: We have three items of information given to us. Let’s express these in terms of number part and unit part, where our basic units are “acres” (of pastureland), “cows” (this is a very reasonable unit for working this problem—units do not have to be just mathy things), “pounds” (of forage; for historical reasons the abbreviation for pounds is “lb”), and “years.” Then our information is the pasture area (20 ac), the amount of forage production per acre per year ($9{,}500\dfrac{\text{lb}}{\text{ac yr}}$), and the amount of forage consumption per cow per year ($17{,}500\dfrac{\text{lb}}{\text{cow yr}}$). We need to combine these quantities using multiplication and division to get an answer whose units are cows; the acres, pounds, and years have to cancel. There is only one way to do this:

$$\frac{1\ \text{cow}\ \cancel{\text{yr}}}{17{,}500\ \cancel{\text{lb}}} \times \frac{9{,}500\ \cancel{\text{lb}}}{1\ \cancel{\text{ac}}\ \cancel{\text{yr}}} \times 20\ \cancel{\text{ac}} = \frac{9{,}500 \times 20}{17{,}500}\ \text{cow} \approx 11\ \text{cows}.$$

The answer is approximately 11 cows.

How do we arrive at the correct way to arrange this computation? We start by remembering that our objective is to arrive at a solution whose units are cows. Only one of the data involves the unit "cow" at all, and that is the grazing requirement $17{,}500 \dfrac{\text{lb}}{\text{cow yr}}$. But this piece of information has "cow" in the *denominator* (the "downstairs" part of the fraction), whereas to get an answer in units of cows we are going to need "cow" in the *numerator* (the "upstairs" part). So we consider the reciprocal

$$\frac{1\ \text{cow yr}}{17{,}500\ \text{lb}}.$$

This has "cow" in the numerator as we wanted, but it also involves units of pounds and years, which we will need to cancel by multiplying by other pieces of information. If we multiply by the yield per acre, the pounds and years will cancel:

$$\frac{1\ \text{cow}\ \cancel{\text{yr}}}{17{,}500\ \cancel{\text{lb}}} \times \frac{9{,}500\ \cancel{\text{lb}}}{1\ \text{ac}\ \cancel{\text{yr}}} = \frac{9{,}500\ \text{cow}}{17{,}500\ \text{ac}}.$$

This is a meaningful quantity—it tells us how many cows can graze on any given acreage of land, with units "cows per acre"—but it is not our final answer, whose units must be "cows." To cancel the "acre" unit, we multiply by the land area, thus arriving at the final answer

$$\frac{1\ \text{cow}\ \cancel{\text{yr}}}{17{,}500\ \cancel{\text{lb}}} \times \frac{9{,}500\ \cancel{\text{lb}}}{1\ \cancel{\text{ac}}\ \cancel{\text{yr}}} \times 20\ \cancel{\text{ac}} = \frac{9{,}500 \times 20}{17{,}500}\ \text{cow} \approx 11\ \text{cows}.$$

1.1.3 Standard Units

We already have some knowledge of the standard units for time and distance, area and volume, and other common physical measurements. Let's review them.

Units of Time

Each of us is familiar with the idea of time. In the introduction we saw how sustainability questions can explicitly involve the measurement of time ("*how long* can this last?"). Table 1 lists some common units of time.

Table 1: Units of Time

Unit (abbr)	Definition	Example
Second (s)	–	Snapping your fingers
Minute (min)	60 s	Singing five verses of "Row, Row, Row Your Boat"
Hour (hr)	60 min	About the time you spend during a casual lunch
Day (day)	24 hr	Time between one sunrise and the next
Week (wk)	7 day	About the time it takes to ship a package by regular mail from the U.S. to Europe
Year (yr)	365 day	The time from one birthday to the next
Century	100 yr	About 4 generations in a family (from you to your great-grandmother)

Units of Length

Units of length come from one of two systems of measurement: **the U.S. system** and **the metric system** (used in much of the rest of the world). We will review both systems of measurement and discuss how to go back and forth between the two systems.

First, let us list some U.S. measurements of length. These measurements will be more familiar to you if you grew up in the United States.

Table 2: Units of Length (U.S.)

Unit (abbr)	Definition	Example
Inch (in)	–	About the length of a paper clip
Foot (ft)	12 in	About the length of a large human foot
Yard (yd)	3 ft	About the width of a single (twin) bed
Mile (mi)	1,760 yd	About the distance traveled walking for 20 minutes

If you grew up outside the United States, you will be more familiar with metric units of length. Some of these are listed in the table below.

Table 3: Units of Length (Metric)

Unit (abbr)	Definition	Example
Micron (μm)	$\frac{1}{1{,}000{,}000}$ m	Size of a small bacterium
Millimeter (mm)	$\frac{1}{1{,}000}$ m	About the width of a pencil tip
Centimeter (cm)	$\frac{1}{100}$ m	About the length of a carpenter ant
Meter (m)	–	A little more than the width of a single (twin) bed
Kilometer (km)	1000 m	About the distance traveled walking for 12 minutes

Notice that the metric units all are related by *powers of 10*, like 10, 100, 1000 and their reciprocals $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$. Moreover, the names of the units all have a standardized form, made up of a prefix applied to the basic unit "meter." You can find a more extensive list of these prefixes, and the powers of 10 that they represent, in the "Useful Data" section of this book (Table 2 on page 486).

Problem 4: Use the information in the tables to determine how long would it take you to walk 3 million centimeters.

Solution: From the data in Table 3, we know that it takes about 12 minutes to walk 1 kilometer. We can express this rate of progress as

$$\frac{12\,\text{min}}{1\,\text{km}}.$$

We want to use this information to find out how long it takes to travel 3 million centimeters. Since our rate-of-progress information uses kilometers rather than centimeters, we convert one to the other, using the conversion information $1\,\text{cm} = \frac{1}{100}\,\text{m}$ and $1\,\text{km} = 1{,}000\,\text{m}$ provided by the table. As we learned in the solution to Problem 2 on page 6, the way to do this is to re-express the conversions as fractions equal to 1 (such as $\dfrac{1\,\text{m}}{100\,\text{cm}} = 1$, for example). Thus we can obtain the travel time:

$$3{,}000{,}000\,\cancel{\text{cm}} \times \frac{1\,\cancel{\text{m}}}{100\,\cancel{\text{cm}}} \times \frac{1\,\cancel{\text{km}}}{1000\,\cancel{\text{m}}} \times \frac{12\,\text{min}}{1\,\cancel{\text{km}}} = 360\,\text{min}.$$

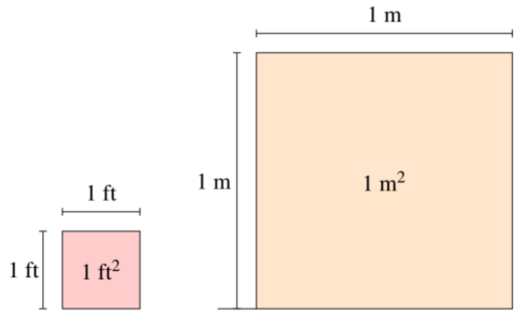


Figure 3: Illustrating the formation of units of area.

Our answer is 360 minutes. That's a lot of minutes, and it will be easier to understand if we convert it to hours, using the fact that 60 minutes make 1 hour:

$$360\,\cancel{\text{min}} \times \frac{1\text{ hr}}{60\,\cancel{\text{min}}} = 6\,\text{hr}.$$

Based on our work above, then, it would take about 6 hours to walk 3 million centimeters. This is an example of expressing a measurement in **human terms**—that is, by reference to familiar human experience. Instead of the mysterious "3 million centimeters," we can now communicate the same information by saying "about six hours' walk." This alternative way of expressing our measurement can be grasped directly, without scientific knowledge.[2]

Remark 2: We separated out the various conversions when we used the unit-factor method in the above solution. It would be equally correct, however, to combine them all in one line:

$$3{,}000{,}000\,\cancel{\text{cm}} \times \frac{1\,\cancel{\text{m}}}{100\,\cancel{\text{cm}}} \times \frac{1\,\cancel{\text{km}}}{1{,}000\,\cancel{\text{m}}} \times \frac{12\,\cancel{\text{min}}}{1\,\cancel{\text{km}}} \times \frac{1\text{ hr}}{60\,\cancel{\text{min}}} = 6\,\text{hr},$$

as we did in the solution to Problem 3 on page 6.

Units of Area and Volume

Area is a unit of measurement created from length. Units like "square feet" (ft^2) and "square meters" (m^2) refer to the area made by multiplying the dimensions. Figure 3 is a diagram demonstrating this idea. From any unit for length, we can get a corresponding unit for area.

Let's work some problems involving unit conversions for areas. As we'll see, we will need to take the *squares* of the conversion ratios to make the units come out right. This works because conversions in the unit-factor method are simply ways of expressing the number 1, and the square of 1 is just 1 again.

Problem 5: What is a square foot in square inches?

Solution: There are 12 inches in a foot; in terms of the unit-factor method, $\dfrac{12\text{ in}}{1\text{ ft}} = 1$. So how many square inches in a square foot, $1\text{ ft}^2 = 1\text{ ft} \times 1\text{ ft}$? We can work this out by the unit-factor method

$$1\text{ ft} \times 1\text{ ft} = \left(1\,\cancel{\text{ft}} \times \frac{12\text{ in}}{1\,\cancel{\text{ft}}}\right) \times \left(1\,\cancel{\text{ft}} \times \frac{12\text{ in}}{1\,\cancel{\text{ft}}}\right) = 12\text{ in} \times 12\text{ in} = 144\text{ in}^2.$$

[2] We'll develop this idea further in Section 1.1.5.

A shorter way to approach this calculation is to take the square of the conversion ratio, $\left(\frac{12\,\text{in}}{1\,\text{ft}}\right)^2 = \frac{(12\,\text{in})^2}{(1\,\text{ft})^2}$. This is still equal to 1 (because the square of 1 is still 1), so we can write

$$1\,\text{ft}^2 = 1\,\text{ft}^2 \times \frac{(12\,\text{in})^2}{(1\,\text{ft})^2} = 1\,\cancel{\text{ft}^2} \times \frac{144\,\text{in}^2}{1\,\cancel{\text{ft}^2}} = 144\,\text{in}^2,$$

getting the same answer as before. Look carefully at both approaches to make sure you understand why they give the same result.

Problem 6: Express 15 square centimeters in square meters.

Solution: Working as in the problem above,

$$15\,\text{cm}^2 = 15\,\text{cm}^2 \times \frac{(1\,\text{m})^2}{(100\,\text{cm})^2} = 15\,\cancel{\text{cm}^2} \times \frac{1\,\text{m}^2}{10{,}000\,\cancel{\text{cm}^2}} = 0.0015\,\text{m}^2.$$

The answer is 0.0015 square meters.

There are a couple of additional units of area that are worth knowing. These are particularly associated with agriculture.

Table 4: U.S. Units of Area

Unit (abbr)	Definition	Example
Acre	$4840\,\text{yd}^2$	Roughly the area of a (U.S.) football field, without the end zones.
Hectare	$10^4\,\text{m}^2$	About 2.5 acres

Volume is also a unit of measurement created from length. This time, though, we are considering cubes instead of squares (Figure 4). Units like "cubic inches" (in^3) and "cubic meters" (m^3) refer to measurement of volume. The technique for carrying out conversions is the same.

Problem 7: What is 1200 cubic inches in cubic feet?

Solution: We now need to take the *cube* of the conversion factor $\frac{1\,\text{ft}}{12\,\text{in}} = 1$. We obtain

$$1{,}200\,\text{in}^3 = 1{,}200\,\text{in}^3 \times \frac{(1\,\text{ft})^3}{(12\,\text{in})^3} = 1{,}200\,\cancel{\text{in}^3} \times \frac{1\,\text{ft}^3}{1{,}728\,\cancel{\text{in}^3}} \approx 0.7\,\text{ft}^3.$$

The answer is approximately 0.7 cubic feet.

Example 1: As well as the "cubic" units for volume, traditional systems of measurement have come up with many other volume units, often specialized to a particular trade or profession: bushels of wheat, gallons of milk, hogsheads of ale, and so on. Table 5 on the opposite page gives a few such volume units worth knowing

Problem 8: How many gallons are in a cubic foot?

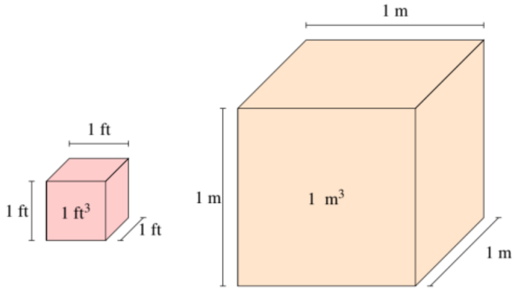


Figure 4: Illustrating the formation of units of volume.

Table 5: Units of Volume

Unit (abbr)	Definition	Example
Gallon (gal)	$231\ \text{in}^3$	Large container of milk
Pint (pt)	$\frac{1}{8}$ gal	Glass of beer
Barrel, of oil (bbl)	42 gal	Barrel about 3 ft high, $1\frac{1}{2}$ ft diameter
Teaspoon (tsp)	$\frac{1}{96}$ pt	The smallest spoon in a set of ordinary silverware
Liter (L)	$1000\ \text{cm}^3$ or $0.0001\ \text{m}^3$	A little over 2 pints

Solution: One gallon is equal to 231 cubic inches (as defined above), and one cubic foot is equal to $12^3 = 1{,}728$ cubic inches. Therefore,

$$1\ \text{ft}^3 = 1{,}728\ \cancel{\text{in}^3} \times \frac{1\ \text{gal}}{231\ \cancel{\text{in}^3}} \approx 7.5\ \text{gal}.$$

Problem 9: Approximately how many cubic centimeters make a teaspoon?

Solution: This problem requires us to relate the U.S. and metric systems of measurement for volume. We can use the unit-factor method and the information in the table above:

$$1\ \text{tsp} = \frac{1}{96}\ \text{pt} \approx \frac{1}{96}\ \cancel{\text{pt}} \times \frac{1\ \cancel{\text{L}}}{2\ \cancel{\text{pt}}} \times \frac{1{,}000\ \text{cm}^3}{1\ \cancel{\text{L}}} \approx 5\ \text{cm}^3.$$

Thus, a teaspoon is approximately equal to 5 cubic centimeters.

Units of Weight or Mass

Table 6: Units of Mass (U.S.)

Unit (abbr)	Definition	Example
Ounce (oz)	–	About the mass of an AA battery
Pound (lb)	16 oz	About the mass of a small bottle of soda
Ton (t)	2,000 lb	About the mass of a subcompact car

In this text, we are going to use the terms "weight" and "mass" interchangeably. Strictly speaking this is not accurate—the "mass" of an object refers to the amount of "stuff" (matter) that it is made of, and "weight" refers to the amount of pull that gravity exerts on that mass. But so long as we stay on the surface of the earth, gravity is pretty much constant, and therefore the two terms "weight" and "mass" can be taken to refer to the same thing. As earthbound people, then, we will not sweat the distinction.

Like distances, masses have two unit systems: a U.S. system and a metric system. If you grew up in the United States, you may be more familiar with the mass units in Table 6 on the previous page. But if you grew up outside of the United States, you are likely more familiar with the measurements in Table 7 instead.

Table 7: Units of Mass (Metric)

Unit (abbr)	Definition	Example
Gram (g)	–	About the mass of a paper clip
Kilogram (kg)	1,000 g	About the mass of a hardback book
Tonne (T)	1,000 kg	About the mass of a subcompact car

Notice that a U.S. "ton" and a metric "tonne" are not quite the same. But they are pretty close (within about 10 percent)—close enough that the difference will not matter for most of our purposes.

Unit Conversions

There are many situations in which we may need to *convert* a measurement from one unit to another. These conversions could be within one unit system (for example, expressing 3 million centimeters as 30 kilometers) or from one unit system to another (for example, expressing 30 kilometers as just under 19 miles).

Here is a table of some useful conversions. (More extensive and precise tables can be found in Section 8.2.)

Table 8: Unit Conversions

Metric Unit	U.S. Conversion	U.S. Unit	Metric Conversion
1 cm	0.39 in	1 in	2.54 cm
1 m	3.3 ft	1 ft	0.3 m
1 km	0.62 mi	1 mi	1.61 km
1 g	0.035 oz	1 oz	28 g
1 kg	2.2 lb	1 lb	0.45 kg
1 T	1.1 t	1 t	0.91 T
1 gal	3.8 L	1 L	0.26 gal

We have already seen how the unit-factor method allows us to handle these conversions efficiently. Let's do a couple more examples as a reminder.

Problem 10: The weight limit for a checked bag on Untied Airlines is 50 lb. If my bag weighs 32 kg, will I be able to check it without paying the overweight charge?

Solution: The fact that 32 is less than 50 does not answer this question! We must *convert* 32 kg to pounds. To do this, we use the fact (from Table 8) that 1 kg equals 2.2 lb. Using the unit-factor method we express this by saying that the fraction $\frac{2.2\,\text{lb}}{1\,\text{kg}}$ equals 1. Therefore,

$$32\,\text{kg} = 32\,\cancel{\text{kg}} \times \frac{2.2\,\text{lb}}{1\,\cancel{\text{kg}}} \approx 70\,\text{lb}.$$

It looks as though I definitely will have to pay an overweight charge!

Problem 11: I am pulled over on the freeway for driving at a speed of 100 feet per second. If the speed limit is 65 miles per hour, was I speeding or not?

Solution: We need to express 100 feet per second in miles per hour. Using Tables 1 and 2 (pages 7–8),

$$100\,\frac{\text{ft}}{\text{sec}} = 100\,\frac{\cancel{\text{ft}}}{\cancel{\text{sec}}} \times \frac{1\,\cancel{\text{yd}}}{3\,\cancel{\text{ft}}} \times \frac{1\,\text{mi}}{1{,}760\,\cancel{\text{yd}}} \times \frac{60\,\cancel{\text{sec}}}{1\,\cancel{\text{min}}} \times \frac{60\,\cancel{\text{min}}}{1\,\text{hr}} = \frac{100 \times 60 \times 60\,\text{mi}}{3 \times 1{,}760\,\text{hr}} \approx 68\,\frac{\text{mi}}{\text{hr}}.$$

So I was speeding, though not by very much.

The moral is: *be mindful of your units!* Forgetting to convert to the appropriate unit system is an easy mistake that can lead to disastrous results—such as the loss of a spacecraft! According to news reports from the late 1990s, "NASA lost a $125 million Mars orbiter because a Lockheed Martin engineering team used English units of measurement while the agency's team used the more conventional metric system for a key spacecraft operation" [199].

Here's a slightly different example

Problem 12: Write the height "5 feet 7 inches" in centimeters.

Solution: "5 feet 7 inches" means an *addition*: five feet *plus* seven inches. But when we want to add two quantities, they must be expressed in the *same* units. In this example, we need to re-express the five feet in units of inches before we add the seven inches:

$$5\,\text{ft} + 7\,\text{in} = \left(5\,\cancel{\text{ft}} \times \frac{12\,\text{in}}{1\,\cancel{\text{ft}}}\right) + 7\,\text{in} = 60\,\text{in} + 7\,\text{in} = 67\,\text{in}.$$

Now that we have expressed the length in units of inches, we can convert to centimeters:

$$67\,\cancel{\text{in}} \times \frac{1\,\text{cm}}{0.39\,\cancel{\text{in}}} \approx 170\,\text{cm}.$$

Remark 3: A calculator will give $67/0.39$ as $171.7048\ldots$. So why did we "round off" the answer above to "approximately 170 centimeters"? To understand this, take a look at the discussion of *precision* in Section 1.3.1.

Units of Temperature

Our final example of unit conversions concerns the measurement of *temperature*. In the United States, temperatures are measured using the *Fahrenheit* scale. On this scale, the freezing point of water is $32\,^\circ$F and the boiling point of water is $212\,^\circ$F, so that there are 180 Fahrenheit degrees between freezing and boiling.

In most other countries, the *Celsius* scale is used. On the Celsius scale, the freezing point of water is $0\,^\circ$C and the boiling point of water is $100\,^\circ$C. Thus there are 100 Celsius degrees between freezing and boiling.

Remark 4: Converting between temperature scales is more complicated than converting between unit systems for the other kinds of measurements (length, time, mass, and so on) that we have discussed so far. The reason is that not only do the units have different sizes (one Celsius degree represents more of a temperature jump than one Fahrenheit degree) but also the *zero points* of the two scales are different. That problem does not come up for other sorts of measurements: zero feet and zero meters both represent the same length, zero! The differences between Fahrenheit and Celsius are illustrated in Figure 5.

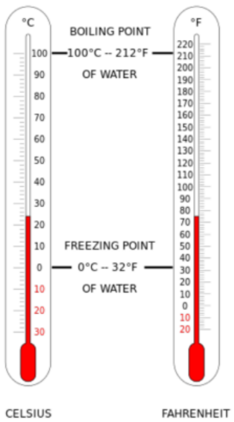


Figure 5: Celsius and Fahrenheit comparison.

What's more, neither the Celsius nor the Fahrenheit scales have their zero points rooted in any fundamental physical reality. (According to legend, 0 °F was simply the coldest temperature that Fahrenheit could reach using the technology available to him at the time.) Modern physics tells us, though, that heat is the result of the disordered motion of atoms and molecules, and this means that there is truly a "coldest possible" temperature, when all atoms and molecules would be at rest. This *absolute zero of temperature* is *much* colder than anything in our ordinary experience: it is about −273 °C, or −460 °F. In physics calculations the *Kelvin* temperature scale is used: Kelvin temperature is simply equal to Celsius temperature plus 273, so that absolute zero is 0 K, water freezes at 273 K, and water boils at 373 K. Table 9 gives the algebraic formulas for converting between these three temperature scales.

Table 9: Temperature Scale Conversions

	°C	°F	K
°C	$C = C$	$C = \frac{5}{9}(F - 32)$	$C = K - 273$
°F	$F = 32 + \frac{9}{5}C$	$F = F$	$F = \frac{9}{5}K - 460$
K	$K = C + 273$	$K = \frac{5}{9}F + 255$	$K = K$

For instance, if you want to convert a temperature from Fahrenheit to Celsius, let the Fahrenheit temperature be F. Look in the table to the intersection of the °F column and the °C row, where you will find the algebraic formula $C = \frac{5}{9}(F - 32)$. This tells you how to compute the Celsius temperature C in terms of the Fahrenheit temperature.

Problem 13: Normal human body temperature is about 98 °F. Express this on the Kelvin scale.

Solution: Looking in the table above, we see the formula $K = \frac{5}{9}F + 255$ to convert from Fahrenheit to Kelvin. Plugging in $F = 98\,^\circ\text{F}$, we obtain

$$K = \frac{5 \times 98}{9} + 255 \approx 55 + 255 = 320\,\text{K}$$

for the Kelvin equivalent.

1.1.4 Percentages and Other Pure Numbers

We've stressed that physical quantities have both a number part and a unit part, but we sometimes have a use for **pure numbers**. These are numerical quantities that have no unit. The most familiar example is a *percentage*.

Example 2: In 2015 the population of California was estimated to be 39 million people while the population of the United States as a whole was 320 million people [261]. Let's express the California population as a *percentage* of the total U.S. population. A percentage is a *proportion*; we divide the population of California by the total population,

$$\frac{39{,}000{,}000\ \cancel{\text{people}}}{320{,}000{,}000\ \cancel{\text{people}}} \approx 0.12 = \frac{12}{100} = 12\%.$$

The units (people) cancel, leaving us with the pure number 0.12, which we can also write as 12% (read as 12 *percent*).

Remark 5: Percent comes from the Latin *per centum*, meaning out of a hundred; 12 out of every 100 people in the U.S. live in California. Notice how you can think of the identity $0.12 = 12\%$ as another example of the unit-factor method:

$$0.12 = 0.12 \times 100\% = 12\%,$$

since 100% is another way of writing 1, and multiplying by 1 makes no difference!

Remark 6: In the previous example, we used the unit "people," but we could have chosen "million people" as our unit without changing the result:

$$\frac{39 \text{ million people}}{320 \text{ million people}} \approx 0.12 = 12\%.$$

If a calculation yields a pure number (or percentage) result, then all the unit parts must cancel. That means we will get the *same* answer whatever units we use in the numerator and denominator (as long as we use the same units for both).

Definition 2

A numerical quantity without a unit is called a **pure number**. A ratio $\frac{Q_1}{Q_2}$ of two quantities Q_1 and Q_2 having the *same units* will produce a pure number.

Example 3: The mathematical constant $\pi \approx 3.14$, which is the ratio of a circle's circumference to its diameter, is a pure number. It appears (among many other places) in the formula for the area of a circle, πr^2, where r is the radius.

Example 4: The *conversion factors* in Section 1.1.3 are also pure numbers. For example,

$$\frac{1 \text{ yd}}{1 \text{ in}} = \frac{36 \text{ in}}{1 \text{ in}} = 36$$

is a pure number.

Example 5: A **concentration** is a pure number: when we say "the concentration of oxygen in the atmosphere by volume is about 21%," we are referring to the ratio

$$\frac{\text{Volume of atmospheric oxygen}}{\text{Volume of entire atmosphere}}.$$

The units cancel, giving us a pure number, about $0.21 = 21\%$.

A percentage is a multiple of $\frac{1}{100}$, as we said earlier. Very small (or trace) concentrations may be expressed not as percentages but as *parts per million* (ppm, that is, multiples of $\frac{1}{1{,}000{,}000}$) or even as *parts per billion* (ppb, that is, multiples of $\frac{1}{1{,}000{,}000{,}000}$). For example, the present concentration of CO_2 in the atmosphere is about 0.04%. We typically express this in parts per million. To do so, use the unit-factor method again: $1{,}000{,}000 \text{ ppm} = 1$, so

$$0.04\% = 0.04\% \times 1{,}000{,}000 \text{ ppm} = 0.04 \times \frac{1}{100} \times 1{,}000{,}000 \text{ ppm} = 400 \text{ ppm},$$

that is, 400 parts per million (by volume).

Remark 7: In Example 5 on the previous page, we expressed the concentration of atmospheric carbon dioxide in parts per million *by volume*. Trace concentrations are also frequently expressed in parts per million (ppm) or parts per billion (ppb) *by weight*. In the case of atmospheric CO_2, 400 parts per million by volume corresponds to about 600 parts per million by weight. The weight of a carbon dioxide molecule is about 1.5 times the weight of an average air molecule and this accounts for the difference in these two measures.

For concentrations of CO_2 and other trace gasses in the atmosphere ppm is almost always intended as parts per million *by volume*, while for concentrations of trace substances in soil and water ppm is usually intended to mean parts per million *by weight*. To avoid confusion we will usually use the abbreviation ppmv when referring to parts per million by volume.

Example 6: Toluene is a byproduct of gasoline production that is used as a solvent in paint thinners and in industrial applications. Toluene is present in the discharge from petroleum refineries and poses a concern for groundwater and drinking water supplies. The U.S. Environmental Protection Agency (EPA) enforces a limit of 1 ppm for toluene in drinking water [20].

1 kilogram (kg) is 1,000 grams and 1 gram is equivalent to 1,000 milligrams (mg). A concentration of 1 ppm (by weight) is thus equivalent to 1 mg/kg:

$$\frac{1\,\cancel{\text{mg}}}{1\,\cancel{\text{kg}}} \times \frac{1\,\cancel{\text{gram}}}{1{,}000\,\cancel{\text{mg}}} \times \frac{1\,\cancel{\text{kg}}}{1{,}000\,\cancel{\text{gram}}} \times 1{,}000{,}000\,\text{ppm} = \frac{1{,}000{,}000\,\text{ppm}}{1{,}000{,}000} = 1\,\text{ppm}$$

See Exercise 18 on page 60 for examples expressing ppm and ppb in more familiar terms.

Problem 14: Suppose that your electric bill is \$70 each month. After following the advice contained in the EPA's Energy Saving Tips for Renters [17], you manage to cut your electric bill by 15%. How much money are you saving each month?

Solution: The amount we save is 15% of \$70, that is,

$$15\% \times \$70 = 15 \times \frac{1}{100} \times \$70 = \$10.50.$$

Problem 15: Alcohol concentration, ABV, or alcohol by volume, is reported as a percent. Suppose a particular brand of beer has an ABV of 5%. How much alcohol is in a 12 oz beer of this brand?

Solution: Using the same idea as in the previous solution, we get

$$5\% \times 12\,\text{oz} = 5 \times \frac{1}{100} \times 12\,\text{oz} = 0.60\,\text{oz},$$

or a little more than half an ounce.

There are many circumstances in which a percentage may be the most meaningful way of measuring some kind of change (for more about why this might be, see Section 4.1.1). For example, changes in population or the size of the economy are usually reported as percentages.

Think about it...

According to recent estimates, roughly 10 million tons of plastic makes its way into the oceans each year to become "marine plastic debris." Although the U.S. generates more plastic waste overall than most other countries, it is responsible for only 1% of plastic flowing into the oceans each year [170], because a much smaller proportion of U.S. plastic waste is "mismanaged" than plastic waste generated in less well-off countries. How much responsibility does the U.S. share, in your opinion, for dealing with the problem of plastic pollution in the world's oceans and with the global management of plastic waste more generally?

1.1.5 Measurements in Human Terms

A measurement, as we have seen, has both a number part and a unit part. But we human beings are not always able to grasp the significance of very large or very small numbers, or of units of measurement that don't relate to a familiar scale. If we can express a measurement in terms of more familiar quantities, it can help us a lot. Let's make the following definition.

Definition 3

We say that a measurement is expressed in **human terms** if it is expressed in a way that allows us to relate it directly to our shared everyday experience, without requiring specialized scientific knowledge.

Some units of measurement are intrinsically "in human terms." For example, one *foot* originally was just the length of your foot! Here are two simple rules for keeping measurements in human terms:

Rule 1: Human Terms Measurement Rules

To express a measurement in human terms, try, if possible, to follow *both* of the rules below:

- Choose units to keep the number part reasonably close to 1; say, between 0.01 and 1,000.
- Use "human scale" units (like feet, inches, hours).

When these rules both apply, it is not hard to express something in human terms. For instance, the spacing of the studs in standard house framing (in the U.S.) is 16 inches. That is a small number *and* a "human scale" unit. It would be foolish to express this spacing as 406,400,000 nanometers or as 0.0002525 miles. Even though both of these conversions are technically correct, they both violate both of the rules above: the numbers are huge or tiny (violating the first rule) and the units of measurement (nanometers, miles) are far from human scale (violating the second).

Figure 6: A grown elephant can weigh 4 tons.

Problem 16: The mass of a full-grown elephant is about 4 tons. How should I express this in human terms?

Solution: There is no single correct answer to a question like this. We could refer to Table 6 on page 11 to say that the mass of the elephant is the same as that of 4 compact cars. Or we could look up the average mass of a U.S. American (about 180 pounds; see Section 8.2) and say that the elephant has the mass of

$$4\,\mathrm{t} \times \frac{2{,}000\,\mathrm{lb}}{1\,\mathrm{t}} \times \frac{1\,\text{American}}{180\,\mathrm{lb}} \approx 45\,\text{Americans}.$$

Notice that the notion of "human terms" depends explicitly on *whose* shared everyday experience is taken as a reference point. In expressing the mass of the elephant in terms of a number of compact cars, we are assuming that compact cars are familiar and elephants are unfamiliar. In a different society, where elephants were abundant but automobiles were rare, one might reverse the process and express the mass of a compact car in terms of a fraction of an elephant—and this would be a "human terms" measurement too.

Example 7: Here's another example from a recent real-life sustainability discussion. In August 2015, President Obama addressed a meeting of Arctic nations in Anchorage, Alaska. In his remarks [244] he used the following example to illustrate the shrinking of Arctic ice:

> *Since 1979, the summer sea ice in the Arctic has decreased by more than 40 percent—a decrease that has dramatically accelerated over the past two decades. One new study estimates that Alaska's glaciers alone lose about 75 gigatons—that's 75 billion tons—of ice each year.*
>
> *To put that in perspective, one scientist described a gigaton of ice as a block the size of the National Mall in Washington—from Congress all the way to the Lincoln Memorial, four times as tall as the Washington Monument. Now imagine 75 of those ice blocks. That's what Alaska's glaciers alone lose each year.*

For anyone who has visited Washington, DC, the image of a giant ice block "the size of the National Mall, four times as tall as the Washington Monument" is both striking and accessible, and this speech is an effective example of putting a very large quantity in human terms. For those who aren't so familiar with Washington, though, perhaps not so much. As we said in the previous example, the notion of "human terms" depends very much on *whose* experience is our reference point.

More difficult situations arise in which we *can't* follow both parts of the Human Terms Measurement Rules at the same time. For example, consider the following data relating to the volume of water on Earth (Section 8.2). "Fresh surface water" refers to fresh water in lakes, rivers, and streams, as well as in mountain snow, glaciers, ice caps, and ice sheets, but not underground freshwater (as in aquifers) nor atmospheric freshwater (as in clouds).

Table 10: Volume of Water on Earth

Type of Water	Volume (cubic meters)
All Water	1,400,000,000,000,000,000
Fresh Surface Water	100,000,000,000,000

A cubic meter can perhaps be considered as a "human terms" unit (you can envisage a cube of water one meter on each side, or just think about the amount of water it might take to fill an average hot tub),

Solution: The energy delivery rate from the gas pump is

$$\frac{120{,}000{,}000\ \mathrm{J}}{10\,\mathrm{sec}} = 12{,}000{,}000\ \mathrm{J/sec}.$$

The corresponding rate for the electrical outlet is 3,000 J/sec. Thus the ratio of the energy delivery rates is

$$\frac{12{,}000{,}000\,\cancel{\mathrm{J/sec}}}{3{,}000\,\cancel{\mathrm{J/sec}}} = 4{,}000;$$

that is, the gas pump delivers energy 4,000 times faster than the domestic electrical outlet. Or to put the matter in terms of time, the gas pump delivers in one second the same amount of energy that the electrical outlet takes 4,000 seconds (a little over an hour) to supply. Or again, we can convert to teaspoons (as in the previous problem) and say that the domestic outlet takes over five seconds to supply the energy equivalent of a teaspoon of gas. All of these can be thought of as simple rescalings.

A more striking "rescaling" is to express the matter in the following way: *if the hose of the gas pump were shrunk to the diameter of a drinking straw, it would deliver energy at the same rate as the electrical outlet*. The striking image of gasoline dribbling slowly out of a ten-foot hose as thin as a drinking straw certainly reinforces the contrast between the gas and electric "pumps," and highlights a major issue for the acceptance of electric cars: it is extremely difficult to achieve the same "refueling rates" with electricity as the ones gasoline has gotten us accustomed to. For the details of how the rescaling is calculated, see Exercise 22 on page 60 at the end of the chapter.

Example 11: The *Sagan Planet Walk*, a public sculpture in Ithaca, NY, is a marvelous example of the power of rescaling to convey an important idea in dramatically visual terms. The Sagan Walk is a scale model, scale 1 : 5,000,000,000, of our solar system and of the various objects (the Sun and planets) that are parts of it. It cuts the huge size of the Solar System down to something that we can appreciate visually.

Figure 8: Standing by the Sun on the Sagan Planet Walk.

In Figure 8 you can see the circular opening in the granite obelisk, almost 11 inches in diameter, that represents the Sun to scale. In each of the other obelisks in the sculpture, representing the planets, there is a Sun-sized opening that has a glass window into which the model of the corresponding planet is embedded. This allows the viewer to experience directly how much bigger the Sun is than even the largest planets.

You can see the Mercury monolith just above the boy's hand in Figure 8, to the right of the Sun monolith. It is just over 10 yards away from the Sun, and another 10 yards will get you to Venus and another 10 more to our Earth. The first four planets are really close together, no further than the trees in our picture. What's more, the models of the corresponding planets are tiny, the size of a pea or smaller. Then, suddenly, the scale starts to grow. It will take you ten minutes of brisk walking to get to Neptune from the Sun, passing on the way by Jupiter, the largest planet—model size 2.9 cm, greater than an inch but not quite the size of a Ping-Pong ball. The Planet Walk honors Carl Sagan's work in science communication by giving visitors a direct sensory engagement with the size and shape of the solar system.

In 2012 the Planet Walk was expanded. A monolith at the Astronomy Center of the University of Hawai'i now represents the nearest star to Earth—Alpha Centauri—on the same scale. Imagine walking the distance from Ithaca to Hilo, Hawai'i (where the Astronomy Center is located), to get some idea of the difficulties of interstellar travel compared to the journeys of our various spacecraft *within* the solar system.

If you want to learn more about the Planet Walk and the rescaling involved, we encourage you to read a beautiful article by mathematician Steve Strogatz, titled "Visualizing Vastness" [313].

Summary of Ideas: Units and Measurement

- A measurement, such as "3.2 million centimeters" consists of a **number part**, 3.2 million, and a **unit part**, centimeters.
- There are two main systems of units: the U.S. system and the metric system. Some examples of U.S. units are **feet**, **pounds**, **gallons**. Some examples of metric units are **meters**, **kilograms**, **liters**.
- We can use the conversion tables in this chapter to convert units within each system and between them.
- To help us with these conversions, we can use the **unit-factor method** to make sure that units cancel out. This tells us which numbers to divide and which to multiply: for example,
$$12\,\cancel{\text{km}} \times \frac{0.62\,\text{mi}}{1\,\cancel{\text{km}}} \approx 7.4\,\text{mi}.$$
- The ratio of two quantities having the same units or dimensions is a **pure number**. Pure numbers can also be expressed as **percentages**.
- To communicate clearly, it helps to express measurements or comparisons in **human terms**. This means that we express them in a way that is relatable to ordinary human experience without requiring scientific knowledge.

1.2 Scientific Notation

Objectives

- ☐ I can explain why **scientific notation** is used.
- ☐ I can convert numbers between decimal and scientific notation.
- ☐ I can recognize different ways of writing the same number using scientific notation, including in **standard form**.
- ☐ I can add, subtract, multiply, and divide numbers written in scientific notation.

1.2.1 Scientific Notation

In the last section we dealt with some very big numbers. When numbers become very large or very small, the standard way of expressing them (called **decimal notation**) takes up a lot of space and becomes hard to read. That is because very large and small numbers are mostly made up of zeros!

For example, two large volumes appeared in Example 10 on page 20:

- 1,400,000,000,000,000,000 cubic meters (the total volume of water on Earth), and
- 100,000,000,000,000 cubic meters (the volume of fresh surface water).

What matters most about these numbers is that the first is in the *millions of trillions* and the second is in the *hundreds of trillions*. To see that, however, you have to carefully count the number of zeros appearing in each expression. The commas help, but couldn't we find some more straightforward way to convey this information?

The key to doing so is to use the idea of *powers of 10*. Remember that 10 raised to a certain power, say n, means multiplying 10 by itself that number (n) of times. Thus, for example,

$$\begin{aligned}
10^1 &= 10,\\
10^2 &= 10 \times 10 = 100,\\
10^3 &= 10 \times 10 \times 10 = 100 \times 10 = 1{,}000,\\
10^4 &= 10 \times 10 \times 10 \times 10 = 1{,}000 \times 10 = 10{,}000,
\end{aligned}$$

Another way of saying this is that 10^n can be written as the digit 1 followed by n zeros. So, if we are given the number 10^9, we know our number is 1 followed by 9 zeros; that is a billion. Similarly, 10^6 is a million, and 10^{12} is a trillion.

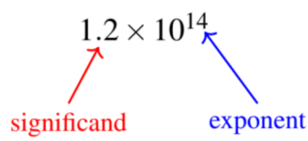
The number 7,000,000,000,000 is written as the digit 7 followed by 12 zeros—7 trillion. We can write that as 7 times 1 trillion, or 7×10^{12}. This technique of using a power of 10 to keep track of the zeros is known as **scientific notation**. Here is a definition.

Definition 1

A number is written in **scientific notation** if it is expressed as the product of two parts:

- an ordinary decimal number called the **significand** (like 7 in the example above), and
- a power of 10 (like 10^{12} in the example above). The power to which 10 is raised (the 12 in the example) is called the **exponent**.

Example 1: The expression below represents the number 120,000,000,000,000 in scientific notation.



The *significand* in this expression is 1.2 and the *exponent* is 14.

Remark 1: The same number in the example above could be expressed in several different ways. For instance

$$1.2 \times 10^{14} = 1.2 \times 10 \times 10^{13} = 12 \times 10^{13},$$

because $1.2 \times 10 = 12$. By shifting the exponent in this way, we can multiply or divide the significand by 10—that is, we can move the decimal point to the left or right. Usually we choose to move the decimal point so that the significand is between[3] 1 and 10. When this has been done, the number is said to be expressed in **standard form**. Thus 1.25×10^{14} and 12.5×10^{13} represent the same number, but the first expression is in standard form and the second is not.

Conversion from Decimal to Scientific Notation

Let's look at some examples of converting quantities to scientific notation and standard form.

Problem 1: Write 2,000,000,000,000,000,000 in scientific notation, standard form.

Solution: The number 2,000,000,000,000,000,000 has 18 zeros, so we can write it as 2×10^{18}.

Problem 2: Write 150,000,000,000,000,000,000 m^3 in scientific notation, standard form.

Solution: This quantity has a number part and a unit part (see Section 1.1.1). In converting to scientific notation we re-express only the number part; the unit part should remain unchanged. Now in this example the number part 150,000,000,000,000,000,000 is 15 with 19 zeros. So we could write it as 15×10^{19}. This is a correct expression, but notice that 15 is not between 1 and 10, so this expression is not in standard form. Instead, we write the number part as 1.5×10^{20}, which *is* in standard form. The answer to the complete problem (including the unit part) is therefore

$$1.5 \times 10^{20}\,\text{m}^3,$$

also in standard form.

[3] We allow 1 but not 10 for a significand in standard form: the standard form of 1,000 is 1×10^3, *not* 10×10^2.

Remark 2: When we express a physical measurement in scientific notation (as in the previous problem), it is only the *number part* of the measurement that we are working with. The *unit part* remains unaffected, and must appear in our final solution. Thus, in the last example, the solution is $1.5 \times 10^{20}\,\mathrm{m}^3$, **not** just 1.5×10^{20}.

You can think of these calculations with powers of 10 in terms of "moving the decimal point." First, imagine your number written in decimal notation (if it is a whole number, imagine a decimal point to the right of the ones place, so that you would think of 127 as 127.0). Then moving the decimal point one step to the left corresponds to dividing by 10 (e.g., $12.7 = 127/10$). This gives us a simple rule for putting a number greater than 1 into standard form .

Rule 1

To express a number greater than 1 in standard form, move the decimal point to the left until you obtain a number between 1 and 10. What you obtain is the significand (in standard form), and the number of steps to the left that you moved the decimal point is the exponent.

For example, to express the number 127 in standard form we move the decimal point 2 steps to the left to get a significand of 1.27 and an exponent of 2:

$$\underbrace{127.0}_{\text{2 steps}} \qquad \text{giving us } 127 = 1.27 \times 10^2.$$

Problem 3: The distance from the Earth to the Sun is approximately 93 million miles. Convert this to inches, and express your answer in scientific notation using standard form.

Solution: Using the unit-factor method we can calculate

$$93{,}000{,}000\,\cancel{\mathrm{mi}} \times \frac{1{,}760\,\cancel{\mathrm{yd}}}{1\,\cancel{\mathrm{mi}}} \times \frac{36\,\mathrm{in}}{1\,\cancel{\mathrm{yd}}} \approx 5{,}900{,}000{,}000{,}000\,\mathrm{in}.$$

Moving the decimal point 12 steps to the left gives us 5.9, so the answer in standard form is 5.9×10^{12} in.

These examples show how to write *large* numbers using scientific notation. But we can also write *small* numbers in the same way. To do this, we need to remember about *negative* powers of 10. Just as the positive powers of 10 are obtained by successively *multiplying* by 10, so the negative powers are obtained by successively *dividing* by 10:

$$\begin{aligned} 10^0 &= 1, \\ 10^{-1} &= \frac{1}{10} = 0.1, \\ 10^{-2} &= \frac{1}{10 \times 10} = 0.01, \\ 10^{-3} &= \frac{1}{10 \times 10 \times 10} = 0.001. \end{aligned}$$

Thus 10^{-n} is

Remark 4: Whatever notation we use, it is important to remember that $a \div b \neq b \div a$. So when dividing numbers in standard form, be careful to carry out the operations in the correct order!

Now, let us look at some more worked problems.

Problem 7: Calculate $(9.5 \times 10^8) \times (4.1 \times 10^8)$.

Solution:

$$(9.5 \times 10^8) \times (4.1 \times 10^8) = (9.5 \times 4.1) \times 10^{8+8} = 38.95 \times 10^{16} \approx 3.9 \times 10^{17}.$$

Problem 8: Calculate $(3.2 \times 10^4) \div (4.5 \times 10^8)$.

Solution:

$$(3.2 \times 10^4) \div (4.5 \times 10^8) = (3.2 \div 4.5) \times 10^{4-8}) \approx 0.71 \times 10^{-4} = 7.1 \times 10^{-5}.$$

If units are involved, we multiply the number parts according to Rule 3 on the previous page, and the unit parts according to the unit-factor method (Section 1.1.2).

Problem 9: According to [136], a cow emits about 0.3 kilograms of methane per day.[4] There are thought to be about 1.2×10^9 cows on Earth. Compute the total mass of methane emitted by cows over one year, expressing your answer in scientific notation.

The mass of the entire atmosphere is about 5×10^{18} kg (see Section 8.2). Methane lasts about 8 years in the atmosphere before breaking down to other gases. What proportion (by mass) of the whole atmosphere is made up of bovine methane emissions?

Solution: We solve the first problem using the unit-factor method:

$$0.3 \frac{\text{kg}}{\cancel{\text{cows}}\,\cancel{\text{days}}} \times (1.2 \times 10^9)\cancel{\text{cows}} \times 365 \frac{\cancel{\text{days}}}{\text{yr}} \approx 130 \times 10^9 \frac{\text{kg}}{\text{yr}} = 1.3 \times 10^{11} \frac{\text{kg}}{\text{yr}}.$$

Bovine methane emissions are about 1.3×10^{11} kg/ yr.

For the second problem, the atmosphere contains about 8 years worth of bovine methane. That is,

$$8\,\text{yr} \times (1.3 \times 10^{11})\,\text{kg/ yr} \approx 10 \times 10^{11}\,\text{kg} = 10^{12}\,\text{kg}.$$

Since the mass of the whole atmosphere is about 5×10^{18} kg, the proportion of bovine methane (which is a pure number) is

$$\frac{10^{12}\,\cancel{\text{kg}}}{5 \times 10^{18}\,\cancel{\text{kg}}} = \frac{1}{5} \times 10^{-6} = 0.2 \times 10^{-6} = 2 \times 10^{-7},$$

or 0.2 parts per million. Though this is a small number, it is in fact a substantial fraction of the total amount of methane in the atmosphere (which is about 1 part per million by weight).

Addition and subtraction

Now we will think about adding and subtracting numbers in scientific notation. There is one situation in which we can do this directly:

[4] Often referred to as "cow farts" online, though [136] showed that the great majority of the methane is burped rather than farted.

Rule 4

In order to directly add (or subtract) two numbers in scientific notation, they need to have the same exponent. Once two numbers have the same exponent, we add (or subtract) their significands.

The two numbers below can be added automatically because they have the same exponent, 8.

$$(2.32 \times 10^8) + (3.1 \times 10^8) = (2.32 + 3.1) \times 10^8 = 5.42 \times 10^8.$$

But you won't always be adding two numbers with the same exponent. Suppose, for example, that you need to compute the following sum:

$$(2 \times 10^9) + (4.3 \times 10^{10}).$$

These numbers do not have the same exponent and therefore cannot be added together directly. We must *rewrite* one number to have the same exponent as the other. Let's rewrite 2×10^9 to have an exponent of 10. We've already seen how to do this in the previous section; let's express it by rules.

Rule 5

- For every increase in the exponent, we have to move the decimal point in the significand to the left.
- For every decrease in the exponent, we have to move the decimal point in the significand to the right.

This means that $2 \times 10^9 = 0.2 \times 10^{10}$. Now we can add using Rule 4:

$$(0.2 \times 10^{10}) + (4.3 \times 10^{10}) = 4.5 \times 10^{10}.$$

Alternatively, we could have rewritten 4.3×10^{10}, moving the decimal point in the other direction to write 4.3×10^{10} as 43×10^9. If we add using this approach, we get another situation in which we can apply Rule 4:

$$(2 \times 10^9) + (43 \times 10^9) = 45 \times 10^9.$$

This is equal to 4.5×10^{10}, the same answer as before. Notice that depending on how we choose to perform the calculation, we may need to re-express our final answer to make sure that it is in standard form.

Critical Thinking

What does the method we've illustrated here have in common with the method we used to solve Problem 12 on page 13?

Here are some more worked examples.

Problem 10: Calculate $9.5 \times 10^8 + 4.1 \times 10^8$.

Solution:

$$9.5 \times 10^8 + 4.1 \times 10^8 = 13.6 \times 10^8 = 1.36 \times 10^9.$$

Problem 11: Calculate $8.1 \times 10^7 - 3 \times 10^6$.

Solution:

$$8.1 \times 10^7 - 3 \times 10^6 = 8.1 \times 10^7 + 0.3 \times 10^7 = (8.1 - 3) \times 10^7 = 7.8 \times 10^7.$$

Problem 12: Calculate $5.75 \times 10^{27} + 2 \times 10^{25}$.

Solution:

$$5.75 \times 10^{27} + 2 \times 10^{25} = 5.75 \times 10^{27} + 0.02 \times 10^{27} = (5.75 + 0.02) \times 10^{27} = 5.77 \times 10^{27}.$$

Problem 13: Calculate

$$4.23 \times 10^{23} - 9.1 \times 10^6.$$

Solution:

$$4.23 \times 10^{23} - 9.1 \times 10^6 \approx 4.23 \times 10^{23}.$$

Although we might consider $9.1 \times 10^6 = 9{,}100{,}000$ to be a large number, it is *tiny* compared to 4.23×10^{23}, which is

$$423{,}000{,}000{,}000{,}000{,}000{,}000{,}000$$

A value in the *millions* has little impact on a number that is in the *thousands of sextillions*. So the subtraction is a negligible change.

Think about it...

A student suggests that the rule for adding two numbers in scientific notation should be "add the significands and add the exponents." How will you convince this student that he is wrong? Avoid *appeal to authority* ("The textbook says that you should do it this other way") and try to find an approach, maybe by means of examples, that helps the student understand why the correct rule makes sense, whereas his proposal does not.

the flow arrows stand for whatever process controls the rate of the corresponding flow. Finally, the "cloud" symbol (gray) signifies a **source** (arrow coming out of the cloud) or **sink** (arrow going into the cloud) of material that is outside the scope of the system currently being discussed.

Remark 2: A "cloud," which indicates a stock "outside the system," is not an absolute thing, but is relative to the specific system that we are trying to understand. Later, we might perhaps return and consider that specific system (maybe call it "A") as a part of some larger system "B." For instance, if we were looking at a wastewater recycling system for a home in an arid climate, the "cloud" to which the drain leads in Figure 3 might be replaced by another "stock," the contents of a wastewater holding tank. On a larger scale, the stock-flow system for the whole house is part of the larger system that represents the hydrological cycle for the whole neighborhood, which itself is part of the water cycle for the whole planet.

Think about it...

The second of Barry Commoner's "four laws of ecology" (from [68]) is "Everything must go somewhere. There is no *away*." What do you think this slogan means? Relate it to the discussion of "clouds" in the preceding remark. Do you agree that "there is no away?"

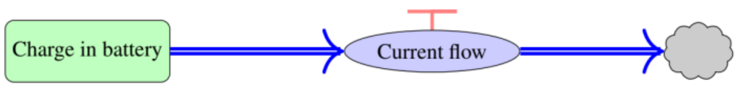


Figure 4: Stock-flow system for a primary cell, such as an AA battery.

Example 2: Consider a system that includes an *AA battery* as an energy source. If this is a non-rechargeable type,[1] then the system diagram for the battery (by itself) is very simple (see Figure 4). In this instance, the *flow* of electrical current out of the battery is measured in *amperes* (sometimes abbreviated to "amps") or *milliamperes* (10^{-3} of an ampere). An ampere is a measure of the *rate of flow* of electrical charge—it is in fact a *speed*, even though it does not have a "per hour" or "per second" in its name to remind you of that fact. Correspondingly, the electrical capacity of the battery, the stock that corresponds to the flow of current, is measured in *ampere-hours*—current *multiplied by* time. For example, we would expect a battery with a capacity of 1500 mAh (milliampere-hours), which is fairly typical for an AA, to be able to supply 150 mA for ten hours, or 30 mA for fifty hours, or 500 mA for three hours, and so on.

Remark 3: The "cloud" in Figure 4 could be replaced, on a larger view, by a diagram containing the other components of the electrical system that the battery supplies. However, it is useful to look at the very simple diagram in Figure 4 by itself. A diagram that looks like this—with outflow but no inflow—will arise whenever we are dealing with a *nonrenewable resource*, a finite quantity of some commodity that we can use or mine or capture while it exists, but that is not replenished to a significant extent on any human time scale. The oil in a particular oilfield, the mineral deposit in a mine, even the water in a deep aquifer like the Ogallala aquifer [196] that supplies irrigation for much of the farmland of the Great Plains, can be regarded as nonrenewable resources. When the battery has run down, the oil wells have run dry, or the mineral deposit has been mined out, their usefulness is exhausted. Abandoned mines and oil fields around the world testify that this is a real possibility.

[1] Also sometimes called a *single-use* battery or *primary cell*.

2.1.2 Equilibrium

Example 3: Figure 5, which is taken from NASA's *Earth Observatory*, illustrates the *carbon cycle* of planet Earth. This is in fact a system diagram with stocks and flows, though the graphics are more lively than the spigots and pipes that we have been drawing so far.

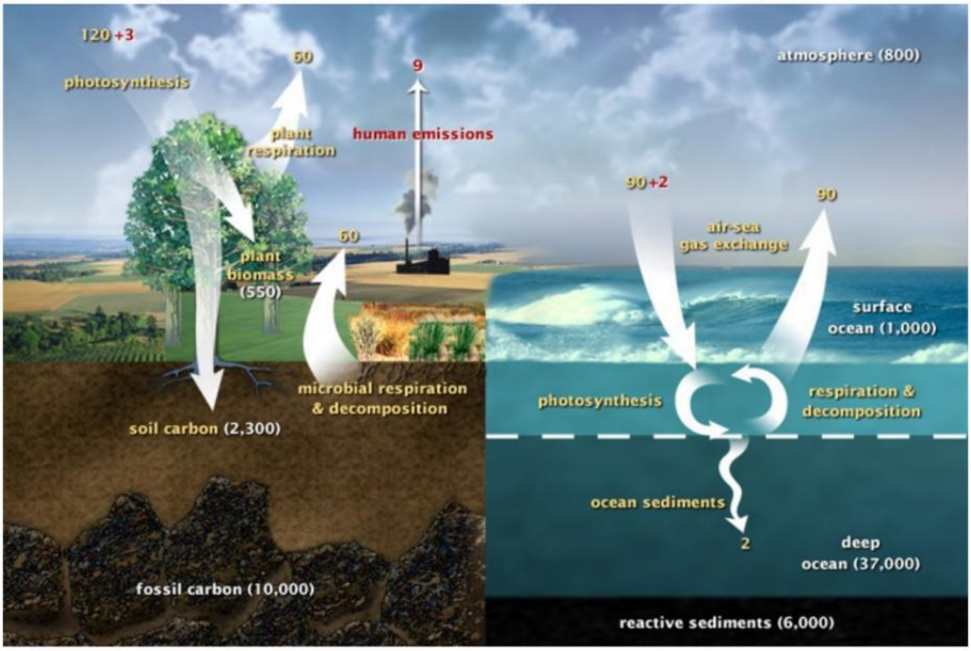


Figure 5: The Earth's carbon cycle from [271]. The white numbers in the figure refer to stock levels, measured in gigatons (billions of tons) of carbon. The yellow numbers refer to natural flows, in billions of tons of carbon per year. The red numbers refer to human-caused flows, or human-caused changes to the natural flows.

The figure contains seven different carbon stocks, as well as many different flows between them. It is much more typical for a system diagram to contain multiple stocks and flows like this; the single-stock diagrams we looked at in the previous two examples were especially simple ones.

Problem 1: From the carbon cycle diagram, compute the total *inflow* of carbon into the atmosphere, and the total *outflow*, in the absence of human-caused contributions.

Solution: The diagram contains three natural inflows of carbon to the atmosphere, labeled "plant respiration" (60 GT per year), "air-sea gas exchange" (90 GT per year), and "microbial respiration and decomposition" (60 GT per year). (For this calculation we are ignoring the red figures like "human emissions.") These total 210 GT per year.

There are also two natural outflows: "photosynthesis" (120 GT per year) and "air-sea gas exchange" (90 GT per year). These also total 210 GT per year. Thus, in the absence of human-caused contributions, the total carbon stock in the atmosphere does not change over time.

The balance between inflows and outflows that we see in this example is so important that there is a special name for it.

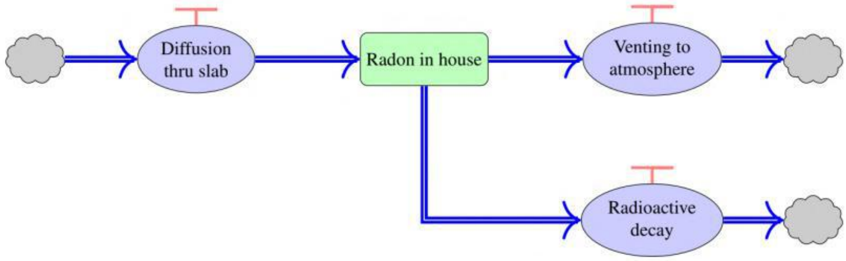


Figure 6: Simple stock-flow model of radon concentration in a home.

The U.S. Environmental Protection Agency (EPA) estimates [16] that radon exposure is the second biggest source of lung cancer deaths in the U.S. after smoking, being responsible for something like 20,000 deaths per year. New homes in high radon areas are typically equipped with active ventilation systems that extract the gas from underneath the foundation slab and vent it to the outside air before it can enter the home and do harm.

Remark 6: Polonium is also a highly radioactive element, though it is less dangerous than radon because it emits a different kind of radioactivity and is a solid rather than a gas. However, a more thorough modeling of radiation risks would have the "radioactive decay" spigot running to a second stock, the stock of polonium, which itself would then decay and run to a third stock, and so on... You can see how these models can quickly become complicated!

2.1.3 Residence Time

Imagine a stock-flow system in equilibrium. If we consider a particular stock S there are two important numbers related to it:

- The quantity of S that is present in the system. (Because we are assuming that the system is in equilibrium, this quantity is a *constant*—it does not change with time.)

- The total flow rate of S in the system. (We can consider either the total *inflow* to S or the total *outflow* from S; because we are assuming equilibrium, these must be the same.)

Figure 7: A student ready to graduate. "Residence time" in college is the *average* length of time from matriculation to graduation.

Example 4: Consider a college as a stock-flow system, with a "stock" of students flowing in through admission and flowing out through graduation. The equilibrium "stock" might be 20,000 students, and the flow rate (admissions or graduations) might be 4,000 students per year.

In general, the "stock" will be measured in certain units (students in the above example) and the "flow" will be measured in "stock" units divided by "time" units (students per year in the example). If we divide the stock by the flow we will therefore get a time, which is called the **residence time** associated with the particular stock:

Definition 5

The **residence time** associated with a stock that is at equilibrium in a stock-flow system is

$$\frac{\text{Level of the stock}}{\text{Flow rate of the stock through the system}}.$$

The residence time is the average amount of time that a unit of the stock will spend between inflow and outflow.

In the example above, the residence time of the students is 20,000 (students) divided by 4,000 (students per year), which equals 5 years. Using the unit-factor method we would write this calculation as follows:

$$20{,}000\ \cancel{\text{students}} \times \frac{1\ \text{yr}}{4{,}000\ \cancel{\text{students}}} = 5\ \text{yr}.$$

That doesn't mean that *each* student will take exactly this length of time to graduate: some will graduate in 4 years, some in 6 or more. Five years is the *average* amount of time spent by a student as an undergraduate at this college.

Problem 4: We open the faucet of the bathtub of Example 1 on page 70, to deliver 2.2 gallons of water per minute, and we also open the drain. The tub fills until the rate of outflow via the drain equals the rate of inflow via the faucet. At this point of dynamic equilibrium, the tub contains 24 gallons of water. Find the residence time of water in the tub.

Solution: The residence time is obtained by dividing the stock level by the flow rate

$$24\ \cancel{\text{gal}} \times \frac{1\ \text{min}}{2.2\ \cancel{\text{gal}}} \approx 11\ \text{min}.$$

This tells us that, *on average*, a molecule of water spends about 11 minutes in the bathtub before going down the drain. However, any individual water molecule arriving through the faucet may run down the drain instantly, or may stay in the tub for hours!

Remark 7: In Problem 4 we did not specify where the water came from, or where it went; this information is not part of the concept "residence time *in the bathtub*." We could even imagine that the waste water is *recirculated* to the faucet via some kind of pump. In that case the same water would stay in the bathtub forever. Nevertheless, we would still say that the residence time was 11 minutes: that would be the average length of time a water molecule spent in the tub before being recirculated via the pump. Be sure you have this picture clear before moving on to the next example.

Problem 5: Estimate the residence time of carbon in the atmosphere, using data from the NASA carbon cycle diagram (Figure 5 on page 72).

Solution: Not all the numbers in the diagram are needed for this calculation. What matters is the total *stock* of carbon in the atmosphere (about 800 gigatons, from the diagram) and the *flow rate* in (and out) of the atmosphere (about 210 gigatons per year, as we calculated in answering Problem 1 on page 72). Thus the residence time is about $800/210 \approx 4$ years.

Question 1: We've just shown that the residence time for carbon in the atmosphere is about 4 years; yet carbon dioxide emissions from the burning of fossil fuels and emissions from other human sources are expected to impact the climate for thousands of years to come. How can these two things be simultaneously true?

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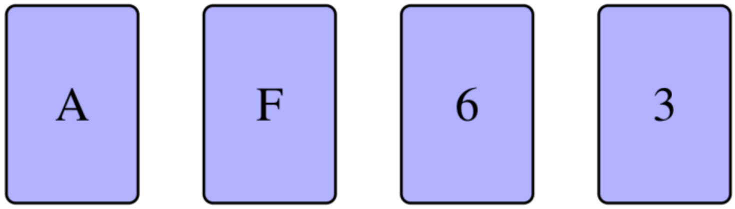

Figure 9: The Wason four-card task.

What if we do *not* obtain a good match? Then the model must be modified, or perhaps rejected entirely. The **scientific method** demands that all models must pass the test of reality; when applied to a real situation, they must describe it accurately. Scientists "stress test" their models by applying them to many different real situations; the more such tests a model passes, the more confidence we can have in it. (Section 5.4 describes this process of assimilating new information in greater detail.)

Question 2: "Stress testing" means that having formed a model—a picture of how reality works—we should look for evidence that might *disconfirm* it, which might prove our model incorrect. Many of us find it intuitively difficult to do this; we are biased to look for evidence that will *confirm* our ideas. A famous demonstration of this is the so-called **Wason 4-card task** [341]. You are shown four cards on a table as in Figure 9. You want to test the following model: *if there is a vowel on one side of the card, then there is an even number on the other side*. (You could think of this as a stock-flow model if you like, with input stock the letters recorded on one side of the card, the flow being some process that converts numbers to letters, which are the output stock recorded on the other side. And your model is that if vowels go *in* to the process, then even numbers come *out*.) Question, then: Which cards should you flip in order to "stress test" your model as quickly as possible?

Answer: You should flip the cards labeled A and 3. The model predicts that the A card should have an even number on the other side. But the model also predicts that the 3 card should have a consonant on *its* other side. Indeed, if you flipped the 3 card and saw a vowel, the proposed model would be invalidated—because it claims that a vowel "input" should always produce an even "output."

Many people faced with the Wason task immediately suggest flipping the A card and the 6 card. But flipping the 6 card cannot *disconfirm* the model. If you flipped the 6 and saw an E, that would agree with the model, true. But if you flipped the 6 and saw an F, that would not *disconfirm* the model: the model does not say anything about the "output" that consonants may produce. The Wason task is contrived, of course, but it is helpful in thinking about the scientific mind-set of always stressing your model as much as possible.

Suppose though that a model (say the radon model that we were talking about before) *has* passed many "stress tests." Then we can cautiously begin to use it for the second function described above, that of *prediction*. In the case of the radon model, we might use it to predict the radon concentration in a new house before it is built, based on information about the house's construction, the local geology, and so on. We can have some confidence that the predictions will be accurate, because they are based on a model that has been tested in a wide variety of real situations. Again in the radon example, a builder might use the information provided by the model to determine whether to install a radon mitigation system during construction. (This will help save money and time—if it is later discovered that a mitigation system is needed, retrofitting one in an already-built home is going to be much more complicated and expensive.)

Stocks, Flows, and Public Discussion

While the distinction between stocks and flows may seem clear, public discussion on human-caused climate change and other topics often confuses them. Thus, such discussions present an important opportunity for *critical thinking*. Here are some examples.

Critical Thinking

Consider the following quotation from Representative Michele Bachmann from her 2009 Earth Day speech on the House floor. Can you identify any stock/flow confusion in this statement?

> "What part of human activity creates carbon dioxide? If carbon dioxide is a negligible gas and it's only three percent of Earth's atmosphere, what part is human activity?
>
> "Human activity contributes perhaps three percent of the three percent. In other words, human activity is maybe 3 percent contributing to the 3 percent of carbon dioxide that's in Earth's atmosphere. It's so negligible—it's a fraction of a fraction of a fraction of a percent—that it can hardly be quantified."

It is indeed true that human-caused flows of carbon are small compared with natural flows (see Figure 5 on page 72 again). However this ignores the fact that the natural inflows and outflows are equal and therefore lead to no change in the stock (dynamic equilibrium). The human-caused flows, even if small, disturb this equilibrium and cause an increase in the stock of atmospheric carbon dioxide which accumulates over time. This increase can be quantified and, as we have seen (Figure 18 on page 50), can be measured quite directly.

Critical Thinking

Consider the following quotation from President Barack Obama from a press conference in 2012. Can you identify any stock/flow confusion in this statement?

> "Now, in my first term, we doubled fuel efficiency standards on cars and trucks. That will have an impact. That will take a lot of carbon out of the atmosphere."

Increasing fuel efficiency standards will mean that cars and trucks will generate less carbon dioxide per mile. In other words, the rate of *flow* of CO_2 into the atmosphere will decrease. But that is not the same as "taking carbon out of the atmosphere" (reducing the *stock*). The stock will continue to increase, just less quickly than it otherwise would have.

Definition 1

Mechanical work is done when something (like the handle of the rowing machine) is moved against a resisting force. The *amount of work* is the product of the force and the distance moved.

With each stroke, the handle of the rowing machine is moved about 3 feet against a resistance of maybe 50 pounds of force. This means that about $3 \times 50 = 150$ *foot-pounds* of work is done. Foot-pounds is a unit of energy, though as we will see there are other units that are more common and useful.

Definition 2

Energy is the capacity to do work.

As we said, energy comes in various forms. Let's hope the athlete on the rowing machine had a good breakfast. By doing so, she stored up *chemical energy* in her body. During her workout, she is converting that chemical energy to *mechanical energy* in moving the handle of the machine. The handle then spins a resistance fan inside the body of the machine, stirring and warming the nearby air. The mechanical energy is converted to *heat energy* (and a little sound energy as well). In fact, the whole system (athlete plus rowing machine) is a *stock-flow system* of the kind we looked at in the previous section, as shown in Figure 11.

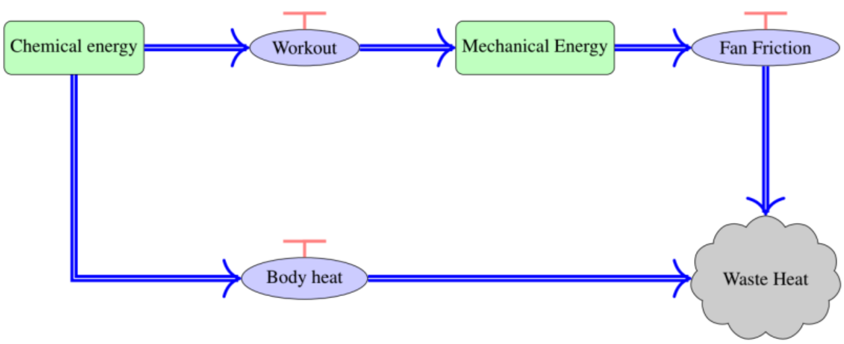

Figure 11: Energy stock-flow diagram for athlete and rowing machine.

The chemical energy stored in the athlete's body is transformed to mechanical energy in the rowing machine, which is transformed to heat by the fan. Notice that we have also shown a direct transfer of chemical energy to heat. When you work out, you warm up!

Remark 1: Why have we shown the heat as a "cloud" rather than as another stock? As we will see a little later, this kind of "waste" heat is an example of *low-grade energy*—it is there in the room, but it is virtually impossible to gather it together for a useful purpose. That's in contrast to the athlete's chemical energy (which can lift weights or move the rowing machine) or even the spinning flywheel's mechanical energy (which could be used, for example, to generate electricity—in fact, the rowing machine's on-board computer is powered in exactly this way.) For more about this see Section 2.2.4 below, "The Laws of Thermodynamics."

2.2.3 Units of Energy and Power

We are going to learn about the units that are used to express energy and power. Remember from Section 1.1.3 that in the U.S. two different unit systems are commonly used: the *metric* system and the *U.S. conventional* system. The most important energy measure is the one that belongs to the metric system. It is called the *joule* (denoted by the letter J).

Since the fundamental metric units are meters (for distance), kilograms (for mass), and seconds (for time), the joule must be defined in terms of these. In fact, a joule can be defined as *the kinetic energy of a two-kilogram mass moving at a speed of one meter per second*. However, a formal definition of this sort is not too important to us. Much more relevant is to have an idea of how large various everyday quantities of energy are, in joules. Some examples are contained in Table 1 on the previous page.

As you can see from the table, one joule is a rather small quantity of energy. Because the joule is a metric unit, we can make larger units by adding the standard prefixes. *Kilojoules* (10^3 J), *megajoules* (10^6 J) and even *gigajoules* (10^9 J) are useful.

Remark 2: The energy (about 10 joules) needed to raise 1 kg through 1 m, which is the same as the energy released when 1 kg falls through 1 m, is a measure of the strength of the Earth's gravity. It is sometimes called the **gravitational constant** and denoted by g.

The U.S. conventional system does not have a single standard unit of energy. Instead, it has a variety of different units that were developed and used for different purposes. Some of these units are listed in the table below, together with their approximate equivalents in joules, which allow you to convert energy measurements from conventional units to metric units or vice versa.

Table 2: Conventional Units of Energy

Unit	Equivalent	Definition
Foot-pound	1.4 J	Work done in moving a distance of one foot against resistance of one pound
British thermal unit (Btu)	1050 J	Energy needed to heat 1 pound of water through 1 degree Fahrenheit
Food calorie (Cal)	4200 J	Energy needed to heat 1 kilogram of water through 1 degree Celsius
Kilowatt-hour (kWh)	3.6×10^6 J	Energy that flows for 1 hour at a rate (power) of 1 kilowatt
Quad	About 10^{18} J	A quadrillion British thermal units (used only for national-scale data)

Finally, let us discuss units of *power*. Power is the *rate of transfer of energy*—it is the rate of *flow* corresponding to the energy *stock*. The fundamental metric unit of power will therefore be a flow rate of one joule per second. This unit is so important that it has a special name.

Definition 4

A power (rate of flow of energy) of one joule per second is called one **watt** (W).

You are probably familiar with the way light bulbs and other electrical appliances are rated in terms of the number of watts they consume. The computer on the rowing machine can show the athlete how many watts of energy she is expending. The rate of *production* as well as *consumption* of energy can be measured in watts. For example, the electrical power output of the Hoover Dam generators is about 2 gigawatts (2×10^9 W). A large coal-fired or nuclear generating station has a similar output.

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for one day. The most stressful moment occurs when their subject decides to take an electrically powered instant-heat shower. This consumes a lot of energy rather quickly (in other words, it requires a lot of power), and as a result it's "all hands on deck" for the cycling team. See the video at [39].

Remark 3: Remember that the 120-watt figure is a 24-hour average: the power output of the body will be higher during vigorous exercise and lower during sleep or rest. Even during exercise, not all of the body's power output will be available as *mechanical* power (that is, available to do work); some will inevitably be lost as heat. Modern competitive cyclists carefully measure their mechanical power output [195]; an elite cyclist might produce 350 watts for a 1-hour period of intense exercise.

2.2.4 The Laws of Thermodynamics

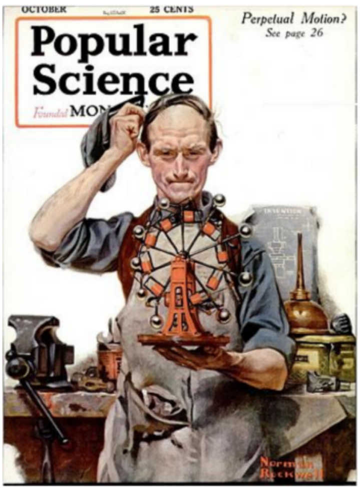

Figure 14: *Perpetual Motion* by Norman Rockwell.

Heat is a special form of energy. As we have seen in examples, whenever we do work, some, and eventually all, of the energy that is expended ends up as low-grade heat, a form in which we can no longer exploit it. On the other hand, *high-grade* heat is a significant component of many of our energy production processes. In eighteenth-century Britain, James Watt burned wood or coal to boil water and used the steam pressure to do mechanical work. The fuel in a modern nuclear power plant is very different, but it is still used to produce high-grade heat, which is then converted to useful energy by boiling water and using the resulting steam to drive turbines. This two-way relationship between heat and other forms of energy is therefore involved in all the energy processes of our modern world—both those that generate useful energy and those that consume it.

The laws of *thermodynamics* govern the two-way relationship between energy and heat. They set limits on what we can and can't do with energy. One thing these laws tell us is that the universe does not provide a "free lunch" in the energy sector: to the disappointment of generations of inventors, there is no way to build a "perpetual motion machine"—the name given to a mythical gadget that would go through a cycle (like turning a wheel) allowing energy to be continually extracted and yet return to its starting position at the end of the cycle. See Figure 14 for Norman Rockwell's vision of one such disappointed inventor (his "invention," called the *overbalanced wheel*, has in fact been "discovered" several times in the fruitless history of perpetual motion).

The first law of thermodynamics is also called the *law of conservation of energy*. It states that energy cannot be created or destroyed, merely *transformed* from one form to another. This idea is implicit in the way we model energy systems as stock-flow systems: the total amount of energy does not change, it just flows from one stock to another.

Rule 2: First Law of Thermodynamics

- Energy can be neither created nor destroyed, although it can be changed from one form to another.
- Heat is a form of energy.

Remark 4: Energy can be neither created nor destroyed—that is what scientists mean by saying that energy is "conserved." But when environmentalists or people in the media discuss the importance of "conserving energy," they are talking about something different: keeping energy in a *usable form*. When, for example, cars or light bulbs produce motion or light from chemical or electrical energy, that energy ultimately ends up as low-grade heat: still there, but no longer usable by us. So "conserving energy" is about *keeping as much energy in a usable (high-grade) form as possible.*

The money analogy is helpful here. If you use money to buy a meal or a textbook or a gallon of gas, that money has not disappeared, but it is no longer usable by you; it is in the seller's pocket. If your bank charges you a fee every time you use your credit card, that money becomes unavailable to you too—it is a "loss," something like the waste heat from a car or from human exertion. It wasn't destroyed; it was transferred to the bank and is no longer available to you. Conserving energy (in the ordinary day-to-day sense) is like reducing expenses and transaction fees and keeping as much money in the bank as possible.

As far as the first law of thermodynamics is concerned, all forms of energy are on the same level. The distinction between "high-grade" and "low-grade" energy is the province of the second law of thermodynamics. This has a precise mathematical formulation [113], but for now we'll be content to state the second law in a quite informal way:

Rule 3: Second Law of Thermodynamics

- Energy comes in high-grade and low-grade forms.
- In any energy transformation, the overall grade goes down.

Let's say a bit more about this business of high-grade versus low-grade. First we agree to classify **all kinds of energy except heat energy** (that includes mechanical, electrical, electromagnetic, chemical, nuclear energy, and so on) as **high-grade**.

To understand the "grade" of a quantity of heat energy is more complicated. The key principle is this: there needs to be a *temperature difference* between an object and its surroundings in order for heat energy contained in the object to do useful work. One of the clearest examples of this is given by *thermoelectric* materials [159]: materials that can convert heat directly into electricity. In order for a thermoelectric material to function, it needs a temperature difference between its "hot side" and "cold side," and the amount of energy produced depends on this temperature difference. See Figure 15, which shows the nuclear-thermoelectric power system for NASA's Mars rover undergoing tests at Idaho National Laboratory. The heat on the "hot side" of the thermoelectric material is provided by the radioactive decay of plutonium, and the "cold side" is at the temperature of Mars's atmosphere. If *both* sides were hot, the device would not work.

Plutonium-powered space vehicles are rather exotic, but James Watt's steam engine illustrates the same principle. The "hot side" is provided by the furnace, which turns water into steam. The "cold side" is provided by the atmosphere, where steam condenses back to water. We make this a definition:

Figure 15: Nuclear thermoelectric power system for Mars rover.

Critical Thinking

"One of these things is not like the other." Conventional and hybrid cars are powered by gasoline, but fully electric cars are powered by electricity, which must itself be generated from other sources—often from burning fossil fuels. Table 3 does not take into account the efficiency of generating electricity from its primary sources. Do you find this misleading? Try to figure out how the numbers would change if this information were also included in the calculation.

The engine in a gasoline-powered car is a complicated machine, but from the point of view of thermodynamics it, like the steam turbine or the thermoelectric generator, is simply a device for turning *heat* (generated by the exploding gasoline-air mixture in the cylinders) into *usable energy* (mechanical in this case). There is a general name for such devices:

Definition 7

A **heat engine** is any energy process that produces usable energy out of the temperature difference between a hot object and its cooler surroundings.

Our discussion of the second law of thermodynamics (Rule 3 on page 89) has so far taken a qualitative form: the greater the temperature difference, the higher the "grade" of the heat energy in the hot object and so the more efficiently we can expect to extract useful work from it. Sometimes, though, we need to know the *quantitative* form of this law: exactly *how* does the efficiency depend on the temperatures? The answer is called the **Carnot limit**:

Rule 4

Consider a heat engine that has a "hot side" and a "cold side." Let T_h equal the temperature of the hot side of the engine and T_c the temperature of the cold side, both measured in *kelvins* (see page 13). Then the efficiency with which the engine converts heat energy into useful work can be no greater than the fraction

$$\frac{T_h - T_c}{T_h},$$

which is called the **Carnot limit** for the given temperatures T_h, T_c.

The Carnot limit is the *theoretical maximum possible* efficiency: actual efficiencies will be less.

Problem 2: What is the Carnot limit on the efficiency of a coal-fired power plant? Assume that the superheated steam enters the turbines at a temperature of 800 K. (This is a reasonable figure in practice.)

Solution: From the data of the question we have $T_h = 800\,\text{K}$. The "cool side" temperature T_c will be the temperature of the cooling water, roughly equal to the surface temperature of the Earth, which is about $15\,^\circ\text{C} = 288\,\text{K}$, let us say (near enough) 300 K. Thus the Carnot limit on efficiency is

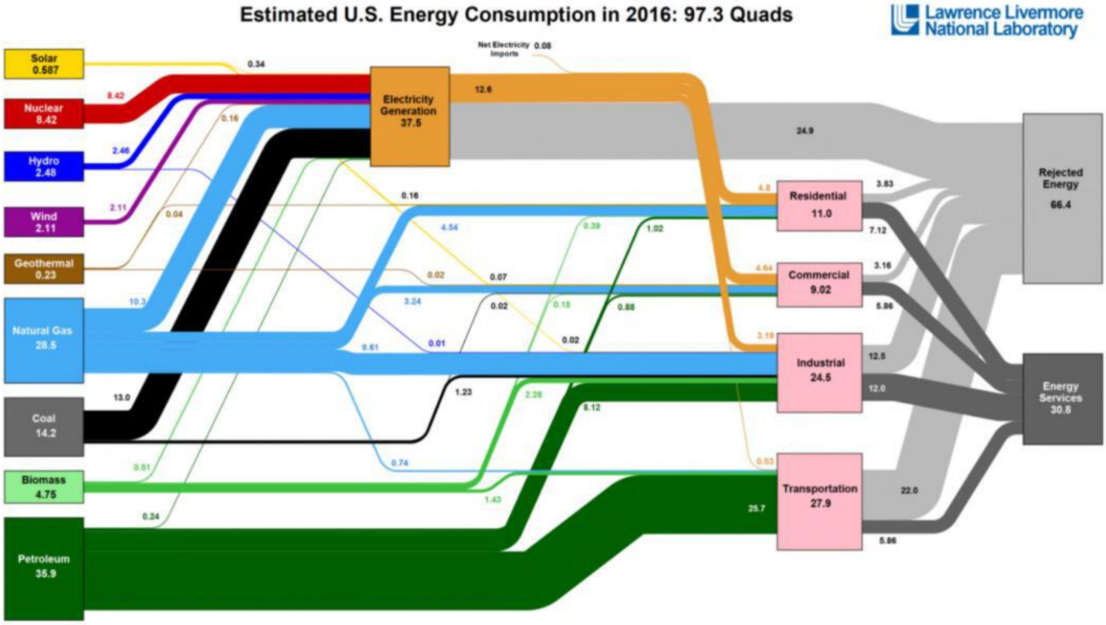

Figure 18: Energy flows in the U.S. in 2016.

as perceived by the eye, and we can convert to watts of light energy by using the *luminous coefficient* which is approximately 683 lumens per watt (see Case Study 7.7). The efficiency of the bulb is therefore

$$\frac{\text{Light power output}}{\text{Power input}} = \frac{1690\,\cancel{\text{lumens}}}{100\,\cancel{\text{W}}} \times \frac{1\,\cancel{\text{W}}}{683\,\cancel{\text{lumens}}} \approx 0.025.$$

The efficiency of the bulb is about $2\frac{1}{2}$ percent. We'll look at alternative, more efficient kinds of lighting in Case Study 7.7.

Example 2:

A great many of the concepts that we've introduced so far are illustrated in the flow chart (Figure 18) produced by the Lawrence Livermore National Laboratory [191]—see the bibliography for a link to an enlargeable graphic. We see, for instance, that the total U.S. energy consumption for the year 2016 was 97.3 quads—remember from Table 2 on page 86 that a "quad" is a quadrillion Btu, or about 10^{18} joules. Of this, 30.8 quads went to "energy services," that is, *useful energy*, and 66.4 quads went to "rejected energy," that is, *waste*. The overall efficiency of U.S. energy use is therefore

$$\frac{30.8}{97.3} \times 100\% \approx 32\%.$$

Looking at the left of the chart, we see that fossil fuels (coal, oil, natural gas) account for $14.2 + 35.9 + 28.5 = 78.6$ of the total 97.3 quads, or about 81% (notice also, by the way, that the next biggest source of energy is nuclear power). As explained in Section 2.2.2, fossil fuel consumption is deeply embedded in our present way of life, but in its present form it is not sustainable. That fundamental tension is something that humanity will have to resolve very soon.