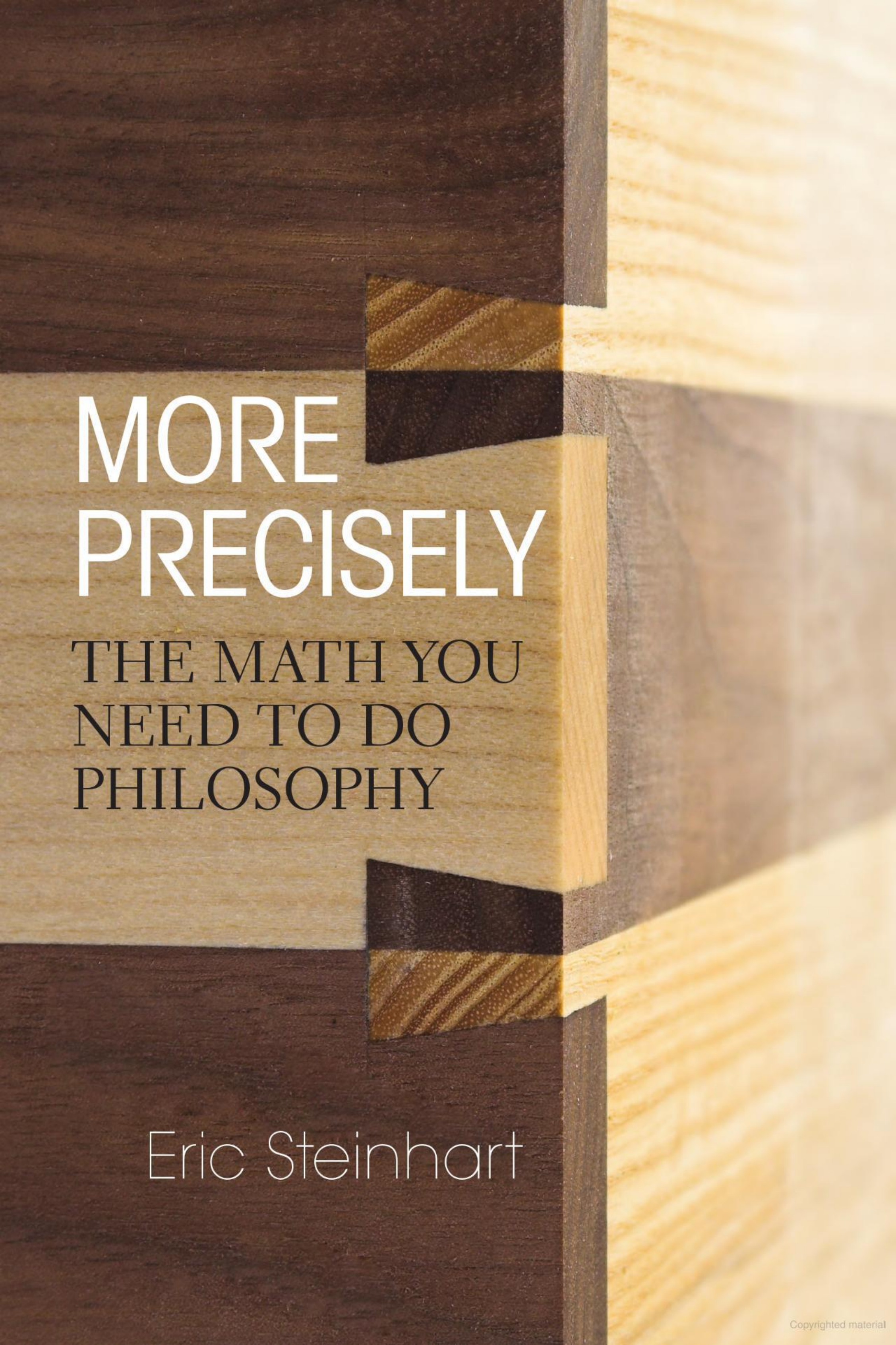

MORE PRECISELY

THE MATH YOU NEED TO DO PHILOSOPHY

Eric Steinhart

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Preface

Anyone doing philosophy today needs to have a sound understanding of a wide range of basic mathematical concepts. Unfortunately, most applied mathematics texts are designed to meet the needs of scientists. And much of the math used in the sciences is not used in philosophy. You're looking at a *mathematics book* that's designed to meet the needs of philosophers. *More Precisely* introduces the mathematical concepts you need in order to do philosophy today. As we introduce these concepts, we illustrate them with many classical and recent philosophical examples. This is math for *philosophers*.

It's important to understand what *More Precisely* is and what it isn't. *More Precisely* is *not* a philosophy of mathematics book. It's a mathematics book. We're not going to talk philosophically about mathematics. We are going to teach you the mathematics you need to do philosophy. We won't enter into any of the debates that rage in the current philosophy of math. Do abstract objects exist? How do we have mathematical knowledge? We don't deal with those issues here. *More Precisely* is *not* a logic book. We're not introducing you to any logical calculus. We won't prove any theorems. You can do a lot of math with very little logic. If you know some propositional or predicate logic, it won't hurt. But even if you've never heard of them, you'll do just fine here. *More Precisely* is an *introductory* book. It is not an advanced text. It aims to cover the basics so that you're prepared to go into the depths on the topics that are of special interest to you. To follow what we're doing here, you don't need anything beyond high school mathematics. We introduce all the technical notations and concepts gently and with many examples. We'll draw our examples from many branches of philosophy – including metaphysics, philosophy of mind, philosophy of language, epistemology, ethics, and philosophy of religion.

It's natural to start with some set theory. All branches of philosophy today make some use of set theory. If you want to be able to follow what's going on in philosophy today, you need to master at least the basic language of set theory. You need to understand the specialized notation and vocabulary used to talk about sets. For example, you need to understand the concept of the intersection of two sets, and to know how it is written in the specialized notation of set theory. Since we're not doing philosophy of math, we aren't going to get into any debates about whether or not sets exist. Before getting into such debates, you need to have a clear understanding of the objects you're arguing about. Our purpose in Chapter 1 is to introduce you to the language of sets and the basic ideas of set theory. Chapter 2 introduces relations and functions. Basic set-theoretic notions, especially relations and functions, are used extensively in the later chapters. So if you're not familiar with those notions, you've got to start with Chapters 1 and 2. Make sure you've really mastered the ideas in Chapters 1 and 2 before going on.

After we discuss basic set-theoretic concepts, we go into concepts that are used in various branches of philosophy. Chapter 3 introduces machines. A machine (in the special sense used in philosophy, computer science, and mathematics) isn't an industrial device. It's a formal structure used to describe some lawful pattern of activity. Machines are often used in philosophy of mind – many philosophers model minds as machines. Machines are sometimes used in metaphysics – simple universes can be modeled as networks of interacting machines. You can use these models to study space, time, and causality. Chapter 4 introduces some of the math used in the philosophy of language. Sets, relations, and functions are extensively used in formal semantic theories – especially possible worlds semantics. Chapter 5 introduces basic probability theory. Probability theory is used in epistemology and the philosophy of science (e.g., Bayesian epistemology, Bayesian confirmation theory). Mathematical concepts are often used in ethics. Utilitarians make great use of sums and products – the utility of a possible world is the sum of the happinesses of the persons in that world. So Chapter 6 discusses some of the math used in various utilitarian theories. Finally, the topic of infinity comes up in many philosophical discussions. Is the mind finitely or infinitely complex? Can infinitely many tasks be done in finite time? What does it mean to say that God is infinite? Chapter 7 introduces the notion of recursion and countable infinity. Chapter 8 shows that there is an endless progression of bigger and bigger infinities. It introduces transfinite recursion.

We illustrate the mathematical concepts with philosophical examples. We aren't interested in the philosophical soundness of these examples. As mentioned, we devote a chapter to the kinds of mathematics used in some utilitarian theories. Is this because utilitarianism is right? It doesn't matter. What matters is that utilitarianism uses lots of math, and that you need to know that math before you can really understand utilitarianism. As another example, we'll spend many pages explaining the mathematical apparatus behind various versions of possible worlds semantics. Is this because possible worlds really exist? We don't care. We do care that possible worlds semantics makes heavy use of sets, relations, and functions. As we develop the mathematics used in philosophy, we obviously talk about lots and lots of mathematical objects. We talk about sets, numbers, functions, and so on. Our attitude to these objects is entirely uncritical. We're engaged in exposition, not evaluation. We leave the interpretations and evaluations up to you. Although we aim to avoid philosophical controversies, *More Precisely* is not a miscellaneous assortment of mathematical tools and techniques. If you look closely, you'll see that the ideas unfold in an orderly and connected way. *More Precisely* is a conceptual narrative.

Our hope is that learning the mathematics we present in *More Precisely* will help you to do philosophy. You'll be better equipped to read technical philosophical articles. Articles and ideas that once might have seemed far too formal will become easy to understand. And you'll be able to apply these concepts in your own philosophical thinking and writing. Of course, some philosophers might object: why should philosophy use mathematics at all? Shouldn't philosophy avoid technicalities? We agree that technicality for its own sake ought to be avoided. As Ansel Adams once said, "There's nothing worse than a sharp image of a fuzzy concept." A bad idea doesn't get any better by expressing it in formal terms. Still, we think that philosophy has a lot to gain from becoming more mathematical. As science became more mathematical, it became more successful. Many deep and ancient problems were solved by making mathematical models of various parts and aspects of the universe. Is it naive to think that philosophy can make similar progress? Perhaps. But the introduction of formal methods into philosophy in the last century has led to enormous gains in clarity and conceptual power. Metaphysics, epistemology, ethics, philosophy of language, philosophy of science, and many other branches of philosophy, have made incredible advances by using the best available mathematical tools. Our hope is that this conceptual progress, slow and uncertain as it may be, will gain even greater strength.

Additional resources for *More Precisely* are available on the World Wide Web. These resources include extra examples as well as exercises. For more information, please visit

<http://broadviewpress.com/moreprecisely>

or

<http://www.ericsteinhart.com>.

Many thanks are due to the philosophers who helped with this project. Jim Moor deserves thanks for helping with the structure of the project. I especially appreciate the help of Chris Daly, Bob Martin, Tara Lowes, and Kathleen Wallace. They carefully read all or part of the manuscript and made very valuable suggestions. And I'm grateful to Gillman Payette for an extremely close reading. His suggestions made this a much better text! Finally, I'd like to thank Ryan Chynces and Alex Sager for being wonderful editors.

1

SETS

1. Collections of Things

As the 19th century was coming to a close, many people began to try to think precisely about collections. Among the first was the Russian-German mathematician Georg Cantor. Cantor introduced the idea of a *set*. For Cantor, a set is the collection of many things into a whole (1955: 85). It's not hard to find examples of sets: a crowd of people is a set of people; a herd of cows is a set of cows, a fleet of ships is a set of ships, and so on. The things that are collected together into a set are the *members* of the set. So if a library is a set of books, then the books in the library are the members of the library. Likewise, if a galaxy of stars is a set of stars, then the stars in the galaxy are the members of the galaxy.

As time went by, Cantor's early work on sets quickly became an elaborate theory. Set theory went through a turbulent childhood (see van Heijenoort, 1967; Hallett, 1988). But by the middle of the 20th century, set theory had become stable and mature. Set theory today is a sophisticated branch of mathematics. Set theorists have developed a rich and complex technical vocabulary – a network of special terms. And they have developed a rich and complex system of rules for the correct use of those terms. Our purpose in this chapter is to introduce you to the vocabulary and rules of set theory. Why study set theory? Because it is used extensively in current philosophy. You need to know it.

Our approach to sets is uncritical. We take the words of the set theorists at face value. If they say some sets exist, we believe them. Of course, as philosophers, we have to look critically at the ideas behind set theory. We need to ask many questions about the assumptions of the set theorists. But before you can criticize set theory, you need to understand it. We are concerned here only with the understanding. You may or may not think that numbers exist. But you still need to know how to do arithmetic. Likewise, you may or may not think that sets exist. But to succeed in contemporary philosophy, you need to know at least some elementary set theory. Our goal is to help you master the set theory you need to do philosophy. Our approach to sets is informal. We introduce the notions of set theory step by step, little by little. A more formal approach involves the detailed study of the *axioms* of set theory. The axioms of set theory are the precisely stated rules of set theory. Studying the axioms of set theory is advanced work. So we won't go into the axioms here. Our aim is to *introduce* set theory. We can introduce it informally. Most importantly, in the coming chapters, we'll show how ideas from set theory (and other parts of mathematics) are applied in various branches of philosophy.

We start with the things that go into sets. After all, we can't have collections of things if we don't have any things to collect. We start with things that aren't sets. An *individual* is any thing that isn't a set. Sometimes individuals are known as *urelemente* (this is a German word pronounced oor-ella-mentuh, meaning *primordial, basic, original elements*). Beyond saying that individuals are not sets, we place no restrictions on the individuals. The individuals that can go into sets can be names, concepts, physical things, numbers, monads, angels, propositions, possible worlds, or whatever you want to think or talk about. So long as they aren't sets. Sets can be inside sets, but then they aren't counted as *individuals*. Given some individuals, we can collect them together to make sets. Of course, at some point we'll have to abandon the idea that every set is a construction of things collected together by someone. For example, set theorists say that there exists a set whose members are all finite numbers. But no human person has ever gathered all the finite numbers together into a set. Still, for the moment, we'll use that kind of constructive talk freely.

2. Sets and Members

Sets have names. One way to refer to a set is to list its members between curly braces. Hence the name of the set consisting of Socrates, Plato, and Aristotle is {Socrates, Plato, Aristotle}. Notice that listing the members is different from listing the names of the members. So

{Socrates, Plato, Aristotle} is a set of *philosophers*; but

{"Socrates", "Plato", "Aristotle"} is a set of *names* of philosophers.

The membership relation is expressed by the symbol $\in$. So we symbolize the fact that Socrates is a member of {Socrates, Plato} like this:

Socrates $\in$ {Socrates, Plato}.

The negation of the membership relation is expressed by the symbol $\notin$. We therefore symbolize the fact that Aristotle is not a member of {Socrates, Plato} like this:

Aristotle $\notin$ {Socrates, Plato}.

And we said that individuals don't have members (in other words, no object is a member of any non-set). So Socrates is not a member of Plato. Write it like this:

Socrates $\notin$ Plato.

Identity. Two sets are identical if, and only if, they have the same members. The long phrase "if and only if" indicates logical equivalence. To say a set S is identical with a set T is equivalent to saying that S and T have the same members. That is, if S and T have the same members, then they're identical; and if S and T are identical, then they have the same members. The phrase "if and only if" is often abbreviated as "iff". It's not a spelling mistake! Thus

S = T if and only if S and T have the same members

is abbreviated as

S = T iff S and T have the same members.

More precisely, a set S is identical with a set T iff for every x, x is in S iff x is in T. One of our goals is to help you get familiar with the symbolism of set theory. So we can write the identity relation between sets in symbols like this:

S = T iff (for every x)(($x \in$ S) iff ($x \in$ T)).

You can easily see that {Socrates, Plato} = {Socrates, Plato}. When writing the name of a set, the order in which the members are listed makes no difference. For example,

{Socrates, Plato} = {Plato, Socrates}.

When writing the name of a set, mentioning a member many times makes no difference. You only need to mention each member once. For example,

{Plato, Plato, Plato} = {Plato, Plato} = {Plato};

{Socrates, Plato, Socrates} = {Socrates, Plato}.

When writing the name of a set, using different names for the same members makes no difference. As we all know, Superman is Clark Kent and Batman is Bruce Wayne. So

{Superman, Clark Kent} = {Clark Kent} = {Superman};

{Superman, Batman} = {Clark Kent, Bruce Wayne}.

Two sets are distinct if, and only if, they have distinct members:

{Socrates, Plato} is not identical with {Socrates, Aristotle}.

3. Set Builder Notation

So far we've defined sets by listing their members. We can also define a set by giving a formula that is true of every member of the set. For instance, consider the set of happy things. Every member in that set is happy. It is the set of all x such that x is happy. We use a special notation to describe this set:

the set of . . .	{ . . . }
the set of all x . . .	{ x . . . }
the set of all x such that . . .	{ x | . . . }
the set of all x such that x is happy	{ x | x is happy }.

Note that we use the vertical stroke "|" to mean "such that". And when we use the variable x by itself in the set builder notation, the scope of that variable is wide open – x can be *anything*. Many sets can be defined using this set-builder notation:

the books in the library = { x | x is a book and x is in the library };

the sons of rich men = { x | x is the son of some rich man }.

A set is never a member of itself. At least not in standard set theory. There are some non-standard theories that allow sets to be members of themselves (see Aczel, 1988). But we're developing standard set theory here. And since standard set theory is used to define the non-standard set theories, you need to start with it anyway! Any definition of a set that implies that it is a member of itself is ill-formed – it does not in fact define any set at all. For example, consider the formula

the set of all sets = { x | x is a set }.

Since the set of all sets is a set, it must be a member of itself. But we've ruled out such ill-formed collections. A set that is a member of itself is a kind of vicious circularity. The rules of set theory forbid the formation of any set that is a member of itself. Perhaps there is a *collection* of all sets. But such a collection can't be a *set*.

4. Subsets

Subset. Sets stand to one another in various relations. One of the most basic relations is the subset relation. A set S *is a subset of* a set T iff every member of S is in T. More precisely, a set S is a subset of a set T iff for every x, if x is in S, then x is in T. Hence

{Socrates, Plato} is a subset of {Socrates, Plato, Aristotle}.

Set theorists use a special symbol to indicate that S is a subset of T:

$S \subseteq T$ means S is a subset of T.

Hence

$\{$Socrates, Plato$\} \subseteq \{$Socrates, Plato, Aristotle$\}$.

We can use symbols to define the subset relation like this:

S $\subseteq$ T iff (for every x)(if $x \in$ S then $x \in$ T).

Obviously, if x is in S, then x is in S; hence every set is a subset of itself. That is, for any set S, S $\subseteq$ S. For example,

$\{$Socrates, Plato$\}$ is a subset of $\{$Socrates, Plato$\}$.

But remember that no set is a member of itself. Being a subset of S is different from being a member of S. The fact that S $\subseteq$ S does *not* imply that S $\in$ S.

Proper Subset. We often want to talk about the subsets of S that are distinct from S. A subset of S that is not S itself is a *proper* subset of S. An identical subset is an *improper* subset. So

$\{$Socrates, Plato$\}$ is an improper subset of $\{$Socrates, Plato$\}$;

while

$\{$Socrates, Plato$\}$ is a proper subset of $\{$Socrates, Plato, Aristotle$\}$.

We use a special symbol to distinguish proper subsets:

S $\subset$ T means S is a proper subset of T.

Every proper subset is a subset. So if S $\subset$ T, then S $\subseteq$ T. However, not every subset is a proper subset. So if S $\subseteq$ T, it does not follow that S $\subset$ T. Consider:

$\{$Socrates, Plato$\} \subseteq \{$Socrates, Plato, Aristotle$\}$	True
$\{$Socrates, Plato$\} \subset \{$Socrates, Plato, Aristotle$\}$	True
$\{$Socrates, Plato, Aristotle$\} \subseteq \{$Socrates, Plato, Aristotle$\}$	True
$\{$Socrates, Plato, Aristotle$\} \subset \{$Socrates, Plato, Aristotle$\}$	False

Two sets are identical iff each is a subset of the other:

S = T iff ((S $\subseteq$ T) & (T $\subseteq$ S)).

We'll use "{}" to denote the empty set. For example,

{} = { x | x is an actual unicorn };
{} = { x | x is a married bachelor }.

It is important not to be confused about the empty set. The empty set isn't nothing or non-being. If you think the empty set exists, then obviously you can't think that it is nothing. That would be absurd. The empty set is exactly what the formalism says it is: it is a set that does not contain any thing as a member. It is a set with no members. According to set theory, the empty set is an existing particular object.

The empty set is a subset of every set. Consider an example: {} is a subset of {Plato, Socrates}. The idea is this: for any x, if x is in {}, then x is in {Plato, Socrates}. How can this be? Well, pick some object for x. Let x be Aristotle. Is Aristotle in {}? The answer is no. So the statement "Aristotle is in {}" is false. And obviously, "Aristotle is in {Plato, Socrates} is false. Aristotle is not in that set. But logic tells us that the only way an *if-then* statement can be false is when the *if* part is true and the *then* part is false. Thus (somewhat at odds with ordinary talk) logicians count an *if-then* statement with a false *if* part as true. So even though both the *if* part and the *then* part of the whole *if-then* statement are false, the whole *if-then* statement "if Aristotle is in {}, then Aristotle is in {Socrates, Plato}" is true. The same reasoning holds for any object you choose for x. Thus for any set S, and for any object x, the statement "if x is in {}, then x is in S" is true. Hence {} is a subset of S.

We can work this out more formally. For any set S, {} $\subseteq$ S. Here's the proof: for any x, it is not the case that ($x \in$ {}). Recall that when the antecedent (the *if* part) of a conditional is false, the whole conditional is true. That is, for any Q, when P is false, (if P then Q) is true. So for any set S, and for any object x, it is true that (if $x \in$ {} then x is in S). So for any set S, it is true that (for all x)(if $x \in$ {} then x is in S). Hence for any set S, {} $\subseteq$ S.

Bear this clearly in mind: the fact that {} is a *subset* of every set does *not* imply that {} is a *member* of every set. The subset relation is *not* the membership relation. Every set has the empty set as a subset. But if we want the empty set to be a member of a set, we have to put it into the set. Thus {A} has the empty set as a subset while {{}, A} has the empty set as both a subset and as a member. Clearly, {A} is not identical to {{}, A}.

6. Unions of Sets

Unions. Given any two sets S and T, we can take their *union*. Informally, you get the union of two sets by adding them together. For instance, if the Greeks = {Socrates, Plato} and the Germans = {Kant, Hegel}, then the union of the

3

MACHINES

1. Machines

Machines are used in many branches of philosophy. Note that the term "machine" is a technical term. When we say something is a machine, we mean that it has a certain formal structure; we don't mean that it's made of metal or anything like that. A machine might be made of organic molecules or it might even be made of some immaterial soul-stuff. All we care about is the formal structure. As we talk about machines, we deliberately make heavy use of sets, relations, functions, and so on. Machines are used in metaphysics and philosophy of physics. You can use machines to model physical universes. These models illustrate various philosophical points about space, time, and causality. They also illustrate the concepts of emergence and supervenience. Machines are also used in philosophy of biology. They are used to model living organisms and ecosystems. They are used to study evolution. They are even used in ethics, to model interactions among simple moral agents. But machines are probably most common in philosophy of mind.

A long time ago, thinkers like Hobbes (1651) and La Mettrie (1748) defended the view that human persons are machines. But their conceptions of machines were imprecise. Today our understanding of machines is far more precise. There are many kinds of machines. We'll start with the kind known as *finite deterministic automata*. As far as we can tell, the first philosopher to argue that a human person is a finite deterministic automaton was Arthur Burks in 1973. He argues for the thesis that, "A finite deterministic automaton can perform all natural human functions." Later in the same paper he writes that, "My claim is that, for each of us, there is a finite deterministic automaton that is behaviorally equivalent to us" (1973: 42). Well, maybe there is and maybe there isn't. But before you try to tackle that question, you need to understand finite deterministic automata. And the first thing to understand is that the term *automaton* is somewhat old-fashioned. The more common current term is just *machine*. So we'll talk about machines.

2. Finite State Machines

2.1 Rules for Machines

A *machine* is any object that runs a *program*. A program guides or governs the behavior of its machine. It is a lawful pattern of activity within the machine – it is the *nature* or *essence* of the machine. Suppose that some machine M runs a program P. Any program P is a tuple (I, S, O, F, G). The item I is the set of possible *inputs* to M. The item S is the set of possible *states* of M. The item O is the set of possible *outputs* of M. The item F is a *transition relation* that takes each (input, state) pair onto one or more states in S. It is a relation from I × S to S. The item G is an *output relation* that takes each (input, state) pair onto one or more outputs in O. It is a relation from I × S to O.

A machine is *finite* iff its set of program states is finite. Such a machine is also known as – you guessed it – a *finite state machine* (an FSM). A machine is *infinite* iff its set of states is infinite. A machine is *deterministic* iff the relations F and G are functions. For a deterministic machine, the item F is a transition function that maps each (input, state) pair onto a state. In symbols, F: I × S → S. And the item G is an output function that maps each (input, state) pair onto an output. In symbols, G: I × S → O. A machine is *non-deterministic* iff either F is not a function or G is not a function (they are one-many or many-many). We'll only be talking about deterministic machines.

There are many ways to present a machine. One way is to display its program as a list of *dispositions*. Each disposition is a rule of this form: if the machine gets input *w* while in state *x*, then it changes to state *y* and produces output *z*. For example, consider a simple robot with three emotional states: calm, happy, and angry. One disposition for this emotional robot might look like this: if you get a smile while you're calm, then change to happy and smile back. Of course, the robot may have other dispositions. But it's important to see that the emotional robot has all and only the dispositions that are defined in its program. It has whatever dispositions we give it. We might give it dispositions that allow it to learn – to form new dispositions, and to modify its original programming. But even then, it won't have any undefined dispositions. It is wholly defined by its program.

Another way to present a machine is to display its program as a *state-transition network*. A state-transition network has circles for states and arrows for transitions. Each arrow is labeled with <input / output>. Figure 3.1 shows how a single disposition is displayed in a state-transition network. Figure 3.2 shows some of the state-transition network for the emotional robot.

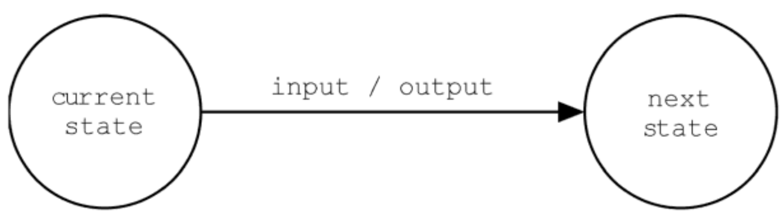

Figure 3.1 Diagram for a single disposition.

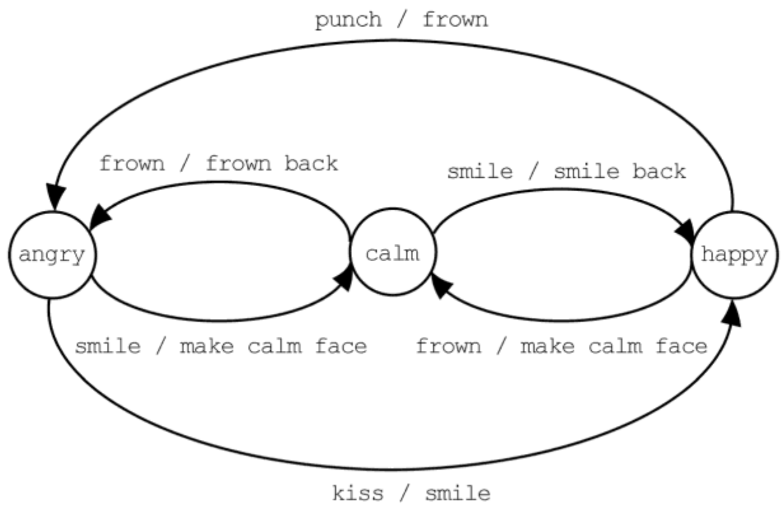

Figure 3.2 State-transition network for an emotional robot.

We can also define the robot by writing its components in set-theoretic notation. So

I = {frown, smile, punch, kiss};
S = {angry, calm, happy};
O = {frown back, smile back, make calm face}.

Table 3.1 details the function F while Table 3.2 details the function G. The functions F and G in these tables are complete – they include all the dispositions of the robot.

(frown, calm)	→	angry
(smile, calm)	→	happy
(punch, calm)	→	angry
(kiss, calm)	→	happy
(frown, happy)	→	calm
(smile, happy)	→	happy
(punch, happy)	→	angry
(kiss, happy)	→	happy
(frown, angry)	→	angry
(smile, angry)	→	calm
(punch, angry)	→	angry
(kiss, angry)	→	happy

Table 3.1 The function F.

(frown, calm)	→	frown back
(smile, calm)	→	smile back
(punch, calm)	→	frown back
(kiss, calm)	→	smile back
(frown, happy)	→	make calm face
(smile, happy)	→	smile back
(punch, happy)	→	frown back
(kiss, happy)	→	smile back
(frown, angry)	→	frown back
(smile, angry)	→	make calm face
(punch, angry)	→	frown back
(kiss, angry)	→	smile back

Table 3.2 The function G.

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