# MATHS in BITE-SIZED CHUNKS

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# INTRODUCTION

I could start this book by telling you that maths is everywhere and yammer on about how important it is. This is true, but I suspect you've heard that one before and it's probably not the reason you picked this book up in the first place.

I could start by saying that being numerate and good at mathematics is an enormous advantage in the job market, particularly as technology plays an increasingly dominant role in our lives. There are great careers out there for mathematically minded people, but, to be honest, this book isn't going to get you a job.

I want to start by telling you that skill in mathematics can be learnt. Many of us have mathematical anxiety. This is like a disease, since we pick it up from other people who have been infected. Parents, friends and even teachers are all possible vectors, making us feel that mathematics is only for a select group of people who are just lucky, who were born with the right brain. They do mathematics without any effort and generally make the rest of us feel stupid.

This is not true.

Anyone can learn mathematics if they want to. Yes, it takes time and effort, like any skill. Yes, some people learn it faster than others, but that's true of most things worth learning. I know you're busy, so the premise here is that you want some easily digestible snippets. You can learn them piecemeal, each building on the one before, so that without too much effort you can take on board the concepts that really do explain the world around us.

I've divided the book up into several sections. You'll remember doing a lot of the more basic stuff at school, but my aim is to cover this at a brisk pace to get to the really tasty bits of mathematics that maybe you didn't see. You can work through the book from start to finish, or dip in and out as and when the mood takes you – a six-course meal and a buffet at the same time!

I've also included lots of anecdotes to spice things up – stories of how discoveries were made, who discovered them and what went wrong along the way. As well as being interesting and entertaining, these serve to remind us that mathematics is a field with a vibrant history that tells us a lot about how our predecessors approached life. It also shows that the famous, genius mathematicians had to work hard to get where they got, just like we do.

Prepare yourself for a feast. I hope you're hungry.

# 1

# NUMBER

Chapter 1

# TYPES OF NUMBER

Sixty-four per cent of people have access to a supercomputer.

In 2017, according to forecasts, global mobile phone ownership was set to reach 4.8 billion people, with world population hitting 7.5 billion. As the Japanese American physicist Michio Kaku (b. 1947) put it: ‘Today, your cell phone has more computer power than all of NASA back in 1969, when it placed two astronauts on the moon.’

At a swipe, each of us can do any arithmetic we need on our phones – so why bother to learn arithmetic in the first place?

It’s because if you can perform arithmetic, you start to understand how numbers work. The study of how numbers work used to be called arithmetic, but nowadays we use this word to refer to performing calculations. Instead, mathematicians who study the nature of numbers are called number theorists and they strive to understand the mathematical underpinnings of our universe and the nature of infinity.

Hefty stuff.

I’d like to start by taking you on a trip to the zoo.

Humans first started counting *things*, starting with one thing and counting up in whole numbers (or *integers*). These numbers are called the *natural* numbers. If I were to put these numbers into a mathematical zoo with an infinite number of enclosures, we'd need an enclosure for each one:

1, 2, 3, 4, 5, 6 . . .

The ancient Greeks felt that zero was not natural as you couldn't have a pile of zero apples, but we allow zero into the natural numbers as it bridges the gap into *negative* integers – minus numbers. If I add zero and the negative integers to my zoo, it will look like this:

. . . −6, −5, −4, −3, −2, −1, 0, 1, 2, 3, 4, 5, 6 . . .

My zoo now contains all the negative integers, which when combined with the natural numbers make up the group of numbers called, imaginatively, the *integers*. As each positive integer matches a negative one, my zoo needs twice as many enclosures as before, with one extra room for zero. However, my infinite mathematical zoo does not need to expand, as it is already infinite. This is an example of the hefty stuff I referred to earlier.

There are other types of numbers that are not integers. The Greeks were happy with piles of apples, but we know an apple can be divided and shared among a number of people. Each person gets a fraction of the apple and I'd like to have an example of each fraction in my zoo.

If I want to list all the fractions between zero and one, it would make sense to start with halves, then thirds, then quarters, etc. This methodical approach should ensure I get all the fractions without missing any. So, you can see that I'm going to have to go through all the natural numbers as denominators (the numbers on the bottom of the fraction). For each different denominator, I'll need all the different numerators (the numbers on the top of the fraction), starting from one and going up to the value of the denominator.

## Fractions

Fractions show numbers that are between whole integers and are written as one number (the numerator) above another (the denominator) separated by a fraction bar. For example, a half looks like:

$$\frac{1}{2}$$

One is the numerator, two is the denominator. The reason it is written this way is that its value is one divided by two. It tells you what fraction of something you get if you share one thing between two people. $\frac{3}{4}$ is three things shared between four people – each person gets three quarters.

Once I've worked out all the fractions between zero and one, I can use this to fill in all the fractions between all the natural numbers. If I add one to all the fractions between zero and one, this will give me all the fractions between one and two. If I add one to all of them, I'll have all the fractions between three and four. I can do this to fill in the fractions between all the natural numbers, and I could subtract to fill in all the fractions between the negative integers too.

So, I have infinity integers and I now need to build infinity enclosures between each of them for the fractions. That means I need infinity times infinity enclosures altogether. Sounds like a big job, but luckily I still have enough enclosures.

As the fractions can all be written as a ratio as well, the fractions are called the *rational* numbers. I now have all the rational numbers, which contain the integers (as integers can be written as fractions by dividing them by one), which contain the natural numbers in the zoo. Finished.

Just a moment – some mathematicians from India 2,500 years ago are saying that there are some numbers that can't be written as fractions. And when they say 'some', they actually mean infinity. They discovered that there is no number that you can square (multiply by itself) to get two, so the square root of two is not a rational number. We can't actually write down the square root of two as a number without rounding it, so we just show what we did to two by using the *radix* symbol: $\sqrt{2}$. There are other really important numbers that are not rational that have been given symbols instead as it is a bit of a faff to write down an unwritedownable number: $\pi$, $e$ and $\varphi$ are three examples that we'll look at later. We call such numbers *irrational*, and I need to put these into the zoo as well. Guess how many irrational numbers there are between consecutive rational numbers? That's right – infinity! However, I can still squeeze these into my infinite zoo without having to build any more enclosures, although Cantor might have a thing or two to say about that (see here).

## Squares and Square Roots

When you multiply a number by itself, we say the number has been *squared*. We show this with a little two called a power or index:

$$3 \times 3 = 3^2$$

Three squared is nine. This makes three the *square root* of nine. Square rooting is the opposite of squaring. The square root of sixteen is four because four squared is sixteen:

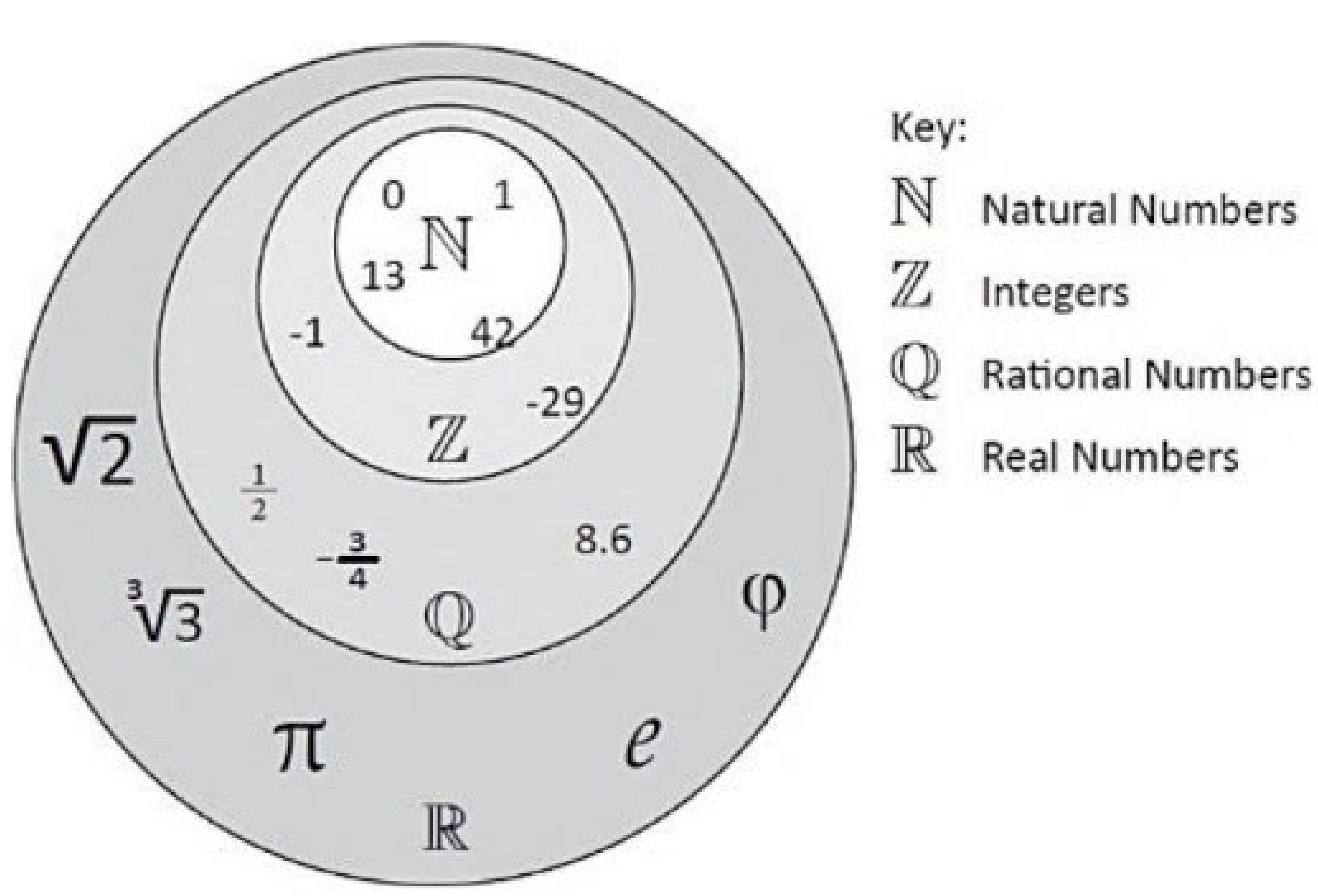
$$\sqrt{16} = 4$$

Numbers like nine and sixteen are called *perfect* squares, because their square root is an integer. Any number, including fractions and decimals, can be squared. Any positive number can be square rooted.

For much more information about this, see here.

When we put the irrational numbers together with the rational numbers we have what mathematicians call the *real* numbers. If you've spotted a pattern in what went before, you'll suspect that there are also not-real numbers and you'd be right. However, I'm going to stop there and name my zoo *The Infinite Real Number Zoo*. Most zoos sort their animals out by type, so I could organize mine into overlapping groups of types. The map might look like this, and I've put a few must-sees in to help you plan your day out:

## The Infinite Real Number Zoo



I must own up to the fact that my zoo owes a lot to the German mathematician David Hilbert (1862–1943). He made great contributions to mathematics but is best known for his advocacy and leadership of the subject. In 1900 he produced a list of twenty-three unsolved problems – now known as the Hilbert problems – for the International Congress of Mathematicians, three of which are still unresolved to this day. The thought experiment *Hilbert's Hotel*, the source for my zoo, concerns Hilbert's musings on a hotel with an infinite number of rooms filled with an infinite number of guests. Hilbert shows that we can still fit another infinite number of guests into the hotel if we can persuade all the initial guests to move to the room with a number double their current room number. The current guests would all now be in even-numbered rooms, leaving the odd-numbered rooms (of which there are infinitely many) for the new arrivals.

Chapter 2

# COUNTING WITH CANTOR

Galileo Galilei (1564–1642) came up with a nice puzzle known as *Galileo's paradox* while under house arrest in Italy for his heretical belief that the earth went around the sun.

It says that while some natural numbers are perfect squares (see here), most are not, so there must be more not-squares than squares. However, every natural number can be squared to produce a perfect square, so there must be the same number of squares as natural numbers. Hence, a paradox: two logical statements that cannot both be true.

Number theorists, as I've said, tackle the nature of infinity and its bizarre arithmetic. Set theory, which is what we were doing when we looked at the infinite mathematical zoo, was invented by the German mathematician Georg Cantor (1845–1918). He figured out that there are actually different types of infinity. He worked on the *cardinality* of sets, which means how many members of the set there are. For instance, if I define set A as being the planets of the solar system, the cardinality of set A is eight. (For more information about why Pluto is no longer a planet, see here.)

Cantor looked at infinite sets too. The natural numbers are infinite, but Cantor said that they are *countably* infinite because as we count upwards from one, we are moving towards infinity, making progress. We'll never get to infinity, but we can approach it. Cantor defined the set of natural numbers as having a cardinality of aleph-zero, or $\aleph_0$ (aleph being the first letter of the Hebrew alphabet). Any other set of numbers where you can make progress also has cardinality $\aleph_0$. So if I include the negative integers with the natural numbers, I can still make progress counting through them, so the set of integers also has cardinality of $\aleph_0$.

If my set were all the rational numbers from zero to one, I could start on zero and try to work through all the fractions towards one. If I consider all the possible denominators for these fractions, I get the natural numbers again. The numerators would also be various parts of the natural numbers, so even the rational numbers from zero to one have a cardinality of $\aleph_0$. This can be extended to show that the set of all the rational numbers has cardinality $\aleph_0$.

Going back to Galileo's paradox, we can see that the set of natural numbers and the set of perfect squares both have cardinality $\aleph_0$ and hence are, in fact, the same size. Paradox no more – thanks, Cantor!

Essentially, sets with cardinality $\aleph_0$ can be methodically listed, even if that list is infinitely long. Cantor was able to think of sets which cannot be methodically listed when he considered the irrational numbers. His *diagonal argument* showed that if you write down all the irrational numbers as decimals, you can always make a new irrational number out of the ones you've written down. When you add this to the set, you can make a new irrational number from the new set.

This loop means that you can never list all the irrational numbers methodically, as you keep finding ones that have been left out. Cantor said that sets like this were *uncountably* infinite and said their cardinality was $\aleph_1$.

Cantor, and many subsequent mathematicians, spent a lot of time trying to work out the relationship between $\aleph_0$ and $\aleph_1$. Cantor proposed the *continuum hypothesis*, which states that there is no set with a cardinality that is between $\aleph_0$ and $\aleph_1$ – there is nothing between countable and uncountable sets. It has since been shown that the continuum hypothesis cannot be proved, or disproved, using set theory.

What can be proved is that Cantor took a concept (infinity) that had only been considered seriously by philosophers and theologians up to that point and kick-started a new way of thinking about the very foundations of mathematics. However, the disagreements and arguments his ideas provoked caused Cantor great distress and provoked bouts of depression that plagued him for the second half of his life. We can only hope that the continuum hypothesis' inclusion as a Hilbert problem (see here) gave him some awareness of the greatness he had achieved. Certainly the idea that even infinities have differences is awe-inspiring stuff.

Chapter 3

# ARITHMETIC

I'm going to work on the principle that you know how to count. I've never met an adult who could not count. It is the first part of mathematics that we learn, often before we go to school. Many small children can even parrot off the numbers from one to ten by rote before they have any understanding of what numbers are.

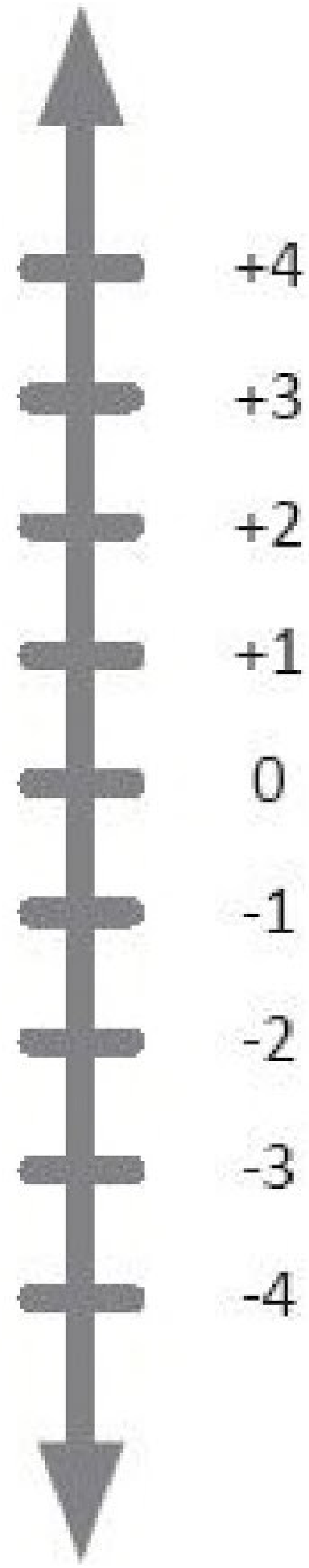
One way of looking at mathematics would be to say that it is based on understanding certain principles which can then be used to achieve certain results. Understanding and processes. However, many of us never quite get the understanding part (or it may not be offered to us in the first place) and we are left with only the process to learn. The problem with this is that, like any skill, it gets worse with neglect. Understanding also fades, but not in the same way. What I love about mathematics is that I, an unremarkable human who lives on a small island in the northern hemisphere, am at the apex of a pyramid of understanding that goes back through thousands of years, people and cultures. There are many people whose maths pyramid is far taller than mine, but I have chosen to spend my career helping other people build up their pyramids. And I know from experience that it doesn't matter how good you are at memorizing facts, algorithms and processes. Without the understanding as a foundation, at some point your pyramid is going to fall over.

Before we look at paper methods of arithmetic, I'd like to take a brief look at the dual nature of the symbols + and –. These were first introduced to the Western world in Germany from the late

1400s onwards. Johannes Widmann (*c.* 1460–98) wrote a book called, in English, *Neat and Nimble Calculation in All Trades* in 1489 which is the earliest printed use of these symbols. From the beginning, the symbols had two meanings each, which some people struggle to differentiate.

Each symbol can be either an *operation*, to add or subtract, or a *sign* to denote positive or negative. They are simultaneously an instruction and a description, a verb and a noun. +3 can mean 'add three' or 'positive three' – how do you know which is meant?

It's fairly common in mathematics education to introduce the concept of a number line – an imaginary line that helps you to perform mental arithmetic and to understand the concepts of 'greater than' and 'less than'. I often ask my students whether they see their number line as horizontal or vertical and which direction the numbers go in. I am sure that there could be some very interesting research here! For the sake of my analogy, our number line will be vertical like a thermometer.




Here we can see the use of + and – in their descriptive form, telling us whether the number is positive or negative. We don't usually include the descriptive + on positive numbers, but I've put them on here to highlight the positive part of the number line. Zero, we can see, is exactly in the middle and so is neither positive or negative.

Now, imagine you are the captain of a mathematical hot-air balloon. You have two ways of changing the height of the balloon – changing the amount of heat in the balloon and changing the amount of ballast in the balloon. We'll treat the heat as positive as it makes the balloon go upwards. You can change the amount of heat in the balloon in two ways. You can add more by using the burner, or take some away by opening a vent at the top of the balloon, allowing hot air to escape. We'll treat the ballast as negative as it makes the balloon go downwards. You can change the amount of ballast in the balloon by throwing some over the side or by having your friend with a drone deliver some more to your basket. We can represent each of these four ideas with a mathematical operation:

| **Action** | **Effect** | | **Balloon goes. . .** |
|---|---|---|---|
| Use Burner | Add<br>+ | Heat<br>+ | ↑ |
| Open Vent | Subtract<br>- | Heat<br>+ | ↓ |
| Add Ballast | Add<br>+ | Ballast<br>- | ↓ |
| Drop Ballast | Subtract<br>- | Ballast<br>- | ↑ |

## Hindu-Arabic Numerals

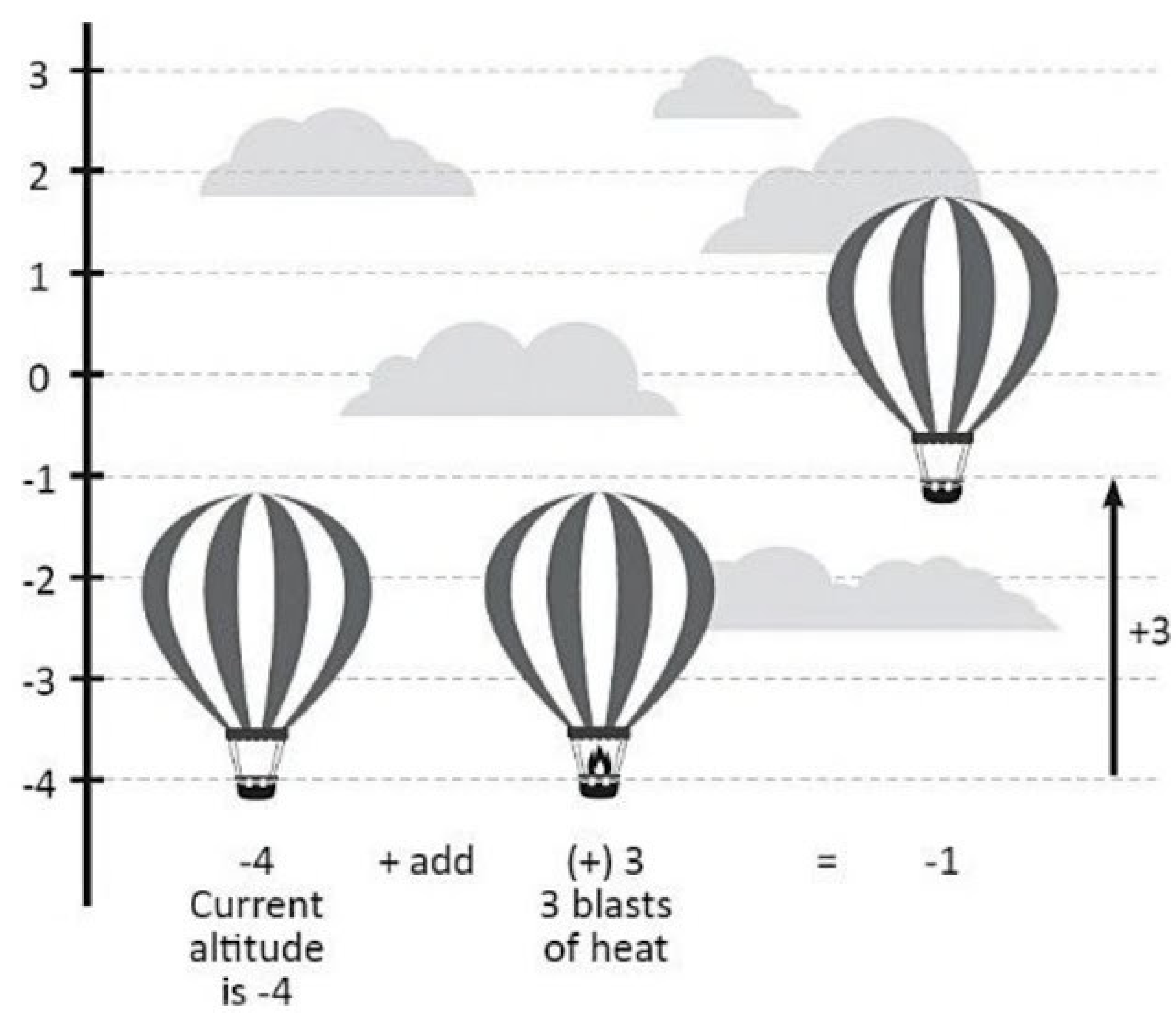
Our way of writing numbers is called the *Hindu-Arabic* system as it combines several breakthroughs from both these cultures. An Indian astronomer called Aryabhata (475–550) was among the first to use a place-value system from about 500 CE, specifying a decimal system where each column was worth ten times the previous. Another Indian astronomer, Brahmagupta (598–670), embellished the system by using nine symbols for the numbers and a dot to represent an empty column, which went on to evolve into our symbol for zero: 0.

The efficiency of calculation that the new system allowed made it popular and it spread across the world. By the ninth century it reached an Arabic mathematician called Muhammad al-Khwarizmi (*c.* 780 to *c.* 850) – from whose name we get the word 'algorithm' – who wrote a treatise on it. This was subsequently translated into Latin, which gave the Western world access to these numbers for the first time.

Sadly, the system didn't gain much traction in Europe. Leonardo of Pisa (*c.* 1175 to *c.* 1240), aka Fibonacci, who was educated in the Arabic world, used it in his book *Liber abaci* in 1202. The book was influential in persuading shopkeepers and mathematicians away from using the abacus for calculation and towards the awesome potential of the Hindu-Arabic system. However, it too was written in Latin, which excluded many people from understanding it. In 1522, Adam Ries (1492–1559) wrote a book in his native German explaining how to use these numerals, which finally enabled literate but not classically educated folk to exploit the system.

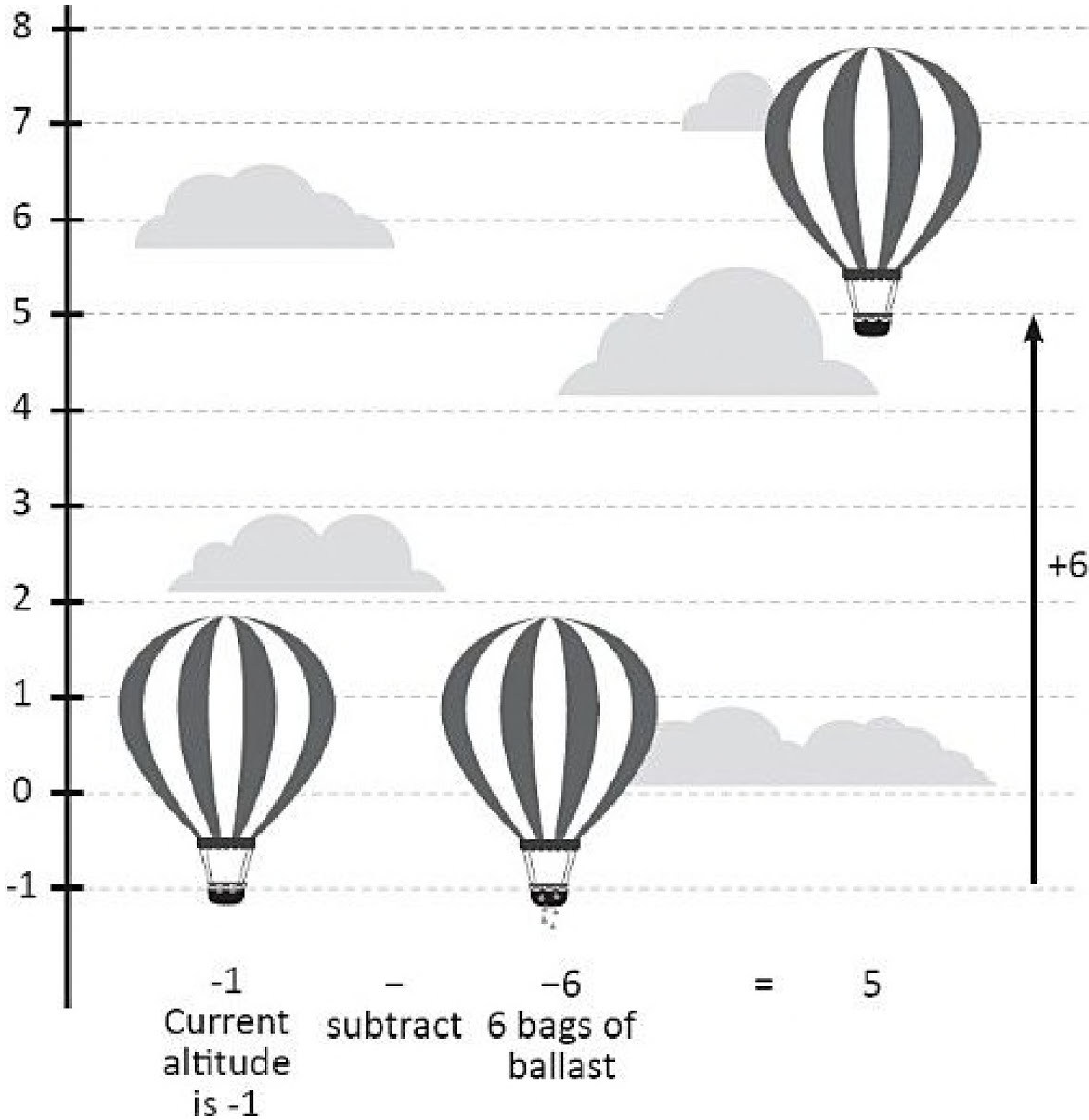
The last row of the table is one that many people accept (or have learnt by rote) but don't really understand why – hopefully the balloon analogy is some help!

We have now sorted out how to make our balloon go up and down, what mathematicians call an operation. If we want to calculate our altitude, our position on the number line, we need to do a calculation, which combines our current place on the number line with an operation. The first number in the calculation tells us our current altitude, and the rest of the calculation tells us what action to take. For example, we could translate −4 + 3 as:




Clearly, this means the balloon will go up three places on the number line, from −4 to −1.[1] Therefore:−4 + 3 = −1

A slightly trickier example, with lots of negatives in it, would be −1 – −6, which we can translate as:




Dropping six bags of ballast over the side is going to make the balloon go up six, so: −1 – −6 = 5

Now that we know when your balloon will go up and when it will go down, we can look at more complicated arithmetic and the rest of the four operations.

Chapter 4

# ADDITION AND MULTIPLICATION

When it comes to doing addition with larger numbers on paper, the methods we use all rely on the information encoded in the number by *place value*. We know that the number represented by the digits 1234 is one thousand, two hundred and thirty-four. This is because each position in the number has a corresponding value. From the right, these are ones (usually called units), tens, hundreds, thousands, tens of thousands, etc., getting ten times bigger every step to the left. So the number 1234 is four units (4), three tens (30), two hundreds (200) and one thousand (1000). I can write 1234 as:

$$1234 = 4 + 30 + 200 + 1000$$

This is called *expanded form* by maths teachers and it's really helpful for understanding how sums work. Imagine the sum 1234 + 5678. If I write each number in expanded form thus:

$$1234 = 4 + 30 + 200 + 1000$$
$$5678 = 8 + 70 + 600 + 5000$$

I can then add each matching value together easily:

1234 + 5678 : 4 + 8 = 12 (units)
30 + 70 = 100 (tens)
200 + 600 = 800 (hundreds)
1000 + 5000 = 6000 (thousands)

From here I can see that 1234 + 5678 = 12 + 100 + 800 + 6000 = 6912.

The way we were taught at school is merely a shorthand of this process. We set up the sum with the columns matching and add through, right to left:

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| + | 5 | 6 | 7 | 8 |

The first calculation is 4 + 8 = 12. We can't write 12 in the one-digit answer box but 12 = 10 + 2, so we leave the 2 in that box and carry the 10 to the next calculation:

| | | | | |
|---|---|---|---|---|
| | | | 1 | |
| | 1 | 2 | 3 | 4 |
| + | 5 | 6 | 7 | 8 |
| | | | | 2 |

Technically, the next column addition is 10 + 30 + 70 = 110, but as we are working in the tens column we can just look at how many tens we have: 1 + 3 + 7 = 11 tens altogether. So again we have too many digits to fit in. 11 = 10 + 1, so we write a 1 in the tens column and carry 1 into the hundreds column:

| | | | | |
|---|---|---|---|---|
| | | 1 | 1 | |
| | 1 | 2 | 3 | 4 |
| + | 5 | 6 | 7 | 8 |
| | | | 1 | 2 |

100 + 200 + 600 = 900:

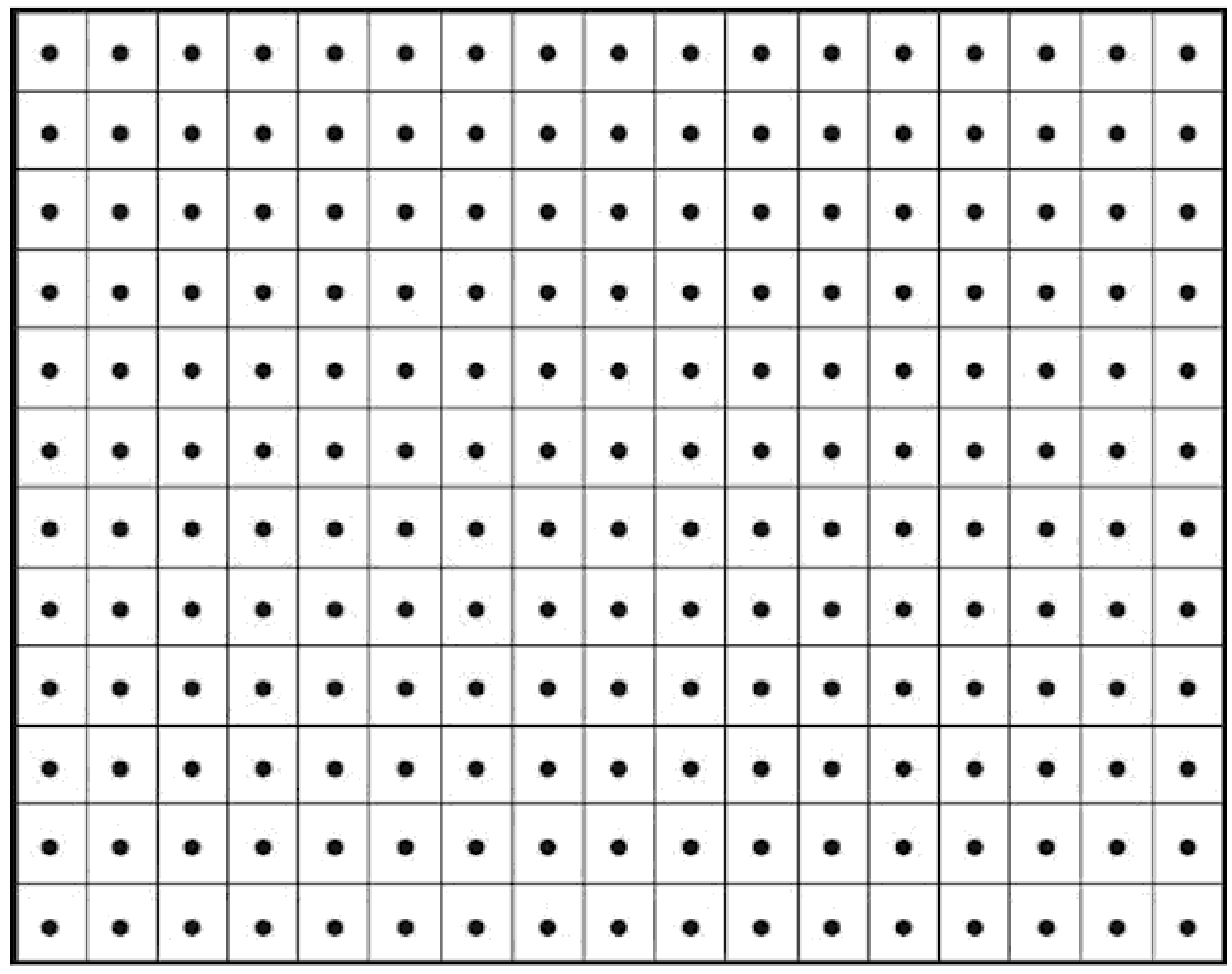
| | | | | |
|---|---|---|---|---|
| | | 1 | 1 | |
| | 1 | 2 | 3 | 4 |
| + | 5 | 6 | 7 | 8 |
| | | 9 | 1 | 2 |

And finally, 1000 + 5000 = 6000:

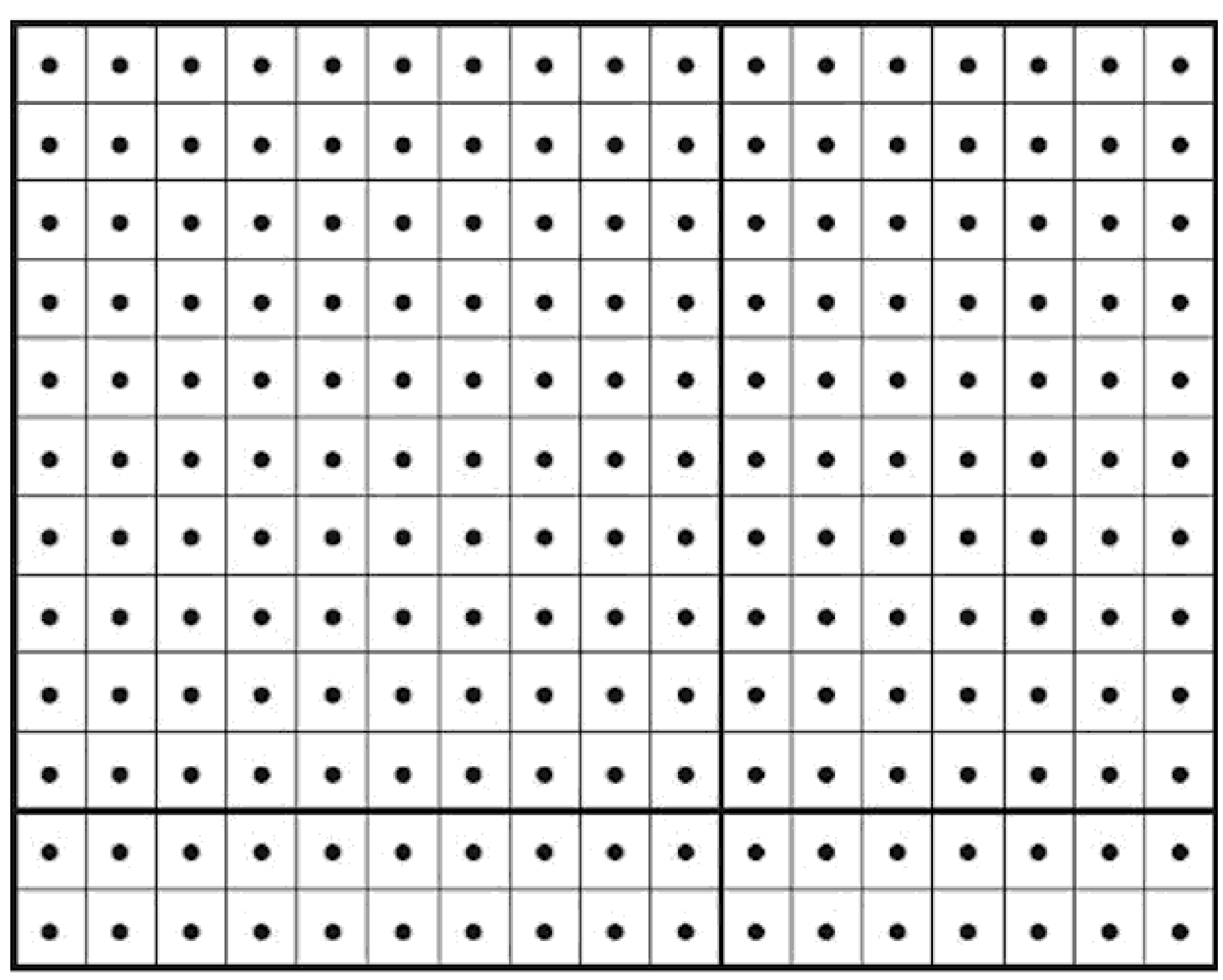
| | | | | |
|---|---|---|---|---|
| | | 1 | 1 | |
| | 1 | 2 | 3 | 4 |
| + | 5 | 6 | 7 | 8 |
| | 6 | 9 | 1 | 2 |

Multiplication is a quick way of doing repeated addition. 12 × 17 asks the question: 'how much is twelve lots of seventeen?' I could work out the answer by adding twelve seventeens together, or seventeen twelves, but multiplying is much faster, provided you have learned your times tables in advance.

Imagine I had a lot of counters. I could solve the 12 × 17 problem by putting out twelve rows of seventeen counters, and then counting them:



However, if I think about 12 as 10 + 2 and 17 as 10 + 7 then I can group the counters:



As I know my times tables, I know how many counters must be in each subdivision:

| | 10 | 7 |
|---|---|---|
| 10 | 10 × 10 = 100 | 10 × 7 = 70 |
| 2 | 2 × 10 = 20 | 2 × 7 = 14 |

So now I know that 12 × 17 = 100 + 70 + 20 + 14 = 204. This method (minus putting out 204 counters) is called the grid method. Here's a slightly more advanced version for solving 293 × 157:

| | 200 | | 90 | | 3 | |
|---|---|---|---|---|---|---|
| 100 | 100 × 200 = | 20000 | 100 × 90 = | 9000 | 100 × 3 = | 300 |
| 50 | 50 × 200 = | 10000 | 50 × 90 = | 4500 | 50 × 3 = | 150 |
| 7 | 7 × 200 = | 1400 | 7 × 90 = | 630 | 7 × 3 = | 21 |
| | | 31400 | | 14130 | | 471 |

You might be asking how I did all the multiplications in my head when they are much larger than what we find in our times tables. Well, there's a nifty hack for that. Every time I multiply an integer by ten, I add a zero to the end of the number. For 100 × 200, I know that 100 must be 1 × 10 × 10 and that 200 must be 2 × 10 × 10. If I put this all together:

## Decimals

It's worth noting that I can extend the idea of place value in both directions. Going to the right of the units column, the columns get ten times smaller each time, giving me tenths, hundredths, thousandths and so on. I use a decimal point to show that the right-most digit is no longer the units. This means I can use the same rules as above to add decimal numbers eg 45.3 + 27.15:

| | | | | | |
|---|---|---|---|---|---|
| | 1 | | | | |
| | 4 | 5 | . | 3 | 0 |
| + | 2 | 7 | . | 1 | 5 |
| | 7 | 2 | . | 4 | 5 |

Notice that I put a zero on the end of 45.3 to make the columns match up, making the calculation clearer (and it's particularly important for subtraction). I can do this as 45.3 is the same as 45.30: three tenths add zero hundredths is still just three tenths. For this reason, mathematicians say 45.30 as forty-five point three zero rather than forty-five point thirty.

$$
\begin{aligned}
100 \times 200 &= \underline{1} \times 10 \times 10 \times \underline{2} \times 10 \times 10 \\
&= \underline{1 \times 2} \times 10 \times 10 \times 10 \times 10 \\
&= \underline{2} \times 10 \times 10 \times 10 \times 10
\end{aligned}
$$

Remembering that every '×10' means putting a zero after the 2, I get 100 × 200 = 20000. I don't go through this entire process whenever I'm doing a grid multiplication. I just multiply the front digits and then add however many zeroes there are in the calculation to the right of it. So for 50 × 200, my thought process was 5 × 2 = 10, and then put three zeroes on. Therefore 50 × 200 = 10000. Bingo.

Back to my grid – you can see I've totalled each column. My final answer is 31400 + 14130 + 471, which I'll do an addition sum for:

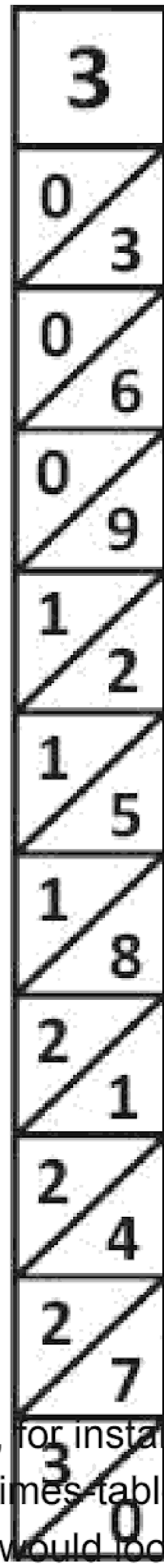
| | | | | | |
|---|---|---|---|---|---|
| | | 1 | 1 | | |
| | 3 | 1 | 4 | 0 | 0 |
| | 1 | 4 | 1 | 3 | 0 |
| + | | | 4 | 7 | 1 |
| | 4 | 6 | 0 | 0 | 1 |

Final answer: 293 × 157 = 46001.

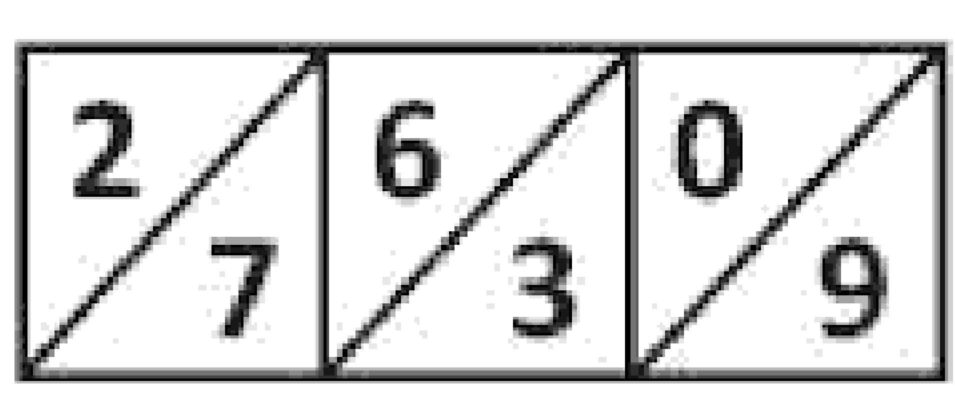
There are other methods, including long multiplication, but as long as you have a working method then stick to it. Let's move on to addition and multiplication's alter egos, subtraction and division.

## Napier's Bones

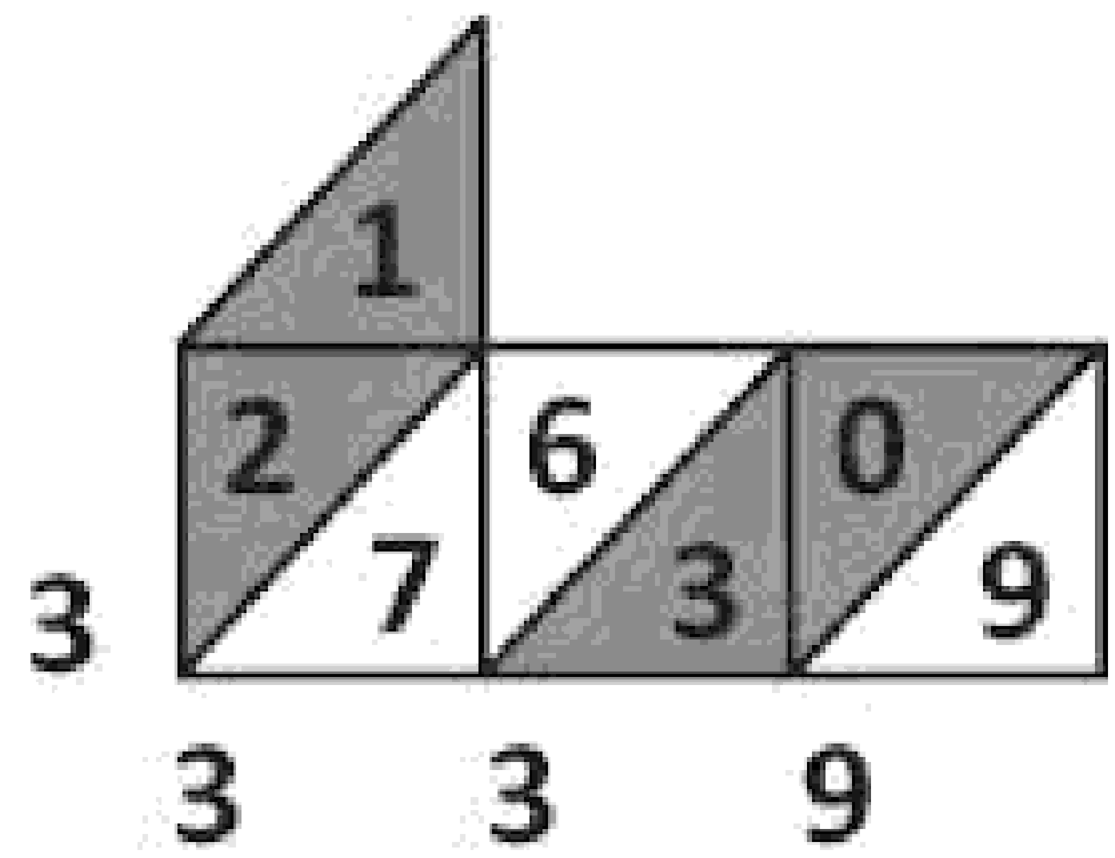
John Napier (1550–1617) was a Scottish mathematician, astronomer and alchemist who invented a set of rods, known as *Napier's bones*, for doing multiplication. These contained a rod for each times table – for instance, the three-times-table rod would look like this:




If you wanted to calculate, for instance, 9 × 371, you would set the three-, seven- and one-times-table rods side by side and read across the ninth row, which would look like this:




You then add together the numbers in each diagonal stripe, starting from the right. If the total is more than nine, I carry into the next stripe:




Hence 9 × 371 = 3339.

Napier was rumoured to dabble in sorcery, having a black rooster as his familiar. He would periodically command his servants to enter a room alone with the bird and stroke it, saying that this would allow the bird to sense the servant's honesty. In fact, Napier put soot on the bird's feathers. Anyone with a guilty conscience would not stroke the bird, their hands would remain sootless and they would be found guilty by the cunning Napier.

Chapter 5

# SUBTRACTION AND DIVISION

Subtraction works very similarly to addition. For instance, 6543 − 5678 is:

$$
\begin{aligned}
6543 - 5678: \quad & 3 - 8 = -5 \\
& 40 - 70 = -30 \\
& 500 - 600 = -100 \\
& 6000 - 5000 = 1000
\end{aligned}
$$

This leaves me with −5 + −30 + −100 + 1000 = −135 + 1000 = 865. We can use our column method again, but whereas in addition we used carrying to cope with having too much in a column, we face the opposite problem with subtraction. If I proceed as I did above:

$$
\begin{array}{rrrrr}
 & 6 & 5 & 4 & 3 \\
- & 5 & 6 & 7 & 8 \\
\hline
 & 1 & -1 & -3 & -5
\end{array}
$$

This doesn't make a lot of sense. To find the correct answer I need to use *borrowing*, although one of my students pointed out that since the borrowed amount never gets returned, *stealing* might be a better word for it. When I notice that 3 – 8 will give me a negative, I boost the 3 by borrowing from the next column. I cross out the 4 and reduce it by one. The 'one' I have borrowed is actually worth ten, so it increases my 3 to 13. 13 – 8 = 5:

| | | | | |
|---|---|---|---|---|
| | | | 3 | |
| | 6 | 5 | ~~4~~ | 13 |
| − | 5 | 6 | 7 | 8 |
| | | | | 5 |

The next column again would leave a negative result as 3 – 7 = –4. Again I borrow one of the hundreds from next door. One hundred is ten tens, boosting my 3 tens up to 13 tens so that I can proceed:

| | | | | |
|---|---|---|---|---|
| | | 4 | 13 | |
| | 6 | ~~5~~ | ~~4~~ | 13 |
| − | 5 | 6 | 7 | 8 |
| | | | 6 | 5 |

Yet another borrow required for the hundreds column before I can carry on, and I can see that my thousands column will be zero:

| | | | | |
|---|---|---|---|---|
| | 5 | 14 | 13 | |
| | ~~6~~ | ~~5~~ | ~~4~~ | 13 |
| − | 5 | 6 | 7 | 8 |
| | 0 | 8 | 6 | 5 |

So we can now see that 6543 – 5678 = 865.

We saw in the previous section that addition and multiplication are closely related. The same is true of subtraction and division. The calculation 3780 ÷ 15 is asking us ‘how many times does fifteen go into 3780?’, i.e. ‘how many times can I subtract fifteen from 3780?’ Indeed, this way of thinking is the key to a method of division called *chunking*. In it, I keep subtracting multiples of the divisor until I get down to zero.

In the first place, I know that 2 × 15 = 30, so 200 × 15 must be 3000. I’ll start by subtracting this from 3780:

| | | | | | |
|---|---|---|---|---|---|
| | 3 | 7 | 8 | 0 | |
| − | 3 | 0 | 0 | 0 | 200 |
| | | 7 | 8 | 0 | |

This leaves 780. Thinking about fifteen, I can see that 4 × 15 = 60, so 40 × 15 = 600. I’ll take this off next:

| | | | | | |
|---|---|---|---|---|---|
| | 3 | 7 | 8 | 0 | |
| − | 3 | 0 | 0 | 0 | 200 |
| | | 7 | 8 | 0 | |
| − | | 6 | 0 | 0 | 40 |
| | | 1 | 8 | 0 | |

Finally, I’ll take off a further twelve fifteens in two goes:

| | | | | | |
|---|---|---|---|---|---|
| | 3 | 7 | 8 | 0 | |
| − | 3 | 0 | 0 | 0 | 200 |
| | | 7 | 8 | 0 | |
| − | | 6 | 0 | 0 | 40 |
| | | 1 | 8 | 0 | |
| − | | 1 | 5 | 0 | 10 |
| | | | 3 | 0 | |
| − | | | 3 | 0 | 2 |
| | | | | 0 | |

Now I know that I took away 200 + 40 + 10 + 2 = 252 lots of 15, so 3780 ÷ 15 = 252. You can see that the better you are at multiplying, the fewer steps you can chunk in.

The feared method of *long division* works along very similar principles. I set up the problem in what I call a *bus stop*:

```
    _______
15 | 3 7 8 0
```

I start from the left. As 15 has two digits, I look at the 3 and the 7 – how many times does 15 go into 37? Twice, giving 30, and I calculate the remainder using subtraction:

```
       2
    _______
15 | 3 7 8 0
 -   3 0
    ----
       7
```

I now shift my attention to the 7 I have just calculated and the 8, which I'll rewrite alongside the 7. Fifteens into 78? Well, five fifteens are 75 . . .

```
       2 5
    _______
15 | 3 7 8 0
 -   3 0 ↓
    ----
       7 8
     - 7 5
    ------
         3
```

Finally, I bring the zero down alongside and consider how many fifteens are in 30:

$$
\begin{array}{rrrrr}
 & & 2 & 5 & 2 \\ \cline{2-5}
15 \; | & 3 & 7 & 8 & 0 \\
- & 3 & 0 & \downarrow & | \\ \cline{2-3}
 & & 7 & 8 & | \\
 & - & 7 & 5 & \downarrow \\ \cline{2-4}
 & & & 3 & 0 \\
 & & - & 3 & 0 \\ \cline{3-5}
 & & & & 0
\end{array}
$$

*Short division* is the same as long division except that we calculate the remainders in our heads and write them in as carries. Short division is handy for converting fractions into decimals. If I want to know what $\frac{5}{8}$ is as a decimal, I can calculate 5 ÷ 8:

$$
\begin{array}{r|l}
 & \\ \cline{2-2}
8 & 5 \quad\quad\quad\quad
\end{array}
$$

Eight goes into 5 zero times with a remainder of 5. I don't have anywhere to write this remainder until I write a decimal point and another zero. I'm allowed to do this because 5 = 5.0, and I'll write a matching decimal point above:

$$
\begin{array}{r|lll}
 & 0. & & \\ \cline{2-4}
8 & 5. & {}^{5}0 & \quad\quad
\end{array}
$$

Eight into 50 goes 6, remainder 2 (I can keep adding zeroes after the decimal point as needed):

$$
\begin{array}{r|lll}
 & 0. & 6 & \\ \cline{2-4}
8 & 5. & {}^{5}0 & {}^{2}0
\end{array}
$$

Eight into 20 goes twice, remainder 4:

$$\begin{array}{r|llll} & 0. & 6 & 2 & \\ \hline 8 & 5. & {}^{5}0 & {}^{2}0 & {}^{4}0 \end{array}$$

Eight into 40 goes five times exactly:

$$\begin{array}{r|llll} & 0. & 6 & 2 & 5 \\ \hline 8 & 5. & {}^{5}0 & {}^{2}0 & {}^{4}0 \end{array}$$

So we now know that $\frac{5}{8} = 5 \div 8 = 0.625$. This method works with any fraction, although you may wish to use long division for harder ones. In the next section we'll take a look at some fractions that don't work out quite so nicely.

Chapter 6

# FRACTIONS AND PRIMES

We just saw how to work out a fraction as a decimal. Let's take a look at $\frac{1}{3}$ – something interesting happens:

| | | | | |
|---|---|---|---|---|
| | 0. | 3 | 3 | 3 . . . |
| 3 | 1. | $^{1}0$ | $^{1}0$ | $^{1}0$ . . . |

We quickly notice that a loop has formed – three into ten is three, remainder one – which will repeat for ever. Decimals that do this are called *recurring* and we use a dot to represent the digit that repeats:

$$\frac{1}{3} = 0.\dot{3}$$

The sevenths are even more interesting:

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0. | 1 | 4 | 2 | 8 | 5 | 7 | 1 | 4 | 2 | 8 | 5 | 7 | 1 | 4 | 2 | 8 | 5 | 7... |
| 7 | 1. | $^{1}0$ | $^{3}0$ | $^{2}0$ | $^{6}0$ | $^{4}0$ | $^{5}0$ | $^{1}0$ | $^{3}0$ | $^{2}0$ | $^{6}0$ | $^{4}0$ | $^{5}0$ | $^{1}0$ | $^{3}0$ | $^{2}0$ | $^{6}0$ | $^{4}0$ | $^{5}0$... |

Here I get a repeating sequence of digits. I can show this by using a pair of dots at the beginning and end of the sequence:

$$\frac{1}{7} = 0.\dot{1}4285\dot{7}$$

What is more, every seventh uses the same sequence, just with different start and end points:

$$\frac{2}{7} = 0.\dot{2}8571\dot{4}$$
$$\frac{3}{7} = 0.\dot{4}2857\dot{1}$$
$$\frac{4}{7} = 0.\dot{5}7142\dot{8}$$
$$\frac{5}{7} = 0.\dot{7}1428\dot{5}$$
$$\frac{6}{7} = 0.\dot{8}5714\dot{2}$$

If you fancy a challenge, take a look at the nineteenths!

By looking at the denominator of a fraction I can tell whether it will recur or terminate. It all depends on whether I can take the denominator and multiply it by something to make it into a power of ten (10, 100, 1000, etc.). If I can, I can get it to sit nicely in the decimal columns when I convert it.

Before we do this, it's worth taking a look at a very important mathematical concept called *the equivalence of fractions*. It says that we can have different fractions with the same value. One way of thinking about it is with pizza. If we share a pizza, half each, we might each cut our pizza into a different number of slices, but we still have the same amount of pizza. Likewise, we all pick up the idea fairly early on at school that a half is two quarters, which is also three sixths, and so on:

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$$

You'll have probably been told 'whatever you do to the top, do to the bottom' by a maths teacher at some stage. What they may not have said is that this preserves equivalence.

This does give me another way to convert some fractions into decimals. For instance, $\frac{51}{250}$ is not something I fancy doing the division for. However, if I multiply the numerator and denominator by four, I get:

$$\frac{51}{250} = \frac{51 \times 4}{250 \times 4} = \frac{204}{1000} = 0.204$$

Job done. The next thing to ponder is how can I tell whether I'll be able to multiply the denominator to get a power of ten?

To examine this, you need to understand the concept of *prime* numbers. These have fascinated mathematicians for a long time. To put it succinctly, a prime number is a natural number with exactly two factors. Eight, for instance, is divisible by one, two, four, and eight itself; four factors mean it is not prime. Five has two factors, one and five, so is prime. One has one factor – one – so is not prime. Forget that nonsense about 'a number divisible only by itself and one' for this very reason. So the first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23.

One of the reasons that prime numbers are so awesome is called *the fundamental theorem of arithmetic*, which says that every natural number can be written as the product of prime numbers, but only in one way. E.g.:

$$30 = 2 \times 3 \times 5$$

There is no other combination of prime numbers multiplied together that will make thirty. Two, three and five are called the *prime factors* of thirty. For me, this makes prime numbers like mathematical DNA – every number is unique, and in numbers we don't have twins or clones to worry about!

Even a huge number like 223,092,870 can only be made up one way with prime numbers (it's $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23$, in fact).

How does this help me with fractions? Well, I said that for a fraction to terminate I must be able to convert its denominator into a power of ten. The prime factors of ten are given by:

$$10 = 2 \times 5$$

To get the prime factors of one hundred, it helps if I recognize that:

$$\begin{aligned} 100 &= \underline{10} \times \underline{10} \\ &= \underline{2 \times 5} \times \underline{2 \times 5} \end{aligned}$$

So the prime factors of ten are two and five, and the same for one hundred (just more of them). We can see that the only prime factors of any power of ten will be two and five. Therefore, if my denominator's prime factors are some combination of twos or fives, there will be a way to multiply it to get a power of ten. My example above had a denominator of 250, and:

$$250 = 2 \times 5 \times 5 \times 5$$

So only twos and fives. I multiplied by four above, which is $2 \times 2$, to get 1000. If the denominator had been 240:

$$240 = 2 \times 2 \times 2 \times 2 \times 3 \times 5$$

This time we have a three in there, so any fraction in its simplest form that has a denominator of 240 will recur. E.g.:

$$\frac{73}{240} = 0.3041\dot{6}$$

On the other hand:

$$\frac{120}{240} = \frac{120 \div 120}{240 \div 120} = \frac{1}{2} = 0.5$$

This fraction, in its simplest form, no longer has a denominator with something other than two or five, so terminates.

## Adding and Subtracting Fractions

While we are on the subject of fractions, a recap of their arithmetic is in order. To add or subtract, we need to convert the fractions so that they have the same denominator. To do this most efficiently, we look for the lowest number that both denominators are factors of – the *lowest common multiple*. For example, if I want to add five-eighths and seven-twelfths, I need to identify the lowest number that eight and twelve both go into. We quickly spot that twenty-four is on both the eight and the twelve times table:

$$\begin{aligned} & \frac{5}{8} + \frac{7}{12} \\ &= \frac{5\times3}{8\times3} + \frac{7\times2}{12\times2} \\ &= \frac{15}{24} + \frac{14}{24} \\ &= \frac{29}{24} \end{aligned}$$

This is a top-heavy or *improper* fraction as the numerator is larger than the denominator. This, for some reason, is unacceptable in mathematics until you reach A Level or equivalent. I believe this is because mixed fractions are easier to understand at a glance, though improper fractions are easier to perform calculations with. To convert an improper fraction to a mixed number, I need to recognise that $\frac{24}{24} = 1$ .

This means that:

$$\frac{29}{24} = \frac{24}{24} + \frac{5}{24} = 1\frac{5}{24}$$

Subtraction works in a similar way:

$$\frac{5}{9} - \frac{1}{4}$$ 36 is the lowest common denominator

$$= \frac{5\times4}{9\times4} - \frac{1\times9}{4\times9}$$ Use equivalence to convert to 36ths

$$= \frac{20}{36} - \frac{9}{36}$$

$$= \frac{11}{36}$$

## Multiplying and Dividing Fractions

Multiplying is straightforward – I multiply the numerators and I multiply the denominators. For example:

$$\frac{3}{5} \times \frac{1}{2} = \frac{3\times1}{5\times2} = \frac{3}{10}$$

It's worth noting that when you multiply by a fraction the total gets smaller. Also, I chose a half here to highlight something that helps us divide fractions. We see above that multiplying by a half is the same as dividing by two, and likewise multiplying by a third would be the same as dividing by three. This relationship is called a *reciprocal.* Two and a half are reciprocals of each other, and it is clear if I write two as a fraction exactly how it works:

$\frac{1}{2}$ is the reciprocal of $\frac{2}{1}$

This is really handy, as it means that dividing by a number is the same as multiplying by its reciprocal:

$$5 \div 3 = 5 \times \frac{1}{3}$$

I can use this to divide fractions:

$$\frac{2}{3} \div \frac{5}{8}$$
$$= \frac{2}{3} \times \frac{8}{5}$$
$$= \frac{2\times 8}{3\times 5}$$
$$= \frac{16}{15}$$
$$= 1\frac{1}{15}$$

Fifteen's prime factors are three and five and so $1\frac{1}{15}$ will be recurring as a decimal.

## Finding Prime Numbers

One of the reasons prime numbers have received a lot of attention from mathematicians, apart from the fundamental theorem of arithmetic, is that no one has discovered a pattern or formula for the prime numbers yet. Many have tried. For instance, the French priest Marin Mersenne (1588–1648) calculated a sequence of numbers using this formula:

$$M_n = 2^n - 1$$

You find the first number by setting n as one, the second by setting n as two, and so on. This gives you:

1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047 . . .

Mersenne noticed that some of the numbers given by the formula are prime numbers, such as 3, 7, 31 and 127, which are the second, third, fifth and seventh numbers in the sequence. Two, three, five and seven are prime numbers themselves, so it seems that if you use a prime number for n, you get a prime number from the formula. But the next prime number after seven is eleven, and the formula gives $M_{11}$ = 2047, which is not a prime as 2047 = 23 × 89.

It is difficult to identify large prime numbers by hand. For instance, $M_{107}$ is a 33-digit number, so it is very time-consuming to check whether anything divides into it and therefore is a factor.

Enter the digital age with computers that can calculate flawlessly and tirelessly. In the 1950s early computers were finding Mersenne primes, as they are known, with hundreds of digits. In 1999 the first million-digit Mersenne prime was discovered. The current record is over 22 million digits for $M_{74,207,281}$.

## Looong Multiplication

At a lecture in 1903, the American mathematician Frank Nelson Cole (1861–1926) wrote the following fact about $M_{67}$, which was believed to be prime:

$$147,573,952,589,676,412,927 = 193,707,721 \times 761,838,257,287$$

He then proceeded to multiply this out, by hand, to prove the result. This took him an hour and was conducted in total silence. At the end of his 'lecture' Cole returned wordlessly to his seat while receiving a standing ovation from his peers.

Why bother? Well, mathematicians will always investigate anything for the sheer love of their subject. But prime numbers are also the backbone of modern-day encryption methods. If I want to transmit a number, such as my credit card details, across the internet, it is easy for people who know what they are doing to intercept this number and spend my money.

To avoid this, the internet uses a method of encryption where a *public key* is used to change the number being transmitted. This key is a combination of very large, apparently random numbers that are, in fact, created from very large prime numbers. Only the intended recipient, who has the *private key*, can reverse the process in any sort of sensible time frame.

The ‘https’ at the beginning of web addresses means that the website uses Hypertext Transfer Protocol with Transport Layer Security (the ‘s’ on the end) to encrypt the information going to and from your computer. So you can happily order things online thanks to some very clever mathematics.