implies $x^R < x^{RR} \leqslant x$, a contradiction, and the latter is prohibited by the definition of number.

(ii) The inequality $x \not\geqslant y$ implies either some $x^R \leqslant y$ or $x \leqslant$ some y^L, whence either $x < x^R \leqslant y$ or $x \leqslant y^L < y$.

Summary. Numbers are totally ordered.

PROPERTIES OF ADDITION

Definition. $0 = \{\,|\,\}$.
We recall that $x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}$.

THEOREM 3. *For all x, y, z we have*

$$x + 0 \equiv x, \qquad x + y \equiv y + x, \qquad (x + y) + z \equiv x + (y + z).$$

Proof.

$$x + 0 \equiv \{x^L + 0 \mid x^R + 0\} \equiv \{x^L \mid x^R\} \equiv x$$

$$x + y \equiv \{x^L + y, x + y^L \mid x^R + y, x + y^R\} \equiv$$
$$\equiv \{y + x^L, y^L + x \mid y + x^R, y^R + x\} \equiv y + x.$$

$$(x + y) + z \equiv \{(x + y)^L + z, (x + y) + z^L \mid \ldots\} \equiv$$
$$\equiv \{(x^L + y) + z, (x + y^L) + z, (x + y) + z^L \mid \ldots\} \equiv$$
$$\equiv \{x^L + (y + z), x + (y^L + z), x + (y + z^L) \mid \ldots\} \equiv$$
$$\equiv \ldots \equiv x + (y + z).$$

In each case the middle identity follows from the inductive hypothesis. Proofs like these we call *1-line proofs* even when as here the "line" is too long for our page. We shall meet still longer 1-line proofs later on, but they do not get harder—one simply transforms the left-hand side through the definitions and inductive hypotheses until one gets the right hand side.

Summary. Addition is a commutative Semigroup operation with 0 as zero, even when we demand identity rather than equality.

PROPERTIES OF NEGATION

Recall the definition $-x = \{-x^R \mid -x^L\}$.

THEOREM 4. (i) $-(x + y) \equiv -x + -y$
(ii) $-(-x) \equiv x$
(iii) $x + -x = 0$

Proof. (i) and (ii) have easy 1-line proofs. Note that (iii) is an equality rather than an identity. If, say, $x + -x \not\geqslant 0$, we should have some $(x + -x)^R \leqslant 0$, that is, $x^R + -x \leqslant 0$ or $x + -x^L \leqslant 0$. But these are false, since we have by induction $x^R + -x^R \geqslant 0$, $x^L + -x^L \geqslant 0$.

Summary. With equality rather than identity, addition is a commutative Group operation, with 0 for zero, and $-x$ for the negative of x. All this is true for general games.

PROPERTIES OF ADDITION AND ORDER

THEOREM 5. *We have $y \geqslant z$ iff $x + y \geqslant x + z$.*

Proof. If $x + y \geqslant x + z$, we cannot have

$$x + y^R \leqslant x + z \text{ or } x + y \leqslant x + z^L,$$

and so by induction we cannot have $y^R \leqslant z$ or $y \leqslant z^L$, so that $y \geqslant z$.

Now supposing $x + y \not\geqslant x + z$, we must have one of

$$x^R + y \leqslant x + z, \quad x + y^R \leqslant x + z, \quad x + y \leqslant x^L + z, \quad x + y \leqslant x + z^L,$$

and if we further suppose $y \geqslant z$, we deduce one of

$$x^R + y \leqslant x + y, \quad x + y^R \leqslant x + y, \quad x + z \leqslant x^L + z, \quad x + z \leqslant x + z^L,$$

all of which imply contradictions by cancellation.

Theorem 5 implies in particular that we have $y = z$ iff $x + y = x + z$, justifying replacement by equals in addition.

THEOREM 6. (i) 0 *is a number,*
(ii) *if x is a number, so is $-x$,*
(iii) *if x and y are numbers, so is $x + y$.*

Proofs. (i) we cannot have $0^L \geqslant 0^R$, since there exists neither a 0^L nor a 0^R.
(ii) From $x^L < x < x^R$ and x^L, x^R numbers, we inductively deduce $-x^R < -x < -x^L$ and $-x^R$, $-x^L$ numbers.
(iii) We deduce inductively that each of

$$x^L + y, x + y^L < x + y < \text{each of } x^R + y, x + y^R,$$

all of $x^L + y$, etc., being numbers.

Summary. Numbers form a totally ordered Group under addition.

PROPERTIES OF MULTIPLICATION

Definition. $1 = \{0 \mid \}$

We recall the definition of multiplication

$$xy = \{x^L y + xy^L - x^L y^L,\quad x^R y + xy^R - x^R y^R \mid \mid x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}.$$

THEOREM 7. *For all x, y, z we have the identities*

$$x0 \equiv 0,\quad x1 \equiv x,\quad xy \equiv yx,\quad (-x)y \equiv x(-y) \equiv -xy,$$

and the equalities

$$(x + y)z = xz + yz,\quad (xy)z = x(yz).$$

Proof. The identities have easy 1-line proofs. The equalities also have 1-line proofs, as follows:

$$\begin{aligned}(x + y)z &\equiv \{(x + y)^L z + (x + y)z^L - (x + y)^L z^L, \ldots \mid \ldots\} \equiv \\ &\equiv \{(x^L + y)z + (x + y)z^L - (x^L + y)z^L, \\ &\quad (x + y^L)z + (x + y)z^L - (x + y^L)z^L, \ldots \mid \ldots\} = \\ &= \{(x^L z + xz^L - x^L z^L) + yz,\quad xz + (y^L z + yz^L - y^L z^L), \ldots \mid \ldots\} \\ &\equiv xz + yz.\end{aligned}$$

[This fails to yield an identity since the law $x + -x = 0$ is invoked.] The central expression for xyz has four expressions like

$$x^L yz + xy^L z + xyz^L - x^L y^L z - x^L yz^L - xy^L z^L + x^L y^L z^L$$

(with perhaps some even number of x^L, y^L, z^L replaced by x^R, y^R, z^R) on the left, and four similar expressions (with an odd number of such replacements) on the right.

Note. We now have the more illuminating form

$$\{xy - (x - x^L)(y - y^L),\quad xy - (x^R - x)(y^R - y) \mid \mid xy + (x - x^L)(y^R - y),\quad xy + (x^R - x)(y - y^L)\}$$

for the product xy.

THEOREM 8. (i) *If x and y are numbers, so is xy*

(ii) *If $x_1 = x_2$, then $x_1 y = x_2 y$*

(iii) *If $x_1 \leqslant x_2$, and $y_1 \leqslant y_2$, then $x_1 y_2 + x_2 y_1 \leqslant x_1 y_1 + x_2 y_2$, the conclusion being strict if both the premises are.*

Proof. We shall refer to the inequality of (iii) as $P(x_1, x_2 : y_1, y_2)$. Note that if $x_1 \leqslant x_2 \leqslant x_3$, then we can deduce $P(x_1, x_3 : y_1, y_2)$ from the inequalities $P(x_1, x_2 : y_1, y_2)$ and $P(x_2, x_3 : y_1, y_2)$ by adding these and cancelling common terms from the two sides.

Now to prove (i), we observe first that inductively, all options of xy are numbers, so that we have only to prove a number of inequalities like

$$x^{L_1}y + xy^L - x^{L_1}y^L < x^{L_2}y + xy^R - x^{L_2}y^R.$$

But if $x^{L_1} \leqslant x^{L_2}$ we have

$$x^{L_1}y + xy^L - x^{L_1}y^L \leqslant x^{L_2}y + xy^L - x^{L_2}y^L < x^{L_2}y + xy^R - x^{L_2}y^R$$

(these two inequalities reducing respectively to $P(x^{L_1}, x : y^L, y)$ and $P(x^{L_2}, x : y^L, y^R)$), while if $x^{L_2} \leqslant x^{L_1}$ we have instead

$$x^{L_1}y + xy^L - x^{L_1}y^L < x^{L_1}y + xy^R - x^{L_1}y^R \leqslant x^{L_2}y + xy^R - x^{L_2}y^R.$$

(these being $P(x^{L_1}, x : y^L, y^R)$ and $P(x^{L_2}, x^{L_1} : y, y^R)$).

Now to prove (ii). This implication follows immediately from the fact that every Left option of either is strictly less than the other, and every Right option strictly greater, the relevant inequalities all being easy.

If $x_1 = x_2$ or $y_1 = y_2$ we can use (ii) to show that the terms on the Left of (iii) are equal to those on the Right.

So we need only consider the case $x_1 < x_2$, $y_1 < y_2$. Since $x_1 < x_2$, we have either $x_1 < x_1^R \leqslant x_2$ or $x_1 \leqslant x_2^L < x_2$, say the former. But then $P(x_1, x_2 : y_1, y_2)$ can be deduced from $P(x_1, x_1^R : y_1, y_2)$ and $P(x_1^R, x_2 : y_1, y_2)$, of which the latter is strictly simpler than the original. A similar argument now reduces our problem to proving strict inequalities of the four forms

$$P(x^L, x : y^L, y), \quad P(x^L, y : y, y^R), \quad P(x, x^R : y^L, y), \quad \text{and} \quad P(x, x^R : y, y^R)$$

which merely assert that xy has the right order relations with its options.

THEOREM 9. *If x and y are positive numbers, so is xy.*

Proof. This follows from $P(0, x : 0, y)$.

Summary. Numbers form a totally ordered Ring. Note that in view of Theorem 8 and the distributive law, we can assert, for example, that $x \geqslant 0$, $y \geqslant z$ together imply $xy \geqslant xz$, and that if $x \neq 0$, we can deduce $y = z$ from $xy = xz$.

PROPERTIES OF DIVISION

We have just shown that if there is any number y such that $xy = t$, then y is uniquely determined by x and t provided that $x \neq 0$. We must now show how to produce such a y. It suffices to show that for positive x there is a number y such that $xy = 1$. We first put x into a sort of standard form.

since we cannot have any inequality $y^L \geqslant y^R$. The typical form of an option of xy is $x'y + xy' - x'y'$, which can be written as $1 + x'(y - y'')$ with the above definition of y'', and this suffices to prove (iii). For (iv), we observe first that $z = xy$ has a left option 0 (take $x^L = y^L = 0$), and that (iii) asserts that $z^L < 1 < z^R$ for all z^L, z^R. Then

$z \geqslant 1$, since no $z^R \leqslant 1$, and $z \leqslant$ no 1^L (since some $z^L = 0$), and also

$1 \geqslant z$, since no $1^R \leqslant z$, and $1 \leqslant$ no z^L,

so that indeed $z = 1$.

Summary. The Class **No** of all numbers forms a totally ordered Field.

Clive Bach has found a similar definition for the square root of a non-negative number x. He defines

$$\sqrt{x} = y = \left\{ \sqrt{x^L}, \frac{x + y^L y^R}{y^L + y^R} \,\middle|\, \sqrt{x^R}, \frac{x + y^L y^{L*}}{y^L + y^{L*}}, \frac{x + y^R y^{R*}}{y^R + y^{R*}} \right\}$$

where x^L and x^R are non-negative options of x, and y^L, y^{L*}, y^R, y^{R*} are options of y chosen so that no one of the three denominators is zero. We shall leave to the reader the easy inductive proof that this is correct.

Martin Kruskal has pointed out that the options of $1/x$ can be written in the form

$$\frac{1 - \Pi\left(1 - \dfrac{x}{x_i}\right)}{x}$$

where the denominator x cancels formally, the x_i denote positive options of x, and the product may be empty. This is a Left option of $1/x$ just when an *even* number of the x_i are Left options of x. There is a similar closed form for Bach's definition of $\sqrt{x}$.

THEOREM 19. *Each positive number is commensurate with some* ω^y.

Proof. We can write x in the form $\{0, x^L \mid x^R\}$, where x^L and x^R now denote positive numbers. Each x^L is commensurate with some ω^{y^L} (say) and each x^R with ω^{y^R}. If x is commensurate with one of its options, we are done. If not, we can add all numbers $r\omega^{y^L}$ as Left options and all $r\omega^{y^R}$ as Right options, and we then see that $x = \omega^y$, where y is the number $\{y^L \mid y^R\}$.

THEOREM 20. $\omega^0 = 1, \omega^{-x} = 1/\omega^x, \omega^{x+y} = \omega^x . \omega^y$.

Proof. The first part is trivial, and the second follows from the first and third. Let $X = \omega^x$, $Y = \omega^y$, and let X' and Y' be the typical options of X and Y. Then the typical option of XY is $X'Y + XY' - X'Y'$. If Y' is 0, this is $X'Y$, and if X' is 0, it is XY'. Otherwise we can suppose $X' = r\omega^{x'}$, $Y' = s\omega^{y'}$, when the formula becomes

$$r\omega^{x'+y} + s\omega^{x+y'} - rs\omega^{x'+y'}$$

by induction.

When this is positive, it lies between two positive real multiples of ω^z, where z is the largest of the three indices, which is always one of $x' + y$ and $x + y'$. We have said enough to show that

$$\omega^x . \omega^y = \{0, r\omega^{x^L+y}, s\omega^{x+y^L} \mid r\omega^{x^R+y}, s\omega^{x+y^R}\} = \omega^{x+y}.$$

Summary. ω^x does indeed behave like the xth power of ω. Those familiar with the normal arithmetic of ordinals will have no difficulty in showing that ω^α is the ordinal usually so called.

THE NORMAL FORM OF x

Let x be an arbitrary positive number, and ω^{y_0} the unique leader commensurate with x. Then we can divide the reals into two classes by putting t into L or R according as $\omega^{y_0} . t \leqslant x$ or $\omega^{y_0} . t > x$. Then L and R are non-empty, since for suitably large n we have $-n \in L$, $n \in R$, and so by the theory of real numbers, one of L and R has an extremal point r_0, say. Write

$$x = \omega^{y_0} . r_0 + x_1.$$

It follows that x_1 is *small compared to* x, that is, that nx_1 is between x and $-x$ for all integers n. If x_1 is not zero, we can produce in a similar way numbers r_1, y_1 such that $x_1 = \omega^{y_1} . r_1 + x_2$, where x_2 is small compared to x_1.

If again x_2 is non-zero, we can continue, producing an expansion

$$x = \omega^{y_0} . r_0 + \omega^{y_1} . r_1 + \ldots + \omega^{y_{n-1}} . r_{n-1} + x_n$$

which will terminate painlessly if any x_n is zero. But usually the expansion