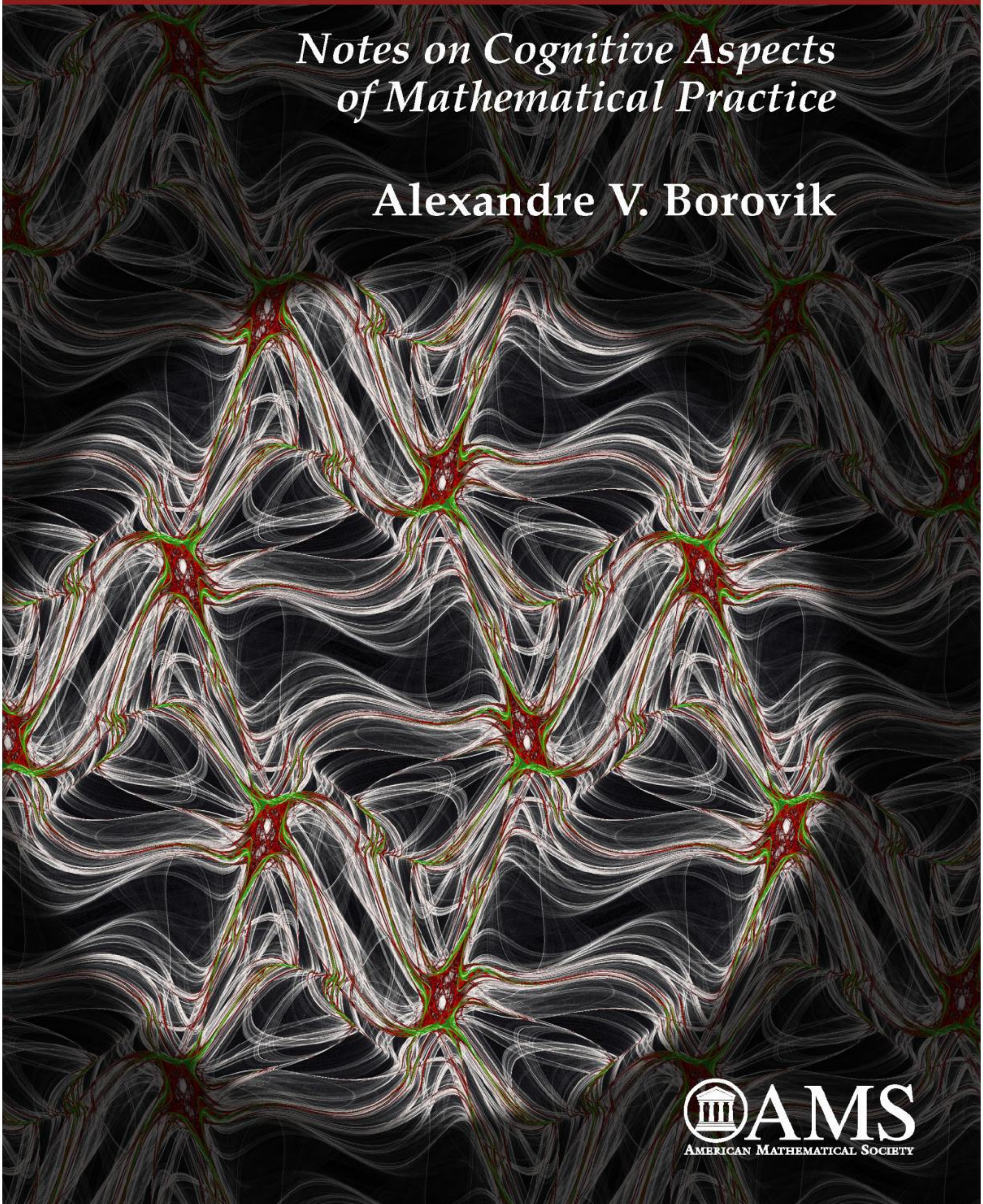
MATHEMATICS
under the Microscope



Notes on Cognitive Aspects of Mathematical Practice

Alexandre V. Borovik

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AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island

The image on the cover, "NeuralNet Quilt", is a section of a planar repeating pattern of type pgg and was created by Mike Field, University of Houston. The pattern was generated by a determinstic, but chaotic, torus map, and the colors represent probabilities of visiting points on the pattern under the iteration.

2000 *Mathematics Subject Classification.* Primary 00A30, 00A35, 97C50.

For additional information and updates on this book, visit
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Library of Congress Cataloging-in-Publication Data

Borovik, Alexandre.
Mathematics under the microscope : notes on cognitive aspects of mathematical practice / Alexandre V. Borovik.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-8218-4761-9 (alk. paper)
1. Mathematics—Psychological aspects. 2. Mathematical ability. I. Title.

BF456.N7B67 2009
510.1′9—dc22

2009029174

10 9 8 7 6 5 4 3 2 1 15 14 13 12 11 10

Contents

Part III
History and Philosophy

Preface

Сквозь волшебный прибор Левенгука ...
Николай Заболоцкий

The portrayal of human thought has rarely been more powerful and convincing than in Vermeer's *Astronomer*. The painting creates the illusion of seeing the movement of thought itself—as an embodied action, as a physical process taking place in real space and time.

I use the *Astronomer* as a visual metaphor for the principal aim of the present book. I attempt to write about mathematical thinking as an objective, real-world process, something which is actually moving and happening in our brains when we do mathematics. Of course, it is a challenging task; inevitably, I have to concentrate on the simplest, atomic activities involved in mathematical practice—hence "the microscope" in the title.

Among other things,

- I look at simple, minute activities, like placing brackets in the sum
$$a + b + c + d + e.$$
- I analyze everyday observations so routine and self-evident that their mathematical nature usually remains unnoticed: for example, when you fold a sheet of paper, the crease for some reason happens to be a perfectly straight line.
- I use palindromes, like MADAM, I'M ADAM, to illustrate how mathematics deals with words composed of symbols—and how it relates the word symmetry of palindromes to the geometric symmetry of solid bodies.
- I even discuss the problem of dividing 10 apples among 5 people!

Why am I earnestly concerned with such ridiculously simple questions? Why do I believe that the answers are important for our understanding of mathematics as a whole?

We cannot seriously discuss mathematical thinking without taking into account the limitations of our brains.

In this book, I argue that we cannot seriously discuss mathematical thinking without taking into account the limitations of the information-processing capacity of our brains. In our conscious and totally controlled reasoning we can process about 16 bits per second. In activities related to mathematics this miserable bit rate is further reduced to 12 bits per second in the addition of decimal numbers and to 3 bits in counting individual objects. Meanwhile the visual processing module of our brains easily handles 10,000,000 bits per second! (See [211, pp. 138 and 143].) We can handle complex mathematical constructions only because we repeatedly *compress* them until we reduce a whole theory to a few symbols which we can then treat as something *simple*, also because we *encapsulate* potentially infinite mathematical processes, turning them into finite objects, which we then manipulate on a par with other much simpler objects. On the other hand, we are lucky to have some mathematical capacities directly wired into the powerful subconscious modules of our brains responsible for visual and speech processing and *powered* by these enormous machines.

As you will see, I pay special attention to *order, symmetry*, and *parsing* (that is, bracketing of a string of symbols) as prominent examples of *atomic* mathematical concepts or processes. I put such "atomic particles" of mathematics at the focus of the study. My position is diametrically opposite to that of Martin Krieger who said in his recent book *Doing Mathematics* [61] that he aimed at

> a description of some of the work that mathematicians do, employing *modern and sophisticated* examples.

Unlike Krieger, I write about "simple things". However, I freely use examples from modern mathematical research, and my understanding of "simple" is not confined to the elementary-school classroom. I hope that a professional mathematician will find in the book sufficient non-trivial mathematical material.

The book inevitably asks the question, "How does the mathematical brain work?" I try to reflect on the explosive development of *mathematical cognition*, an emerging branch of neurophysiology which purports to locate structures and processes in the human brain responsible for mathematical thinking [159, 171]. However, I am not a cognitive psychologist; I write about the cognitive mechanisms of mathematical thinking from the position of a practicing mathematician who is trying to take a very close look through the magnifying glass at his own everyday work. I write not so much about discoveries of cognitive science as of their implications for our understanding of mathematical practice. I do not even insist on the ultimate correctness of my interpretations of findings of cognitive psychologists and neurophysiologists. With science developing at its present pace, the current understanding of the internal working of the brain is no more than a preliminary sketch; it is likely to be overwritten in the future by deeper works.

Instead, I attempt something much more speculative and risky. I take, as a working hypothesis, the assumption that mathematics is produced by our brains and therefore bears imprints of some of the intrinsic structural patterns of our minds. If this is true, then a close look at mathematics might reveal some of these imprints—not unlike the microscope revealing the cellular structure of living tissue.

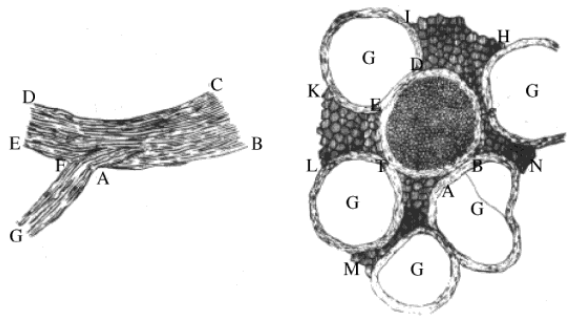
I try to bridge the gap between mathematics and mathematical cognition by pointing to structures and processes of mathematics which are sufficiently non-trivial to be interesting to a mathematician, while being deeply integrated into certain basic structures of our minds and which may lie within reach of cognitive science. For example, I pay special attention to *Coxeter Theory*. This theory lies at the very heart of modern mathematics and could be informally described as an algebraic expression of the concept of symmetry; it is named after H. S. M. Coxeter who laid its foundations in his seminal works [336, 337]. Coxeter Theory provides an example of a mathematical theory where we occasionally have a glimpse of the inner working of our minds. I suggest that Coxeter Theory is so natural and intuitive because its underlying cognitive mechanisms are deeply rooted in both the visual and verbal processing modules of our minds. Moreover, Coxeter Theory itself has clearly defined geometric (visual) and algebraic (verbal) components which perfectly match the great visual/verbal divide of mathematical cognition.

Mathematics is the study of mental objects with reproducible properties.

However, in paying attention to the "microcosm" of mathematics, I try not to lose the large-scale view of mathematics. One of the principal points of the book is the essential *vertical* unity of mathematics, the natural integration of its simplest objects and concepts into the complex hierarchy of mathematics as a whole.

One of the principal points of the book is the essential vertical *unity of mathematics.*

The *Astronomer* is, again, a useful metaphor. The celestial globe, the focal point of the painting, boldly places it into a cosmological perspective. The Astronomer is reaching out to the Universe—but, according to the widely held attribution of the painting, he is Vermeer's neighbor and friend Antonij van Leeuwenhoek, the inventor of the microscope and the discoverer of the *microcosm*, a beautiful world of tiny creatures which no one had ever seen before. Van Leeuwenhoek also discovered the cellular structure of living organisms, the basis of the unity of life.



Microstructure of nerve fibers: a drawing by Antonij van Leeuwenhoek, circa 1718. Public domain.

The next principal feature of the book is that I center my discussion of mathematics as a whole—in all its astonishing unity—around the thesis, due to Davis and Hersh [21], that mathematics is

the study of mental objects with reproducible properties.

In this book, the Davis–Hersh thesis works at three levels.

First, it allows us to place mathematics in the wider context of the evolution of human culture. Chapter 11 of the book is a brief diversion into *memetics*, an emerging interdisciplinary area of research concerned with the mechanisms of the evolution of human culture. The term *meme*, an analogue of "gene", was made popular by Richard Dawkins [167] and was introduced into mainstream philosophy and cultural studies by Daniel Dennett [25]. It refers to elementary units of cultural transmission. I discuss the nature and role of "mathematical" memes in detail sufficient, I hope, for making the claim that mathematical memes play a crucial role in many meme complexes of human culture: they increase the precision of reproduction of the complex, thus giving it an evolutionary advantage. Remarkably, the memes may remain invisible, unnoticed for centuries and not recognized as rightly belonging to mathematics. In this book, I argue that this is a characteristic property of "mathematical" memes:

> If a meme has the intrinsic property that it increases the precision of reproduction and error correction of the meme complexes it belongs to and if it does that without resorting to external social or cultural restraints, then it is likely to be an object or construction of mathematics.

So far research efforts in mathematical cognition have been concentrated mostly on brain processes during quantification and counting (I refer the reader to the book *The Number Sense: How the Mind Creates Mathematics* by Stanislas Dehaene [171] for a first-hand account of the study of number sense and numerosity). Important as they are, these activities occupy a very low level in the hierarchy of mathematics. Not surprisingly, the remarkable achievements of cognitive scientists and neurophysiologists are mostly ignored by the mathematical community. This situation may change fairly soon, since conclusions drawn from neurophysiological research could be very attractive to policymakers in mathematics education, especially since neurophysiologists themselves do not shy away from making direct recommendations. I believe that hi-tech "brain scan" cognitive psychology and neurophysiology will more and more influence policies in mathematics education. If mathematicians do not pay attention now, it may very soon be too late; we need a dialogue with the neurophysiological community. The development of neurophysiology and cognitive psychology has reached the point where mathematicians should start some initial discussion of the issues involved. Furthermore, the already impressive body of literature on mathematical cognition might benefit from a critical assessment by mathematicians.

Cognitive psychology and neurophysiology will more and more influence policies in mathematics education. If mathematicians do not pay attention now, it may very soon be too late; we need a dialogue with the neurophysiological community.

Second, the Davis–Hersh thesis puts the underlying cognitive mechanisms of mathematics into the focus of the study.

Finally, the Davis–Hersh thesis is useful for understanding the mechanisms of learning and teaching mathematics: it forces us to analyze the underlying processes of interiorization and reproduction of the mental objects of mathematics.

In my book, I try to respond to the sudden surge of interest in mathematics education which can be seen in the mathematical research community. It appears that it has finally dawned on us that we are a dying breed, that the very reproduction of mathematics as a social institution and a professional community is under threat. I approach the problems of mathematical education from this viewpoint which should not be easily set aside: what kind of mathematics teaching allows for the production of future professional mathematicians? What is it that makes a mathematician? What are the specific traits which need to be encouraged in a student if we want him or her to be capable of a rewarding career in mathematics? I hope that my observations and questions might be interesting to all practitioners and theorists of general mathematical education. But I refrain from any critique of, or recommendations for, school mathematics teaching.

Alexandre Borovik, aged 11

The *unity* of mathematics means that there are no boundaries between "recreational", "elementary", "undergraduate", and "research" mathematics; in my book, I freely move throughout the whole range. Nevertheless, I try to keep the book as non-technical as possible. I hope that the book will find readers among school teachers as well as students.

In a few instances, the mathematics used appears to be more technical. This usually happens when I have to resort to *metamathematics*, a mathematical description of the structure and role of mathematical theories. But even in such cases, mathematical concepts are no more than a presentation tool for a very informal description of my observations.

Occasionally I could not resist the temptation to include some comments on matters of my own professional interest; however, such comments are indicated in the text by smaller print.

Photographs in this book

I come from childhood as from a homeland.
Antoine de Saint-Exupéry, *Pilot de guerre*

I tried to place in the margins of the book a photograph of every living mathematician/computer scientist/historian of mathematics/philosopher of mathematics/scholar of mathematics mentioned or quoted in the book. The catch is, I am using *childhood* photographs. In my book, I write a lot about children and early mathematical education, and I wish my book to bear a powerful reminder that we all were children once. I hope that the reader agrees that the photographs make a fascinating gallery—and my warmest thanks go to everyone who contributed his or her photograph.

I tried to place a photograph of a particular person in that section of the book where his/her views had some impact on my writing. The responsibility for my writing is my own, and including a photograph of a person should not be construed as his or her tacit endorsement of my views.

Apologies

This book may need more than one preface, and in the end there would still remain room for doubt whether anyone who had never lived through similar experiences could be brought closer to the experience of this book by means of prefaces.
Friedrich Nietzsche

I hope that the reader will forgive me that the book reflects my personal outlook on mathematics. To preempt criticism of my sweeping generalizations (and of the even greater sin of using introspection as a source of empirical data), I quote Sholom Aleichem:

> Man's life is full of mystery, and everyone tries to compare it to something simple and easier to grasp. I knew a carpenter, and he used to say: "A man is like a carpenter. Look at the carpenter; the carpenter lives, lives and then dies. And so does a man."

And to ward off another sort of criticism, I should state clearly that I understand that, by writing about mathematics instead of doing mathematics, I am breaking a kind of taboo. As G. H. Hardy famously put it in his book *A Mathematician's Apology* [45, p. 61]:

> The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done. Statesmen despise publicists, painters despise art-critics, and physiologists, physicists, mathematicians have similar feelings; there is no scorn more profound, or on the whole justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation is work for second-rate minds.

Having broken a formidable taboo of my own tribe, I can only apologize in advance if I have disregarded, inadvertently or through ignorance, any sacred beliefs of other disciplines and professions. To reduce the level of offence, I ask the discerning reader to treat my book not so much as a statement of my beliefs but as a list of questions which have puzzled me throughout my professional career in mathematics and which continue to puzzle me.

Perhaps, my questions are naive. However, I worked on the book for several years and kept the text on the Web, returning to it from time to time to add some extra polish or to correct the errors. So far, the changes in the book were limited to expanding and refining the list of questions, not inserting answers—I cannot find any in the existing literature. This is one of the reasons why I believe that perhaps at least some of my questions deserve a thorough discussion in the mathematical, educational, and cognitive science communities.

My last apology concerns the use of terminology. Some terms and expressions which attained a specialized meaning in certain mathematics-related disciplines are used in this book in their (original) wider and vaguer sense and therefore are more friendly to the readers. To fend off a potential criticism from nit picking specialists, I quote a fable which I heard from one of the great mathematicians of our time, Israel Gelfand:

> A student corrected an old professor in his lecture by pointing out that a formula on the blackboard should contain cotangent instead of tangent. The professor thanked the student, corrected the formula and then added:
>
> "Young man, I am old and no longer see much difference between tangent and cotangent—and I advise that you do so as well."

Indeed, when mathematicians informally discuss their work, they tend to use a very flexible language—exactly because the principal technical language of their profession is exceptionally precise. I follow this practice in my book; I hope it allows me to be friendly towards all my readers and not only my fellow mathematicians.

Acknowledgements: Inspiration and Help

The gods have imposed upon my writing
the yoke of a foreign tongue
that was not sung at my cradle.
Hermann Weyl

I thank my children, Sergey and Maria, who read a much earlier version of the book and corrected my English (further errors introduced by me are not their responsibility) and who introduced me to the philosophical writings of Terry Pratchett. I am grateful to my wife, Anna, the harshest critic of my book; this book would never have appeared without her. She also provided a number of illustrations.

As the reader may notice, Israel Gelfand is the person who most influenced my outlook on mathematics. I am most grateful to him for generously sharing with me his ideas and incisive observations.

I am indebted to Gregory Cherlin and Reuben Hersh and to my old friend Owl for most stimulating conversations and many comments on the book; some of the topics in the book were included on their advice.

Almost everyday chats with Hovik Khudaverdyan about mathematics and the teaching of mathematics seriously contributed to my desire to proceed with this project.

During our conversation in Paris, the late Paul Moszkowski forcefully put forth the case for the development of the theory of Coxeter groups without reference to geometry and pointed me toward his remarkable paper [388].

Jeff Burdges, Gregory Cherlin, David Corfield, Chandler Davis, Ed Dubinsky, Erich Ellers, Tony Gardiner, Ray Hill, Chris Hobbs, David Pierce, John Stillwell, Robert Thomas, Ijon Tichy, and Neil White carefully read and corrected the whole or parts of the book.

My thanks are due to a number of people for their advice and comments on the specific areas touched upon in the book: to David Corfield—on the philosophy of mathematics, to Susan Blackmore—on memetics, to Vladimir Radzivilovsky—for explaining to me the details of his teaching method, to Satyan Devadoss—on diagrams and drawings used in this book, to Ray Hill—on the history of coding theory, to Péter Pál Pálfy—on universal algebra, to Sergey Utyuzhnikov—on chess, turbulence, and dimensional analysis, to Alexander Jones and Jeremy Gray—on the history of Euclidean geometry, to Victor Goryunov—on multivalued analytic functions, to Thomas Hull—on the history of Origami, to Gordon Royle—on Sudoku, to Alexander Kuzminykh and Igor Pak—on convex geometry, to Dennis Lomas—on visual thinking, to Semen Kutateldaze—on philosophy and convex geometry, and, finally, to Paul Ernest and Inna Korchagina for general encouraging comments.

Jody Azzouni, Barbara Sarnecka, and Robert Thomas sent me the texts of their papers [5, 6], [163, 225], [92].

David Petty provided diagrammatic instructions for the Origami Chinese Junk (Figures 11.2 and 11.3). Dougald Dunham allowed me to use his studies of hyperbolic tessellations in M. C. Escher's engravings (Figures 5.4 and 5.5). Ali Nesin made illustrations for Chapter 10. Simon Thomas provided me with diagrams used in Section 12.8.

I am lucky that my university colleagues David Broomhead, Paul Glendinning, Bill Lionheart, and Mark Muldoon are involved in research in mathematical imaging and/or mathematical models of neural activity and perception; their advice has been invaluable.

Paul Glendinning gave me permission to quote large fragments of his papers [183, 185].

Anonymous, age unknown

My work on genetic algorithms shaped my understanding of the evolution of algorithms; I am grateful to my collaborator Rick Booth who shared with me the burden of the project. Also, the very first seed which grew into this book can be found in our joint paper [108].

Finally, my thanks go to the blogging community—I have picked from the blogosphere some ideas and quite a number of references—and especially to numerous anonymous commentators on my blog.

Acknowledgements: Hospitality

I developed some of the ideas of Section 7.1 in a conversation with Maria do Rosário Pinto; I thank her and Maria Leonor Moreira for their hospitality in Porto.

Parts of the book were written during my visits to University Paris VI in January 2004 and June 2005 on invitation from Michel Las Vergnas, and I use this opportunity to tell Janette and Michel Las Vergnas how enchanted I was by their hospitality.

Section 10.5 of the book is a direct result of a mathematical tour of Cappadocia in January 2006, organized by my Turkish colleagues Ayşe Berkman, David Pierce, and Şükrü Yalçınkaya—my warmest thanks to them for their hospitality in Turkey on that and many other occasions.

Acknowledgements: Institutional

An invitation to the conference *The Coxeter Legacy: Reflections and Projections* at the University of Toronto had considerable influence on my work on this book, and I am most grateful to its organizers.

My work on genetic algorithms was funded by EPSRC (grant GR/R29451).

While working on the book, I used, on several occasions, the facilities of Mathematisches Forschungsinstitut Oberwolfach, The Fields Institute for Research in Mathematical Sciences, and the Isaac Newton Institute for Mathematical Sciences.

Chapter 7 of this book was greatly influenced by the Discussion Meeting *Where will the next generation of UK mathematicians come from?* held in March 2005 in Manchester. The meeting was supported by the Manchester Institute for Mathematical Sciences, by the London Mathematical Society, by the Institute of Mathematics and Applications, and by the UK Mathematics Foundation.

It was during the MODNET Conference on Model Theory in Antalya, November 2–11, 2006, that I placed the first chapter of the book on the Internet. MODNET (Marie Curie Research Training Network in Model Theory and Applications) is funded by the European Commission under contract no. MRTN-CT-2004-512234.

In July 2007 and July 2008 I enjoyed the hospitality of Mathematical Village in Şirince, Turkey, built and run by Ali Nesin.

I started writing this book in *Café de Flore*, Paris—an extreme case of vanity publishing! Since then, I continued my work in many fine establishments, among them *Airbräu, das Brauhaus im Flughafen* in Munich, *Cafe del Turco* in Antalya, *L'authre Bistro* on rue des Ecoles, and *Café des Arts* on place de la Contrescarpe in Paris—I thank them all.

Alexandre Borovik
July 10, 2009
Didsbury

List of Figures

List of Photographs

Part I

Simple Things:
How Structures of Human Cognition Reveal Themselves in Mathematics

1

A Taste of Things to Come

This is the opening chapter of the book, and I use it to set the tone of my narrative: I start with some simple mathematical observations and briefly discuss what they possibly say about the inner workings of our minds. Surprisingly, this discussion very naturally involves some non-trivial ideas and results from the frontier of mathematical research. But it is better to see it for yourself.

1.1 Simplest possible example

Simplicity, simplicity, simplicity!
I say, let your affairs be as two or three,
and not a hundred or a thousand;
instead of a million count half a dozen,
and keep your accounts on your thumb-nail.

Henry David Thoreau, *Walden*

In my account, I am not afraid to be very personal, almost sentimental, and have decided to start the discussion of the "simple things" of mathematics by turning to my memories from my school years.

Always test a mathematical theory on the simplest possible example—and explore the example to its utmost limits.

I had my most formative mathematical experiences at the tender age of thirteen, when I still lived in my home village on the shores of Lake Baikal in Siberia. I learned elementary calculus from two thin booklets sent to me from a mathematics correspondence school: *The Method of Coordinates* [266] and *Functions and Graphs* [267]. Much later in my life I met one of the authors of the books, the famous mathematician Israel Gelfand, and had a chance to do some mathematics with him.

Once I mentioned to Gelfand that I read his *Functions and Graphs*; in response, he rather sceptically asked me what I had learned from the book. He was delighted to hear my answer: "The general principle of always looking at the simplest possible example". "Yes!" exclaimed Gelfand in his usual manner, "yes, this is my most important discovery in mathematics teaching!" He proceeded by saying how proud he was that, in his famous seminars, he always pressed the speakers to provide simple examples, but, as a rule, he himself was able to suggest a simpler one. [1]

So, let us look at the principle in more detail:

> Always test a mathematical theory on the simplest possible example...

This is a banality, of course. Everyone knows it; therefore almost no one follows it. So let me continue:

> ...and explore the example to its utmost limits.

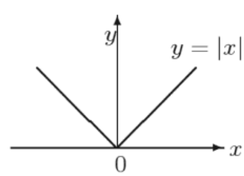
This book contains a number of examples pushed to their intrinsic limits. See, in particular, Section 2.6 and the discussion of Figure 2.11 on page 40 for some examples from the theory of Coxeter groups and mirror systems. What could be simpler than that?

But it is even more instructive to look at an example from *Functions and Graphs*.

What is the simplest graph of a function? Of course, that of a linear function,

$$y = ax + b.$$

But what are the simplest non-linear elementary functions? The apparent answer is quadratic polynomials. Well, *Functions and Graphs* suggests something different. The simplest non-linear function is the *magnitude*, or *absolute value*, $y = |x|$.



Indeed, it allows

- easy plotting and interpolation;

I claim that the difference between the "switch" and "flow" modes of computation is felt and recognized by almost every mathematician. Most undergraduate students of mathematics in their second or third year of study can judge—and with a surprising degree of certainty and immediacy in their answers—what kind of mathematics is more suitable for them, discrete or continuous. They just know, even if they have never before given any thought to the issue. Perhaps, we should tell them that there is a difference.[3]

Not being a professional neurophysiologist, I can only conjecture that the two types of mathematical activities should be reflected in two different patterns of brain activity, perhaps even easily noticeable with the help of modern brain scan techniques. Meanwhile, within mathematics itself the two modes of calculation are recognized as being intrinsically different and are analyzed to considerable depth. In the next section I briefly describe the findings of mathematicians.

1.3 Choiceless computation

We choose our joys and sorrows long before we experience them.

Kahlil Gibran

So, we started with the absolute value function $y = |x|$ as an example of "the simplest possible example" and are now moving to a mathematical description of the difference between the "switch" and "flow" modes of computation.

As I will frequently do in this book, I use a concept from computer science as a pointer to possible structures of human cognition responsible for particular ways of manipulating mathematical objects. In this case, a possible indicator is the concept of *choiceless polynomial time computation* [313].

Some terminology ought to be explained.

1.3.1 Polynomial time complexity

An algorithm is said to have *polynomial time* complexity (of degree d) if, when working with inputs of size l, it requires $O(l^d)$ elementary operations (see the endnote [4] for an explanation of $O(\ )$-notation). Let us look, for example, at the addition of two integers. The input size here is the number of digits required to write the integers down; if both summands are smaller than n, then each needs at most

$$l = [\log_{10}(n)] + 1$$

digits (here, $[\log_{10}(n)]$ denotes $\log_{10}(n)$ rounded down to the nearest integer). To add the integers, we need, in each position, to add despite the immense complexity and power of the brain, the mental processes of mathematics appear to be surprisingly resource-limited. Therefore I have a feeling that branches of logic developed for the needs of complexity theory might provide better metaphors than the general theory of computation.

1.4 Analytic functions and the inevitability of choice

AEROFLOT *flight attendant: "Would you like a dinner?"*
Passenger: "And what's the choice?"
Flight attendant: "Yes—or no."

We have mentioned at the beginning of our discussion that $|x|$ is a non-analytic function. It can be written by a single algebraic formula

$$|x| = \sqrt{x^2},$$

with the only glitch being that of the two values of the square root $\pm\sqrt{x^2}$ we have *to choose* the positive one, namely, $\sqrt{x^2}$.

One may argue that in the case of the absolute value function the choice is artificial and is forced on us by the function's awkward definition. But let us turn to solutions of algebraic equations, which give more natural examples of the inevitability of choice.

The classical formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Chris Hobbs, aged 6

for the roots of the quadratic equation is the limit of what we can do with analytic functions without choosing branches of multivalued analytic functions—but even here, beware of complications and read an interesting comment from Chris Hobbs.[7] Recall that the inverse of the square function $x = y^2$ is a two-valued function $y = \pm\sqrt{x}$ whose graph has two branches, positive $y = \sqrt{x}$ and negative $y = -\sqrt{x}$. Similarly, the cube root function $y = \sqrt[3]{x}$ has three distinct branches, but they become visible only in the complex domain, since only one cube root of a real number is real; the other two are obtained from it by multiplying it by complex factors

$$-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

The classical formula—which can be traced back to Gerolamo Cardano (1501–1576) and Niccolò Tartaglia (1499–1557)—for the roots of the cubic equation

$$x^3 + ax^2 + bx + c = 0$$

gives its three roots as

$$-\frac{a}{3} + \sqrt[3]{\frac{-2a^3 + 9ab - 27c + \sqrt{(2a^3 - 9ab + 27c)^2 + 4(a^2 + 3b)^3}}{54}} + \sqrt[3]{\frac{-2a^3 + 9ab - 27c - \sqrt{(2a^3 - 9ab + 27c)^2 + 4(a^2 + 3b)^3}}{54}},$$

$$-\frac{a}{3} + \frac{-1 - i\sqrt{3}}{2}\sqrt[3]{\frac{-2a^3 + 9ab - 27c + \sqrt{(2a^3 - 9ab + 27c)^2 + 4(a^2 + 3b)^3}}{54}} + \frac{-1 + i\sqrt{3}}{2}\sqrt[3]{\frac{-2a^3 + 9ab - 27c - \sqrt{(2a^3 - 9ab + 27c)^2 + 4(a^2 + 3b)^3}}{54}},$$

$$-\frac{a}{3} + \frac{-1 + i\sqrt{3}}{2}\sqrt[3]{\frac{-2a^3 + 9ab - 27c + \sqrt{(2a^3 - 9ab + 27c)^2 + 4(a^2 + 3b)^3}}{54}} + \frac{-1 - i\sqrt{3}}{2}\sqrt[3]{\frac{-2a^3 + 9ab - 27c - \sqrt{(2a^3 - 9ab + 27c)^2 + 4(a^2 + 3b)^3}}{54}}.$$

Please notice the carefully choreographed choice of the branches of the square root $\sqrt{\ }$ and the cube root function $\sqrt[3]{\ }$, the rhythmic dance of pluses and minuses. Without that choice, Cardano's formula produces too many values, only three of which are true roots.

Indeed, if we work with multivalued functions without making any distinction between their branches, we have to accept that the superposition of an m-valued function and an n-valued function has mn values. We cannot collect like terms: an innocent looking expression like

$$\sqrt{x} + \sqrt{9x}$$

defines, if we interpret "$\sqrt{x}$" as two-valued, a function with *four* branches

$$\pm\sqrt{x} \pm \sqrt{9x} = \{\,-4\sqrt{x}, -2\sqrt{x},\, 2\sqrt{x},\, 4\sqrt{x}\,\}.$$

It is a rigorous mathematical fact [307] that solutions of equations of degree higher than two cannot be analytically expressed by choiceless multivalued formulae (even if we allow for more sophisticated analytic functions than radicals); see a discussion of the topological nature of this fact by Vladimir Arnold [3, p. 38].

This last observation is especially interesting in the historic context. At the early period of development of symbolic algebra, mathematicians were tempted to introduce functions more general than roots. The following extract from Pierpaolo Muscharello's *Algorismus* from 1478 is taken from Jens Høyrup [54]:

> Pronic root is as you say, 9 times 9 makes 81. And now take the root of 9, which is 3, and this 3 is added above 81, so that the pronic root of 84 is said to be 3.

In effect, Muscharello wanted to introduce the inverse of the function

$$z \mapsto z^4 + z.$$

Arnold's theorem explains why such tricks could not lead to an easy solution of cubic and quadric equations and why it had been abandoned.

1.5 You name it—we have it

This section is more technical and can be skipped.

As I have already said on several occasions, this book is about simple atomic objects and processes of mathematics. However, mathematics is huge and immensely rich; even the simplest observations about its simplest objects may already have been developed into sophisticated and highly specialized theories. Mathematics' astonishing cornucopian richness and its bizarre diversity are not frequently mentioned in works on philosophy and methodology of mathematics—but this point has to be emphasized, since its makes the question about *unity* of mathematics much more interesting.

In this section, I will briefly describe a "mini-mathematics", a mathematical theory concerned with a close relative of the absolute value function, the *maximum* function of two variables

$$z = \max(x, y).$$

Of course, the absolute value function $|x|$ can be expressed as

$$|x| = \max(x, -x).$$

Similarly, the maximum $\max(x, y)$ can be expressed in terms of the absolute value $|x|$ and arithmetic operations—I leave it to the reader as an exercise. [?]

Oh yes, do it.

The theory is known by the name of *tropical mathematics*. The strange name has no deep meaning: the adjective "tropical" was coined by French mathematicians in honor of their Brazilian colleague Imre Simon, one of the pioneers of the new discipline. Tropical mathematics works with the usual real numbers but uses only two operations: addition, $x + y$, and taking the maximum, $\max(x, y)$—therefore it is one of the extreme cases of "switch-flipping", choice-based mathematics. Notice that addition is distributive with respect to taking maximum:

$$a + \max(b, c) = \max(a + b, a + c).$$

Index