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Praefatio
Unus non sufficit orbis

In my mentor Robert Rosen's iconoclastic masterwork *Life Itself* [1991], which dealt with the epistemology of life, he proposed a Volume 2 that was supposed to deal with the ontogeny of life. As early as 1990, before *Life Itself* (i.e., 'Volume 1') was even published (he had just then signed a contract with Columbia University Press), he mentioned to me in our regular correspondence that Volume 2 was "about half done". Later, in his 1993 Christmas letter to me, he wrote:

> ...I've been planning a companion volume [to *Life Itself*] dealing with ontology. Well, that has seeped into every aspect of everything else, and I think I'm about to make a big dent in a lot of old problems. Incidentally, that book [*Life Itself*] has provoked a very large response, and I've been hearing from a lot of people, biologists and others, who have been much dissatisfied with prevailing dogmas, but had no language to articulate their discontents. On the other hand, I've outraged the "establishment". The actual situation reminds me of when I used to travel in Eastern Europe in the old days, when everyone was officially a Dialectical Materialist, but unofficially, behind closed doors, nobody was a Dialectical Materialist.

When Rosen died unexpectedly in 1998, his book *Essays on Life Itself* [published posthumously in 2000] was in the final stages of preparation. But this collection of essays is *not* 'Volume 2', as he explained in its Preface:

> Thus this volume should be considered a supplement to the original volume. It is not the projected second volume, which deals with ontogenetics rather than with epistemology, although some chapters herein touch on ideas to be developed therein.

We see, therefore, that the "projected second volume" was then still a potentiality. I have, however, never seen any actualization of this 'Volume 2', and no part of its manuscript has ever been found.

Rosen did, nevertheless, leave behind a partially completed manuscript tentatively entitled "Complexity". This was a work-in-progress, with only a few sections (mostly introductory material) finished. It may or may not be what he had in mind for the projected second volume of *Life Itself*. My opinion is that it is not. To me, its contents are neither sufficiently extent nor on-topic enough for it to be a more-than-half-done Volume 2 on the ontogeny of life. In the years since, I had begun an attempt to extend the manuscript into *Life Itself Volume 2*, but this effort of raising his orphan, as it were, was abandoned for a variety of reasons — one of them was that I did not want to be Süssmayr to Mozart's *Requiem*.

The book that you are now reading, *More Than Life Itself*, is therefore *not* my completion of the anticipated Volume 2 of Robert Rosen's *Life Itself*, and has not incorporated any of his text from the "Complexity" manuscript. It is entirely my own work in the Rashevsky-Rosen school of relational biology. The inheritance of Nicolas Rashevsky (1899–1972) and Robert Rosen (1934–1998), my academic grandfather and father, is, of course, evident (and rightly and unavoidably so). Indeed, some repetition of what Rosen has already written first (which is worthy of repetition in any case) may occasionally be found. After all, he was a master of *les mots justes*, and one can only rearrange a precise mathematical statement in a limited number of ways. As Aristotle said, "When a thing has been said once, it is hard to say it differently."

The crux of *relational biology*, a term coined by Nicolas Rashevsky, is

"*Throw away the matter and keep the underlying organization.*"

The characterization of life is not what the underlying physicochemical *structures* are, but by its entailment *relations*, what they *do*, and to what *end*. In other words, life is not about its material cause, but is intimately linked to the other three Aristotelian causes, formal, efficient, and final. This is, however, not to say that structures are not biologically important: structures and functions are intimately and synergistically related. Our slogan is simply an emphatic statement that we take the view of 'function dictates structure' over 'structure implies function'. Thus *relational biology* is the operational description of our endeavour, the characteristic name of our approach to our subject, which is *mathematical biology*. Note that 'biology' is the noun and 'mathematical' is the adjective: the study of living organisms is the subject, and the abstract deductive science that is mathematics is the tool. Stated otherwise, biology is the final cause and mathematics is the efficient cause. The two are indispensable ingredients, indeed complementary (and complimentary) halves of our subject. Relational biology can no more be done without the mathematics than without the biology. Heuristic, exploratory, and expository discussions of a topic, valuable as they may be, do not become the topic itself; one must distinguish the science from the meta-science.

The Schrödinger question "What is life?" is an abbreviation. A more explicitly posed expansion is "What distinguishes a *living system* from a non-living one?"; alternatively, "What are the defining characteristics of a natural system for us to perceive it as being alive?" This is the epistemological question Rosen discusses and answers in *Life Itself*. His answer, in a nutshell, is that an *organism* — the term is used in the sense of an 'autonomous life form', i.e., any living system — admits a certain kind of relational description, that it is 'closed to efficient causation'. (I shall explain in detail these and many other somewhat cryptic, very Rosen terms in this monograph.) The epistemology of biology concerns what one learns about life by looking at the living. From the epistemology of life, an understanding of the relational model of the inner workings of what is alive, one may move on to the ontogeny of life. The ontology of biology involves the existence of life, and the creation of life out of something else. The ontogenetic expansion of Schrödinger's question is "What makes a natural system alive?"; or, "What does it take to fabricate an organism?" This is a hard question. This monograph *More Than Life Itself* is my first step, a *synthesis* in every sense of the word.

With the title that I have chosen for the book, I obviously intend it to be a *continuation* of Robert Rosen's conception and work in *Life Itself*. But (as if it needs to be explicitly written) I am only Robert Rosen's student, not Robert Rosen himself. No matter how sure I am of my facts, I cannot be so presumptuous as to state that, because I know my mentor-colleague-friend and his work so well, what I write is what he would have written. In other words, I cannot, of course, claim that I speak for Robert Rosen, but in his absence, with me as a 'torch-bearer' of the school of relational biology, my view will have to suffice as a surrogate.

But surrogacy implicitly predicates nonequivalence. My formulations occasionally differ from Rosen's, and this is another reason why I find it more congenial to not publish my *More Than Life Itself* as 'Volume 2 of Robert Rosen's *Life Itself* '. I consider these differences *evolutionary* in relational biology: as the subject develops from Rashevsky to Rosen to me, each subsequent generation branches off on the *arbor scientiae*. Any errors (the number of which I may fantasize to be zero but can only hope to be small, and that they are slight and trivially fixable) that appear in this book are, naturally, entirely mine. The capacity to err is, in fact, the real marvel of evolution: the processes of metabolism-repair-replication are ordained from the very beginning to make small mistakes. Thus through mutational blunders progress and improvements are made. The Latin root for 'error', the driving force of evolution, is *erratio*, which means roving, wandering about looking for something, quest.

In complex analysis (the theory of functions of a complex variable), *analytic continuation* is a technique used to extend the domain of a given holomorphic (*alias* analytic) mapping. As an analogue of this induction, I use the term *synthetic continuation* in the subtitle of this monograph that is the song of our *synthetic* journey. Analytic biology attempts to model specific fragments of natural phenomena; synthetic biology begins with categories of mathematical objects and morphisms, and seeks their realizations in biological terms. Stated otherwise, in relational biology, mathematical tools are used synthetically: we do not involve so much in the making of particular models of particular biological phenomena, but rather invoke the entailment patterns (or lack thereof) from certain mathematical theories and interpret them biologically. Nature is the realization of the simplest conceivable mathematical ideas. I shall have a lot more to say on analysis versus synthesis in this monograph.

Someone once said to Rosen: "The trouble with you, Rosen, is that you keep trying to answer questions nobody wants to ask." [Rosen 2006]. It appears that his answers themselves cause even more self-righteous indignation in some people, because the latter's notions of truth and Rosen's answers do not coincide. Surely only the most arrogant and audacious would think that the technique they happen to be using to engage their chosen field is the be-all and end-all of that subject, and would be annoyed by any alternate descriptions, Rosen's or otherwise. One needs to remember that the essence of a complex system is that a single description does not suffice to account for our interactions with it. Alternate descriptions are fundamental in the pursuit of truth; plurality spices life.

"*One world is not enough.*"

Uncritical generalizations about what Rosen said are unhelpful. For example, according to Rosen, one of the many corollaries of being an organism is that it must have noncomputable models. The point is that *life itself* is not computable. This in no way means that he somehow implies that computable models are useless, and therefore by extension people involved with biological computing are wasting their time! There are plenty of useful computing models of biological processes. The simple fact is that computing models (an indeed any models whatsoever) will be, by definition, *incomplete*, but they may nevertheless be fruitful endeavours. One learns a tremendous amount even from partial descriptions.

Along the same vein, some impudent people take great offence in being told by Rosen that their subject area (*e.g.* the physics of mechanisms), their 'niche', is *special* and hence *nongeneric*. Surely it should have been a compliment! An algebraic topologist, say, would certainly take great pride that her subject area is indeed not run-of-the-mill, and is a highly specialized area of expertise. I am a mathematical biologist. I would not in my wildest dream think that mathematics can provide *almost all* (in the appropriately mathematical sense) the tools that are suitable for the study of biology. A mathematical biologists is a specialist in a very specialized area. There is nothing wrong in being a specialist; what is wrong is the reductionistic view that the specialization is in fact general, that all (or at least all in the 'territory' of the subject at hand) should conform to the specialization. Why is being nongeneric an insult? Are some people, in their self-aggrandizement, really pretentious enough to think that the subject they happen to be in would provide answers to *all* the questions of life, the universe, and everything?

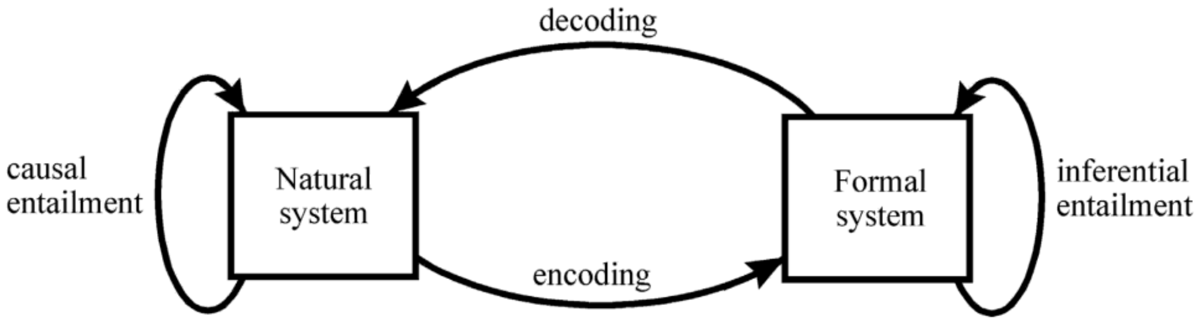
Rosen's revelations hit particular hard those who believe in the 'strong' Church-Turing thesis, that for every physically realizable process in nature there exists a Turing machine that provides a complete description of the process. In other words, to them, *everything* is computable. Note that Rosen only said that life is not computable, not that artificial life is impossible. However one models life, natural or artificial, one cannot succeed by computation alone. Life is not definable by an algorithm. Artificial life does not have to be limited to what a computing machine can do algorithmically; computing is but one of a multitude of available tools. But for the 'strong' Church-Turing thesis believers, they would have the syllogism

> Rosen says life is not computable.
> Everything is computable.
> Therefore Rosen says artificial life is impossible.

Compare and contrast this to Alan Turing's psychotic syllogism, a *non sequitur* that is so iconic of his demise

> Turing believes machines think.
> Turing lies with men.
> Therefore machines cannot think.

The following is a diagram of the modelling relation. (I shall have a lot more to say about it in Chapter 4.)



Natural systems from the external world generate in us our percepts of 'reality'. While causal entailments *themselves* may be universal truths, *perceived* causal entailments are *not*. All we have are our own *observations*, *opinions*, *interpretations*, our individual *alternate descriptions* of 'reality' that are our personal *models* of 'truth'. Causal entailments are interpreted, not proven.

A mathematical proof is absolute; it is categorically more than a scientific 'proof' (and a judicial 'proof') of 'beyond a reasonable doubt'. A scientific theory can never be proven to the same absolute certainty of a mathematical theorem. This is because a scientific 'proof' is merely considered 'highly likely based on the evidence available'; it depends on observation, experimentation, perception, and interpretation — all of which are fallible, and are in any case approximations of truth. Sometimes the minimized doubt later turns out to be errors, and paradoxically, in the same spirit of 'errors drive evolution', the inherent weakness in scientific proofs leads to scientific revolutions, when, 'based on new evidence', 'proven' theories are refined, updated, surpassed, or replaced.

Modelling is the art of bringing entailment structures into congruence. The essence of an art is that it rests on the heuristic, the unentailed, the intuitive leap. The encoding and the decoding arrows in the modelling relation diagram are themselves unentailed. Theoretical scientists are more artists than artisans. Natural Law assures them only that their art is not in vain, but it in itself provides not the slightest clue how to go about it. There is no right or wrong in art, only similarities and differences, the congenial and the uncongenial.

Among the four arrows in the diagram of the modelling relation, only inferential entailment may be *proven* in the rigorous mathematical sense. Absolute statements about the truth of statements validated by proofs cannot be disputed. Rosen proved the theorems that he stated in *Life Itself*, although his presentations are not in the orthodox form of definition-lemma-theorem-proof-corollary that one finds in conventional mathematics journals and textbooks. His book is, after all, not a text in pure mathematics. While the presentation may be 'Gaussian', with all scaffolding removed, there are nevertheless enough details in Rosen's prose that any reasonably competent mathematician can 'fill in the blanks' and rewrite the proofs in full, if one so wishes. But because of the unorthodox heuristic form, people have contended Rosen's theorems. Since the dispute is over form rather than substance, it is not surprising that the contentions are mere grumbles, and no logical fallacies in the theorems have ever been found. A common thread running in many of the anti-Rosen papers that I have encountered is the following: they simply use definitions of terms different from Rosen's, whence resulting in consequences different from Rosen's, and thereby concluding that Rosen must be wrong! I shall in the present monograph recast Rosen's theorems in as rigorously mathematical a footing as possible, using the algebraic theory of lattices. It is an interesting exercise in itself, but it is most unlikely to convert any skeptics with their preconceived ideas of truth.

The other three arrows in the modelling relation diagram — causal entailment, encoding, and decoding — all have intuitive elements in their art and science. As such, one if so inclined may claim that another's interpretations are *uncongenial*, but cannot conclude that they are *wrong*: there are no *absolute truths* concerning them.

Rosen closed his monograph *Anticipatory Systems* with these words:

> For in a profound sense, the study of models is the study of man; and if we can agree about our models, we can agree about everything else.

Agree with our models and partake our synodal exploration. Else agree to disagree, then we shall amicably part company.

In the Preface of *Life Itself*, Rosen identified his intended readership by quoting from Johann Sebastian Bach's *Clavierübung III*:

Denen Liebhabern,
und besonders denen Kennern von vergleichen Arbeit,
zur Gemüths Ergezung...

[Written for those who love,
and most especially those who appreciate such work,
for the delight of their souls...]

Let me add to that sentiment by quoting a couplet from a Chinese classic:

勞者自歌
非求傾聽

[The diligent one sings for oneself,
not for the recruitment of an audience.]

The same readers who took delight in *Life Itself* should also enjoy this *More Than Life Itself*. Be our companions on our journey and join us in our songs.

A. H. Louie
22 February, 2009

Nota bene

Many references in this monograph are drawn from Robert Rosen's Trilogy:

- [*FM*] *Fundamentals of Measurement and Representation of Natural Systems* [1978]

- [*AS*] *Anticipatory Systems: Philosophical, Mathematical, and Methodological Foundations* [1985a], and

- [*LI*] *Life Itself: A Comprehensive Inquiry into the Nature, Origin, and Fabrication of Life* [1991].

Additional references are in

- [*NC*] "Organisms as Causal Systems which are not Mechanisms: an Essay into the Nature of Complexity" [1985b] and

- [*EL*] *Essays on Life Itself* [2000].

My thesis

- [*CS*] "Categorical System Theory" [1985] contains much of the background material on the category theory of natural and formal systems. (See the Bibliography for publication details of these references.) Familiarity with our previous work is not a prerequisite; it would, however, make simpler a first reading of this monograph. I strive to make it as self-contained as possible, but because of the subjects' inherent complexity, the entailment patterns of the many concepts cannot be rendered unidirectional and sequential. Some topics I present herein depend not only upon material on previous pages but also upon material on following pages. So in a sense this monograph is an embodiment of a relational diagram in graph-theoretic form, a realization of the branches and cycles of the entailment patterns.

In this book I assume that the reader is familiar with the basic facts of *naive set theory*, as presented, for example, in Halmos [1960]. Set theory from the naive point of view is the common approach of most mathematicians (other than, of course, those in mathematical logic and the foundations of mathematics). One assumes the existence of a suitable universe of sets (*viz*. the universe of *small* sets) in which the set-theoretic constructions, used in contexts that occur naturally in mathematics, will not give rise to paradoxical contradictions. In other words, one acknowledges these paradoxes, and moves on. This is likewise the position I take in this monograph. In the Prolegomena I present some set-theoretic and logical preliminaries, but more for the clarity of notations than for the concepts themselves. For example, the relative complement of a set A in a set B may be variously denoted as $B \sim A$, $B - A$, $B \backslash A$, etc.; I use the first one.

I often used the language of *category theory* as a metalanguage in my text. The definitive reference on this branch of abstract algebra is Mac Lane [1978]. I give a concise summary in the Appendix of those category-theoretic concepts that appear in my exposition.

Prolegomenon
Concepts from Logic

道可道
非常道
名可名
非常名

The principle that can be stated
Cannot be the absolute principle.
The name that can be given
Cannot be the permanent name.

— Lao Tse (6th century BC)
Tao Te Ching
Chapter 1

In principio...

0.1 ***Κοιναι εννοιαι α*** Book I of Euclid's *Elements* begins with a list of twenty-three definitions and five postulates in plane geometry, followed by five *common notions* that are general logical principles. Common Notion 1 states

"*Things equal to the same thing are also equal to one another.*"

Equality is a primitive: such proclamation of its self-evident property without proof is the very definition of *axiom*. Thus formally begins mathematics...

It may be argued that *equality* is the most basic property in any mathematical subject. In set theory, equality of sets is formulated as the

0.2 Axiom of Extension *Two sets are equal if and only if they have the same elements.*

Subset

I shall assume that the reader has a clear intuitive idea of the notions of a *set* and of *belonging to a set*. I use the words 'set', 'collection', and 'family' as synonyms. The elementary operations that may be performed with and on sets are used without further comments, save one:

0.3 Definition If A and B are sets and if every element of A is an element of B, then A is a *subset* of B.

The wording of the definition implies that there are two possibilities: either $A = B$, or B contains at least one element that is not in A, in which case A is called a *proper subset* of B. It has been increasingly popular in the mathematical literature to use $A \subseteq B$ as notation, seduced by the ordering relation $\leq$. This usage, unfortunately, almost always ends here. $\subseteq$-users rarely use the then-consistent notation $A \subset B$, analogous to $<$, to mean proper subset, but often resort to the idiosyncratic $\subsetneqq$ instead. The few exceptions that do employ $\subset$ to mean 'proper subset' invariably lead to confusion, because of the well-established standard notation

(1) $A \subset B$

for '*either* $A = B$ *or* A is a proper subset of B'. The notation I use in this book is this standard which is inclusive of both senses of the containment of set A in set B. Sometimes $A \subset B$ is reversely described as 'B is a *superset* of A'.

If A and B are sets such that $A \subset B$ and $B \subset A$, then the two sets have the same elements. It is equally obvious *vice versa*. The Axiom of Extension 0.2 may, therefore, be restated as the

0.4 Theorem *Two sets A and B are equal if and only if* $A \subset B$ *and* $B \subset A$.

On account of this theorem, the proof of set equality $A = B$ is usually split into two parts: first prove that $A \subset B$, and then prove that $B \subset A$.

Conditional Statements and Variations

0.5 Conditional Many statements, especially in mathematics, are of the form '*If* p, *then* q.' We have already encountered some in this prologue. These are called *conditional statements*, and are denoted in the predicate calculus of formal logic by

(2) $$p \to q.$$

The if-clause p is called the *antecedent* and the then-clause q is called the *consequent*. Note that the conditional form (2) may be translated equivalently as 'q *if* p.' So the clauses of the sentence may be written in the reverse order, when the antecedent does not in fact 'go before', and the conjunction 'then' does not explicitly appear in front of, the consequent.

If the antecedent is true, then the conditional statement is true if the consequent is true, and the conditional statement is false if the consequent is false. If the antecedent is false, then the conditional statement is true regardless of whether the consequent is true or false. In other words, the conditional $p \rightarrow q$ is false if p is true and q is false, and it is true otherwise.

0.6 I Say What I Mean

> "Then you should say what you mean," the March Hare went on.
>
> "I do," Alice hastily replied; "at least — at least I mean what I say — that's the same thing, you know."
>
> "Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"
>
> "You might just as well say," added the March Hare, "that 'I like what I get' is the same thing as 'I get what I like'!"
>
> "You might just as well say," added the Dormouse, which seemed to be talking in his sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe'!"
>
> "It is the same thing with you," said the Hatter, and here the conversation dropped, and the party sat silent for a minute,...

— Lewis Carroll (1865)
Alice's Adventures in Wonderland
Chapter VII A Mad Tea Party

Alice's "I do" is the contention "I say what I mean". This may be put as the conditional statement

"If I mean it, then I say it.".

which is form (2), $p \rightarrow q$, with p = 'I mean it' and q = 'I say it'. It is equivalent to the statement

"I say it if I mean it.".

The conditional $p \to q$ may also be read as 'p *only if* q'. Alice's statement is then

"I mean it only if I say it.".

The adverb 'only' has many nuances, and in common usage 'only if' is sometimes used simply as an emphasis of 'if'. But in mathematical logic 'only if' means 'exclusively if'. So 'p *only if* q.' means 'If q does not hold, then p cannot hold either.' In other words, it is logically equivalent to 'If not q, then not p.', which in the predicate calculus is

(3) $\neg q \to \neg p$

(where $\neg$ denotes *negation*, the logical *not*). The conditional form (3) is called the *contrapositive* of the form (2). The contrapositive of Alice's "I mean it only if I say it." (= "If I mean it, then I say it.") is the equivalent conditional statement

"If I do not say it, then I do not mean it.".

0.7 I Mean What I Say The conditional form

(4) $q \to p$

is called the *converse* of the form (2), and the equivalent contrapositive of the converse, *i.e.* the conditional form

(5) $\neg p \to \neg q$,

is called the *inverse* of the original form (2). A conditional statement and its converse or inverse are *not* logically equivalent. For example, if p is true and q is false, then the conditional $p \to q$ is false, but its converse

$q \to p$ is true. The confusion between a conditional statement and its converse is a common mistake. Alice thought "I mean what I say." (*i.e.* the converse statement "If I say it, then I mean it.") was the same thing as "I say what I mean." (the original conditional statement "If I mean it, then I say it."), and was then thoroughly ridiculed by her Wonderland acquaintances.

0.8 Biconditional The conjunction

(6) $$(p \to q) \wedge (q \to p)$$

(where $\wedge$ is the logical *and*) is abbreviated into

(7) $$p \leftrightarrow q,$$

called a *biconditional statement*. Since $q \to p$ may be read 'p if q' and $p \to q$ may be read 'p only if q', the biconditional statement $p \leftrightarrow q$ is 'p *if and only if* q', often abbreviated into 'p *iff* q'. If p and q have the same truth value (*i.e.* either both are true or both are false), then the biconditional statement $p \leftrightarrow q$ is true; if p and q have opposite truth values, then $p \leftrightarrow q$ is false.

Mathematical Truth

> "Pure mathematics consists entirely of such asseverations as that, if such and such a proposition is true of *anything*, then such a such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true. … If our hypothesis is about *anything* and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be de-

fined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

— Bertrand Russell (1901)
Recent work on the principles of Mathematics

In mathematics, theorems (also propositions, lemmata, and corollaries) assert the truth of statements. Grammatically speaking, they should have as their subjects the statement (or the name of, or some other reference to, the statement), and as predicates the phrase 'is true' (or 'holds', or some similar such). For example, the concluding Rosen theorem in Section 9G of *LI* is

0.9 Theorem *There can be no closed path of efficient causation in a mechanism.*

(The word 'mechanism' has a very specific meaning in the Rosen lexicon:

0.10 Definition A natural system is a *mechanism* if and only if all of its models are simulable.

I shall have a lot more to say on this in Chapter 8.) Theorem 0.9 should be understood as

0.9' Theorem *'There can be no closed path of efficient causation in a mechanism.' is true.*

Or, what is the same,

0.9' Theorem *Theorem 0.9 is true.*

But, of course, this Theorem 0.9' really means

0.9" Theorem *Theorem 0.9' is true.*

Or, equivalently,

0.9" Theorem *"Theorem 0.9 is true." is true.*

This "statement about a statement" idea may, alas, be iterated *ad infinitum*, to

0.9 $^{\omega}$ Theorem *…… " " " " "Theorem 0.9 is true." is true " is true " is true " is true" ……*.

Lewis Carroll wrote about this hierarchical 'reasoning about reasoning' paradox in a witty dialogue *What the Tortoise said to Achilles* [1895]. Efficiency and pragmatism dictate the common practice that the predicate is implicitly assumed and hence usually omitted. A theorem, then, generally consists of just the statement itself, the truth of which it asserts.

0.11 Implication An *implication* is a *true* statement of the form

(8) " '$p \to q$' is true."

It is a statement about (the truth of) the conditional statement

(9) '$p \to q$'.

The implication (8) is denoted in formal logic by

(10) $p \Rightarrow q$,

which is read as 'p *implies* q'. When a conditional statement is expressed as a theorem in mathematics, *viz.*

Theorem *If* p, *then* q.

it is understood in the sense of (8), that it is an implication.

The difference between $\rightarrow$ and $\Rightarrow$, *i.e.* between a conditional statement and an implication, is that of syntax and semantics. Note that $p \rightarrow q$ is just a proposition in the predicate calculus, which may be true or false. But $p \Rightarrow q$ is a statement *about* the conditional statement $p \rightarrow q$, asserting that the latter is a true statement. In particular, when $p \Rightarrow q$, the situation that p is true and q is false (which is the only circumstance for which the conditional $p \rightarrow q$ is false) *cannot* occur.

0.12 Modus tollens Since a conditional statement and its contrapositive are equivalent, when $p \rightarrow q$ is true, $\neg q \rightarrow \neg p$ is also true. The contrapositive inference

(11) $$(p \Rightarrow q) \Rightarrow (\neg q \Rightarrow \neg p)$$

is itself an implication, called *modus tollens* in mathematical logic.

Most mathematical theorems are stated, or may be rewritten, as implications. The Rosen Theorem 0.9, for example, is $p \Rightarrow q$ with $p =$ 'N is a mechanism' and $q =$ 'there is no closed path of efficient causation in N', where N is a natural system. Stated explicitly, it is the

0.13 Theorem *If a natural system N is a mechanism, then there is no closed path of efficient causation in N.*

The equivalent contrapositive implication $\neg q \rightarrow \neg p$ is the

0.14 Theorem *If a closed path of efficient causation exists in a natural system N, then N cannot be a mechanism.*

0.15 Equivalence A true statement of the form

(12) " '$p \leftrightarrow q$' is true.",

which asserts the truth of a biconditional statement, is called an *equivalence*. It is denoted as

(13) $p \Leftrightarrow q$,

and is read as 'p and q are *equivalent*'. It is clear from the definitions that the equivalence (13) is equivalent to the conjunction

(14) $p \Rightarrow q$ and $q \Rightarrow p$.

When $p \Leftrightarrow q$, either both p and q are true, or both are false.

When a biconditional statement is expressed as a theorem in mathematics, *viz.*

Theorem *p if and only if q*.

it is understood in the sense of (12) that the biconditional statement $p \leftrightarrow q$ is in fact true, that it is the equivalence $p \Leftrightarrow q$.

0.16 Definition A *definition* is trivially a theorem — by definition, as it were. It is also often expressed as an equivalence, *i.e.*, with an 'if and only if' statement. See, for example, Definition 0.10 of 'mechanism'.

Occasionally a definition may be *stated* as an implication (*e.g.* Definition 0.3 of 'subset'), but in such cases the converse is *implied* (by convention, or, indeed, by definition). Stated otherwise, a definition is always an equivalence, whether it is expressed as such or not, between the term being defined and the defining conditions. Definition 0.3 is the implication $p \Rightarrow q$ where p = 'every element of A is an element of B' and q = 'set A is a *subset* of set B'. But since this is a definition, implicitly entailed is the converse $q \Rightarrow p$:

0.3' Definition If a set A is a *subset* of a set B then every element of A is an element of B.

So the definition is really

0.3'' Definition A set A is a *subset* of a set B *if and only if* every element of A is an element of B.

Note that this implicit entailment is *not* a contradiction to the fact, discussed above in 0.7, that a conditional statement is not logically equivalent to its converse. The propositions $p \to q$ and $q \to p$ will always remain logically distinct, and in general the implication $p \Rightarrow q$ says nothing about $q \Rightarrow p$. The previous paragraph only applies to definitions, and its syntax is

(15) '*If* $p \Rightarrow q$ *is a definition*,
then also $q \Rightarrow p$,
whence $p \Leftrightarrow q$.'

Necessity and Sufficiency

0.17 Modus ponens The law of inference

(16) 'If $p \Rightarrow q$ and p is true, then q is true.'

is called *modus ponens*.

This inference follows from the fact that when $p \Rightarrow q$, $p \to q$ is true; so the situation that p is true and q is false (the only circumstance for which the conditional $p \to q$ is false) cannot occur. Thus the truth of p predicates q. Incidentally, modus ponens is the 'theorem' that begins the propositional canon in Lewis Carroll's *What the Tortoise said to Achilles*

[1895]. Note that the truth of $p \to q$ is required for the truth of p to entail the truth of q. In a general (not necessarily true) conditional statement $p \to q$, the truth values of p and q are independent.

Because of its inferential entailment structure (that the truth of p is sufficient to establish the truth of q), the implication $p \Rightarrow q$ may also be read 'p is *sufficient* for q'. Contrapositively (hence equivalently), the falsehood of q is sufficient to establish the falsehood of p. In other words, if q is false, then p cannot possibly be true; *i.e.* the truth of q is necessary (although some additional true statements may be required) to establish the truth of p. Thus the implication $p \Rightarrow q$ may also be read 'q is *necessary* for p'. The equivalence $p \Leftrightarrow q$ (*i.e.* when 'p iff q' is true so that p and q predicate each other) may, therefore, be read 'p is *necessary and sufficient* for q'.

0.18 Membership The concepts of necessity and sufficiency are intimately related to the concept of subset. Definition 0.3" is the statement

(17) $$A \subset B \text{ iff } \forall x \left(x \in A\right) \Rightarrow \left(x \in B\right).$$

Stated otherwise, when A is a subset of B (or, what is the same, B *includes* A), membership in A is sufficient for membership in B, and membership in B is necessary for membership in A. Similarly, the Axiom of Extension 0.2 is the statement

(18) $$A = B \text{ iff } \forall x \left(x \in A\right) \Leftrightarrow \left(x \in B\right);$$

i.e. membership in A and membership in B are necessary and sufficient for each other.

The major principle of set theory is the

0.19 Axiom of Specification For any set U and any statement $p(x)$ about x, there exists a set P the elements of which are exactly those $x \in U$ for which $p(x)$ is true.

It follows immediately from the Axiom of Extension that the set P is determined uniquely. To indicate the way P is obtained from U and $p(x)$, the customary notation is

(19) $$P = \{x \in U : p(x)\}.$$

The '$p(x)$' in (19) is understood to mean "'$p(x)$' is true" (with the conventional omission of the predicate); it may also be read as 'x has the property p'.

For example, let **N** be the set of all natural systems, and let $s(N)$= 'all models of N are simulable'. Then one may denote the set of all mechanisms **M** (*cf.* Definition 0.10)] as

(20) $$\mathbf{M} = \{N \in \mathbf{N} : s(N)\}.$$

When the 'universal set' U is obvious from the context (or inconsequential), it may be dropped, and the notation (19) abbreviates to

(21) $$P = \{x : p(x)\}.$$

As a trivial example, a set A may be represented as

(22) $$A = \{x : x \in A\}.$$

0.20 Implication and Inclusion Statement (17) connects set inclusion with implication of the membership property. Analogously, if one property implies another, then the set specified by the former is a subset of the set specified by the latter (and conversely). Explicitly, if x has the property p

implies that x has the property q, *i.e.* if $\forall x\, p(x) \Rightarrow q(x)$, then $P = \{x : p(x)\}$ is a subset of $Q = \{x : q(x)\}$ (and conversely):

(23) $$P \subset Q \text{ iff } \forall x\, p(x) \Rightarrow q(x).$$

The equivalence (23) may be read as $P \subset Q$ if and only if p is sufficient for q, and also $P \subset Q$ if and only if q is necessary for p.

For example, let **N** be the set of all natural systems, let $t(N)$= 'there is no closed path of efficient causation in N', and let

(24) $$\mathbf{T} = \{N \in \mathbf{N} : t(N)\}.$$

Let **M** be the set of all mechanisms as specified in (20). Theorem 0.9 (the proof of which is the content of Chapter 9 of *LI*, and is given an alternate presentation later on in Chapter 8 of this monograph) is the statement

(25) $$\forall N \in \mathbf{N} \;\; s(N) \Rightarrow t(N),$$

whence equivalently

(26) $$\mathbf{M} \subset \mathbf{T}.$$

Complements

> "I think that it would be reasonable to say that no man who is called a philosopher really understands what is meant by the complementary descriptions."
>
> — Niels Bohr (1962)
> *Communication 1117*

"Some believe the Principle of Complementarity, but the rest of us do not."

— Anonymous

0.21 Definition The *relative complement* of a set A in a set B is the set of elements in B but not in A:

(27) $$B \sim A = \{x \in B : x \notin A\}.$$

When B is the 'universal set' U (of some appropriate universe under study, *e.g.* the set of all natural systems **N**), the set $U \sim A$ is denoted A^c, *i.e.*

(28) $$A^c = \{x \in U : x \notin A\},$$

and is called simply the *complement* of the set A. An element of U is either a member of A, or not a member of A, but not both. That is, $A \bigcup A^c = U$, and $A \bigcap A^c = \varnothing$.

The set specified by the property p, $P = \{x : p(x)\}$, has as its complement the set specified by the property $\neg p$; *i.e.*

(29) $$P^c = \{x : \neg p(x)\}.$$

In the predicate calculus, there are these

0.22 Laws of Quantifier Negation

(30) $$\neg \forall x \; p(x) \Leftrightarrow \exists x \; \neg p(x)$$

(31) $$\neg \exists x \; p(x) \Leftrightarrow \forall x \; \neg p(x)$$

The negation of the statement $s(N)$ = '*all* models of N are simulable' is thus $\neg s(N)$ = '*there exists* a model of N that is *not* simulable' This characterizes the collection of natural systems that are *not* mechanisms as those that have at least one nonsimulable model.

The predicate calculus also has this trivial tautology:

0.23 Discharge of Double Negation

(32) $\neg\neg p \Leftrightarrow p$

The negation of the statement $t(N)$= 'there is no closed path of efficient causation in N' is therefore $\neg t(N)$= 'there exists a closed path of efficient causation in N'. The equivalent contrapositive statement of (25) is hence

(33) $\forall N \in \mathbf{N}\ \neg t(N) \Rightarrow \neg s(N),$

which gives the

0.24 Theorem *If there exists a closed path of efficient causation in a natural system, then it has at least one model that is not simulable* (*whence it is not-a-mechanism*).

I shall explore the semantics of Theorems 0.9, 0.13, 0.14, and 0.24 (instead of just their sample syntax used in this prologue to illustrate principles of mathematical logic) in Chapter 8 *et seq*.

Neither More Nor Less

0.25 Nominalism

> "I don't know what you mean by 'glory'," Alice said.
>
> Humpty Dumpty smiled contemptuously. "Of course you don't — till I tell you. I meant 'there's a nice knock-down argument for you'!"
>
> "But 'glory' doesn't mean 'a nice knock-down argument'," Alice objected.
>
> "When *I* use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean — neither more nor less."
>
> "The question is," said Alice, "whether you *can* make words mean so many different things."
>
> "The question is," said Humpty Dumpty, "which is to be master — that's all."
>
> — Lewis Carroll (1871)
> *Through the Looking-Glass,*
> *and What Alice Found There*
> Chapter VI Humpty Dumpty

Humpty's point of view is known in philosophy as *nominalism*, the doctrine that universals or abstract concepts are mere names without any corresponding 'reality'. The issue arises because in order to perceive a particular object as belonging to a certain class, say 'organism', one must have a prior notion of 'organism'. Does the term 'organism', described by this prior notion, then have an existence independent of particular organisms? When a word receives a specific technical definition, does it have to reflect its prior notion, the common-usage sense of the word? Nominalism says no.

0.26 Semantic Equivocation A closely related issue is a fallacy of misconstrual in logic known as *semantic equivocation*. This fallacy is quite common, because words often have several different meanings, a condition known as *polysemy*. A polysemic word may represent any one of several concepts, and the semantics of its usage are context-dependent. Errors arise when the different concepts with different consequences are mixed together as one. For a word that has a technical definition in addition to its everyday meaning, non sequitur may result when the distinction is blurred.

Confusion often ensues from a failure to clearly understand that words mean "neither more nor less" than what they are defined to mean, not what they are perceived to mean. This happens even in mathematics, where terms are usually more precisely defined than in other subjects. The most notorious example is the term 'normal', which appears in numerous mathematical subject areas to define objects with specific properties. In almost all cases (*e.g.* normal vector, normal subgroup, normal operator), the normal subclass is nongeneric within the general class of objects; *i.e.* what is defined as 'normal' is anything but normal in the common-usage sense of 'standard, regular, typical'.

While it is not my purpose in this monograph to dwell into nominalism and semantic equivocation themselves, they do make occasional appearances in what follows as philosophical and logical undertones.

0.27 Structure 'Extreme' polysemous words, those having two current meanings that are opposites, are called *amphibolous*. For example, the word 'structure', which means 'a set of interconnecting parts of a thing' (its Latin root is *struere*, to build), has antonymous usage in biology and mathematics: 'concrete' in one, and 'abstract' in the other.

In biology, 'structure' means *material* structure, the constituent physicochemical parts. In our subject of relational biology, our slogan is 'function dictates structure'. Entailment relations within living systems are their most important characteristics.

In mathematics, on the other hand, 'structure' (as in *set-with-structure*) in fact means the *relations* defined on the object. A *structure* on a set is a collection of nullary, unary, binary, ternary ... operations satisfying as axioms a variety of identities between composite operations. Thus a partially ordered set (which I shall introduce in Chapter 1) is a set equipped with a binary operation $\leq$ having certain specified properties. A group is a set equipped with a binary (the group multiplication), a nullary (the unit element), and a unary (the inverse) operation, which together satisfy certain identities. A topological space's structure is a collection of its subsets (the open sets) with certain prescribed properties. And so forth. It is perhaps this 'relations are structure' concept in mathematics that inspired Nicolas Rashevsky on his foundation of relational biology.

0.28 Function We should also note the polysemy of the word 'function'. The Latin *functio* means 'performance'. An activity by which a thing fulfils a purpose, the common meaning of 'function', may be considered a performance. This is the word's biological usage, although the teleologic 'fulfils a purpose' sense is regularly hidden. (I shall have much more to say on this later.) A mathematical function may be considered as a set of operations that are performed on each value that is put into it. Leibniz first used the term function in the mathematical context, and Euler first used the notation $f(x)$ to represent a function, because the word begins with the letter f.

Since in mathematics 'function' has a synonym in 'mapping', in this book I shall use *mapping* for the mathematical entity (*cf.* Definition 1.3), and leave *function* to its biological sense.

PART I

Exordium

No one really understood music unless he was a scientist, her father had declared, and not just a scientist, either, oh, no, only the real ones, the theoreticians, whose language was mathematics. She had not understood mathematics until he had explained to her that it was the symbolic language of relationships. "And relationships," he had told her, "contain the essential meaning of life."

— Pearl S. Buck (1972)
The Goddess Abides
Part I

Equivalence relation is a fundamental building block of epistemology. The first book of the Robert Rosen trilogy is *Fundamentals of Measurement and Representation of Natural Systems* [*FM*]. It may equally well be entitled 'Epistemological Consequences of the Equivalence Relation'; therein one finds a detailed mathematical exposition on the equivalence relation and its linkage to *similarity*, the pre-eminent archetypal concept in all of science, in both the universes of formal systems and natural systems.

Equivalence relation is also a fundamental building block of mathematics. The concept of equivalence is ubiquitous. Many of the theorems in this book depend on the fact that the collection of equivalence relations on a set is a mathematical object known as a *lattice*. In this introductory Part I, I present a précis of the algebraic theory of lattices, with emphasis, of course, on the topics that will be of use to us later on. Some theorems will only be stated in this introduction without proofs. Their proofs may be found in books on lattice theory or universal algebra. The standard reference is *Lattice Theory* written by Garrett Birkhoff, a founder of the subject [Birkhoff 1967].

1

Praeludium: Ordered Sets

Mappings

1.1 Definition
(i) If X is a set, the *power set* $\mathcal{P}X$ of X is the family of all subsets of X.
(ii) Given two sets X and Y, one denotes by $X \times Y$ the set of all *ordered pairs* of the form (x, y) where $x \in X$ and $y \in Y$. The set $X \times Y$ is called the *product* (or *cartesian product*) of the sets X and Y.

1.2 Definition A *relation* is a set R of ordered pairs; i.e. $R \subset X \times Y$, or equivalently $R \in \mathcal{P}(X \times Y)$, for some sets X and Y.

The collection of *all* relations between two sets X and Y is thus the power set $\mathcal{P}(X \times Y)$.

1.3 Definition A *mapping* is a set f of ordered pairs with the property that, if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Note the requirement for a subset of $X \times Y$ to qualify it as a mapping is in fact quite a stringent one: most, i.e., *common*, members of $\mathcal{P}(X \times Y)$ do not have this property. A mapping is therefore a *special*, i.e., *nongeneric*, kind of relation. But genericity is not synonymous with importance: general relations and mappings are both fundamental mathematical objects of study.

1.4 Definition Let f be a mapping. One defines two sets, the *domain* of f and the *range* of f, respectively by

(1) $$\operatorname{dom}(f) = \{x : (x, y) \in f \text{ for some } y\}$$

and

(2) $$\operatorname{ran}(f) = \{y : (x, y) \in f \text{ for some } x\}.$$

Thus f is a subset of the product $\operatorname{dom}(f) \times \operatorname{ran}(f)$. If $\operatorname{ran}(f)$ contains exactly one element, then f is called a *constant mapping*.

Various words, such as 'function', 'transformation', and 'operator', are used as synonyms for 'mapping'. The mathematical convention is that these different synonyms are used to denote mappings having special types of sets as domains or ranges. Because these alternate names also have interpretations in biological terms, to avoid semantic equivocation, in this book I shall — unless convention dictates otherwise — use *mapping* (and often *map*) for the mathematical entity.

1.5 Remark The traditional conception of a mapping is that of something that assigns to each element of a given set a definite element of another given set. I shall now reconcile this with the formal definition given above. Let f be a mapping and let X and Y be sets. If $\operatorname{dom}(f) = X$ and $\operatorname{ran}(f) \subset Y$, whence f is a subset of $X \times Y$, one says that f is a *mapping of X into Y*, denoted by

(3) $$f : X \to Y,$$

and occasionally (mostly for typographical reasons) by

(4) $$X \xrightarrow{f} Y.$$

The collection of *all* mappings of X into Y is a subset of the power set $\mathcal{P}(X \times Y)$; this subset is denoted Y^X (see A.3, in the Appendix).

To each element $x \in X$, by Definition 1.3, there corresponds a *unique* element $y \in Y$ such that $(x, y) \in f$. Traditionally, y is called the *value of the mapping* f *at the element* x, and the relation between x and y is denoted by $y = f(x)$ instead of $(x, y) \in f$. Note that the $y = f(x)$ notation is only logically consistent when f is a mapping — for a general relation f, it is possible that $y \neq z$ yet both $(x, y) \in f$ and $(x, z) \in f$; if one were to write $y = f(x)$ and $z = f(x)$ in such a situation, then one would be led, by Euclid's Common Notion 1 (*cf.* 0.0 and also the Euclidean property 1.10(*e*) below), to the conclusion that $y = z$: a direct contradiction to $y \neq z$.

With the $y = f(x)$ notation, one has

(5) $$\operatorname{ran}(f) = \{y : y = f(x) \text{ for some } x\},$$

which may be further abbreviated to

(6) $$\operatorname{ran}(f) = \{f(x) : x \in \operatorname{dom}(f)\}.$$

One then also has

(7) $$f = \{(x, f(x)) : x \in X\}.$$

From this last representation, we observe that when $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$ (where $\mathbb{R}$ is the set of real numbers), my formal definition of a mapping coincides with that of the 'graph of f' in elementary mathematics.

Sometimes it is useful to trace the path of an element as it is mapped. If $a \in X$, $b \in Y$, and $b = f(a)$, one uses the 'maps to' arrow (note the short vertical line segment at the tail of the arrow) and writes

(8) $$f : a \mapsto b.$$

Note that this 'element-chasing' notation of a mapping *in no way* implies that there is somehow only one element a in the domain $\operatorname{dom}(f) = \{a\} = X$, mapped by f to the only element b in the range $\operatorname{ran}(f) = \{b\} \subset Y$. a is a symbolic representation of *variable* elements in the domain of f, while b denotes its corresponding image $b = f(a)$ defined by f. The notation $f : a \mapsto b$ is as general as $f : X \to Y$, the former emphasizing the elements while the latter emphasizing the sets. One occasionally also uses the 'maps to' arrow to define the mapping f itself:

(9) $$x \mapsto f(x).$$

1.6 Definition Let f be a mapping of X into Y. If $E \subset X$, $f(E)$, the *image* of E under f, is defined to be the set of all elements $f(x) \in Y$ for $x \in E$; i.e.,

(10) $$f(E) = \{f(x) : x \in E\} \subset Y.$$

In this notation, $f(X)$ is the range of f.

1.7 Definition If f is a mapping of X into Y, the set Y is called the *codomain* of f, denoted by $\operatorname{cod}(f)$.

The range $f(X) = \operatorname{ran}(f)$ is a subset of the codomain $Y = \operatorname{cod}(f)$, but they need not be equal. When they are, i.e. when $f(X) = Y$, one says that f is a *mapping of X onto Y*, and that $f: X \to Y$ is *surjective* (or is a *surjection*). Note that every mapping maps onto its range.

1.8 Definition If $E \subset Y$, $f^{-1}(E)$ denotes the set of all $x \in X$ such that $f(x) \in E$:

$$(11) \qquad f^{-1}(E) = \{x : f(x) \in E\} \subset X,$$

and is called the *inverse image* of E under f. If $y \in Y$, $f^{-1}(\{y\})$ is abbreviated to $f^{-1}(y)$, so it is the set of all $x \in X$ such that $f(x) = y$.

Note that $f^{-1}(y)$ may be the empty set, or may contain more than one element. If, *for each* $y \in Y$, $f^{-1}(y)$ consists of *at most* one element of X, then f is said to be a *one-to-one* (*1-1*, also *injective*) *mapping* of X into Y. Other commonly used names are '$f: X \to Y$ is an *injection*', and '$f: X \to Y$ is an *embedding*'. This may also be expressed as follows: f is a one-to-one mapping of X into Y provided $f(x_1) \neq f(x_2)$ whenever $x_1, x_2 \in X$ and $x_1 \neq x_2$.

If $A \subset X$, then the mapping $i: A \to X$ defined by $i(x) = x$ for all $x \in A$ is a one-to-one mapping of A into X, called the *inclusion map* (of A in X).

If $f: X \to Y$ is both one-to-one and onto, i.e. both injective and surjective, then f is called *bijective* (or is a *bijection*), and that it establishes a *one-to-one correspondence* between the sets X and Y.

While the domain and range of f are specified by f as in Definition 1.4, the codomain is not *yet* uniquely determined — all that is required so far is that it contains the range of f as a subset. One needs to invoke a category theory axiom (see Appendix: Axiom A.1(c1)), and *assigns* to each mapping f a unique set $Y = \operatorname{cod}(f)$ as its codomain.

Equivalence Relations

Recall Definition 1.2 that a relation R is a set of ordered pairs, $R \subset X \times Y$ for some sets X and Y. Just as for mappings, however, there are traditional terminologies for relations that were well established before this formal definition. I shall henceforth use these traditional notations, and also concentrate on relations with $X = Y$.

1.9 Definition If X is a set and $R \subset X \times X$, one says that R is a *relation on* X, and write $x\,R\,y$ instead of $(x, y) \in R$.

1.10 Definition A relation R on a set X is said to be
(r) *reflexive* if for all $x \in X$, $x\,R\,x$;
(s) *symmetric* if for all $x, y \in X$, $x\,R\,y$ implies $y\,R\,x$;
(a) *antisymmetric* if for all $x, y \in X$, $x\,R\,y$ and $y\,R\,x$ imply $x = y$;
(t) *transitive* if for all $x, y, z \in X$, $x\,R\,y$ and $y\,R\,z$ imply $x\,R\,z$;
(e) *Euclidean* if for all $x, y, z \in X$, $x\,R\,z$ and $y\,R\,z$ imply $x\,R\,y$.

1.11 Definition A relation R on a set X is called an *equivalence relation* if it is reflexive, symmetric, and transitive; *i.e.* if it satisfies properties (r), (s), and (t) in Definition 1.10 above.

The *equality* (or *identity*) relation I on X, defined by $x\,I\,y$ if $x = y$, is an equivalence relation. As a subset of $X \times X$, I is the *diagonal* $I = \{(x, x) : x \in X\}$. Because of reflexivity (r), any equivalence relation $R \subset X \times X$ must have $(x, x) \in R$ for all $x \in X$; thus $I \subset R$. The *universal* relation U on X, defined by $x\,U\,y$ if $x, y \in X$, is also an equivalence relation. Since $U = X \times X$, for any equivalence relation R on X one has $R \subset U$.

The equality relation I is Euclidean. Indeed, when $R = I$, the Euclidean property (e) is precisely Euclid's Common Notion 1 (*cf.* 0.0):

"Things equal to the same thing are also equal to one another." One readily proves the following

1.12 Theorem *If a relation is Euclidean and reflexive, it is also symmetric and transitive* (*hence it is an equivalence relation*).

1.13 Definition Let R be an equivalence relation on X. For each $x \in X$ the set

$$(12) \qquad [x]_R = \{ y \in X : xRy \}$$

is called the *equivalence class of* x *determined by* R, or the *R-equivalence class of* x. The collection of all equivalence classes determined by R is called the *quotient set* of X under R, and is denoted by X/R; i.e.

$$(13) \qquad X/R = \{ [x]_R : x \in X \}.$$

1.14 All or Nothing By reflexivity (*r*), one has $x \in [x]_R$ for all $x \in X$. The equivalence classes determined by R are therefore all nonempty, and

$$(14) \qquad X = \bigcup_{x \in X} [x]_R .$$

Also, the members of X/R are pairwise disjoint. For suppose $x, y \in X$ and $[x]_R \cap [y]_R \neq \varnothing$. Choose $z \in [x]_R \cap [y]_R$, whence xRz and yRz. By symmetry (*s*) and transitivity (*t*) one has yRx. Now if $w \in [x]_R$, xRw, so together with yRx just derived, transitivity (*t*) gives yRw, whence $w \in [y]_R$. This shows that $[x]_R \subset [y]_R$. By symmetry (of the argument) $[y]_R \subset [x]_R$, and consequently $[x]_R = [y]_R$.

Stated otherwise, every element of X belongs to exactly one of the equivalence classes determined by R.

One usually uses the notation $\leq$ instead of R when it is a partial order.

1.21 Definition A *partially ordered set* (often abbreviated as *poset*) is an ordered pair $\langle X, \leq \rangle$ in which X is a set and $\leq$ is a partial order on X.

When the partial order $\leq$ is clear from the context, one frequently for simplicity omits it from the notation, and denote $\langle X, \leq \rangle$ by the underlying set X. Each subset of a poset is itself a poset under the same partial order.

1.22 Definition Let $\langle X, \leq \rangle$ be a poset and $x, y \in X$. If $x \leq y$, one says 'x is *less than or equal to* y', or 'y is *greater than or equal to* x', and write '$y \geq x$'. One also writes '$x < y$' (and '$y > x$') for '$x \leq y$ and $x \neq y$', whence reads 'x is *less than* y' (and 'y is *greater than* x').

The simplest example of a partial order is the equality relation I. The equality relation, as we saw above, is also an equivalence relation, and it is in fact the only relation which is *both* an equivalence relation and a partial order.

The relation $\leq$ on the set of all integers $\mathbb{Z}$ is an example of a partial order. As another example, the *inclusion* relation $\subset$ is a partial order on the power set $\mathcal{P}A$ of a set A.

Morphisms in the category of posets are order-preserving mappings:

1.23 Definition A mapping f from a poset $\langle X, \leq_X \rangle$ to a poset $\langle Y, \leq_Y \rangle$ is called *order-preserving*, or *isotone*, if

(16) $$x \leq_X y \text{ in } X \quad \text{implies} \quad f(x) \leq_Y f(y) \text{ in } Y.$$

(The somewhat awkward symbols $\leq_X$ and $\leq_Y$ are meant to indicate the partial orders on X and Y may be different. With this clearly understood, I shall now simplify the notation for the next part of the definition.) Two posets X and Y are *isomorphic*, written $X \cong Y$, if there exists a bijective map $f : X \to Y$ such that both f and its inverse $f^{-1} : Y \to X$ are order-preserving; i.e.

(17) $\quad x \le y$ in X iff $f(x) \le f(y)$ in Y.

Any poset may be represented as a collection of sets ordered by inclusion:

1.24 Theorem *Let $\langle X, \le \rangle$ be a poset. Define $f : X \to \mathcal{P}X$, for $x \in X$, by*

(18) $\quad f(x) = \{ y \in X : y \le x \}$.

Then X is isomorphic to the range of f ordered by set inclusion $\subset$; i.e. $\langle X, \le \rangle \cong \langle f(X), \subset \rangle$.

1.25 Definition The *converse* of a relation R is the relation $\breve{R}$ such that $x \breve{R} y$ if $y R x$.

Thus the converse of the relation 'is included in' $\subset$ is the relation 'includes' $\supset$; the converse of 'less than or equal to' $\le$ is 'greater than or equal to' $\ge$. A simple inspection of properties (*r*), (*a*), and (*t*) leads to the

1.26 Duality Principle *The converse of a partial order is itself a partial order.*

The *dual* of a poset $X = \langle X, \le \rangle$ is the poset $\tilde{X} = \langle X, \ge \rangle$ defined by the converse partial order. Definitions and theorems about posets are dual in pairs (whenever they are not self-dual). If any theorem is true for all posets, then so is its dual.

1.27 Definition Let $\le$ be a partial order on a set X and let $A \subset X$. The subset A is *bounded above* if there exists $x \in X$ such that $a \le x$ for all

$a \in A$; such $x \in X$ is called an *upper bound* for A. An upper bound x for A is called the *supremum* for A if $x \leq y$ for all upper bounds y for A. A subset A can have at most one supremum (hence the article *the*), and if it exists it is denoted by $\sup A$.

The terms *bounded below*, *lower bound*, and *infimum* (notation $\inf A$) are defined analogously. A set that is bounded above and bounded below is called *bounded*.

1.28 The Greatest and the Least Every element of X is an upper bound for the empty set $\varnothing \subset X$. So if $\varnothing$ has a supremum in X, then $\sup \varnothing$ is an element such that $\sup \varnothing \leq y$ for all $y \in X$; such an element (if it exists) is called the *least element* of X, and this is the element $\sup \varnothing = \inf X$.

Dually, $\inf \varnothing$, if it exists, is an element such that $y \leq \inf \varnothing$ for all $y \in X$; such an element is called the *greatest element* of X, and this is the element $\inf \varnothing = \sup X$.

Let A be a set and consider the poset $\langle \mathfrak{P}A, \subset \rangle$. Each subset $\mathfrak{S} \subset \mathfrak{P}A$ (i.e. each family of subsets of A) is bounded above (trivially by the set A) and bounded below (trivially by the empty set $\varnothing$), hence bounded. A subset B of A (i.e. $B \in \mathfrak{P}A$) is an upper bound for $\mathfrak{S}$ if and only if $\bigcup_{S \in \mathfrak{S}} S \subset B$, and a lower bound for $\mathfrak{S}$ if and only if $B \subset \bigcap_{S \in \mathfrak{S}} S$. Thus $\bigcup_{S \in \mathfrak{S}} S = \sup \mathfrak{S}$ and $\bigcap_{S \in \mathfrak{S}} S = \inf \mathfrak{S}$. The least element of $\langle \mathfrak{P}A, \subset \rangle$ is $\varnothing$, and the greatest element of $\langle \mathfrak{P}A, \subset \rangle$ is A.

The greatest and least elements of a poset X are only considered to 'exist' if they are members of X. It is important to note, however, that an upper bound, a lower bound, the supremum, and the infimum (if any exists) for a subset $A \subset X$ are only required to be elements of the original poset X. They may or may not be in the subset A itself. In the example in the previous paragraph, $\bigcup_{S\in\mathfrak{S}} S = \sup\mathfrak{S}$ and $\bigcap_{S\in\mathfrak{S}} S = \inf\mathfrak{S}$ are members of $\mathfrak{P}A$, but they are not necessarily members of $\mathfrak{S}$.

Even in cases where there is no greatest or least element, there may be elements in a poset that have no other elements greater than or less than they are:

1.29 Definition Let $\leq$ be a partial order on a set X and let $A \subset X$. An element $x \in A$ is *maximal* if whenever $y \in A$ and $x \leq y$ one has $x = y$. Stated otherwise, $x \in A$ is maximal if $x < y$ for no $y \in A$. Dually, an element $x \in A$ is *minimal* if whenever $y \in A$ and $y \leq x$ one has $x = y$, or equivalently if $y < x$ for no $y \in A$.

Note that maximal and minimal elements of A are required to be members of A.

The greatest element (if it exists) must be maximal, and the least element (if it exists) must be minimal. But the converse is not true. As an example, let A be the three-element set $\{1,2,3\}$. Its power set is

$$\mathfrak{P}A = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},A\},$$

partially ordered by $\subset$. The element A is the greatest element (hence a maximal element) of this poset $\langle \mathfrak{P}A,\subset\rangle$, and the element $\varnothing$ is the least element (hence a minimal element). Now consider $\mathfrak{S} \subset \mathfrak{P}A$ with

$$\mathfrak{S} = \{\{1\},\{1,2\},\{1,3\}\}.$$

$\langle \mathfrak{S},\subset\rangle$ is a poset in its own right. One has

$$\sup\mathfrak{S} = \bigcup_{s\in\mathfrak{S}} S = \{1\}\cup\{1,2\}\cup\{1,3\} = \{1,2,3\} = A$$

and

$$\inf \mathfrak{S} = \bigcup_{s \in \mathfrak{S}} S = \{1\} \cap \{1,2\} \cap \{1,3\} = \{1\}.$$

So both $\sup \mathfrak{S}$ and $\inf \mathfrak{S}$ exist in $\mathcal{P}A$, but $\sup \mathfrak{S} \notin \mathfrak{S}$ while $\inf \mathfrak{S} \in \mathfrak{S}$. $\langle \mathfrak{S}, \subset \rangle$ has no greatest element, but both $\{1,2\}$ and $\{1,3\}$ are maximal elements. $\langle \mathfrak{S}, \subset \rangle$ has $\{1\}$ as its least element (which is therefore also a minimal element).

1.30 Theorem *Any (nonempty) finite subset of a poset has minimal and maximal elements.*

PROOF Let $\langle X, \le \rangle$ be a poset and $A = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of X. Define $m_1 = x_1$, and for $k = 2, \ldots, n$, define

$$m_k = \begin{cases} x_k & \text{if } x_k < m_{k-1} \\ m_{k-1} & \text{otherwise} \end{cases}. \tag{19}$$

Then m_n will be minimal. Similarly, A has a maximal element. □

1.31 Poset as Category A partially ordered set $\langle X, \le \rangle$ may *itself* be considered as a category, in which the objects are elements of X, and a hom-set $X(x, y)$ for $x, y \in X$ has either a single element or is empty, according to whether $x \le y$ or not. Product in this category corresponds to infimum, and coproduct corresponds to supremum. This is a single-poset-as-a-category, and is completely different from 'the category of all posets and isotone mappings' considered in 1.23 above. Note the analogy to a single-set-as-a-category (i.e. a discrete category) versus the category **Set** of all sets and mappings (see A.2 and A.3).

2

Principium: The Lattice of Equivalence Relations

Lattices

2.1 Definition A *lattice* is a nonempty partially ordered set $\langle L, \leq \rangle$ in which each pair of elements x, y has a supremum and an infimum in L. They are denoted by $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$, and are often called, respectively, the *join* and *meet* of x and y.

For any elements x, y of a lattice,

(1) $$x \leq y \quad \Leftrightarrow \quad x \vee y = y \quad \Leftrightarrow \quad x \wedge y = x.$$

If the lattice L (as a poset) has a least element 0, then $0 \wedge x = 0$ and $0 \vee x = x$ for all $x \in L$. If it has a greatest element 1, then $x \wedge 1 = x$ and $x \vee 1 = 1$ for all $x \in L$.

It is trivially seen from condition (1), known as *consistency*, that every totally ordered set is a lattice, in which $x \vee y$ is simply the larger and $x \wedge y$ is the smaller of x and y. The poset $\langle \text{P}A, \subset \rangle$, of the power set of a set A with the partial order of inclusion, is also a lattice; for any $X, Y \in \text{P}A$, $X \vee Y = X \bigcup Y$ and $X \wedge Y = X \bigcap Y$.

The families of open sets and closed sets, respectively, of a topological space are both lattices. In these lattices the partial orders, joins, and meets are the same as those for the power set lattice.

The collection of all subgroups of a group G is a lattice. The partial order $\leq$ is set-inclusion restricted to subgroups, i.e. the relation 'is a subgroup of'. For subgroups H and K of G, $H \wedge K = H \bigcap K$, but $H \vee K$ is the smallest subgroup of G containing H and K (which is generally not their set-theoretic union).

A lattice may also be regarded as a set with two binary operators, $\vee$ and $\wedge$, *i.e.* the triplet $\langle L, \vee, \wedge \rangle$. Again, for simplicity of notation, we often abbreviate to the underlying set and denote the lattice as L. The two operators satisfy a number of laws that are similar to the laws of addition and multiplication, and these laws may be used to give an alternative definition of lattices.

2.2 Theorem *Let L be a lattice, then for any $x, y, z \in L$,*

(*a*) [*associative*] $x \vee (y \vee z) = (x \vee y) \vee z$, $x \wedge (y \wedge z) = (x \wedge y) \wedge z$;

(*b*) [*commutative*] $x \vee y = y \vee x$, $x \wedge y = y \wedge x$;

(*c*) [*absorptive*] $x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$;

(*d*) [*idempotent*] $x \vee x = x$, $x \wedge x = x$.

Conversely, if L is a set with two binary operators $\vee$ and $\wedge$ satisfying (*a*)–(*c*), *then* (*d*) *also holds, and a partial order may be defined on L by the rule*

(*e*) $x \leq y$ *if and only if* $x \vee y = y$

[*whence if and only if $x \wedge y = x$*]. *Relative to this ordering, L is a lattice such that $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$.*

We have already seen the duality principle for partial orders in 1.26. As one may deduce from Theorem 2.2, this duality is expressed in lattices by interchanging $\vee$ and $\wedge$ (hence interchanging $\leq$ and $\geq$), resulting in its *dual*. Any theorem about lattice remains true if the join and meet are interchanged.

2.3 Duality Principle *The dual of a lattice is itself a lattice*.

2.4 The Greatest and the Least The greatest element of a poset (if it exists) must be maximal, and the least element (if it exists) must be minimal. I illustrated with an example in 1.29 that the converse is not necessarily true. But for a lattice, one has

Theorem *In a lattice, a maximal element is the greatest element* (*and hence unique*)*; dually, a minimal element is the least element* (*and hence unique*).

PROOF Let x_1 and x_2 be two maximal elements. Their join $x_1 \vee x_2$ is such that $x_1 \leq x_1 \vee x_2$ and $x_2 \leq x_1 \vee x_2$ (by definition of $\vee$ as the supremum). Because x_1 is maximal, it cannot be less than another element, so $x_1 \leq x_1 \vee x_2 \quad \Rightarrow \quad x_1 = x_1 \vee x_2$; similarly, because x_2 is maximal, $x_2 \leq x_1 \vee x_2 \quad \Rightarrow \quad x_2 = x_1 \vee x_2$. Therefore $x_1 = x_2$. Thus there can only be one maximal element.

Now let x be the only maximal element, and y be an arbitrary element of the lattice. One must have $x \leq y \vee x$ by definition of $\vee$; but x is maximal, so $x \leq y \vee x \quad \Rightarrow \quad x = y \vee x$, whence $y \leq x$ (by property 2.2(e) above). Thus x is the greatest element. □

Note that this theorem does not say a lattice necessarily has the greatest and the least elements, only that *if* a maximal (respectively, minimal) element exists, *then* it is the greatest (respectively, least).

2.5 Inequalities *Let L be a lattice, and let $x, y, z \in L$. Then*

(a) [*isotone*] *if $x \le z$, then $x \wedge y \le y \wedge z$ and $x \vee y \le y \vee z$;*

(b) [*distributive*] $x \wedge (y \vee z) \ge (x \wedge y) \vee (x \wedge z)$ *and*

$$x \vee (y \wedge z) \le (x \vee y) \wedge (x \vee z);$$

(c) [*modular*] *if $x \le z$, then $x \vee (y \wedge z) \le (x \vee y) \wedge z$.*

While any subset of a poset is again a poset under the same partial order, a subset of a lattice need not be a lattice (because $x \vee y$ or $x \wedge y$ may not be members of the subset even if x and y are). So one needs to make the explicit

2.6 Definition A *sublattice* of a lattice L is a subset M which is closed under the operators $\vee$ and $\wedge$ of L; i.e. $M \subset L$ is a sublattice if $x \vee y \in M$ and $x \wedge y \in M$ for all $x, y \in M$.

The empty set is a sublattice, and so is any singleton subset. Let $a, b \in L$ and $a \le b$. The (*closed*) *interval* is the subset $[a, b] \subset L$ consisting of all elements $x \in L$ such that $a \le x \le b$. An interval $[a, b]$ need not be a chain (Definition 1.33), but it is always a sublattice of L, and it has the least element a and the greatest element b.

Let A be a set and let $*$ be a fixed element of A, called the *base point*. A *pointed subset of A* is a subset X of A such that $* \in X$ (*cf.* Example A.6(ii) in the Appendix). The *pointed power set* $\mathcal{P}_* A$ is the subset of the power set $\mathcal{P}A$ containing all pointed subsets of A; *i.e.* $\mathcal{P}_* A = \{X \subset A : * \in X\}$. $\mathcal{P}_* A$ is a sublattice of $\mathcal{P}A$.

Note that it is possible for a subset of a lattice L to be a lattice without being a sublattice of L. For example, as we saw above, the collection $\Sigma(G)$ of all subgroups of a group G is a lattice. When G is considered a set (forgetting the group structure), $\mathcal{P}G$ is a lattice with $\vee = \bigcup$ and $\wedge = \bigcap$. $\Sigma(G)$ is a subset of $\mathcal{P}G$, but it is not a sublattice of $\mathcal{P}G$.

While $\Sigma(G)$ and $\mathcal{P}G$ have the same partial order $\leq = \subset$ and the same meet operator $\wedge = \bigcap$, the join operator $\vee$ of $\Sigma(G)$ is not inherited from that of $\mathcal{P}G$.

For another example, let X be a set, whence the power set $\mathcal{P}(X \times X)$ is a lattice with $\vee = \bigcup$ and $\wedge = \bigcap$. A relation on X is a subset of $X \times X$, so the collection $\mathcal{Q}X$ of all equivalence relations on X is a subset of $\mathcal{P}(X \times X)$. As we shall soon see, $\mathcal{Q}X$ is again a lattice, but not usually a sublattice of $\mathcal{P}(X \times X)$. In particular, the union of two equivalence relations need not be an equivalence relation. $\mathcal{Q}X$ is another example of a lattice with a join operator different from the standard set-theoretic union.

A morphism in the category of lattices is defined in the obvious structure-preserving fashion:

2.7 Lattice Homomorphism A mapping f from a lattice L to a lattice L' is called a (*lattice*) *homomorphism* if for all $x, y \in L$

(2) $$f(x \vee y) = f(x) \vee f(y) \quad \text{and} \quad f(x \wedge y) = f(x) \wedge f(y).$$

A lattice homomorphism preserves the ordering: $x \leq y \Rightarrow f(x) \leq f(y)$. But not every order-preserving mapping (i.e. poset homomorphism) between lattices is a lattice homomorphism.

2.8 Lemma *If f is a homomorphism from a lattice L into a lattice L', then the image $f(L)$ is a sublattice of L'.*

If a lattice homomorphism $f : L \to L'$ is one-to-one and onto, then f is called an *isomorphism* of L onto L', and the two lattices are said to be *isomorphic*. If $f : L \to L'$ is one-to-one, then f is an *embedding*, L and (where S° denotes the *interior* of the set S) are the join and meet that make $\mathbb{O}$ into a complete lattice. Similarly, for $\{G_a : a \in A\} \subset \mathbb{C}$, the operations

$$(5) \qquad \bigvee_{a \in A} G_a = \left(\bigcup_{a \in A} G_a \right)^- \quad \text{and} \quad \bigwedge_{a \in A} G_a = \bigcap_{a \in A} G_a$$

(where S^- denotes the *closure* of the set S) are the join and meet that make $\mathbb{C}$ into a complete lattice. [Note that when the index set A is finite, the $(\ )^\circ$ and $(\ )^-$ of the definitions are redundant, and these new $\vee$ and $\wedge$ become identical to set-theoretic union and intersection respectively.] This is another example that shows it is possible for a subset of a lattice L to be a lattice without being a sublattice of L. With the operators defined as in (4) and (5), both $\mathbb{O}$ and $\mathbb{C}$ are themselves lattices, and they are both subsets of $\mathbb{P}X$. But neither is a sublattice of $\mathbb{P}X$, because in each case, the partial order, join, and meet are not *all* identical to the $\subset$, $\bigcup$, and $\bigcap$ of $\mathbb{P}X$.

The Lattice $\mathbb{Q}X$

Let X be a set and let $\mathbb{Q}X$ denote the collection of all equivalence relations on X. A relation on X is a subset of $X \times X$, so $\mathbb{Q}X$ is a subset of $\mathbb{P}(X \times X)$. An equivalence relation as a subset of $X \times X$ has a very special structure (Lemma 1.19), so an arbitrary subset of an equivalence relation is not necessarily itself an equivalence relation. The partial order $\subset$ of set inclusion, *when restricted to* $\mathbb{Q}X$, implies more. When two equivalence relations $R_1, R_2 \in \mathbb{Q}X$ are such that $R_1 \subset R_2$, it means that in fact every R_1-equivalence class is a subset of some R_2-equivalence class. This also means, indeed, that the blocks in the partition defined by R_1 are obtained by further partitioning the blocks in the partition defined by R_2. Stated otherwise, the blocks of R_2 are obtained from those of R_1 by taking set-theoretic unions of them. I shall henceforth use the notation $R_1 \le R_2$

when $R_1, R_2 \in \mathfrak{Q}X$ are such that $R_1 \subset R_2$. An alternate description of $R_1 \le R_2$ is

2.13 Definition Let R_1 and R_2 be equivalence relations on a set X. One says that R_1 *refines* R_2 (and that R_1 is a *refinement* of R_2) if for all $x, y \in X$,

(6) $$x R_1 y \quad \Rightarrow \quad x R_2 y.$$

When R_1 refines R_2, i.e. when $R_1 \le R_2$, one says that R_1 *is finer than* R_2, and that R_2 *is coarser than* R_1.

One may verify that the relation of refinement on $\mathfrak{Q}X$ is a partial order. Thus

2.14 Theorem $\langle \mathfrak{Q}X, \le \rangle$ *is a partially ordered set.*

The equality relation I is the least element, and the universal relation U is the greatest element in the poset $\langle \mathfrak{Q}X, \le \rangle$. Stated otherwise, the equality relation I, which partitions X into a collection of singleton sets, is the finest equivalence relation on X; the universal relation U, which has one single partition that is X itself, is the coarsest equivalence relation on X. Contrast this with the fact that $\varnothing$ is the least element in $\langle \mathfrak{P}(X \times X), \subset \rangle$, while the largest element is the same $U = X \times X$.

2.15 Definition Let R_1 and R_2 be equivalence relations on a set X. Their *meet* $R_1 \wedge R_2$ is defined as

(7) $$x(R_1 \wedge R_2)y \quad \text{iff} \quad x R_1 y \text{ and } x R_2 y.$$

It is trivial to verify that $R = R_1 \wedge R_2$ is an equivalence relation on X, and that R refines both R_1 and R_2, i.e.

(8) $R \leq R_1$ and $R \leq R_2$,

and is the coarsest member of $\mathbb{Q}X$ with this property. In other words,

(9) $R = R_1 \wedge R_2 = \inf\{R_1, R_2\}$.

One also has

2.16 Lemma *The equivalence classes of $R_1 \wedge R_2$ are obtained by forming the set-theoretic intersection of each R_1-equivalence class with each R_2-equivalence class. As subsets of $\mathbb{P}(X \times X)$, $R_1 \wedge R_2 = R_1 \cap R_2$.*

Since the collection of equivalence classes form a partition, the R_1-class and R_2-class that intersect to form $R_1 \wedge R_2$-class are uniquely determined.

The definition of meet may easily be extended to an arbitrary index set A and a collection of equivalence relations $\{R_a : a \in A\}$:

(10) $x\left(\bigwedge_{a \in A} R_a\right)y$ iff $x\,R_a\,y$ for all $a \in A$.

And one has

(11) $\bigwedge_{a \in A} R_a = \inf\{R_a : a \in A\}$.

The set-theoretic union of two equivalence relations does not necessarily have the requisite special structure as a subset of $X \times X$ (Lemma 1.19) to make it an equivalence relation. The join has to be defined thus:

2.17 Definition Let R_1 and R_2 be equivalence relations on a set X. Their *join* $R_1 \vee R_2$ is defined as follows: $x(R_1 \vee R_2)y$ iff there is a finite sequence of elements $x_1, \ldots, x_n \in X$ such that

(12) $x\,R_1 x_1,\ x_1\,R_2 x_2,\ x_2\,R_1 x_3,\ \ldots, x_n\,R_1 y$.

One readily verifies that $R = R_1 \vee R_2$ is an equivalence relation on X, and that R is refined by both R_1 and R_2, i.e.

(13) $R_1 \le R$ and $R_2 \le R$,

and is the finest member of $\mathbb{Q}X$ with this property. In other words,

(14) $R = R_1 \vee R_2 = \sup\{R_1, R_2\}$.

One concludes from (13) that, as subsets of $\mathbb{P}(X \times X)$, $R_1 \subset R_1 \vee R_2$ and $R_2 \subset R_1 \vee R_2$, whence $R_1 \bigcup R_2 \subset R_1 \vee R_2$. The set (and relation) $R_1 \vee R_2$ is called the *transitive closure* of the union $R_1 \bigcup R_2$.

For an *arbitrary* index set A and a collection of equivalence relations $\{R_a : a \in A\}$, the definition of the join $\bigvee_{a\in A} R_a$ (i.e. the transitive closure of the union $\bigcup_{a\in A} R_a$), that corresponds to the binary join in (12), is: $x\left(\bigvee_{a\in A} R_a\right)y$ iff there exist a *finite* sequence of elements $x_1,\ldots,x_n \in X$ and indices $a_1,\ldots,a_n \in A$ such that

(15) $x\,R_{a_1} x_1,\ x_1\,R_{a_2} x_2,\ x_2\,R_{a_3} x_3,\ \ldots, x_n\,R_{a_n} y$.

With the meet and join as defined in 2.15 and 2.17, $\mathbb{Q}X$ is a lattice. In fact,

2.18 Theorem $\mathbb{Q}X$ *is a complete lattice.*

Because of the one-to-one correspondence between equivalence relations and partitions (Lemma 1.18), any sublattice of the lattice of equivalence relations is also called a *partition lattice*.

Mappings and Equivalence Relations

2.19 Definition Given a mapping $f : X \to Y$, one calls two elements $x_1, x_2 \in X$ *f-related* when $f(x_1) = f(x_2)$, and denotes this relation by R_f; i.e.

$$x_1 R_f x_2 \quad \text{iff} \quad f(x_1) = f(x_2). \tag{16}$$

Then R_f is an equivalence relation on X, whence the equivalence classes determined by R_f form a partition of X. f is a constant mapping on each R_f-equivalence class. R_f is called the *equivalence relation on* X *induced by* f, and f is called a *generator* of this equivalence relation.

The equivalence relation induced on a set X by a constant mapping is the universal relation U (with only one single partition block which is all of X). The equivalence relation induced on a set X by a one-to-one mapping is the equality relation I (with each partition block a singleton set).

Any mapping with domain X induces an equivalence relation on X. It is a very important fact that *all* equivalence relations on X are of this type:

2.20 Theorem *If R is an equivalence relation on* X, *then there is a mapping* f *with domain* X *such that* $R = R_f$.

PROOF Consider the mapping from X to the quotient set of X under R, $\pi : X \to X/R$, that maps an element of X to its equivalence class; i.e.

$$\pi(x) = [x]_R \quad \text{for} \quad x \in X. \tag{17}$$

This mapping π is called the *natural mapping* (*projection*) of X onto X/R, and has the obvious property that $R_\pi = R$. □

2.23 Definition Let X be a set. An *observable* of X is a mapping with domain X. The collection of all observables of X, i.e. the union of Y^X for all **Set**-objects Y, may be denoted $\bullet^X$.

2.24 Equivalent Observables We just saw that a mapping f with domain X induces an equivalence relation $R_f \in \mathbb{Q}X$. Dually, equivalence relations on a set X induce an (*algebraic*) *equivalence* relation $\sim$ on the set of all mappings with domain X (i.e. on the set $\bullet^X$ of observables of X), as follows. If f and g are two mappings with domain X, define $f \sim g$ if $R_f = R_g$, i.e. if and only if

(20) $$f(x) = f(y) \quad \Leftrightarrow \quad g(x) = g(y) \qquad \text{for all } x, y \in X.$$

This means the equivalence relations induced by f and g partition their common domain the same way. Stated otherwise, $f \sim g$ iff f and g are generators of the same equivalence relation in $\mathbb{Q}X$. By definition, an observable cannot distinguish among elements lying in the same equivalence class of its induced equivalence relation. Two algebraically equivalent mappings 'convey the same information' about the partitioning of the elements of X — one cannot distinguish the elements of X further by employing equivalent observables. Succinctly, one has

(21) $$\bullet^X / \sim \;\cong\; \mathbb{Q}X.$$

Note that the algebraic equivalence $f \sim g$ only means that $X/R_f \cong X/R_g$; in other words, there is a one-to-one correspondence between $f(X)$ and $g(X)$, but there may be no relation whatsoever between the *values* $f(x)$ and $g(x)$ for $x \in X$. Indeed, the two mappings f and g may even have codomains that do not intersect. In particular, if their codomains are equipped with metrics, the fact that $f(x)$ may be 'close' to $f(y)$ in $\mathrm{cod}(f)$ says nothing about the closeness between $g(x)$ and $g(y)$ in $\mathrm{cod}(g)$. So in this sense, even equivalent mappings give 'alternate information' about the elements of X, when the codomains are taken into account.

2.25 Qualitative versus Quantitative An *observable* of X, as I define it, may have any set Y as codomain. The difference between 'qualitative' and 'quantitative' thus becomes in degree and not in kind. Indeed, an observable measures a 'quantity' when Y is a set of numbers [*e.g.* when Y is a subset of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or even $\mathbb{C}$ (respectively the sets of natural numbers, integers, rational numbers, real numbers, and complex numbers), without for now straying into the territories of quaternions and Cayley numbers], and measures a 'quality' when Y is not a 'numerical set'.

Seen in this light, quantitative is in fact a meagre subset of qualitative. The traditional view of reductionism is (among other things) that every perceptual quality can and must be expressible in numerical terms. Consider Ernest Rutherford's infamous declaration "Qualitative is nothing but poor quantitative." For us, the features of natural systems in general, and of biological systems in particular, that are of interest and importance are precisely those that are *unquantifiable*. Even though the codomains of qualitative observables can only be described ostensively, the observables themselves do admit rigorous formal definitions. Rosen has discussed much of this in earlier work. See, for example, *AS*, *NC*, *LI*, *EL*.

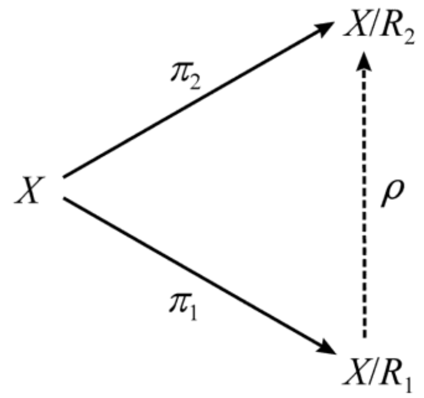
Linkage

Let R_1 and R_2 be equivalence relations on a set X. Recall (Definition 2.13) the partial order of *refinement* in the lattice $\mathfrak{Q}X$: $R_1 \leq R_2$ (R_1 *refines* R_2) if

(22) $$x R_1 y \quad \Rightarrow \quad x R_2 y.$$

2.26 Lemma *If R_1 is a refinement of R_2, then there is a unique mapping $\rho : X/R_1 \to X/R_2$ that makes the diagram*

(23)



commute.

PROOF Define $\rho\left(\left[x\right]_{R_1}\right) = \left[x\right]_{R_2}$. □

The mapping ρ induces an equivalence relation on X/R_1. By Lemma 2.21, one sees that $\left(X/R_1\right)/R_\rho \cong X/R_2$. In other words, when R_1 refines R_2, one may regard X/R_2 as a quotient set of X/R_1.

The refinement relation between two equivalence relations may be defined through their generators:

2.27 Lemma *Let f and g be two mappings with domain X. $R_f \leq R_g$ in $\mathbb{Q}X$ if and only if*

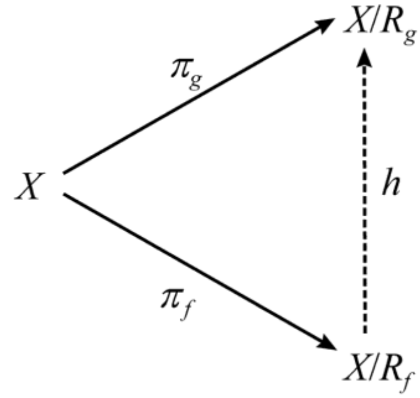
(24) $$f(x) = f(y) \;\Rightarrow\; g(x) = g(y) \quad \text{for all } x, y \in X.$$

2.28 Definition If $R \leq R_g$ in $\mathbb{Q}X$, then g is called an *invariant of* R.

An invariant of an equivalence relation R is an invariant of every refinement of R. R_g is the largest equivalence relation of which g is invariant. An invariant of R is constant on the equivalence classes of R. An invariant g of R will in general take the same value on more than one R-class; it takes on distinct values on distinct R-classes iff $R = R_g$, i.e., iff g is a generator of R.

If $R_f \le R_g$, then by Lemma 2.26 there is a unique mapping $h : X/R_f \to X/R_g$ that makes the diagram

(25)



commute. This says the value of g at every $x \in X$ is completely determined by the value of f through the relation

(26) $$g(x) = h(f(x)).$$

Thus, in the obvious sense of 'is a function of', one has

2.29 Lemma *If $R_f \le R_g$ in $\mathbb{Q}X$, then g is a function of f.*

2.30 Definition Let f and g be observable of X. Let $\pi_f : X \to X/R_f$ and $\pi_g : X \to X/R_g$ be the natural quotient maps. For the R_f-equivalence class $[x]_{R_f} \in X/R_f$, consider the set of R_g-equivalence classes that intersect $[x]_{R_f}$; i.e. consider the set

$$\pi_g \circ \pi_f^{-1}\left([x]_{R_f}\right) = \left\{[y]_{R_g} \in X/R_g : f(x) = f(y)\right\}$$
$$(27) \qquad = \left\{[y]_{R_g} \in X/R_g : [y]_{R_g} \cap [x]_{R_g} \neq \varnothing \right\}.$$

Note that $[x]_{R_g} \in \pi_g \circ \pi_f^{-1}\left([x]_{R_f}\right)$, so the set (27) is necessarily nonempty, containing at least one R_g-equivalence class. One says

(a) g is *totally linked to* f *at* $[x]_{R_f}$ if the set (27) consists of a single R_g-class;

(b) g is *partially linked to* f *at* $[x]_{R_f}$ if the set (27) consists of more than one R_g-class, but is not all of X/R_g;

(c) g is *unlinked to* f *at* $[x]_{R_f}$ if the set (27) is all of X/R_g.

Further, one says that g is *totally linked to* f if g is totally linked to f at each $[x]_{R_f} \in X/R_f$, and that g is (*totally*) *unlinked to* f if g is unlinked to f at each $[x]_{R_f} \in X/R_f$.

It is immediate from the definition that $R_f \leq R_g$ has another characterization:

2.31 Lemma *g is totally linked to f if and only if R_f refines R_g.*

And therefore

2.32 Corollary *f and g are totally linked to each other iff $R_f = R_g$, i.e. $f \sim g$.*

In 1953, B. Jónsson found a simpler proof that gave a stronger result.

2.38 Theorem *Every lattice has a type 3 representation.*

The proofs involved transfinite recursion, and produced (non-constructively) an infinite set X in the representation, even when the lattice L is a finite set. For several decades, one of the outstanding questions of lattice theory was whether every *finite* lattice can be embedded into the lattice of equivalence relations on a *finite* set. An affirmative answer was finally given in 1980 by P. Pudlák and J. Tůma:

2.39 Theorem *Every finite lattice has a representation* $\langle X, f \rangle$ *with a finite set* X.

A representation of a lattice L induces an embedding of L into the lattice of subgroups of a group. Given a representation $\langle X, f \rangle$ of L, let G be the group of all permutations on X that leave all but finitely many elements fixed, and let $\Sigma(G)$ denote the lattice of subgroups of G. Define $h : L \to \Sigma(G)$ by

(31) $$h(a) = \{\phi \in G : x\, f(a)\, \phi(x) \text{ for all } x \in X\}.$$

[Note that $f(a) \in \mathbb{Q}X$, so $x\, f(a)\, \phi(x)$ in (31) is the statement 'x is $f(a)$-related to $\phi(x)$'; i.e. $(x, \phi(x)) \in f(a)$.] One sees that h is an embedding (i.e. a one-to-one lattice homomorphism), thus

2.40 Theorem *Every lattice can be embedded into the lattice of subgroups of a group.*

3

Continuatio: Further Lattice Theory

Modularity

3.1 Definition Let L be a lattice, and let $x, y, z \in L$. The *modular identity* (which is self-dual) is

(m) if $x \leq z$ then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Not all lattices satisfy property (m); but if a lattice does, it is said to be *modular*.

Recall the *modular inequality* 2.5(c) [*if* $x \leq z$, *then* $x \vee (y \wedge z) \leq (x \vee y) \wedge z$], which is satisfied by *all* lattices.

Let G be a group and let $\Sigma(G)$ denote the lattice of subgroups of G. Let $\mathrm{N}(G)$ be the set of all *normal* subgroups of G. $\mathrm{N}(G)$ is a sublattice of $\Sigma(G)$, inheriting the same $\vee$ and $\wedge$. Recall (2.1) that for subgroups H and K of G, $H \wedge K = H \cap K$, and $H \vee K$ is the smallest subgroup of G containing H and K. For $H, K \in \mathrm{N}(G)$, the join becomes the simpler $H \vee K = HK$ in $\mathrm{N}(G)$. Note that this is not a 'different' $\vee$, but a consequent property because H and K are normal subgroups. $\mathrm{N}(G)$ is a modular lattice, while $\Sigma(G)$ in general is not.

3.2 Transposition Principle *In any modular lattice, the intervals* $[b, a \vee b]$ *and* $[a \wedge b, a]$ *are isomorphic, with the inverse pair of isomorphisms* $x \mapsto x \wedge a$ *and* $y \mapsto y \vee b$.

Two intervals of a lattice are called *transposes* when they can be written as $[b, a \vee b]$ and $[a \wedge b, a]$ for suitable a and b, hence the name of Theorem 3.2.

A natural question to ask after having Theorem 2.38, that every lattice has a type 3 representation, is whether all lattices have, in fact, representations of either type 1 or type 2 (*cf.* Definition 2.36). The answer is negative for general lattices, and is positive only for lattices with special properties, which serve as their characterizations.

3.3 Theorem *A lattice has a type 2 representation if and only if it is modular.*

The lattice $\mathrm{N}(G)$ of normal subgroups of a group G is modular, whence by Theorem 3.3 it has a type 2 representation. It, indeed, has a natural representation $\langle X, f \rangle$ with $X = G$ (as the underlying set) and, for $H \in \mathrm{N}(G)$, $f(H) = \{(x, y) \in G \times G : xy^{-1} \in H\}$. (This representation is in fact type 1.)

While type 2 representation is completely characterized by the single modular identity (*m*), the characterization of lattices with type 1 representations is considerably more complicated. The question of whether a set of properties exists that characterizes lattices with type 1 representations (i.e. such that a lattice has a type 1 representation if and only if it satisfies this set of properties) is an open question. It has been proven thus far that even if such a set exists, it must contain infinitely many properties.

Distributivity

3.4 Lemma *In any lattice L, the following three conditions are equivalent:*

($d1$) *for all* $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

($d2$) *for all* $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

(m') *for all* $x, y, z \in L$, $(x \vee y) \wedge z \leq x \vee (y \wedge z)$.

3.5 Definition A lattice L is *distributive* if it satisfy one (hence all three) of the conditions ($d1$), ($d2$), and (m').

Recall the *distributive inequalities* 2.5(b) [($d1$) with $\geq$ in place of $=$ and ($d2$) with $\leq$ in place of $=$], which are satisfied by *all* lattices; but the conditions ($d1$), ($d2$), and (m') are not. Note also that the 'for all $x, y, z \in L$' quantifier is essential for their equivalence. In an arbitrary (non-distributive) lattice L, when one of ($d1$), ($d2$), and (m') is true for three *specific elements* $x, y, z \in L$, it does not necessarily imply that the other two are true for the same three elements.

Any chain (or totally ordered set) is distributive. The dual of a distributive lattice is distributive, and any sublattice of a distributive lattice is distributive. The power set lattice is distributive; it is in fact the canonical distributive lattice. Every distributive lattice has a representation in a power set lattice:

3.6 Theorem *A distributive lattice can be embedded into the power set lattice* $\langle \wp X, \subset \rangle$ *of some set* X.

Combining condition (m') with the *modular inequality* 2.5(c), one has

3.7 Theorem *Every distributive lattice is modular.*

A distributive lattice also has the nice 'cancellation law':

3.8 Theorem *For a lattice to be distributive, it is necessary and sufficient that*

(1) *if* $x \wedge z = y \wedge z$ *and* $x \vee z = y \vee z$, *then* $x = y$.

Complementarity

3.9 Definition Let L be a lattice with least element 0 and greatest element 1 [whence $0 = \inf L = \sup \varnothing$ and $1 = \sup L = \inf \varnothing$]. A *complement* of an element $x \in L$ is an element $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$.

The relation 'is a complement of' is clearly symmetric. Also, 0 and 1 are complements of each other.

3.10 Definition A lattice L (with 0 and 1) is said to be *complemented* if all its elements have complements.

Let L be a lattice, $a, b \in L$, and $a \leq b$. The interval $[a,b] \subset L$ is itself a lattice, with least element a and greatest element b. A complement of $x \in [a,b]$ is thus a $y \in [a,b]$ such that $x \wedge y = a$ and $x \vee y = b$, in which case one also says x and y are *relative complements in the interval* $[a,b]$. The interval $[a,b]$ is *complemented* if all its elements have complements.

3.11 Definition A lattice L is said to be *relatively complemented* if all its intervals are complemented.

distributive, it is necessary and sufficient that relative complements be unique (if they exist).

3.15 Theorem *In any interval of a distributive lattice, an element can have at most one complement. Conversely, a lattice with unique complements (whenever they exist) in every interval is distributive.*

Complementarity may also be used to characterize modular lattices:

3.16 Theorem *A lattice L is modular if and only if for each interval $[a,b] \subset L$, any two comparable elements of $[a,b]$ that have a common complement are equal; i.e. iff for all $[a,b] \subset L$ and $x_1, x_2, y \in [a,b]$, if*

(i) $x_1 \leq x_2$ *or* $x_2 \leq x_1$,

(ii) $x_1 \wedge y = x_2 \wedge y = a$, *and*

(iii) $x_1 \vee y = x_2 \vee y = b$,

then $x_1 = x_2$.

One may use these theorems to show that a particular lattice is *not* distributive by demonstrating an element with two distinct complements, or *not* modular by demonstrating an element with two distinct comparable relative complements. Thus, in view of the constructions in the proofs of Theorems 3.13 and 3.14, and Theorems 3.15 and 3.16, one may conclude that the full lattice $\mathbb{Q}X$ of *all* equivalence relations on a set X is *not* in general distributive, and *not* in general modular. But because of the representation theorems, it evidently contains distributive and modular sublattices.

3.17 Definition A *Boolean lattice* is a complemented distributive lattice.

In a complemented lattice, every element by definition has at least one complement. In a distributive lattice with 0 and 1, every element by Theorem 3.8 has at most one complement. Thus

3.18 Theorem *In any Boolean lattice L, each element x has one and only one complement x^*. Further, for all $x, y, z \in L$*

(i) $x \vee x^* = 1$, $x \wedge x^* = 0$;

(ii) $(x^*)^* = x$;

(iii) $(x \vee y)^* = x^* \wedge y^*$, $(x \wedge y)^* = x^* \vee y^*$.

A Boolean lattice is self-dual. Its structure may be considered as an algebra with two binary operations $\vee$, $\wedge$, and one unary operation * (satisfying the requisite properties), whence it is called a *Boolean algebra*. Note that a Boolean algebra is required to be closed under the operations $\vee$, $\wedge$, and *. So a proper interval of a Boolean algebra may be a Boolean sublattice, but is not necessarily a Boolean subalgebra. A distributive lattice with 0 and 1, however, has a largest Boolean subalgebra formed by its complemented elements:

3.19 Theorem *The collection of all complemented elements of a distributive lattice with 0 and 1 is a Boolean algebra.*

The power set lattice $\mathcal{P}X$ is a Boolean algebra, called the *power set algebra* of X. A *field of sets* is a subalgebra of a power set algebra.

3.20 Stone Representation Theorem *Each Boolean algebra is isomorphic to a field of sets.*

Equivalence Relations and Products

3.21 Lemma *Let $X = X_1 \times X_2$ and let $R_1, R_2 \in \mathcal{Q}X$ be the equivalence relations on X induced by the natural projections $\pi_1 : X \to X_1$, $\pi_2 : X \to X_2$, i.e. $R_1 = R_{\pi_1}$, $R_2 = R_{\pi_2}$. Then*

(i) *Each R_1-class intersects every R_2-class; each R_2-class intersects every R_1-class;*

(ii) *The intersection of an* R_1*-class with an* R_2*-class contains exactly one element of* X, *whence* $R_1 \wedge R_2 = I$ (*the equality relation*);

(iii) $R_1 \vee R_2 = U$ (*the universal relation*).

The conditions $R_1 \wedge R_2 = I$ and $R_1 \vee R_2 = U$, of course, say that R_1 and R_2 are complements in the lattice $\mathbb{Q}X$ (Definition 3.9 and Theorem 3.13). This lemma follows directly from the definitions and the observation that an R_1-class is of the form $\{(a, y) : y \in X_2\}$ for some fixed $a \in X_1$, and an R_2-class is of the form $\{(x, b) : x \in X_1\}$ for some fixed $b \in X_2$; in other words, each R_1-class $\pi_1^{-1}(a)$ is a copy of X_2, and each R_2-class $\pi_2^{-1}(b)$ is a copy of X_1.

The converse of Lemma 3.21 is also true:

3.22 Lemma *Let* X *be a set and let* $R_1, R_2 \in \mathbb{Q}X$ *satisfy the three conditions* (i)–(iii) *in Lemma 3.21. Then* $X = X_1 \times X_2$, *where* $X_1 = X/R_1$ *and* $X_2 = X/R_2$.

PROOF By Lemma 2.16, the equivalence classes of $R_1 \wedge R_2$ are obtained by forming the set-theoretic intersection of each R_1-equivalence class with each R_2-equivalence class. Given $R_1, R_2 \in \mathbb{Q}X$, a map

$$\phi : X/(R_1 \wedge R_2) \to X/R_1 \times X/R_2 \qquad (4)$$

may therefore be defined, that sends a $R_1 \wedge R_2$-class to the uniquely determined ordered pair of R_1-class and R_2-class of which the $R_1 \wedge R_2$-class is the intersection. This map ϕ is one-to-one.

If condition 3.21(i) is satisfied, then ϕ is onto $X/R_1 \times X/R_2$, whence

(5) $$X/(R_1 \wedge R_2) \cong X/R_1 \times X/R_2 .$$

Condition 3.21(ii) then completes the proof to the requisite $X \cong X/R_1 \times X/R_2$. □

In terms of generating observables, one has

3.23 Lemma *Let X be a set equipped with two observables f, g. Then there is always an embedding (i.e. a one-to-one mapping)*

(6) $$\phi : X/R_{fg} \to X/R_f \times X/R_g .$$

This embedding is onto if and only if f and g are unlinked to each other.

Covers and Diagrams

The notion of 'immediate superior' in a hierarchy may be defined in any poset:

3.24 Definition Let X be a poset and $x, y \in X$. One says y *covers* x, or x *is covered by* y, if $x < y$ and there is no $z \in X$ for which $x < z < y$.

The covering relation in fact determines the partial order in a finite poset: the latter is the smallest reflexive and transitive relation that contains the former.

3.25 Definition Let X be poset with least element 0. An element $a \in X$ is called an *atom* if a covers 0.

In the poset $\langle \mathrm{P}X, \subset \rangle$, for $A, B \subset X$, B covers A if and only if $A \subset B$ and $B \sim A$ contains exactly one element. Any singleton subset of X is an atom. In the poset $\langle \mathrm{Q}X, \leq \rangle$, for two partitions R_1 and R_2, R_2 covers R_1 if and only if one of the blocks of R_2 is obtained by the union of two blocks of R_1, while the rest of the blocks of R_2 are identical to those of R_1. In these two examples, note that we are considering the *full* posets $\mathrm{P}X$ and $\mathrm{Q}X$; with their subsets, the 'gaps' in the covers may of course be larger.

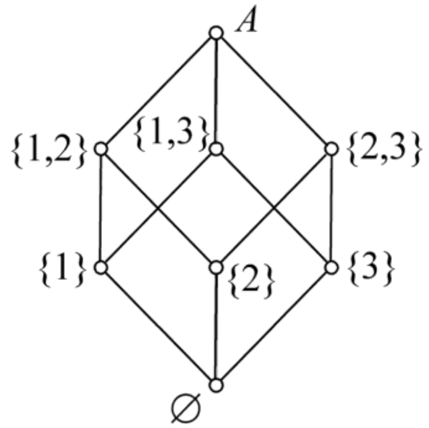
3.26 Hasse Diagram Using the covering relation, one may obtain a graphical representation of a *finite* poset X. Draw a point (or a small circle or a dot) for each element of X. Place y higher than x whenever $x < y$, and draw a straight line segment joining x and y whenever y covers x. The resulting graph is called a (*Hasse*) *diagram* of X.

Let us consider two simple examples. Let A be the three-element set $\{1,2,3\}$. Its power set is

$$\mathrm{P}A = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, A\}.$$

The diagram of $\langle \mathrm{P}A, \subset \rangle$ is

(7)



Semimodularity

3.29 Lemma *In any lattice, if* $x \neq y$ *and both* x *and* y *cover* z, *then* $z = x \wedge y$. *Dually, if* $x \neq y$ *and* z *covers both* x *and* y, *then* $z = x \vee y$.

3.30 Theorem *In a modular lattice,*

(i) *if* $x \neq y$ *and both* x *and* y *cover* z *(whence* $z = x \wedge y$*), then* $x \vee y$ *covers both* x *and* y*;*

(ii) *if* $x \neq y$ *and* z *covers both* x *and* y *(whence* $z = x \vee y$*), then both* x *and* y *cover* $x \wedge y$.

3.31 Corollary *In a modular lattice,* x *covers* $x \wedge y$ *if and only if* $x \vee y$ *covers* y.

3.32 Definition A lattice is (*upper*) *semimodular* if x covers $x \wedge y$ implies $x \vee y$ covers y. The dual property is called *lower semimodular*, for a lattice in which $x \vee y$ covers y implies x covers $x \wedge y$.

Corollary 3.31 says that a modular lattice is both (upper) semimodular and lower semimodular. Upper semimodularity is equivalent to the condition (i) in Theorem 3.30, and dually, lower semimodularity is equivalent to the condition (ii) in Theorem 3.30. Henceforth I shall follow the convention that 'semimodular' by itself means 'upper semimodular'.

The lattice of equivalence relations $\mathfrak{Q}X$ is not modular if X contains four or more elements. However,

3.33 Theorem $\mathfrak{Q}X$ *is a semimodular lattice.*

Chain Conditions

Most of the partially ordered sets and lattices we encounter are infinite, but many of them satisfy certain ‘finiteness conditions’.

3.34 Lemma *In any poset* $\langle X, \leq \rangle$ *the following conditions are equivalent:*

(a) [*ascending chain condition, ACC*] *every ascending chain becomes stationary: if*

(11) $$x_1 \leq x_2 \leq x_3 \leq \cdots,$$

then there exists $n \in \mathbb{N}$ *such that* $x_k = x_n$ *for all* $k \geq n$*;*

(b) *every strictly ascending chain terminates: if*

(12) $$x_1 < x_2 < x_3 < \cdots,$$

then the chain has only finitely many terms;

(c) [*maximum condition*] *every nonempty subset of* X *has a maximal element.*

PROOF $(a) \Rightarrow (b)$ is trivial, because the chain in (b) can become stationary only by terminating.

$(b) \Rightarrow (c)$ follows, because for a nonempty subset $Y \subset X$, one may pick $x_1 \in Y$. If x_1 is not maximal, one may choose $x_2 \in Y$ such that $x_1 < x_2$. Generally, for each $x_k \in Y$, either x_k is maximal, or there exists $x_{k+1} \in Y$ such that $x_k < x_{k+1}$. Thus one obtains a strictly ascending chain of the form in (b), which must then terminate. The last element in the chain is then maximal in Y.

$(c) \Rightarrow (a)$: given an ascending chain of the form in (a), let x_n be maximal in the set $\{x_1, x_2, x_3, \ldots\}$. Then $x_k \leq x_n$ for all $k \in \mathbb{N}$, whence with the ascending chain condition $x_n \leq x_{n+1} \leq x_{n+2} \leq \cdots$, one must have $x_n = x_{n+1} = x_{n+2} = \cdots$; i.e. the chain becomes stationary. □

Dually, one has

3.35 Lemma *In any poset* $\langle X, \leq \rangle$ *the following conditions are equivalent:*

(*a*) [*descending chain condition, DCC*] *every descending chain becomes stationary: if*

(13) $$x_1 \geq x_2 \geq x_3 \geq \cdots,$$

then there exists $n \in \mathbb{N}$ *such that* $x_k = x_n$ *for all* $k \geq n$*;*

(*b*) *every strictly descending chain terminates: if*

(14) $$x_1 > x_2 > x_3 > \cdots,$$

then the chain has only finitely many terms;

(*c*) [*minimum condition*] *every nonempty subset of X has a minimal element.*

Recall Theorem 1.30 that any nonempty *finite* subset of *any* poset has minimal and maximal elements. The maximum and minimum conditions 3.34(*c*) and 3.35(*c*) — for *every* nonempty subset, finite or infinite — are *not* satisfied by *all* posets. The two lemmata say that when a poset satisfies condition (*c*), then it also equivalently satisfies the corresponding conditions (*a*) and (*b*). Note that the proof of the implication $(b) \Rightarrow (c)$ requires the Axiom of Choice (1.37), and it may be shown that this is indispensable. Indeed, the implication $3.34(a) \Rightarrow (c)$ is Zorn's Lemma (1.36). Stated otherwise, without the Axiom of Choice (whence its equivalent Zorn's Lemma), the maximum condition is stronger than the ascending chain condition; but with the Axiom of Choice, both are equivalent.

The poset of natural numbers $\langle \mathbb{N}, \leq \rangle$ satisfies the *DCC*. Condition 3.35(*b*) says that an *infinite* strictly descending chain cannot exist. This fact is the basis of an invention by Pierre de Fermat:

3.36 The Method of Infinite Descent *Suppose that the assumption that a given natural number has a given property implies that there is a smaller natural number with the same property. Then no natural number can have this property.*

Note that the method of infinite descent actually uses the fact that is the 'opposite' of its name: there *cannot* be infinite descent in natural numbers. Stated otherwise, using the method, one may prove that certain properties are impossible for natural numbers by proving that if they hold for any numbers, they would hold for some smaller numbers; then by the same argument, these properties would hold for some still-smaller numbers, and so on *ad infinitum*, which is impossible because a sequence of natural numbers cannot strictly decrease indefinitely. It may even be argued that Fermat used this method in (almost) all of his proofs in number theory. (He might have, perhaps, even used it in the one proof that a margin was too narrow to contain!)

The next two lemmata say that induction principles hold for posets with chain conditions.

3.37 Lemma *Let $\langle X,\leq\rangle$ be a poset satisfying the ACC. If $P(x)$ is a statement such that*

(i) *$P(x)$ holds for all maximal elements x of X;*

(ii) *whenever $P(x)$ holds for all $x>y$ then $P(y)$ also holds;*

then $P(x)$ is true for every element x of X.

PROOF Let $Y=\{x\in X:\neg P(x)\}$ (i.e., Y is the collection of all $x\in X$ for which $P(x)$ is false). I shall show that Y has no maximal element. For if $y\in Y$ is maximal, then consider elements $x\in X$ such that $x>y$. Either no such elements exist, or $P(x)$ has to be true, because y is maximal. But then condition (ii) implies that $P(y)$ is true, contradicting $y\in Y$. (When there are no elements

$x \in X$ with $x > y$, the antecedent of condition (ii) is vacuously satisfied.) Since $\langle X, \leq \rangle$ satisfies the *ACC*, whence by Lemma 3.34 also the maximum condition, the only subset of X that has no maximal element is empty. Thus $Y = \varnothing$, and so $P(x)$ is true for every element x of X. □

Dually, one has

3.38 Lemma *Let $\langle X, \leq \rangle$ be a poset satisfying the DCC. If $P(x)$ is a statement such that*

(i) *$P(x)$ holds for all minimal elements x of X;*

(ii) *whenever $P(y)$ holds for all $y < x$ then $P(x)$ also holds;*

then $P(x)$ is true for every element x of X.

Note that in both Lemmata 3.37 and 3.38, condition (i) is in fact a special case of (ii). If x is a minimal element of X, then there are no elements $y \in X$ with $y < x$. The antecedent of condition 3.38(ii) is vacuously satisfied, whence $P(x)$ is true; i.e. (ii)$\Rightarrow$(i). Condition (i) is included in the statements of the lemmata because maximal and minimal elements usually require separate arguments. Compare Lemma 3.38 with ordinary mathematical induction; we see that (i) is analogous to ordinary induction's 'initial step', and (ii) is analogous to the 'induction step'. Indeed, Lemmata 3.37 and 3.38 are known as Principles of *Transfinite Induction*. Lemma 3.38 is used more often in practice than Lemma 3.37, because it is usually more convenient to use the *DCC* and minimum condition than their dual counterparts.

3.39 Definition A poset $\langle X, \leq \rangle$ is *well-ordered* if every nonempty subset of X has a minimal element.

This is, of course, simply the minimum condition of Lemma 3.35(c). The concept of 'well-ordered set' has a separate set-theoretic history, and is

PART II

Systems, Models, and Entailment

If, then, it is true that the axiomatic basis of theoretical physics cannot be extracted from experience but must be freely invented, can we ever hope to find the right way? Nay, more, has this right way any existence outside our illusions? Can we hope to be guided safely by experience at all when there exist theories (such as classical mechanics) which to a large extent do justice to experience, without getting to the root of the matter? I answer without hesitation that there is, in my opinion, a right way, and that we are capable of finding it. Our experience hitherto justifies us in believing that nature is the realisation of the simplest conceivable mathematical ideas. I am convinced that we can discover by means of purely mathematical constructions the concepts and the laws connecting them with each other, which furnish the key to the understanding of natural phenomena. Experience may suggest the appropriate mathematical concepts, but they most certainly cannot be deduced from it. Experience remains, of course, the sole criterion of the physical utility of a mathematical construction. But the creative principle resides in mathematics. In a certain sense, therefore, I hold it true that pure thought can grasp reality, as the ancients dreamed.

— Albert Einstein (10 June 1933)
On the Methods of Theoretical Physics
Herbert Spencer Lecture, University of Oxford