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REALITY AND IMAGINATION

There are many realities out there. There is, of course, the physical reality we find ourselves in. Then there are those imaginary universes that resemble physical reality very closely, such as the one where everything is exactly the same except I *didn't* pee in my pants in fifth grade, or the one where that beautiful dark-haired girl on the bus turned to me and we started talking and ended up falling in love. There are plenty of *those* kinds of imaginary realities, believe me. But that's neither here nor there.

I want to talk about a different sort of place. I'm going to call it "mathematical reality." In my mind's eye, there is a universe where beautiful shapes and patterns float by and do curious and surprising things that keep me amused and entertained. It's an amazing place, and I really love it.

The thing is, physical reality is a disaster. It's way too complicated, and nothing is at all what it appears to be. Objects expand and contract with temperature, atoms fly on and off. In particular, nothing can truly be measured. A blade of grass has no actual length. Any measurement made in this universe is necessarily a rough approximation. It's not bad; it's just the nature of the place. The smallest speck is not a point, and the thinnest wire is not a line.

Mathematical reality, on the other hand, is imaginary. It can be as simple and pretty as I want it to be. I get to have all those perfect things I can't have in real life. I will never hold a circle in my hand, but I can hold one in my mind. And I can measure it. Mathematical reality is a beautiful wonderland of my own creation, and I can explore it and think about it and talk about it with my friends.

Now, there are lots of reasons people get interested in physical reality. Astronomers, biologists, chemists, and all the rest are trying to figure out how it works, to describe it.

I want to describe mathematical reality. To make patterns. To figure out how they work. That's what mathematicians like me try to do.

The point is I get to have them both—physical reality and mathematical reality. Both are beautiful and interesting (and somewhat frightening). The former is important to me because I am in it, the latter because it is in me. I get to have both these wonderful things in my life and so do you.

My idea with this book is that we will design patterns. We'll make patterns of shape and motion, and then we will try to understand our patterns and measure them. And we will see beautiful things!

But I won't lie to you: this is going to be very hard work. Mathematical reality is an infinite jungle full of enchanting mysteries, but the jungle does not give up its secrets easily. Be prepared to struggle, both intellectually and creatively. The truth is, I don't know of any human activity as demanding of one's imagination, intuition, and ingenuity. But I do it anyway. I do it because I love it and because I can't help it. Once you've been to the jungle, you can never really leave. It haunts your waking dreams.

So I invite you to go on an amazing adventure! And of course, I want you to love the jungle and to fall under its spell. What I've tried to do in this book is to express how math feels to me and to show you a few of our most beautiful and exciting discoveries. Don't expect any footnotes or references or anything scholarly like that. This is *personal*. I just hope I can manage to convey these deep and fascinating ideas in a way that is comprehensible and fun.

Still, expect it to be slow going. I have no desire to baby you or to protect you from the truth, and I'm not going to apologize for how hard it is. Let it take hours or even days for a new idea to sink in—it may have originally taken centuries!

I'm going to assume that you love beautiful things and are curious to learn about them. The only things you will need on this journey are common sense and simple human curiosity. So relax. Art is to be enjoyed, and this is an art book. Math is not a race or a contest; it's just you playing with your own imagination. Have a wonderful time!

ON PROBLEMS

What is a math problem? To a mathematician, a problem is a *probe*—a test of mathematical reality to see how it behaves. It is our way of "poking it with a stick" and seeing what happens. We have a piece of mathematical reality, which may be a configuration of shapes, a number pattern, or what have you, and we want to understand what makes it tick: What does it do and why does it do it? So we poke it—only not with our hands and not with a stick. We have to poke it with our minds.

As an example, let's say you've been playing around with triangles, chopping them up into other triangles and so forth, and you happen to make a discovery:

When you connect each corner of a triangle to the middle of the opposite side, the three lines seem to all meet at a point. You try this for a wide variety of triangles, and it always seems to happen. Now you have a mystery! But let's be very clear about exactly what the mystery is. It's not about your drawings or what looks like is happening on paper. The question of what pencil-and-paper triangles may or may not do is a scientific one about physical reality. If your drawing is sloppy, for example, then the lines won't meet. I suppose you could make an extremely careful drawing and put it under a microscope, but you would learn a lot more about graphite and paper fibers than you would about triangles.

The real mystery is about imaginary, too-perfect-to-exist triangles, and the question is whether these three perfect lines meet in a perfect point in mathematical reality. No pencils or microscopes will help you now. (This is a distinction I will be stressing throughout the book, probably to the point of annoyance.) So how are we to address such a question? Can anything ever really be known about such imaginary objects? What form could such knowledge take?

Before examining these issues, let's take a moment to simply delight in the question itself and to appreciate what is being said here about the nature of mathematical reality.

What we've stumbled onto is a conspiracy. Apparently, there is some underlying (and as yet unknown) structural interplay going on that is making this happen. I think that is marvelous and also a little scary. What do triangles know that we don't? Sometimes it makes me a little queasy to think about all the beautiful and profound truths out there waiting to be discovered and connected together.

So what exactly is the mystery here? The mystery is *why*. Why would a triangle want to do such a thing? After all, if you drop three sticks at random they usually don't meet at a point; they cross each other in three different places to form a little triangle in the middle. Isn't that what we would expect to happen?

What we are looking for is an explanation. Of course, one reason why an explanation may not be forthcoming is that it simply isn't true. Maybe we fooled ourselves by wishful thinking or clumsy drawing. There's a lot of "fudging" in physical reality, so maybe we just couldn't see the little triangle where the lines cross. Perhaps it was so small that it got lost among all the smears and pencil crumbs. On the other hand, it's certainly the kind of thing that *could* be true. It has a lot of elements that mathematicians look for: naturalness, elegance, simplicity, and a certain inevitable quality. So it's probably true. But again, the question is why.

Now here's where the art comes in. In order to explain we have to create something. Namely, we need to somehow construct an argument—a piece of reasoning that will satisfy our curiosity as to why this behavior is happening. This is a very tall order. For one thing, it's not enough to draw or build a bunch of physical triangles and see that it more or less works for them. That is not an explanation; it's more of an "approximate verification." Ours is a much more serious philosophical issue.

Without knowing why the lines meet at a common point, how can we know that they actually do? In contrast with physical reality, there's nothing to observe. How will we ever know anything about a purely imaginary realm? The point is, it doesn't matter so much *what* is true. It matters *why* it's true. The why *is* the what.

Not that I am trying to minimize the value of our ordinary senses —far from it. We desperately need any and all aids to our intuition and imagination: drawings, models, movies, whatever we can get. We just have to understand that ultimately these things are not really the subject of the conversation and cannot really tell us the truth about mathematical reality.

So now we really are in a predicament. We have discovered what we think may be a beautiful truth, and now we need to prove it. This is what mathematicians do, and this is what I hope you will enjoy doing yourself.

Is this such an extraordinarily difficult thing to do? Yes, it is. Is there some recipe or method to follow? No, there isn't. This is abstract art, pure and simple. And art is always a struggle. There is no systematic way to create beautiful and meaningful paintings or sculptures, and there is also no method for producing beautiful and meaningful mathematical arguments. Sorry. Math is the hardest thing there is, and that's one of the reasons I love it.

So no, I can't tell you how to do it, and I'm not going to hold your hand or give you a bunch of hints or solutions in the back of the book. If you want to paint a picture from your heart, there is no "answer painting" on the back of the canvas. If you are working on a problem and you are stuck and in pain, then welcome to the club. We mathematicians don't know how to solve our problems either. If we did, they would no longer be problems! We're always working at the edge of the unknown, and we're always stuck. Until we have a breakthrough. And I hope you have many—it's an incredible feeling. But there is no special procedure for doing mathematics. You just have to think a lot and hope that inspiration comes to you.

But I won't just drop you into the jungle and leave you there. Your intelligence and your curiosity you will have to supply yourself—these are your machete and your canteen. But maybe I can provide you with a compass in the form of a few general words of advice.

The first is that *the best problems are your own*. You are the intrepid mental explorer; it's your mind and your adventure. Mathematical reality is *yours*—it's in your head for you to explore any time you feel like it. What are your questions? Where do you want to go? I've enjoyed coming up with some problems for you to think about, but these are merely seeds I've planted to help you start growing your own jungle. Don't be afraid that you can't answer your own questions—that's the natural state of the mathematician. Also, try to always have five or six problems you are working on. It is very frustrating to keep banging your head against the same wall over and over. (It's much better to have five or six walls to bang your head against!) Seriously, taking a break from a problem always seems to help.

Another important piece of advice: *collaborate.* If you have a friend who also wants to do math, you can work together and share the joys and frustrations. It's a lot like playing music together. Sometimes I will spend six or eight hours working on a problem with a friend, and even if we accomplish next to nothing, we still had fun feeling dumb together.

So let it be hard. Try not to get discouraged or to take your failures too personally. It's not only you that is having trouble understanding mathematical reality; it's all of us. Don't worry that you have no experience or that you're not "qualified." What makes a mathematician is not technical skill or encyclopedic knowledge but insatiable curiosity and a desire for simple beauty. Just be yourself and go where you want to go. Instead of being tentative and fearing failure or confusion, try to embrace the awe and mystery of it all and joyfully make a mess. Yes, your ideas won't work. Yes, your intuition will be flawed. Again, welcome to the club! I have a dozen bad ideas a day and so does every other mathematician.

Now, I know what you're thinking: a bunch of fuzzy, romantic talk about beauty and art and the exquisite pain of creativity is all very well and good, but how on earth am I supposed to do this? I've never created a mathematical argument in my life. Can't you give me a little more to go on?

Let's go back to our triangle and the three lines. How can we begin to cobble together some sort of an argument? One place we could start is by looking at a symmetrical triangle.

This kind of triangle is also called **equilateral** (Latin for "same sides"). Now, I know this is an absurdly atypical situation, but the idea is that if we can somehow explain why the lines meet in this special case, it might give us a clue about how to proceed with a more general triangle. Or it might not. You never know, you just have to mess around—what we mathematicians like to call "doing research."

In any event, we have to start somewhere, and it should at least be easier to figure something out in this case. What we have going for us in this special situation is tons of symmetry. *Do not ignore symmetry!* In many ways, it is our most powerful mathematical tool. (Put it in your backpack with your machete and canteen.)

Here symmetry allows us to conclude that anything that happens on one side of the triangle must also happen on the other. Another way to say this is that if we flipped the triangle across its line of symmetry, it would look exactly the same.

In particular, the midpoints of the two sides would switch places, as would the lines connecting them to their opposite corners.

But this means that the crossing point of these two lines can't be on one side of the line of symmetry, else when we flip the triangle it would move to the other side, and we could tell that it got flipped!

So the crossing point must actually be *on* the line of symmetry. Clearly our third line (the one connecting the top corner to the middle of the bottom side) is simply the line of symmetry itself, and so that is why all three lines meet at a point. Isn't that a nice explanation?

This is an example of a mathematical argument, otherwise known as a *proof*. A proof is simply a story. The characters are the elements of the problem, and the plot is up to you. The goal, as in any literary fiction, is to write a story that is compelling as a narrative. In the case of mathematics, this means that the plot not only has to make logical sense but also be simple and elegant. No one likes a meandering, complicated quagmire of a proof. We want to follow along rationally to be sure, but we also want to be charmed and swept off our feet aesthetically. A proof should be lovely as well as logical.

Which brings me to another piece of advice: *improve your proofs*. Just because you have an explanation doesn't mean it's the best explanation. Can you eliminate any unnecessary clutter or complexity? Can you find an entirely different approach that gives you deeper insight? Prove, prove, and prove again. Painters, sculptors, and poets do the same thing.

Our proof just now, for instance, despite its logical clarity and simplicity, has a slightly arbitrary feature. Even though we made an essential use of symmetry, there's something annoyingly asymmetrical about the proof (at least to me). Specifically, the argument favors one corner. Not that it's so very bad to pick one corner and use its line as our line of symmetry, it's just that the triangle is so symmetrical; we shouldn't have to make such an arbitrary choice.

We could, for instance, use the fact that in addition to having flip-symmetry, our triangle is also *rotationally* symmetric. That is, if we turn it one-third of a full turn around, it looks exactly the same. This means that our triangle must have a *center*.

Now, if we flip the triangle across any of its three lines of symmetry (favoring none of them), the triangle doesn't change, so its center must stay put. This means that the center point lies on all three lines of symmetry. So that's why the lines all meet!

Now, I'm not trying to say that this argument is so much better or even all that different. (And in fact, there are lots of other ways to prove it.) All I'm saying is that deeper insight and understanding can be gained by coming at a problem in more than one way. In particular, the second proof not only tells me that the lines meet, it tells me where—namely, at the center of rotation. Which makes me wonder, where exactly is that? Specifically, how far up an equilateral triangle is its center?

Throughout the book, questions like this will come up. Part of becoming a mathematician is learning to ask such questions, to poke your stick around looking for new and exciting truths to uncover. Problems and questions that occur to me I will put in boldface type. Then you can think about them and work on them as you please and hopefully also come up with problems of your own. So here's your first one:

Where is the center of an equilateral triangle?

Now going back to the original problem, we see that we have barely made a dent. We have an explanation for why the lines meet in an equilateral triangle, but our arguments are so dependent on symmetry, it's hard to see how this will help in the more general situation. Actually, I suppose our first argument still works if our triangle has two equal sides:

The reason is that this kind of triangle, known as **isosceles** (Greek for "same legs"), still possesses a line of symmetry. This is a nice example of generalization—getting a problem or an argument to make sense in a wider context. But still, for the average asymmetrical triangle, our arguments clearly won't work.

This puts us in a place that is all too familiar to mathematicians. It's called stuck. We need a new idea, preferably one that doesn't hinge so much on symmetry. So let's go back to the drawing board.

Is there something else we can do with these characters? We have a triangle, the midpoints of the sides, and the lines drawn to them from the corners.

Here's a thought. What if we connect the midpoints? Does anything interesting happen? This is the kind of thing you have to do as a mathematician: try things. Will they work? Will they yield useful information? Usually not. But you can't just sit there staring at some shapes or numbers. Try anything and everything. As you do more math, your intuition and your instincts will sharpen, and your ideas will get better. How do you know which ideas to try? You don't. You just have to guess. Experienced mathematicians have a great deal of sensitivity to structure, and so our guesses are more likely to be right, but we still have to guess. So guess.

The important thing is not to be afraid. So you try some crazy idea, and it doesn't work. That puts you in some pretty good company! Archimedes, Gauss, you and I—we're all groping our way through mathematical reality, trying to understand what is going on, making guesses, trying out ideas, and mostly failing. And then every once in a while, you succeed. (Perhaps more frequently if you are Archimedes or Gauss.) And that feeling of unlocking an eternal mystery is what keeps you going back to the jungle to get scratched up all over again.

So imagine you've tried this idea and that idea, and one day it occurs to you to connect the midpoints.

What do we notice? Well, we've divided the original triangle into four smaller ones. In the symmetrical case, they are clearly identical. What happens in general?

Are the triangles all the same? Actually, it looks like three of them might just be smaller (half-scale) versions of the original triangle. Could that be true? What about the middle one? Could it also be the same, only rotated upside down? What exactly have we stumbled onto here?

We've stumbled onto a glimmer of truth, pattern, and beauty, that's what. And maybe this will lead to something wholly unexpected, possibly having nothing to do with our original problem. So be it. There's nothing sacred about our three lines problem; it's a question like any other. If your thoughts on one problem lead you to another, then good for you! Now you have *two* problems to work on. My advice: be open-minded and flexible. *Let a problem take you where it takes you*. If you come across a river in the jungle, follow it!

Are these four triangles identical?

Let's suppose this is true. And that, by the way, is a perfectly fine thing to do. Mathematicians are always supposing things and seeing what would happen (the Greeks even had a word for it—they called it *analysis*). There are thousands of apparent mathematical truths out there that we humans have discovered and believe to be true but have so far been unable to prove. They are called *conjectures*. A conjecture is simply a statement about mathematical reality that you believe to be true (usually you also have some examples to back it up, so it is a reasonably educated guess). I hope that you will find yourself conjecturing all over the place as you read this book and do mathematics. Maybe you will even prove some of your conjectures. Then you get to call them *theorems*.

Supposing that our conjecture about the four triangles is true (and, of course, we still want a nice proof of this), the next question would be whether this helps us solve our original problem. Maybe it will, maybe it won't. You just have to see if anything comes to you.

Essentially, engaging in the practice of mathematics means that you are playing around, making observations and discoveries, constructing examples (as well as counterexamples), formulating conjectures, and then—the hard part—proving them. I hope you will find this work fascinating and entertaining, challenging, and ultimately deeply rewarding.

So I will leave the problem of the triangle and its intersecting lines in your capable hands.

Which brings me to my next bit of advice: *critique your work*. Subject your arguments to scathing criticism by yourself and by others. That's what all artists do, especially mathematicians. As I've said, for a piece of mathematics to fully qualify as such, it has to stand up to two very different kinds of criticism: it must be logically sound and convincing as a rational argument, and it must also be elegant, revelatory, and emotionally satisfying. I'm sorry that these criteria are so painfully steep, but that is the nature of the art.

Now, aesthetic judgments are obviously quite personal, and they can change with time and place. Certainly that has happened with mathematics no less than with other human endeavors. An argument that was considered beautiful a thousand or even a hundred years ago might now be looked upon as clumsy and inelegant. (A lot of classical Greek mathematics, for example, appears quite dreadful to my modern sensibilities.)

My advice is not to worry about trying to hold yourself to some impossibly high standard of aesthetic excellence. If you like your proof (and most of us are fairly proud of our hard-won creations), then it is good. If you are dissatisfied in some way (and most of us are), then you have more work to do. As you gain experience, your taste will grow and develop, and you may find later that you are unhappy with some of your earlier work. That is as it should be.

I think the same could be said for logical validity as well. As you do more mathematics, you will literally get smarter. Your logical reasoning will become tighter, and you will begin to develop a mathematical "nose." You will learn to be suspicious, to sense that some important details have been glossed over. So let that happen.

Now, there is a certain obnoxious type of mathematician who simply cannot allow false statements to be made at any time. I am not one of them. I believe in making a mess—that's how great art happens. So your first essays in this craft are likely to be logical disasters. You will believe things to be true, and they won't be. Your reasoning will be flawed. You will jump to conclusions. Well, go ahead and jump. The only person you have to satisfy is yourself. Believe me, you will discover plenty of errors in your own deduction. You will declare yourself a genius at breakfast and an idiot at lunch. We've all done it.

Part of the problem is that we are so concerned with our ideas being simple and beautiful that when we do have a pretty idea, we want so much to believe it. We want it to be true so badly that we don't always give it the careful scrutiny that we should. It's the mathematical version of "rapture of the deep." Divers see such beautiful sights that they forget to come up for air. Well, logic is our air, and careful reasoning is how we breathe. So don't forget to breathe!

The real difference between you and more experienced mathematicians is that we've seen a lot more ways that we can fool ourselves. So we have more nagging doubts and therefore insist on a much higher standard of logical rigor. We learn to play the devil's advocate.

Whenever I am working on a conjecture, I always entertain the possibility that it is false. Sometimes I work to prove it, other times I try to refute it—to prove myself wrong. Occasionally, I discover a counterexample showing that I was indeed misled and that I need to refine or possibly scrap my conjecture. Still other times, my attempts to construct a counterexample keep running into the same barrier, and this barrier then becomes the key to my eventual proof. The point is to keep an open mind and not to let your hopes and wishes interfere with your pursuit of truth.

Of course, as much as we mathematicians may ultimately insist on the most persnickety level of logical clarity, we also know from experience when a proof "smells right," and it is clear that we could supply the necessary details if we wished. The truth of the matter is that math is a human activity, and we humans make mistakes. Great mathematicians have "proved" utter nonsense, and so will you. (It's another good reason to collaborate with other people—they can raise objections to your arguments that you might overlook.)

The point is to get out there in mathematical reality, make some discoveries, and have fun. Your desire for logical rigor will grow with experience; don't worry.

So go ahead and do your mathematical art. Subject it to your own standards of rationality and beauty. Does it please you? Then great! Are you a tormented struggling artist? Even better. Welcome to the jungle!

PART ONE

SIZE AND SHAPE

1

Here is a nice pattern.

Let me tell you why I find this kind of thing so attractive. First of all, it involves some of my favorite shapes.

I like these shapes because they are simple and symmetrical. Shapes like these that are made of straight lines are called **polygons** (Greek for "many corners"). A polygon with all its sides the same length and all its angles equal is called **regular**. So I guess what I'm saying is, I like regular polygons.

Another reason why the design is appealing is that the pieces fit together so nicely. There are no gaps between the tiles (I like to think of them as ceramic tiles, like in a mosaic), and the tiles don't overlap. At least, that's how it appears. Remember, the objects that we're really talking about are perfect, imaginary shapes. Just because the picture looks good doesn't mean that's what is really going on. Pictures, no matter how carefully made, are part of physical reality; they can't possibly tell us the truth about imaginary, mathematical objects. Shapes do what they do, not what we want them to do.

So how can we be sure that the polygons really do fit perfectly? For that matter, how can we know *anything* about these objects? The point is, we need to measure them—and not with any clumsy real-world implements like rulers or protractors, but with our minds. We need to find a way to measure these shapes using philosophical argument alone.

Do you see that in this case what we need to measure are the angles? In order to check that a mosaic pattern like this will work, we need to make sure that at every corner (where the tiles meet) the angles of the polygons add up to a full turn. For instance, the ordinary square tiling works because the angles of a square are quarter turns and it takes four of them to make a full turn.

By the way, I like to measure angles as portions of a full turn instead of using degrees. It seems simpler to me and more natural than using an arbitrary division of a turn into 360 parts (of course *you* may do as you please). So I'm going to say that a square has a corner angle of 1/4.

One of the first things people discovered about angles is the surprising fact that for any triangle (no matter what shape) the sum of the angles is always the same, namely a half turn (or 180 degrees if you must be vulgar).

To get a feel for this, you might want to make some paper triangles and cut off their corners. When you join them together, they will always form a straight line. What a beautiful discovery! But how can we really know that it is true?

One way to see it is to view the triangle as being sandwiched between two parallel lines.

Notice how these lines form Z shapes with the sides of the triangle. (I suppose you might call the one on the right side a backward Z, but it doesn't really matter.) Now, the thing about Z shapes is that their angles are always equal.

This is because a Z shape is symmetrical: it looks exactly the same if you rotate it a half turn around its center point. That means the angle at the top must be the same as the angle at the bottom. Does that make sense? This is a typical example of a symmetry argument. The invariance of a shape under a certain set of motions allows us to deduce that two or more measurements must be the same.

Going back to our triangle sandwich, we see that each angle at the bottom corresponds to an equal angle at the top.

This means that the three angles of the triangle join together at the top to form a straight line. So the three turns add up to a half turn. What a delightful piece of mathematical reasoning!

This is what it means to do mathematics. To make a discovery (by whatever means, including playing around with physical models like paper, string, and rubber bands), and then to explain it in the simplest and most elegant way possible. This is the art of it, and this is why it is so challenging and fun.

One consequence of this discovery is that if our triangle happens to be equilateral (that is, regular) then its angles are all equal, so they must each be 1/6. Another way to see this is to imagine driving around the perimeter of the triangle.

We make three equal turns to get back to where we started. Since we end up making one complete turn, each of these must be exactly 1/3. Notice that the turns we've made are actually the *outside* angles of the triangle.

Since the inside and outside angles combine to make a half turn, the inside angles must be

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

In particular, six of these triangles will fit together at a corner.

Hey, this makes a regular hexagon! So as a bonus, we get that the angles of a regular hexagon must be twice those of the triangle, in other words 1/3. This means that three hexagons fit together perfectly.

So it is possible to have knowledge about these shapes after

This time we have squares and triangles, but instead of lying flat, they are arranged to form a sort of ball shape. This kind of object is called a **polyhedron** (Greek for “many sides”). People have been playing around with them for thousands of years. One approach to thinking about them is to imagine unfolding them flat onto a plane. For example, one corner of my shape would unfold to look like this:

Here, we have two squares and two triangles around a point, but they leave a gap so that the shape can be folded up into a ball. So in the case of polyhedra, we need the angles to add up to less than a full turn.

What happens if the angle sum is more than a full turn?

Another difference between polyhedra and flat mosaics is that the design involves only a finite number of tiles. The pattern will still go on forever (in a sense), but it will not extend indefinitely into space. Naturally, I’m curious about these patterns, too.

What are all the symmetrical polyhedra?

In other words, what are all the different ways to make polyhedra out of regular polygons so that at each corner we see the same pattern? Archimedes figured out all of the possibilities. Can you?

Of course, the most symmetrical kind of polyhedron would be one where all the faces are identical, like a cube. These are called **regular polyhedra**. It is an ancient discovery that there are exactly five of these (the so-called Platonic solids). Can you find all five?

What are the five regular polyhedra?

2

What is measuring? What exactly are we doing when we measure something? I think it is this: we are making a comparison. We are comparing the thing we are measuring to the thing we are measuring it with. In other words, *measuring is relative*. Any measurement that we make, whether real or imaginary, will necessarily depend on our choice of measuring unit. In the real world, we deal with these choices every day—a cup of sugar, a ton of coal, a thing of fries, whatever.

The question is, what sort of units do we want for our imaginary mathematical universe? For instance, how are we going to measure the lengths of these two sticks?

Let's suppose (for the sake of argument) that the first stick is exactly twice as long as the second. Does it really matter how many inches or centimeters they come out to be? I certainly don't want to subject my beautiful mathematical universe to something mundane and arbitrary like that. For me, it's the proportion (that 2:1 ratio) that's the important thing. In other words, I'm going to measure these sticks relative to each other.

One way to think of it is that we simply aren't going to have any units at all, just proportions. Since there isn't a natural choice of unit for measuring length, we won't have one. So there. The sticks are just exactly as long as they are. But the first one is twice as long as the second.

The other way to go is to say that since the units don't matter, we'll choose whatever unit is convenient. For example, I could choose the second stick to be my unit, or ruler, so that the lengths come out nice. The first stick has length 2, the second stick has length 1. I could just as easily say the lengths are 4 and 2, 6 and 3, or 1 and 1/2. It just doesn't matter. When we make shapes or patterns and measure them, we can choose any unit that we want to, keeping in mind that what we are really measuring is a *proportion*.

I guess a simple example would be the perimeter of a square. If we choose our unit to be the side of the square (and why not?), then the perimeter would obviously be 4. What that really means is that for any square, the perimeter is four times as long as the side.

This business of units is related to the idea of scale. If we take some shape and blow it up by a certain factor, say 2, then all of our length measurements on the big shape will come out just as if we were measuring the original shape with a half-size ruler.

Let's call the process of blowing up (or shrinking down) **scaling**. So the second shape is obtained from the first by scaling by a factor of 2. Or, if we like, we could say that the first shape is the second one scaled by a factor of 1/2.

Two figures related by a scaling are called **similar**. All I'm really trying to say here is that if two shapes are similar, related by a certain scaling factor, then all corresponding length measurements are related by that same factor. People say that such things are "in proportion." Notice that scaling doesn't affect angles at all. The shape stays the same, only the size changes.

If two triangles have the same angles, are they necessarily similar? How about four-sided shapes?

Show that if a right triangle is chopped into two smaller ones, they must both be similar to the original triangle.

The nice thing about not having arbitrary units and always choosing to measure relative proportions is that it makes all our questions *scale independent*. To me, this is the simplest and most aesthetically pleasing approach. And given the fact that your shapes are in your head and mine are in mine, I really don't see any other alternative. Is your imaginary circle bigger or smaller than mine? Does that question even have any meaning?

But before we can begin to go about measuring something, we need to know precisely what object it is that we are talking about.

Let's suppose I have a square.

Now, there are some things I know about this shape right off the bat, such as the fact that it has four equal sides. The thing about information like this is that it is not really a discovery, nor does it require any explanation or proof. It's simply part of what I mean by the word *square*. Whenever you create or define a mathematical object, it always carries with it the blueprint of its own construction—the defining features that make it what it is and not some other thing. The questions we are asking as mathematicians then take this form: If I ask for such and such, what else do I get as a consequence? For example, if I ask for four equal sides, does that force my shape to be a square? Clearly, it doesn't.

It could be a diamond shape with equal sides, a so-called **rhombus** (Greek for "spinning top"). In other words, the prescription of having four equal sides contains a certain amount of wiggle room. So one thing to always be aware of is whether you've pinned down your objects enough to get any information out of them. We can't precisely measure the angles of an arbitrary rhombus, because that description still allows the shape the freedom to squirm around and change its angles. We need to be clear about the extent to which we have specified our objects so that we can ask well-posed, meaningful questions.

Are the opposite sides of a rhombus always parallel? Are the diagonals perpendicular?

Suppose we ask that the angles of our rhombus all be right angles. That certainly forces our shape to be a square, because that's what the word *square* means! Now is there any room left for it to wiggle around? There is in fact one more degree of freedom remaining, which is that it could change its size. (This would be relative, of course, to some other object we are considering. If all we had were a square, then size would have no meaning.)

Suppose we cut a triangle from one corner to the middle of the opposite side. Does the area get cut in half?

Some areas are relatively easy to measure. For example, suppose we have a 3 by 5 rectangle.

It is easy to see that we can chop this rectangle into fifteen identical pieces, each of which is a unit square. So the area of the rectangle is 15. That is, it takes up exactly fifteen times as much space as a unit square does. In general, if the sides of a rectangle are nice whole numbers, say m and n, then the area is simply their product, mn. We can just count the m rows of n squares each.

But what if the sides don't come out even? How can we measure the area of a rectangle if we can't chop it up nicely into unit squares?

Here are two rectangles of the same height.

I like to think of the second one as a "stretched" version of the first. Is it clear that their areas are in the same proportion as their lengths? Stretching in one direction is called **dilation**. What we're saying is that dilation of a rectangle by a certain factor multiplies its area by that factor.

In particular, we can think of a rectangle with sides a and b as a unit square that has been dilated twice: by a factor of a in one direction and by *b* in another. This means that the area of the unit square will get multiplied, first by *a* and then by *b*. In other words, it gets multiplied by *ab*. So the area of a rectangle is just the product of its sides. It doesn't matter whether the sides come out even or not.

What about the area of a triangle? My favorite way to think about it is to imagine a rectangular box built around the triangle. It turns out that the area of the triangle is always half that of the rectangle. Do you see why?

Why does a triangle take up exactly half of its box?

What happens to the area of the triangle as we slide the tip horizontally? What if it goes past the sides of the box?

Show that when we connect the midpoints of the sides of any four-sided shape, it forms a parallelogram. What is its area?

Can a polygon always be chopped into pieces and reassembled to form a square?

One interesting feature of area is the way it behaves with respect to scaling. We can think of a scaling as being the result of two dilations by the same factor. If we have a square, and we scale it by a factor of r, then its area will get multiplied by r^2. For example, if you blow up a square by a factor of 2, its perimeter will double, but its area will quadruple.

As a matter of fact, this will be true for any shape. The effect of scaling on area is to multiply by the square of the scaling factor, no matter what shape you're dealing with. A nice way to see this is to imagine a square with the same area as your shape.

After scaling by a factor of r, their areas will still be equal—the two shapes enclose the same amount of space whether or not I change my ruler. Since the area of the square gets multiplied by r^2, so must the area of the other shape.

There is also the question of three-dimensional size. This is usually called **volume**. Naturally, we can take as our unit of volume that of a cube with unit sides. The first question is how to measure a simple three-dimensional box.

How does the volume of a box depend on the lengths of its sides?

What is the effect of scaling on volume?

4

The study of size and shape is called **geometry**. One of the oldest and most influential problems in the history of geometry is this one: How long is the diagonal of a square?

Naturally, what we are really asking about is the proportion of diagonal to side. For convenience, let's take the side of the square to have length 1, and write *d* for the length of the diagonal. Now look at this design.

We have four unit squares coming together to make a 2 by 2 square. Notice that their diagonals also form a square. This square has sides of length *d*, so we can think of it as a unit square scaled by a factor of *d*. In particular, its diagonal must be *d* times as long as that of a unit square, so it must have length d^2. On the other hand, just looking at the design we can see that its diagonal has length 2. This means that whatever *d* is, d^2 must be equal to 2. Another way to see this is to notice that the *d* by *d* square takes up exactly half the area of the big square. Since the area of the big square is 4, this again says that $d^2 = 2$.

So what is *d*? A good guess might be 1-1/2. But no, $\frac{3}{2} \times \frac{3}{2} = \frac{9}{4}$, which is greater than 2. This means *d* must actually be a little smaller. We can try other numbers: 7/5 is too small, 10/7 is too big, 17/12 is very close but still not quite right.

So what are we going to do, keep trying numbers till the cows come home? What we are looking for is a proportion $\frac{a}{b}$ such that

$$\frac{a}{b} \times \frac{a}{b} = 2.$$

The only way this can happen is if the top number *a* when multiplied by itself is exactly twice as big as the bottom number *b* multiplied by itself. In other words, we need to find two whole numbers *a* and *b* so that

$$a^2 = 2b^2.$$

Since we're only interested in the ratio $\frac{a}{b}$, there is no point in looking at numbers *a* and *b* that are both even (we could just cancel any common factors of 2). We can also rule out the possibility that *a* is odd: if *a* were an odd number, then a^2 would also be odd, and there would be no way for it to be double the size of b^2.

Why is the product of two odd numbers always odd?

So the only numbers $\frac{a}{b}$ we need to consider are those where *a* is even and *b* is odd. But then a^2 is not only even but *twice* an even (that is, divisible by 4). Do you see why?

Why is the product of two even numbers always divisible by 4?

Now, since *b* is odd, b^2 must also be odd, and so $2b^2$ is twice an odd. But we need a^2 to be *equal* to $2b^2$. How can twice an even be twice an odd? It can't.

What does this mean? It means that there simply aren't any whole numbers *a* and *b* with $a^2 = 2b^2$. In other words, *there is no fraction whose square is 2.* Our diagonal to side proportion *d* cannot be expressed as a fraction in any way—no matter how many pieces we divide our unit into, the diagonal will never come out evenly.

The amazing discovery is this: the big square takes up exactly as much area as the two smaller squares put together. No matter what shape the rectangle has, its sides and diagonal will always conspire to make these squares add up this way.

But why on earth should *that* be true? Here is a pretty way to see it using mosaic designs.

The first one uses the two smaller squares, together with four copies of the triangle, to make one big square. The second design uses the larger square (the one built on the diagonal) and those same four triangles to make another big square. The point is that these two big squares are identical; they both have sides equal to the two sides of the rectangle added together. In particular, this means that the two mosaics have the same total area. Now, if we remove the four triangles from each, the remaining areas must also match, so the two smaller squares really do take up exactly as much space as the larger one.

Let’s call the sides of the rectangle *a* and *b* and the diagonal *c.* Then the square of side *a* together with the square of side *b* has the same total area as the square of side *c*. In other words,

$$a^2 + b^2 = c^2.$$

This is the famous **Pythagorean theorem** relating the diagonal and sides of a rectangle. It’s named after the Greek philosopher Pythagoras (circa 500 BC), although the discovery is actually far older, dating back to the ancient Babylonian and Egyptian civilizations.

For example, we find that a 1 by 2 rectangle has a diagonal of length $\sqrt{5}$. As usual, this number is hopelessly irrational. Generally speaking, a rectangle whose sides are nice whole numbers will almost always have an irrational diagonal. This is because the Pythagorean relation involves the square of the diagonal rather than the diagonal itself. On the other hand, a 3 by 4 rectangle has a diagonal of length 5, since $3^2 + 4^2 = 5^2$. Can you find any other nice rectangles like that?

Which rectangles have whole number sides and diagonals?

How about the three-dimensional version? Instead of a rectangle, we can ask about a rectangular box.