

# One, Two, Three

*Absolutely Elementary Mathematics*



David Berlinski

*Author of A Tour of the Calculus*

# One, Two, Three

ABSOLUTELY ELEMENTARY MATHEMATICS

David Berlinski



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## INTRODUCTION

This is a little book about absolutely elementary mathematics (**AEM**); and so a book about the natural numbers, zero, the negative numbers, and the fractions. It is neither a textbook, a treatise, nor a trot. I should like to think that this book acts as an anchor to my other books about mathematics.

Mathematicians have always imagined that mathematics is rather like a city, one whose skyline is dominated by three great towers, the state ministries of a powerful intellectual culture—our own, as it happens. They are, these great buildings, devoted to Geometry, Analysis, and Algebra: the study of space, the study of time, and the study of symbols and structures.

Imposing as Babylonian ziggurats, these buildings convey a sacred air.

The common ground on which they rest is sacred too, made sacred by the scuffle of human feet.

This is the domain of absolutely elementary mathematics.

Many parts of mathematics glitter alluringly. They are exotic. Elementary mathematics, on the other hand, evokes the very stuff of life: paying bills, marking birthdays, dividing debts, cutting bread, and measuring distances. It is earthy. Were textbooks to disappear tomorrow, and with them the treasures that they contain, it would take centuries to rediscover the calculus, but only days to recover our debts, and with our debts, the numbers that express them.

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Elementary mathematics as it is often taught and sometimes used requires an immersion into messiness. Patience is demanded, pleasure deferred. Decimal points seem to wander, negative numbers become positive, and fractions stand suddenly on their heads.

And what *is* three-fourths divided by seven-eighths?

The electronic calculator has allowed almost everyone to treat questions such as this with an insouciant indifference. Quick, accurate, and cheap, it does better what one hundred years ago men and women struggled to do well. The sense that in elementary mathematics things are familiar—half remembered, even if half forgotten—is comforting, and so are the calculator and the computer, faithful almost to a fault, but the imperatives of memory and technology do prompt an obvious question: why bother to learn what we already know or at least thought we knew?

The question embodies a confusion. The techniques of elementary mathematics are one thing, but their *explanations* are quite another. Everyone can add two simple natural numbers together—two and two, for instance. It is much harder to say what addition means and why it is justified. Mathematics explains the meaning and provides the justification. The theories that result demand the same combination of art and sophistication that is characteristic of any great intellectual endeavor.

It could so easily have been otherwise. Elementary mathematics, although pressing in its urgency, might have refused to cohere in its theory, so that, when laid out, it resembled a map in which roads diverged for no good reason or ended in a hopeless jumble. But the theory by which elementary mathematics is explained and its techniques are justified is intellectually coherent. It is powerful. It makes sense. It is never counter-intuitive. And so it is appropriate to its subject. If when it comes to the simplest of mathematical operations—addition again—there remains something that we do not understand, that is only because there is nothing in nature (or in life) that we understand as completely as we might wish.

Nonetheless, the theory that results is radical. Do not doubt

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it. The staples of childhood education are gone in the night. One idea is left, and so one idea predominates: *that the calculations and concepts of absolutely elementary mathematics are controlled by the single act of counting by one*. There is in this analysis an economy of effect, and a reduction of experience to its essentials, as dramatic as anything found in the physical sciences.

Until the end of the nineteenth century, this was not understood. A century later, it is still not *widely* understood. School instruction is of little help. “Please forget what you have learned in school,” the German mathematician Edmund Landau wrote in his book *Foundations of Analysis*; “you haven’t learned it.”

From time to time, I am going to ask that readers do some forgetting all their own.

A secret must now be imparted. It is one familiar to anyone writing about (or teaching) mathematics: no one very much likes the subject. It is best to say this at once. Like chess, mathematics has the power to command obsession but not often affection.

Why should this be—the distaste for mathematics, I mean?

There are two obvious reasons. Mathematics confronts the beginner with an aura of strangeness, one roughly in proportion to its use of arcane symbols. There is something about mathematical symbolism, a kind of peevishness, that while it demands patience, seems hardly to promise pleasure.

*Why bother?*

If the symbolic apparatus of mathematics is one impediment to its appreciation, the arguments that it makes are another. Mathematics is a matter of proof, or it is nothing. But certainty does not come cheap. There is often a remarkable level of detail in even a simple mathematical argument, and, what is worse, a maddening difference between the complicated structure of a proof and the simple and obvious thing it is intended to demonstrate. There is no natural number standing between zero and one. Who would

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doubt it? Yet it must be shown, and shown step by step. Difficult ideas are required.

*Why bother?*

A tricky bargain is inevitably involved. In mathematics, something must be invested before anything is gained, and what is gained is never quite so palpable as what has been invested. It is a bargain that many men and women reject.

*Why bother indeed?*

The question is not ignominious. It merits an answer.

In the case of many parts of mathematics, answers are obvious. Geometry is the study of space, the mysterious stuff between points. To be indifferent to geometry is to be indifferent to the physical world. This is one reason that high-school students have traditionally accepted Euclid with the grudging sense that they were being forced to learn something that they needed to know.

And algebra? The repugnance (in high school) that this subject evokes has always been balanced by the sense that its symbols have a magical power to control the flux and flee of things. Farmers and fertilizers were the staple of ancient textbooks. But energy and mass figure in those that are modern. Einstein required *only* high-school algebra in creating his theory of special relativity, but he required high-school *algebra*, and he would have been lost without it.

Mathematical analysis came to the mature attention of European mathematicians in the form of the calculus. They understood almost at once that they had been vouchsafed the first and in some respects the greatest of scientific theories. To wonder at the importance of analysis, or to scoff at its claims, is to ignore the richest and most intensely developed body of knowledge acquired by the human race.

Yes, yes. This is all very uplifting, but absolutely elementary mathematics? Not very long ago, the French mathematician Alain Connes invented the term *archaic mathematics* to describe the place where ideas are primeval and where they have not yet

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separated themselves into disciplines. It is an elegant phrase, an apt description. And it indicates just why elementary mathematics, when seen properly, has the grandeur of what is absolute. It is fundamental, and so, like language, an instinctive gesture of the human race.

A *theory* of absolutely elementary mathematics is an account in modern terms of something deep in the imagination; its development over the centuries represents an extraordinary exercise in self-consciousness.

This is what justifies the bothering, the sense that, by seeing an old, familiar place through the mathematician's eyes, we can gain the power to see it for the first time.

This is no little thing.

—Paris, 2010



# 1

*One sheep, two sheep, three. Wool to follow . . .*

## NUMBERS

The natural numbers 1, 2, 3, . . . , play a twofold role in our ordinary affairs. Without them, there could be no counting, and so no answer to the question *How many?* A man who is unable to tell whether he is looking at one sheep or two of them cannot *identify* sheep. He is left staring at so much wool on the hoof. It is the natural numbers that offer him relief from sheeplessness. “The creation of numbers,” Thierry of Chartres remarked in the twelfth century, “was the creation of *things*.”

As counting endows things with their identity, so it imposes on them their difference. Three sheep make for three *things*. The natural numbers are the expression in nature of division and distinctness. Between the number one and the number two there is, after all, nothing whatsoever, and nothing between things that are distinct either, however much alike they might be in various respects. The discreteness of the natural numbers is as absolute as the one enforced by the surface of our skin, which permits contact but not, alas, commingling.

There are certainly substances in the world that cannot be counted—mud, for example. The word ‘mud’ seems indifferently to designate mud wherever it is and however it may be found. But so strong is the intellectual impulse to subordinate experience to counting that ordinary English provides the tools by which even

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mud can be made numerate—a *spot* of mud, or a *dab*, or a *pile*, whereupon there is *one* spot, *two* dabs, or *three* piles. The same *one*, *two*, and *three* used in counting sheep are also used in sorting them. It is the natural numbers that make it possible for some leather-faced Spanish shepherd, his concave cheeks pursing around two gold teeth, to put his sheep and thus his life in order.

The *first* is mine, *hombre*, as these shepherders say, the *second* yours, and the *third* his.

## A SCRIBAL ART

The Sumerians drilled their children in **AEM** more than five thousand years ago, when the desert sun was new and nothing was yet old. Sumerian children were taught the basics; their teachers had grasped the essentials. They did not find it easy. Sumerian scribes studied for years beyond childhood in order to hen-scratch clay tablets with tax records, business claims, legal codes, real-estate transactions. They left behind a sense of their mathematical intimacy, the first in history.

An inadequate sense of their calling was not among their afflictions. “The scribal art,” one wrote, “is the father of masters.”

Scribes *alone*, he added, could “write [inscribe] a stele; draw a field, settle accounts.”

There is a gap in the text, a break in its flow.

And then a phrase isolated on both sides, one suggestive of the scribe’s intellectual grandeur: . . . *the palace* . . .

At the end of the third millennium B.C., the Sumerian empire ran streaming into the desert sands, defeated at last by time. Carried by the wind, I suppose, or some other current of warm thought, the scribal sense of intimacy with **AEM** was acquired by Chinese mandarins, intoxicated by their new power over pictograms, and acquired again by the Babylonians, so that it appears throughout the ancient world.

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Different societies used **AEM** in their own ways and for their own ends. Every society missed something, and no society, not even our own, knew or knows it all.

### A MAN APART

Leopold Kronecker was born in 1823, his birthplace, the small city of Liegnitz, in East Prussia. Having quaked to the sound of Russian tanks in late 1944, Liegnitz is now known as Legnica. It is a part of Poland. East Prussia has vanished.

Kronecker's face is not easy to read on a photograph. Harsh lighting and extended exposure have darkened and deepened every facial line. The stern creases suggest an unacknowledged blood tie between Leopold Kronecker and General William Tecumseh Sherman. In both men, the forehead is high, and the hair cut short, almost *en brosse*; the eyes are deeply recessed and melancholy. In all this, Kronecker, at least, is completely Prussian and austere, but his nose has undertaken a racially unmistakable life of its own, hooking proudly at the bridge, and then curving toward its fluted tip.

I mention this not in order to make fun of another man's nose—I have a give-away nose all my own, after all—but to convey something of Kronecker's capacity to stand apart from other mathematicians while standing among them. Kronecker was that rare character in the history of thought, a *mathematical* skeptic, unwilling to countenance ideas that he could not completely grasp, and very quick to conclude that he could not grasp most ideas completely. If Kronecker the Glum was notable for saying *no—no* to the negative numbers, *no* to the real numbers, *no* to sets—he was notable for saying *yes* to the natural numbers, a great life-affirming *yes* to these ancient objects of thought and experience, a *yes* spilling over to encompass any mathematical construction that returned to the natural numbers in a finite series of steps.

Kronecker, the man of a thousand *no*'s, and Kronecker, the

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man of a single yes, were combined in a singular personality: suave, supple, self-satisfied.

While still in his twenties, Leopold Kronecker pursued a career in business as the manager of his uncle's estates in East Prussia. He had a remarkable gift for practical affairs, and over the course of eight years, he made himself a wealthy man. Thereafter, he bought a splendid Berlin mansion, and after marrying his uncle's daughter, Fanny Prausnitzer, made it a center of culture and refinement.

Wealth made Kronecker indifferent to the great game of mathematical chairs in which the leading mathematicians of Europe stood looking eagerly at a small number of seat cushions still glowing with the warmth of some departed professor's buttocks. When the music stopped, they scrambled unceremoniously for the vacant seat. Inevitably, most were disappointed. Mathematicians of genius, such as Georg Cantor, spent years waiting for a call from Berlin and were bitterly vexed when it did not come.

Herr Kronecker expressed no very great interest in becoming Herr Professor. He did not need to scramble for his seat. Or for his supper. What he lacked was the right to lecture at the University of Berlin. This he wished very much to have. Devoted to topics in number theory, elliptic functions, and algebra, his papers were in every way remarkable without in any way being revolutionary. When he was elected to the Berlin Academy in 1861, he gained the right to lecture at the university.

Having declined to mount the greasy pole, he found himself at its very top. Once there, he determined to persecute those with whom he disagreed. It was an activity he carried out with never-flagging zeal.

## IN EVERY HUMAN MIND

At the very beginning of human history, a Neolithic hunter chipped a number of slash marks or tallies onto his ax handle.

Was he recording bison killed? I do not know. I like to think that as *my* ancestor, he had a contemplative nature, and regarded the numbers as things in themselves, leaving those bloated bison to his rivals.

If the natural numbers appear at the very beginning of human history, they also appear spontaneously in every human mind. Otherwise, arithmetic could not be taught. Anthropologists are often amazed by the radically incommensurable way in which different societies organize the most basic facts of experience. Seeing this is said to be one of the pleasures of travel. Nonetheless, our own *one, two, three*, the Latin *unus, duo, tres*, and the Akkadian *dis, min, es*, designate precisely the same numbers. If goat eyes are a delicacy in Khartoum but not in New York, it is nonetheless true that three goat eyes is one more than two in both cities.

Because they are universal, the natural numbers very rarely are the cause of introspection. We take them for granted. We would be lost without them.

There they are.

*What they are* is another matter entirely.

The English logician and philosopher Bertrand Russell was a passionate opponent of the First World War, and he took the occasion of his confinement as a conscientious objector to organize his thoughts about the nature of the numbers. It may be imagined that Russell was writing under conditions of personal austerity, but in his *Autobiography* he indicates that, except for his freedom, he was provided every comfort by his jailers.

The book that Russell wrote in prison, his *Introduction to Mathematical Philosophy*, is a work of logical analysis. It has had a very great influence among mathematicians and philosophers, because it offers an account of the natural numbers in terms of something *other* than the natural numbers. Such an account is needed, Russell believed, because the numbers are “elusive” in their nature, and though they make their influence felt in the

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most ordinary of activities—counting sheep, after all—what they are doing is far easier to determine than how they are doing it.

For one thing, the numbers are not physical objects. They are not objects at all. Three sheep are in the pasture. There are not, in addition to the sheep, three numbers loitering around and munching grass.

Nor, however, are the numbers properties of physical objects. Three sheep are three in number, just as they are white in color. This is a step in the right direction. But to argue that being three is just like being white invites the question of just what property makes three sheep three. We know what makes them white: it is their color. To say that what makes them three is their *number* does not seem a step forward. If we knew what the numbers were, we would know what three of them add up to.

To the question of just what makes three sheep three, Russell argued that those three sheep were similar to other sets of three things—triplets, troikas, or trios. This is obviously so. Three sheep and three shepherders *are* alike. There are three of them. Russell next argued that being alike in being three could be defined with no appeal to the number three. This is the crucial step. Three sheep and three shepherders are alike if each shepherd can be matched with one and only one sheep, and vice versa. Numbers are not required. No sheep lacks a herder and no herder a sheep.

This is ingenious, but it is also disappointing. The number three was destined to disappear in favor of similar sets, but what, after all, makes a set of three sheep a set of *three* sheep and not four? Four shepherders and four sheep may also be put into correspondence so that nothing is left over and no sheep or shepherd left out. The obvious answer is that a set of four sheep is larger than a set of three sheep.

It is, in fact, larger by precisely *one* sheep.

. . .

## WIVES, GOATS, NUMBERS

The natural numbers begin at one; they increase by one; and they go on forever. The knowledge that this is so is an inheritance of the race. Anthropologists, it is true, report that certain tribes lack a complete sense of the numbers. Men count in the fashion of *one*, *two*, *many*, and they refer to any number past two by observing glumly that it is many. *One chief, two goats, many wives*, as a great chief might say.

I am skeptical of such reports, because I feel quite certain that stealing one of the chief's wives would prompt the chief to observe that he has *one less* than many wives. If he is capable of determining that he has *one less* wife than he might need, he is equally capable of determining that he has *one more* wife than he might want. Pressed thus by the exigencies of tribal life, he could count up by enumerating his ensuing discontent: *many wives, one more than many wives, one more than one more than many wives*, and so on toward frank domestic nightmare.

Going in the other direction, he could count down until he reached bedrock in the number one, whereupon he could compare the number of his squabbling wives with the number of his chiefs—one in both cases. It is a laborious system, to be sure, but brain workers are often indifferent to practical concerns.

## GOD'S WORK

If it is difficult to say what the numbers are, it is difficult again to say how they are used. The most familiar way of counting a small number of sheep is to match those sheep to the tips of one's fingers as they uncoil from a fist; it is what we all do when we are minded to count sheep. But as an explanation of counting sheep, it suffers the drawback that counting one's fingers, however familiar, is no easier to explain than counting one's sheep. Shall we explain counting three sheep by an appeal to counting sheep one

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at a time, with counting by one explained in terms of a physical act—moving those sheep one by one from pasture to paddock, for example? It has an appeal, this sort of explanation. Something is being done, and when it has been done, someone has done something. But obviously, if we wish to understand what it means to count three sheep, it will hardly improve our grasp of the matter to be told that we must first count sheep one by one *three* times. What holds for counting holds for ordering too, as when a shepherd's ever-useful fingers are used to explain the fact that the first sheep into the paddock comes *before* the second, and the second *before* the third. Fingers are vigorously extended: the first, the second, and the third. Yet, if sheep are following in a certain order, it is hardly to the point to appeal to the *same* order among fingers as among flocks. If the order is not the same between fingers and flocks, of what use are the fingers? If it is the same, of what use the analogy?

At a certain point—*now*, perhaps—it becomes reasonable to suppose that neither the numbers nor the operations they make possible permit an analysis in which they disappear in favor of something more fundamental. It is the numbers that are fundamental. They may be better understood; they may be better described; but they cannot be *bettered*.

The natural numbers, Leopold Kronecker remarked, are a gift from God. Everything else is the work of man. This is a radical position in thought—an admission, on the one hand, that the natural numbers cannot be explained, and a suggestion, on the other, that the mathematician's proper work must be to accept this strange gift and from it derive all others.

It is comforting to realize that in **AEM** we are doing God's work.



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This could not be true; it is not so. Formulas are symbolic forms: marks on paper, sounds on the moving air, or even lines within a computer program. The number four cannot replace anything in any formula. Only symbols can replace symbols.

On the other hand, the number four was equal to itself long before formulas were in existence; and equal to itself, for that matter, before the earth cooled, or the solar system formed, or the universe erupted into being out of nothingness.

Numbers owe their identity to no symbolic contrivance. They are what they are. They have always been what they always were. They are not destined to change. But the numerals (and so the formulas) name; they denote; they designate; they are a part of the apparatus with which we make up a world of symbols in order to represent a world of things. How symbols designate numbers is mysterious, because we do not understand how names designate things. Speaking in *The Book of the Thousand and One Nights*, a Prince offers as good an explanation as any: There is no letter in any language, he remarks, "which is not governed by a spirit, a ray, or an emanation of the virtue of Allāh."

### A MAN FOR ALL SYMBOLS

The notation used to name the natural numbers is Hindu-Arabic, and it seems to have gained currency among mathematicians during the early part of the ninth century.

The man most associated with the transmission of the Hindu-Arabic numerals to the west is Abu Ja'far Muhammad ibn Musa al-Khwarizmi. Born at some time during the latter part of the eighth century, and dying at some time during the middle of the

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sticker 'War is never the answer,' I think that, to the contrary, war most certainly is the answer, if the question is 'What is a three-letter word for organized armed conflict?'"

But war is, of course, not a three-letter word; it is not a word at all.

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What is yet lacking is a way in which these symbols may be used to denote the numbers beyond nine. Left to their own devices, of course, nine numerals can hardly do more than name nine numbers.

An especially ingenious Baghdad merchant might have designated the number ten by  $9 + 1$ . The number after that could be designated by  $9 + 1 + 1$ . This would hardly have been a contribution to a merchant's practical concerns, if only because a bill for one hundred and seventy drachmas would have run on for pages.

The solution to this problem emerged in stages, and it emerged in the way solutions so often emerge in mathematics, a slapped-together strategy followed by a carefully contrived cleanup. Thus merchants with business more pressing than mathematical notation long ago expressed their bills of lading or sale by writing **I plus X** for the number ten, or simply **IX**, where the symbol **I**, by means of its new and unfamiliar *position*, designated the number ten, and where **X** served simply as a placeholder, a symbol signifying that nothing was being added to ten.

Thereafter, precisely the same scheme could encompass the numbers that followed, with the number eleven expressed as **IX plus I**, or **II**. Having the sense, I am hoping, that he might have discovered something profound, that Baghdad merchant might well have noted with satisfaction that by writing **IX** for ten, or **II** for eleven, he had discovered the key to positional notation, the great door swinging open to admit both merchants and mathematicians.

In any compound numeral of the form **ab**, where **a** and **b** stand for the numerals from **1** to **9**, position is king and key, indicating *both* that **b** is to be added to **a**, and that **b** marks things in units of one, while **a** marks them in units of ten. That bill of lading now emerges just as that long-dead merchant might have written it:



*In the name of Allah the Most Powerful, the Most High, the Most Merciful*

Dates:	17 drachmas
Oil:	13 drachmas
Almonds:	1X drachmas
Figs:	1X drachmas

whereupon positional notation and the ever-useful X come into play again in the total of 5X drachmas.

The placeholder X was in time replaced by the symbol 0, so that 5X emerges in its modern and familiar form as the numeral 50.

The cleanup followed years later when the placeholder underwent a further promotion, so that 0 is now regarded as the name of a number in its own right. And for every good reason. The simplest of hygienic routines demands it. The sum of 2 and 0 is 2. Treating 0 as a placeholder, and so as a symbol, renders this identity incoherent. A placeholder cannot be *added* to a number, any more than a horse's name can be entered in a race.

Yet the promotion of 0 from a placeholder to the name of a living number is itself hardly a model of logical scrupulousness, for if 0 is a name, just *what* does it name? The obvious answer, in virtue of the fact that two plus zero is still two, is that it names *nothing*.

But if zero names nothing, then it is difficult to make sense of *adding* zero to two. There is no *adding* nothing to anything.

If, on the other hand, 0 names something, then it is again difficult to see why two plus something should *remain* two. It hardly helps to insist that zero is that unique something that behaves as if it were nothing. Mathematicians have traditionally resolved these difficulties by embracing the thesis that at times something is nothing, a metaphysical achievement that cannot be said to inspire a sense of serenity.

The converse, that at times nothing is something, is, of course, among the most useful declarations of the human race.

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