Oxford Users' Guide to Mathematics

Oxford
Users' Guide to Mathematics

Edited by

Eberhard Zeidler

Max Planck Institute for Mathematics in the Sciences
Leipzig

Translated and typeset by

Bruce Hunt

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship, and education by publishing worldwide in

Oxford New York

Auckland Bangkok Buenos Aires Cape Town Chennai
Dar es Salaam Delhi Hong Kong Istanbul Karachi Kolkata
Kuala Lumpur Madrid Melbourne Mexico City Mumbai
Nairobi São Paulo Shanghai Taipei Tokyo Toronto

Oxford is a registered trade mark of Oxford University Press in the UK and in certain other countries

Published in the United States by Oxford University Press Inc., New York

Translated by Bruce Hunt

Title of the original: Teubner-Taschenbuch der Mathematik, Vol 1

Translation arranged with the approval of the Publisher B. G. Teubner from the original German edition

A catalogue record for this title is available from the British Library

Library of Congress Cataloging in Publication Data
(Data available)

ISBN 0 19 850763 1

10 9 8 7 6 5 4 3 2 1

Typeset in LaTeX by Bruce Hunt
Printed in Hong Kong
on acid-free paper by Nordica Printing Co., Ltd.

Contents

Introduction

The greatest mathematicians like Archimedes, Newton and Gauss have always been able to combine theory and applications into one.

Felix Klein (1849–1925)

Mathematics has more than 5000 years of history. It is the most powerful instrument of the human mind, able to precisely formulate laws of nature. In this way it is possible to dwell into the secrets of nature and into the incredible, unimaginable extension of the universe. Fundamental branches of mathematics are

- algebra,
- geometry, and
- analysis.

Algebra is concerned with, at least in it original form, the solution of equations. Cuneiform writing from the days of King Hammurapi (eighteenth century BC) document that the practical mathematical thinking of the Babylonians was strongly algebra-oriented. On the other hand, the mathematical thought of ancient Greece, whose crowning achievement was the appearance of Euclid's *The Elements* (around 300 BC), was strongly influenced by geometry. Analytical thinking, based on the notion of limit, was not systematically developed until the creation of calculus by Newton and Leibniz in the seventeenth century.

Important branches of applied mathematics are aptly described by the following indications:

- ordinary and partial differential equations (describing the change in time of systems of nature, engineering and society),
- the calculus of variations and optimization,
- scientific computing (the approximation and simulation of processes with more and more powerful computing machines).

Foundations of mathematics are concerned with

- mathematical logic, and
- set theory.

These two branches of mathematics did not exist until the nineteenth century. Mathematical logic investigates the possibilities, but also the limits of mathematical proofs.

Because of its by nature very formal development, it is well-equipped to describe processes in algorithms and on computers, which are free of subjectivity. Set theory is basically a powerful language for formulating mathematics. Instead of dealing in this book with the formal aspects of set theory, we put our emphasis on the liveliness and broad nature of mathematics, something which has fascinated mankind for centuries.

In modern mathematics there are opposing tendencies visible. On the one hand, we observe an increase in the degree of specialization. On the other hand, there are open questions coming from the theory of elementary particles, cosmology and modern technology which have such a high degree of complexity that they can only be approached through a synthesis of quite diverse areas of mathematics. This leads to a unification of mathematics and to an increasing elimination of the non-natural split between pure and applied mathematics.

The history of mathematics is full of the appearance of new ideas and methods. We can safely assume that this tendency with continue on into the future.

0. Important Formulas, Graphical Representations and Tables

Everything should be made as simple as possible, but not simpler.
Albert Einstein (1879–1955)

0.1 Basic formulas of elementary mathematics

0.1.1 Mathematical constants

Table 0.1. Some frequently used mathematical constants.

Symbol	*Approximation*	*Notation*
π	3.14 15 92 65	Ludolf number pi
e	2.71 82 81 83	Euler[1] number e
C	0.57 72 15 67	Euler constant
ln 10	2.30 25 85 09	natural logarithm of the number 10

A table of the most important scientific constants can be found at the end of this handbook.

Factorial: Often the symbol

$$\boxed{n! := 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}$$

is used for the shown product; this product is called *n-factorial.* Moreover, we define $0! := 1$.

Example 1: $1! = 1$, $2! = 1 \cdot 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 24$, $5! = 120$ and $6! = 720$.

In statistical physics, one requires the value of $n!$ for n around 10^{23}. For such astronomical numbers, one has the *Stirling formula*

$$\boxed{n! = \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n}} \tag{0.1}$$

as a good approximation (cf. 0.7.3.2).

[1]Leonhard Euler (1707–1783) was the most productive mathematician of all times. His collected works fill 72 volumes and more than 5000 additional letters. His monumental lifetime work has shaped much of the modern mathematical science. At the end of this handbook there is a table of the history of mathematics, which should help the reader to orient her- or himself in the history of mathematics and its greatest contributors.

Infinite series for π and e: The precise value of π is given as the value of the convergent Leibniz series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \tag{0.2}$$

Because of the alternating sign of the terms, the error of the truncated series is always given by the following term. Thus, the terms listed on the right-hand side of (0.2) give an approximation of π for which the error is at most 1/9. This series, however, is not used for practical computations of values for π on computers, because it converges very slowly. Contemporary approximations of π are accurate up to more than 2 billion decimal places (cf. the more detailed discussion of the number π in 2.7.7). The value of e is the value of the following convergent series

$$\mathrm{e} = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

For large numbers n, for example, one has approximately

$$\mathrm{e} = \left(1 + \frac{1}{n}\right)^n . \tag{0.3}$$

More precisely, the right-hand side of (0.3) approaches for larger and larger values of n the value of the number e. One also writes for this

$$\boxed{\mathrm{e} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .}$$

In words: the number e is the limit of the sequence of numbers $\left(1 + \frac{1}{n}\right)^n$, as n approaches infinity. With the help of the number e one can define the most important function in mathematics:

$$\boxed{y = \mathrm{e}^x .} \tag{0.4}$$

This is the Euler e-function (exponential function, cf. 0.2.5). The inverse function of (0.4) is the natural logarithm

$$\boxed{x = \ln y}$$

(cf. 0.2.6). In particular for powers of 10 one gets

$$\boxed{\ln 10^x = x \cdot \ln 10 = x \cdot 2.30\,25\,85\,09.}$$

Here x can be an arbitrary real number.

Representations of π and e through continued fractions: For more detailed investigations of the structure of numbers, one uses representations in terms of *continued fractions* instead of decimal numbers (cf. 2.7.5). The representations of π and e in terms of continued fractions are displayed in Table 2.7.

The Euler constant C: The precise value of C is given by the formula

$$\mathrm{C} = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln(n+1)\right) = -\int_0^\infty \mathrm{e}^{-t}\ln t\,dt\,.$$

For large natural numbers n, one thus has the approximation formula

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} = \ln(n+1)+\mathrm{C}\,.$$

The Euler constant C appears in a surprisingly large number of mathematical formulas (cf. 0.7).

0.1.2 Measuring angles

Degrees: In Figure 0.1, some of the most often used angles, measured in degrees, are shown. An angle of 90° is also called a *right angle.* In ancient Sumeria near the Euphrates and Tigris rivers, more than 4,000 years ago, a number system with the basis 60 (sexagesimal system) was used. One can trace back to this usage the fact that the numbers 12, 24, 60 and 360 are used in such an important way in our measurement of time and angles. In addition to the degree, other measures for angles used in, for example, astronomy are the following smaller measurements:

$$1' \quad \text{(arc minute)} = \frac{1^\circ}{60}\,,$$
$$1'' \quad \text{(arc second)} = \frac{1^\circ}{3\,600}\,.$$

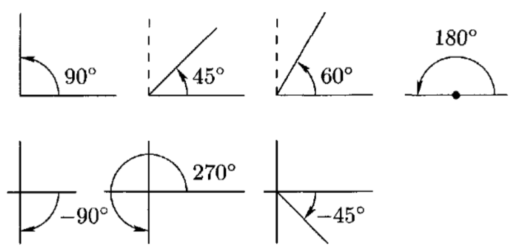


Figure 0.1. Postive and negative angles.

Example 1 (Astronomy): The face of the sun is about 30′ (half a degree) in the sky.

Because of the motion of the earth and the sun, the stars in the sky change their positions. Half the maximal change per year is called a *parallax.* This is equal to the angle α, which the star would appear to see between the earth and the sun when they are at maximal distance from each other (cf. Fig 0.2 and Table 0.2).

Table 0.2. Parallax and distance.

Star	*Parallax*	*Distance*
Proxima Centauri (nearest star)	0.765″	4.2 light years
Sirius (brightest star)	0.371″	8.8 light years

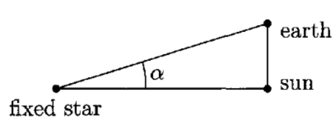


Figure 0.2.

A parallax of one arc second corresponds to a distance of 3.26 light years ($3.1\cdot 10^{13}$ km). This distance is also referred to as a *parsec.*

Radians: A angle of α degrees ($\alpha°$) corresponds to

$$\boxed{\alpha = 2\pi\left(\frac{\alpha°}{360°}\right)}$$

radians. Here α is the length of an arc on the unit circle which is cut out by the angle $\alpha°$ (Figure 0.3). In Table 0.3 one finds often-used values for this measurement.

Convention: Unless stated otherwise, all angles in this book will be measured in radians.

Table 0.3. Angles and radians.

Degrees	1°	45°	60°	90°	120°	135°	180°	270°	360°
Radians	$\frac{\pi}{180}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

$1' = \frac{\pi}{10\,800} = 0.000291, \quad 1'' = \frac{\pi}{648\,000} = 0.000005$

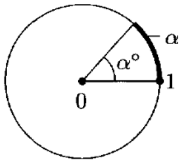


Figure 0.3.

Sum of angles in a triangle: In a triangle, the sum of the angles is always π, i.e.,

$$\boxed{\alpha + \beta + \gamma = \pi}$$

(cf. Figure 0.4).

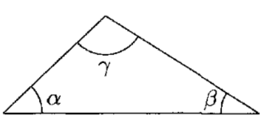


Figure 0.4. Angles in a triangle.

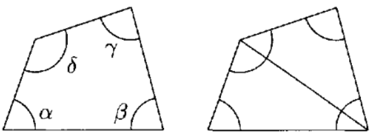
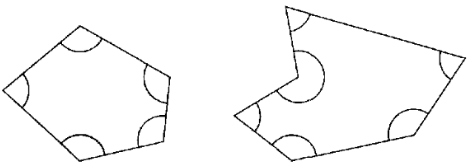


Figure 0.5. Angles in a quadrangle.

Sum of angles in a quadrangle: Since a rectangle can be decomposed into two triangles, the sum of angles must be 2π, i.e.,

$$\boxed{\alpha + \beta + \gamma + \delta = 2\pi}$$

(cf. Figure 0.5).



(a) pentagon **(b)** hexagon

Figure 0.6. Pentagon and hexagon.

Sum of angles in an n-gon: In general one has

$$\boxed{\text{Sum of the inside angles of a } n\text{-gon} = (n-2)\pi\,.}$$

Example 2: For a pentagon (5-gon) (resp. hexagon (6-gon)), the sum of the angels is 3π (resp. 4π) (Figure 0.6).

0.1.3 Area and circumference of plane figures

In Table 0.4 the most important plane figures are illustrated. The meaning of the appearing trigonometric functions $\sin\alpha$ and $\cos\alpha$ is explained in detail in 0.2.8.

Table 0.4. Surface area and circumference of several polygons.

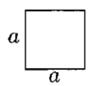
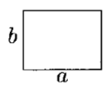
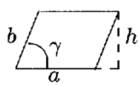
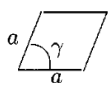
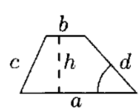
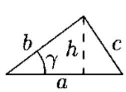
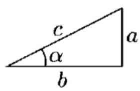
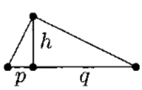
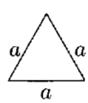
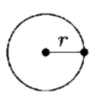
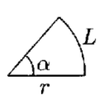
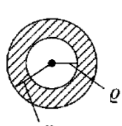
Figure	***Diagram***	***Area A***	***Circumference C***
square		$A = a^2$ (a length of a side)	$C = 4a$
rectangle		$A = ab$ (a, b lengths of the sides)	$C = 2a + 2b$
parallelogram		$\boxed{A = ah = ab\sin\gamma}$ (a length of the base, b length of the side h height)	$C = 2a + 2b$
rhombus (equilateral parallelogram)		$A = a^2 \sin\gamma$	$C = 4a$
trapezoid (quadrangle with two parallel sides)		$A = \frac{1}{2}(a+b)h$ (a, b length of the parallel sides, h height)	$C = a + b + c + d$
triangle		$\boxed{A = \frac{1}{2}ah = \frac{1}{2}ab\sin\gamma}$ (a length of the base, b, c length of the other sides, h height, $s := C/2$) *Heronian formula for the area:* $\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$	$C = a + b + c$

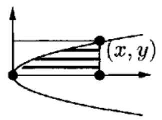
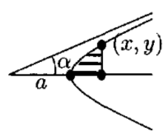
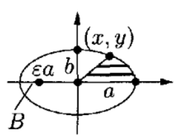
Table 0.4 (continued)

right triangle	 	$\boxed{A = \frac{1}{2}ab}$ *relation between sides and angles:* $\boxed{a = c\sin\alpha, \quad b = c\cos\alpha, \quad a = b\tan\alpha}$ (c hypotenuse [2], a opposite leg, b neighboring leg) *Theorem of Pythagoras*[3]*:* $\boxed{a^2 + b^2 = c^2}$ *Euclidean relation for the height:* $\boxed{h^2 = pq}$ (h height over the hypotenuse, p, q segment lengths)	$C = a + b + c$
equilateral triangle		$A = \frac{\sqrt{3}}{4}a^2$	$C = 3a$
circle		$\boxed{A = \pi r^2}$ (r radius)	$\boxed{C = 2\pi r}$
sector of a circle		$A = \frac{1}{2}\alpha r^2$	$C = L + 2r$, $L = \alpha r$
annulus		$A = \pi(r^2 - \varrho^2)$ (r outer radius, ϱ inner radius)	$C = 2\pi(r + \varrho)$

[2]In a right triangle the side which is opposite the right angle is called the *hypotenuse*. The other sides are called *catheti* or simply *legs*.

[3]Pythagoras from Samos (at 500 BC) is considered to be the founder of the famous school of the Pythagoreans in ancient Greece. The theorem of Pythagoras however was know almost 1,000 years before that, by the Babylonians under the regent King Hammurapi (1728–1686 BC).

Table 0.4 (continued)

parabola sector[4]		$A = \frac{1}{3}xy$	
hyperbola sector		$A = \frac{1}{2}\left(xy - ab \cdot \operatorname{arcosh}\frac{x}{a}\right)$ $(b = a\tan\alpha)$	
ellipse sector		$A = \frac{1}{2}ab \cdot \operatorname{arcosh}\frac{x}{a}$	
ellipse	diagram as above, where B is the focal point	$\boxed{A = \pi ab}$ (a, b lengths of the axi, $b < a$, ε numerical eccentricity)	$\boxed{C = 4aE(\varepsilon)}$ (cf. (0.5))

The meaning of elliptic integrals for the calculation of the circumference of the ellipse: The numerical eccentricity ε of an ellipse is given by the formula

$$\boxed{\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}.}$$

The geometric interpretation of ε is to be found in the fact that the focal point of the ellipse has a distance from the center of the ellipse of εa. For a circle, one has $\varepsilon = 0$. The larger the numerical eccentricity ε is, the flatter the ellipse is.

It was already noticed in the eighteenth century that the *circumference of an ellipse can not* be calculated by elementary means. This circumference is given by $C = 4aE(\varepsilon)$, where we use the notation

$$\boxed{E(\varepsilon) := \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 \varphi}\, \mathrm{d}\varphi} \tag{0.5}$$

for the *complete elliptic integral of the second kind* of Legendre. There are tabulated values for this integral (cf. 0.5.4). For an ellipse we always have that $0 \le \varepsilon < 1$. As

[4]Parabola, hyperbola und ellipse will be considered in 0.1.7. The function arcosh will be introduced in 0.2.12.

approximations for all these values one has the series

$$E(\varepsilon) = 1 - \left(\frac{1}{2}\right)^2 \varepsilon^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{\varepsilon^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{\varepsilon^6}{5} - \ldots$$
$$= 1 - \frac{\varepsilon^2}{4} - \frac{3\varepsilon^4}{64} - \frac{5\varepsilon^6}{256} - \ldots$$

The general theory of elliptic integrals was created in the nineteenth century (cf. 1.14.19).

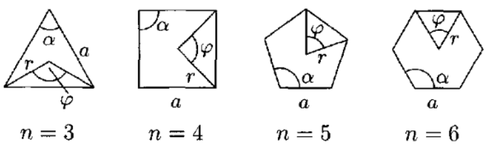


Figure 0.7. Regular n-gons.

Regular n-gons: A n-gon is said to be *regular*, if all the sides and angles are equal (Figure 0.7).

The distance from the center to one of the corners of the n-gon will be denoted by r. Then the geometry of a regular n-gon is determined by the following statements:

center angle	$\varphi = \frac{2\pi}{n}$,
complementary angle	$\alpha = \pi - \varphi$,
length of sides	$a = 2r \sin\frac{\varphi}{2}$,
circumference	$C = na$,
area	$A = \frac{1}{2}nr^2 \sin\varphi$.

Theorem of Gauss: A n-gon with $n \le 20$ can be constructed with the help of a ruler and compass, if and only if

$$\boxed{n = 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ 15,\ 16,\ 17,\ 20.}$$

In particular, such a construction is not possible for $n = 7,\ 9,\ 11,\ 13,\ 14,\ 18,\ 19$. This result is the consequence of Galois theory and will be considered in 2.6.6 in more detail.

0.1.4 Volume and surface area of solids

In Table 0.5 the most important three-dimensional figures are collected.

Table 0.5. Volume and surface area of some solids.

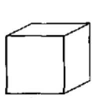
Solid	***Diagram***	***Volume V***	***Surface area O*** ***section area M***
cube		$V = a^3$ (a length of sides)	$O = 6a^2$

Table 0.5 (continued)

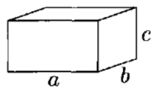
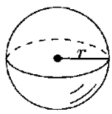
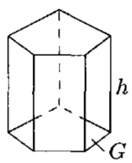
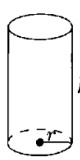
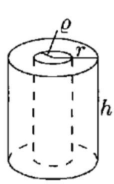
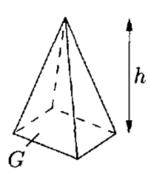
parallelepiped		$V = abc$ (a, b, c lengths of sides)	$O = 2(ab+bc+ca)$
ball		$\boxed{V = \frac{4}{3}\pi r^3}$ (r radius)	$\boxed{O = 4\pi r^2}$
prism		$\boxed{V = Gh}$ (G area of the base, h height)	
cylinder		$V = \pi r^2 h$ (r radius, h height)	$O = M + 2\pi r^2$, $M = 2\pi rh$
solid annulus		$V = \pi h(r^2 - \varrho^2)$ (r outer radius, ϱ inner radius, h height)	
pyramid		$\boxed{V = \frac{1}{3}Gh}$ (G area of the base, h height)	

Table 0.5 (continued)

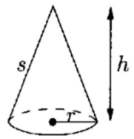
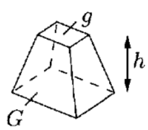
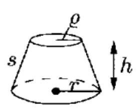
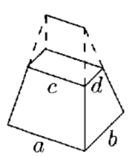
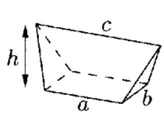
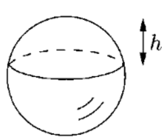
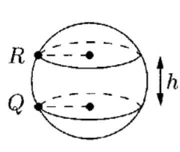
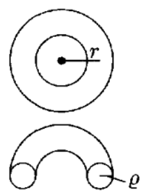
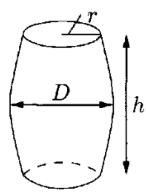
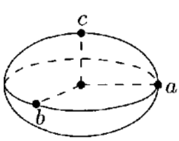
circular cone		$V = \frac{1}{3}\pi r^2 h$ (r radius, h height, s length of a meridian)	$O = M + \pi r^2$, $M = \pi r s$
capped pyramid		$V = \frac{h}{3}(G + \sqrt{Gg} + g)$ (G surface area of the base, g area of the top)	
capped cone		$V = \frac{\pi h}{3}(r^2 + r\varrho + \varrho^2)$ (r, ϱ radii, h height, s length of the side)	$O = M + \pi(r^2 + \varrho^2)$, $M = \pi s(r + \varrho)$
obelisk		$V = \frac{1}{6}(ab + (a+c)(b+d) + cd)$ (a, b, c, d lengths of the sides)	
wedge (the sides are equilateral triangles)		$V = \frac{\pi}{6} bh(2a + c)$ (a, b base side lengths, c upper edge, h height)	
section of a ball (bounded by a meridian)		$V = \frac{\pi}{3} h^2(3r - h)$ (r radius of the ball, h height)	$O = 2\pi r h$ (top part)
slice of a ball (bounded by two meridians)		$V = \frac{\pi h}{6}(3R^2 + 3\varrho^2 + h^2)$ (r radius of the ball, h height, R and ϱ radii of the meridians)	$O = 2\pi r h$ (middle layer)

Table 0.5 (continued)

torus		$V = 2\pi r^2 \varrho$ (r radius of the torus, ϱ radius of the section)	$O = 4\pi^2 r \varrho$
barrel (with circular section)		$V = 0.0873\, h(2D + 2r)^2$ (D diameter, r radius at the top, h height; the formula is an approximation)	
ellipsoid		$V = \frac{4}{3}\pi abc$ (a, b, c lengths of the axi, $c < b < a$)	see the formula of Legendre (L) for O

The meaning of elliptic integrals for the calculation of the surface area of the ellipsoid: The surface of an ellipsoid can not be calculated by elementary means. One requires again elliptic integrals. For this one has the formula of Legendre

$$\boxed{O = 2\pi c^2 + \frac{2\pi b}{\sqrt{a^2 - c^2}} \left(c^2 F(k, \varphi) + (a^2 - c^2) E(k, \varphi)\right)} \tag{L}$$

with

$$k = \frac{a}{b} \frac{\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}}, \qquad \varphi = \arcsin \frac{\sqrt{a^2 - c^2}}{a}.$$

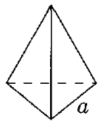
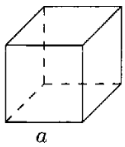
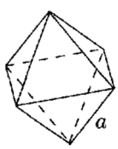
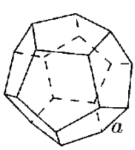
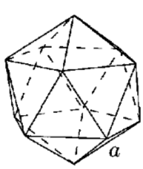
The formulas for the elliptic integrals $E(k, \varphi)$ and $F(k, \varphi)$ can be found in 0.5.4

0.1.5 Volumes and surface areas of regular polyhedra

Polyhedra: A *polyhedron* is a solid which is bounded by elementary parts (plane figures).

The *regular polyhedra* (also called *Platonic solids*) have faces, all of which are congruent, regular n-gons of side length a, in which at all corners the same number of faces meet. There are precisely 5 regular polyhedra, which are listed in Table 0.6.

Table 0.6. The five Platonic solids.[5]

Regular polyhedron		***Faces***	***Volume***	***Surface area***
tetrahedron		4 equilateral triangles	$\frac{\sqrt{2}}{12} \cdot a^3$	$\sqrt{3}a^2$
cube		6 squares	a^3	$6a^2$
octahedron		8 equilateral triangles	$\frac{\sqrt{2}}{3} \cdot a^3$	$2\sqrt{3} \cdot a^2$
dodecahedron		12 equilateral pentagons	$7.663 \cdot a^3$	$20.646 \cdot a^2$
icosahedron[6]		20 equilateral triangles	$2.182 \cdot a^3$	$8.660 \cdot a^2$

Euler's polyhedral formula: The following relation is generally true for regular polyhedra:[7]

$$\text{number of corners } c - \text{ number of edges } e + \text{number of faces } f = 2\,.$$

[5]In this table, the common length of an edge is denoted by a. The fomulas for the volumes and areas of the dodecahedron and the icosahedron are approximations.

[6]The German mathematician Felix Klein wrote an famous book about the symmetries of the icosahedron and its relation to the equations of fifth degree, (cf. [22]).

[7]This formula is a special case of a general topological fact. Since the surfaces of the regular polyhedra are all homeomorph to the sphere, they have genus 0 and the Euler characteristic 2.

Table 0.7 verifies this formula.

Table 0.7. The key numbers for the Platonic solids.

Regular polyhedron	c	e	f	$c+e-f$
tetrahedron	4	6	4	2
cube	8	12	6	2
octahedron	6	12	8	2
dodecahedron	20	30	12	2
icosahedron	12	30	20	2

0.1.6 Volume and surface area of n-dimensional balls

The following formulas are necessary in statistical physics. In these formulas, n is roughly of the size 10^{23}. For such large values of n, one uses the Stirling formula for an approximation to the value of $n!$ (cf. (0.1)).

Characterization of the solid ball by an inequality: The n-dimensional ball $K_n(r)$ of radius r with center at the origin is defined to be the set of all points $(x_1, \ldots, x_n)$ that satisfy the inequality

$$\boxed{x_1^2 + \ldots + x_n^2 \leq r^2.}$$

Here $x_1, \ldots, x_n$ are real numbers with $n \geq 2$. The boundary (surface) of this ball is formed by the set of all $(x_1, \ldots, x_n)$ which satisfy the inequality

$$\boxed{x_1^2 + \ldots + x_n^2 = r^2.}$$

For the volume V_n and the surface area O_n of $K_n(r)$ one has the following formulas of Jacobi:

$$\boxed{\begin{aligned} V_n &= \frac{\pi^{n/2} r^n}{\Gamma\left(\frac{n}{2}+1\right)}, \\ O_n &= \frac{2\pi^{n/2} r^{n-1}}{\Gamma\left(\frac{n}{2}\right)}. \end{aligned}}$$

The gamma function Γ is considered in section 1.14.16. It satisfies the recursion formula

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for all } x > 0$$

with $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. From this one gets for $m = 1, 2, \ldots$ the following formulas:

$$V_{2m} = \frac{\pi^m r^{2m}}{m!}, \qquad V_{2m+1} = \frac{2(2\pi)^m r^{2m+1}}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2m+1)}$$

and

$$O_{2m} = \frac{2\pi^m r^{2m-1}}{(m-1)!}, \qquad O_{2m+1} = \frac{2^{2m+1} m! \pi^m r^{2m}}{(2m)!}$$

Example: In the special case $n = 3$ and $m = 1$, one gets the well-known formulas

$$V_3 = \frac{4}{3}\pi r^3, \qquad O_3 = 4\pi r^2$$

for the volume V_3 and the surface area O_3 of the three-dimensional ball of radius r.

0.1.7 Basic formulas for analytic geometry in the plane

Analytic geometry describes geometric objects like lines, planes and conic sections by means of equations for the coordinates and investigates the geometric properties through manipulations with these inequalities. This process of increased use of arithmetic and algebra in geometry goes back to the philosopher, scientist and mathematician René Descartes (1596–1650), after whom the Cartesian coordinates have their name.

0.1.7.1 Lines

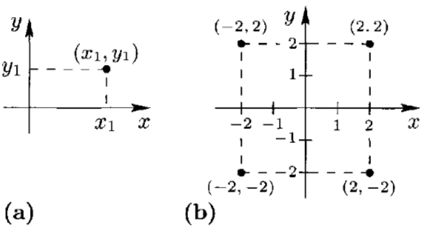


Figure 0.8. Cartesian coordinates.

All of the following formulas are in terms of a Cartesian coordinate system, in which the y-axis is perpendicular to the x-axis. The coordinates of a point (x_1, y_1) are given as in Figure 0.8(a). The x coordinate of a point left of the y-axis is negative, and the y coordinate of a point underneath the x-axis is also negative.

Example 1: The points $(2, 2)$, $(2, -2)$, $(-2, -2)$ and $(-2, 2)$ are found in Figure 0.8(b).

The distance d of the two points (x_1, y_1) and (x_2, y_2):

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

(Figure 0.9). This formula corresponds to the theorem of Pythagoras.

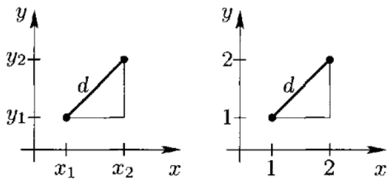


Figure 0.9. The distance between two points.

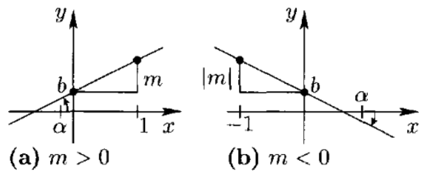


Figure 0.10. The equation of a line.

Example 2: The distance of the two points $(1,1)$ and $(2,2)$ is

$$d = \sqrt{(2-1)^2 + (2-1)^2} = \sqrt{2}\,.$$

The equation of a line:

$$\boxed{y = mx + b\,.} \tag{0.6}$$

Here b is the intersection of the line with the y-axis (*y-intersect)*, and the *slope* of the line is m (Figure 0.10). For the *slope angle* α one has the relation

$$\boxed{\tan\alpha = m\,.}$$

(i) If one knows a point $(x_1\,,\ y_1)$ of the line and the slope m, then one gets the missing value of b as $b = y_1 - mx_1$.

(ii) If one knows two points $(x_1\,,\ y_1)$ and $(x_2\,,\ y_2)$ on the line with $x_1 \neq x_2$, then:

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}\,, \qquad b = y_1 - mx_1\,.} \tag{0.7}$$

Example 3: The equation of the line through the two points $(1,1)$ and $(3,2)$ is

$$y = \frac{1}{2}x + \frac{1}{2}\,,$$

as by (0.7) we get $m = \dfrac{2-1}{3-1} = \dfrac{1}{2}$ and $b = 1 - \dfrac{1}{2} = \dfrac{1}{2}$ (Figure 0.11).

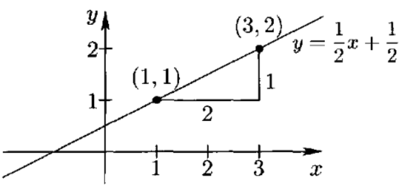


Figure 0.11. The slope of a line.

Abscissa equation of a line: If one divides the equation a line (0.6) by b and sets $\dfrac{1}{a} := -\dfrac{m}{b}$, then one gets:

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1\,.} \tag{0.8}$$

For $y = 0$ (resp. $x = 0$) one can read off from this that the line hits the x-axis at the point $(a, 0)$ (resp. the y-axis at the point $(0, b)$) (Figure 0.12(a)).

Example 4: If we divide the line equation

$$y = -8x + 4$$

by 4, it follows that $\dfrac{y}{4} = -2x + 1$ and consequently

$$2x + \frac{y}{4} = 1\,.$$

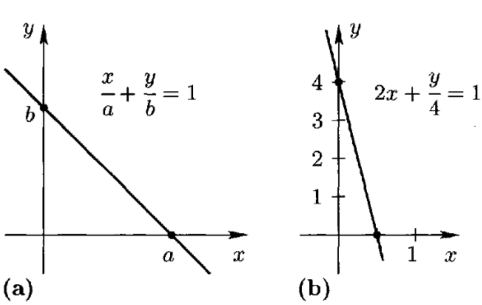


Figure 0.12. The abscissa of a line.

If we set $y = 0$, then we get $x = \dfrac{1}{2}$. Hence the line intersects the x-axis in the point $x = \dfrac{1}{2}$ (Figure 0.12(b)).

Equation of the y-axis:

$$\boxed{x = 0\,.}$$

This equation is not a special case of (0.6). It corresponds formally to a slope of $m = \infty$ (infinite slope).

General equation of a line: All lines are defined as the set of points satisfying the equation

$$\boxed{Ax + By + C = 0}$$

with real constants A, B and C, which satisfy the condition $A^2 + B^2 \neq 0$.

Example 5: For $A = 1$, $B = C = 0$ one gets the equation $x = 0$ of the y-axis.

Applications to linear algebra: A series of problems in analytic geometry are most easily solved by using the language of vectors (linear algebra). This will be considered in section 3.3.

0.1.7.2 Circles

The equation of a circle of radius r with center at the point (c, d):

$$\boxed{(x-c)^2 + (y-d)^2 = r^2} \tag{0.9}$$

(Figure 0.13(a)).

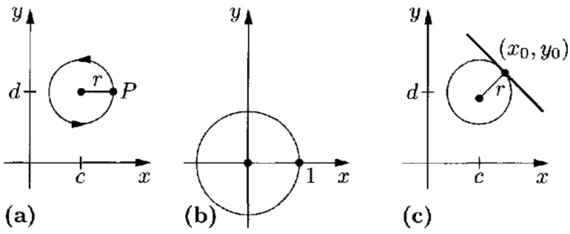


Figure 0.13. Circles in the plane.

Example: The equation of a circle of radius $r = 1$ with center at the origin $(0, 0)$ is (Figure 0.13(b)):

$$x^2 + y^2 = 1\,.$$

Equation of the tangent to a circle:

$$\boxed{(x-c)(x_0-c) + (y-d)(y_0-d) = r^2\,.}$$

This is the equation of the tangent to the circle (0.9) through the point (x_0, y_0) (Figure 0.13(c)).

Parameterization of the circle of radius r with center at the point (c, d):

$$\boxed{x = c + r\cos t\,, \qquad y = d + r\sin t\,, \qquad 0 \le t < 2\pi\,.}$$

If one interprets t as the time, then this starting point at $t = 0$ corresponds to the point P in Figure 0.13(a). In the time given by parameters $t = 0$ to $t = 2\pi$, the circle is transversed exactly once counter-clockwise with constant speed (*mathematical positive direction*).

Curvature K of a circle of radius R: By definition, one has

$$\boxed{K = \frac{1}{R}\,.}$$

0.1.7.3 Ellipse

The equation of an ellipse with center at the origin:

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.} \tag{0.10}$$

We assume $0 < b < a$. Then the ellipse lies symmetrically with respect to the origin. The length of the long (resp. short) axis of the ellipse is equal to a (resp. b) (Figure 0.14(a)). One also introduces the following quantities:

$$\begin{aligned} \text{linear eccentricity} \quad & e = \sqrt{a^2 - b^2}\,, \\ \text{numerical eccentricity} \quad & \varepsilon = \frac{e}{a}\,, \\ \text{half-parameter} \quad & p = \frac{b^2}{a}\,. \end{aligned}$$

The two points $(\pm e, 0)$ are called the *focal points* $B_\pm$ of the ellipse (Figure 0.14(a)).

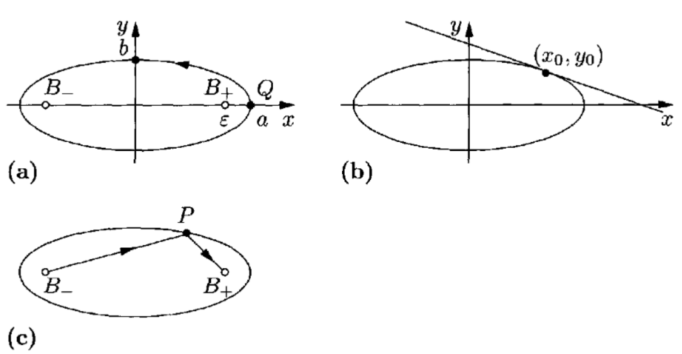


Figure 0.14. The ellipse.

Equation of a tangent to the ellipse:

$$\boxed{\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1\,.}$$

This is the equation of the tangent to the ellipse (0.10) through the point (x_0, y_0) (Figure 0.14(b)).

Parameterization of an ellipse:

$$\boxed{x = a\cos t\,, \qquad y = b\sin t\,, \qquad 0 \le t < 2\pi\,.}$$

When the parameter t runs through the values from 0 to 2π, the ellipse in (0.10) is run through once counter-clockwise. The starting value $t = 0$ corresponds to the point on the curve Q (Figure 0.14(a)).

Geometric characterization of an ellipse: An ellipse is by definition the set of points P, whose sum of distances from two given points B_- and B_+ is constant, equal to $2a$ (cf. Figure 0.14(c)).

These points are called the *focal points.*

Construction: To construct an ellipse, one fixes two points B_- and B_+ which are to serve as focal points. Then one fixes the ends of a piece of string with a thumbtack to these focal points, and moves the pencil with the help of the string, keeping the string taut. The pencil then has drawn an ellipse (Figure 0.14(c)).

Physical property of the focal points: A light ray which is sent from one of the focal points B_- and reflected on the ellipse, meets the other focal point B_+ (Figure 0.14(c)).

Surface area and circumference of an ellipse: See Table 0.4.

The equation of an ellipse in polar coordinates, directrix property and curvature radii: See section 0.1.7.6

0.1.7.4 Hyperbola

The equation of a hyperbola with center at the origin:

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,.} \tag{0.11}$$

Here a and b are positive constants.

Asymptotes of a hyperbola: A hyperbola intersects the x-axis in the points $(\pm a, 0)$. The two lines

$$y = \pm\frac{b}{a}x$$

are called the *asymptotes* of the hyperbola. These lines approach the branches of the hyperbola as one moves out from the origin (Figure 0.15(b)).

Focal points: We define

$$\text{linear eccentricity} \quad e = \sqrt{a^2 + b^2}\,,$$
$$\text{numerical eccentricity} \quad \varepsilon = \frac{e}{a}\,,$$
$$\text{half-parameter} \quad p = \frac{b^2}{a}\,.$$

The two points $(\pm e, 0)$ are called the *focal points* $B_\pm$ of the hyperbola (Figure 0.15(a)).

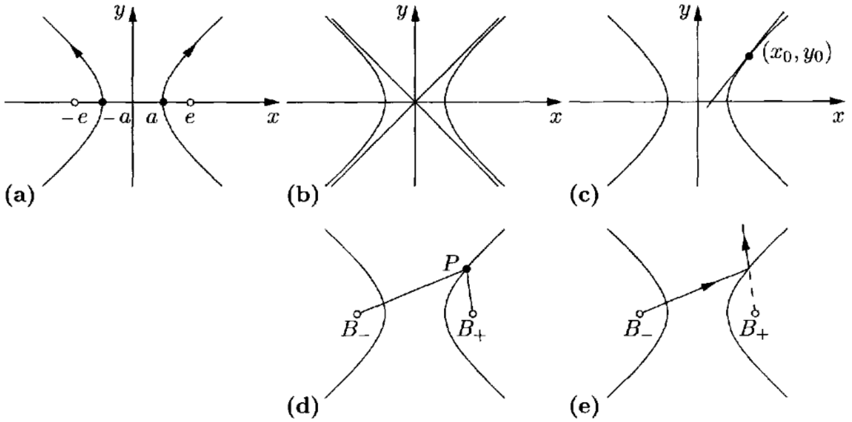


Figure 0.15. Properties of the hyperbola.

Equation for the tangents to a hyperbola:

$$\boxed{\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1\,.}$$

This is the equation of the tangent to the hyperbola (0.11) through the point (x_0, y_0) (Figure 0.15(c)).

Parameterization of a hyperbola[8]**:**

$$\boxed{x = a\cosh t\,, \qquad y = b\sinh t\,, \qquad -\infty < t < \infty\,.}$$

As the parameter t runs through all real values, the right branch of the hyperbola in Figure 0.15(a) is run through once in the direction of the arrow in that picture. The initial point at $t = 0$ is the point $(a, 0)$ on the hyperbola. Similarly, the left hyperbola branch in Figure 0.15(a) is run through once by the parameterization

$$x = -a\cosh t\,, \qquad y = b\sinh t\,, \qquad -\infty < t < \infty\,.$$

Geometric characterization of a hyperbola: By definition, a hyperbola consists of all points P whose difference of distances from two given points B_- and B_+ is constant, equal to $2a$ (cf. Figure 0.15(d)). These points are again called the *focal points.*

[8]The hyperbolic functions $\cosh t$ and $\sinh t$ are treated in detail in 0.2.10 .

Physical property of the focal points: A light ray emerging from B_- is reflected on the hyperbola in such a way that its backward extension passes through the other focal point B_+ (Figure 0.15(e)).

Surface area of a hyperbola section: See Table 0.4.

Equation of hyperbolas in polar coordinates, directrix properties and curvature radii: See section 0.1.7.6

0.1.7.5 Parabola

The equation of a parabola:

$$\boxed{y^2 = 2px\,.} \tag{0.12}$$

Here p is a positive constant (Figure 0.16). We define:

$$\text{linear eccentricity} \quad e = \frac{p}{2}\,,$$
$$\text{numerical eccentricity} \quad \varepsilon = 1\,.$$

The point $(e, 0)$ is called the *focal point* of the parabola (Figure 0.16(a)).

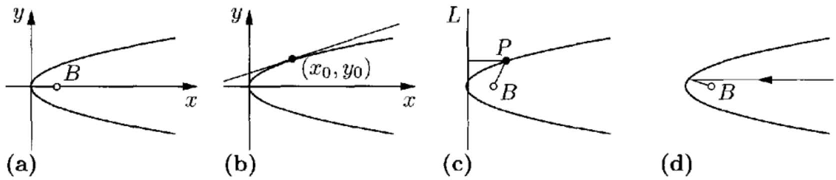


Figure 0.16. Properties of the parabola.

The equation of a tangent to a parabola:

$$\boxed{yy_0 = p(x + x_0)\,.}$$

This is the equation of the tangent to the parabola (0.12) through the point (x_0, y_0) (Figure 0.16(b)).

Geometric characterization of parabolas: By definition, a parabola consists of all points P, whose distance from a fixed point B (focal point) and a fixed line L (directrix) is equal (Figure 0.16(c)).

Physical property of the focal point (parabolic mirror): A light ray, which is parallel to the x-axis and hits the parabola, is reflected in such a way that it passes through the focal point (Figure 0.16(d)).

Surface area of a parabolic sector: See Table 0.4.

Equation of a parabola in polar coordinates and the curvature radii: See section 0.1.7.6

0.1.7.6 Polar coordinates and conic sections

Polar coordinates: Instead of Cartesian coordinates, often *polar coordinates* are used, in order to take advantage of the symmetry of the equations in certain problems. The polar coordinates (r, φ) of a point P in the plane are given as in Figure 0.17 by the distance r of the point P from the origin O and the angle φ of the line segment $\overline{OP}$ with the x-axis. The following relation between the Cartesian coordinates (x, y) and the polar coordinates (r, φ) of a point P hold:

$$\boxed{x = r\cos\varphi\,, \qquad y = r\sin\varphi\,, \qquad 0 \le \varphi < 2\pi\,.} \tag{0.13}$$

Moreover, one has

$$r = \sqrt{x^2 + y^2}\,, \qquad \tan\varphi = \frac{y}{x}\,.$$

Conic sections: By definition, a conic section is obtained by taking the section of a double circular cone with a plane (Figure 0.18). In this way, the following figures occur:

(i) *Regular conic sections:* Circle, ellipse, parabola or hyperbola.

(ii) *Degenerate conic sections:* two lines, one line or a point.

Equation of regular conic sections in polar coordinates:

$$\boxed{r = \frac{p}{1 - \varepsilon\cos\varphi}}$$

(cf. Table 0.8). The regular conic sections are characterized by the geometrical property, that they consists of all points P for which the relation

$$\frac{r}{d} = \varepsilon$$

is constant, equal to ε, where r is the distance from a fixed point B (focal point) and d is the distance from a fixed line L (directrix).

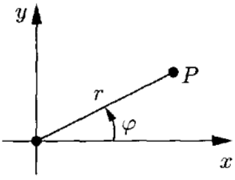


Figure 0.17.

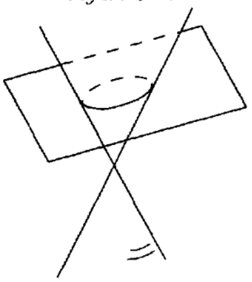
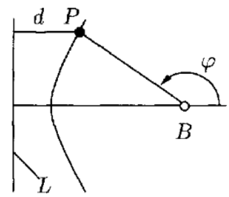
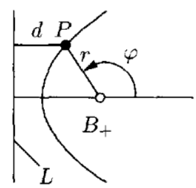
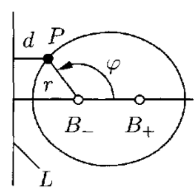
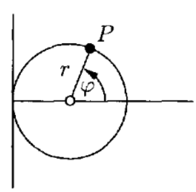


Figure 0.18.

Vertical circle and curvature radius: In the apex S of a regular conic sections, one can inscribe a circle in such a way, that it touches the conic section (i.e., has the same tangent as) at the point S. The radius of this vertical circle is called the *curvature radius* R at the point S. The same construction is possible at an arbitrary point $P(x_0, y_0)$ of the conic section (cf. Table 0.9). The curvature K at the point P is given by definition by the formula

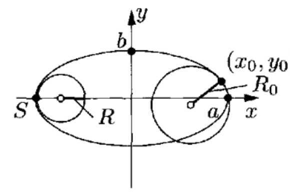
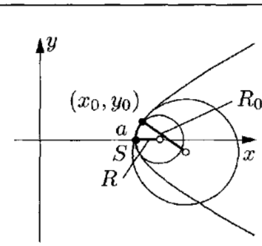
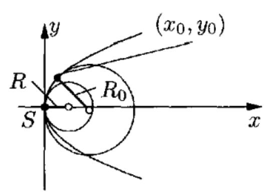
$$\boxed{K = \frac{1}{R_0}\,.}$$

Table 0.8. Regular conic sections.

Conic section	***Numerical eccentricity ε***	***Linear eccentricity e***	***Half-parameter p***	***Directrix-property*** $\frac{r}{d} = \varepsilon$
hyperbola[8]	$\varepsilon > 1$	$e = \frac{\varepsilon p}{(1-\varepsilon)^2}$	$p = \frac{b^2}{a}$	
parabola	$\varepsilon = 1$	$e = \frac{p}{2}$		
ellipse	$0 \leq \varepsilon < 1$	$e = \frac{\varepsilon p}{1-\varepsilon^2}$	$p = \frac{b^2}{a}$	
circle	$\varepsilon = 0$ (limiting case $d = \infty$)	$e = 0$	$p =$ radius r	

[8]Because of the inequalities $\varepsilon > 1$, $\varepsilon = 1$ and $\varepsilon < 1$, the Greek mathematician Appolonius of Perga (roughly 260–190 BC) introduced the nomenclature ὑπερβολή (hyperbolé which means excess), παραβολή (parabolé which means equality) and ἔλλειψις (élleipsis which means deficiency).

Table 0.9. Inscribed circles.

Conic	*Equation*	*Curvature radius*	*Diagram*
ellipse	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$	$R_0=a^2b^2\left(\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}\right)^{3/2}$, $R=\dfrac{b^2}{a}=p$	
hyperbola	$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$	$R_0=a^2b^2\left(\dfrac{x_0^2}{a^4}+\dfrac{y_0^2}{b^4}\right)^{3/2}$, $R=\dfrac{b^2}{a}=p$	
parabola	$y^2=2px$	$R_0=\dfrac{(p+2x_0)^{3/2}}{\sqrt{p}}$, $R=p$	

0.1.8 Basic formulas of analytic geometry of space

Cartesian coordinates in space: A spatial Cartesian coordinate system is given as shown in Figure 0.19 by three axi which are perpendicular to one another, which are denoted as the x-axis, y-axis and z-axis, and which are oriented in the same way as the thumb, the pointing finger and the middle finger of the right hand (right-handed system). The coordinates (x_1, y_1, z_1) of a point are determined by perpendicular projection onto the axi.

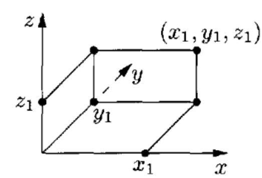


Figure 0.19.

Equation of a line through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$x = x_1 + t(x_2 - x_1)\,, \qquad y = y_1 + t(y_2 - y_1)\,, \qquad z = z_1 + t(z_2 - z_1)\,.$$

The parameter t runs through the real numbers and can be interpreted as the time (Figure 0.20(a)).

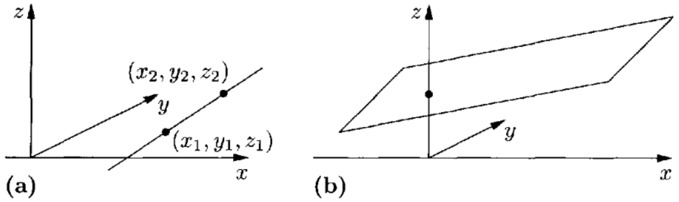


Figure 0.20. Equations for lines and planes in three-space.

Distance d between the two points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$\boxed{d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}\,.}$$

Equation of a plane:

$$\boxed{Ax + By + Cz = D\,.}$$

The real constants A, B and C must fulfill the condition $A^2 + B^2 + C^2 \neq 0$ (Figure 0.20(b)).

Applications of vector algebra to lines and planes in three-space: See 3.3.

0.1.9 Powers, roots and logarithms

Power laws: For all positive real numbers a, b and all real numbers x, y one has:

$$\boxed{\begin{array}{l} a^x a^y = a^{x+y}\,, \quad (a^x)^y = a^{xy}\,, \\ (ab)^x = a^x b^x\,, \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}\,, \quad a^{-x} = \frac{1}{a^x}\,. \end{array}}$$

It wasn't until after a long historical course of development that the notion of powers a^x for *arbitrary real exponents* was realized (cf. 0.2.7).

Important special cases: For $n = 1, 2, \ldots$ one has:

1. $a^0 = 1\,, \quad a^1 = a, \quad a^2 = a \cdot a, \quad a^3 = a \cdot a \cdot a. \quad \ldots$
2. $a^n = a \cdot a \cdot \ldots \cdot a$ (n factors).
3. $a^{-1} = \frac{1}{a}\,, \quad a^{-2} = \frac{1}{a^2}\,, \quad \ldots, \quad a^{-n} = \frac{1}{a^n}.$
4. $a^{\frac{1}{2}} = \sqrt{a}\,, \quad a^{\frac{1}{3}} = \sqrt[3]{a}.$

***n^{th}* roots:** Let a positive real number a be given. Then $x = a^{1/n}$ is the unique solution to the equation

$$\boxed{x^n = a\,, \qquad x \geq 0\,.}$$

In older literature the term $a^{1/n}$ is often denoted by $\sqrt[n]{a}$ (n^{th} root). In manipulations with expressions involving such roots it is better to use the expression $a^{1/n}$, since then one can use the general rules for powers and is not restricted to 'rules for roots'.

Example 1: From $\left(a^{\frac{1}{n}}\right)^{\frac{1}{m}} = a^{\frac{1}{mn}}$ the root law $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$ follows.

Limit relation for general powers: For $x = \dfrac{m}{n}$ with $m, n = 1, 2, \ldots$ the following relation holds:

$$a^x = \left(\sqrt[n]{a}\right)^m .$$

Moreover $a^{-x} = 1/a^x$. Hence the calculation of a^x for arbitrary *rational* exponents x can be *reduced* to the calculation of roots.

Now let an arbitrary real number x be given. We choose a number sequence[9] (x_k) of real numbers x_k with

$$\lim_{k\to\infty} x_k = x .$$

Then we have

$$\boxed{\lim_{k\to\infty} a^{x_k} = a^x .}$$

This is an expression of the *continuity* of the exponential function (cf. 1.3.1.2). If one chooses in particular a sequence x_k of rational numbers x_k, then the expressions a^{x_k} can be expressed in terms of powers of roots, and a^x is approximated for larger and larger k more and more accurately.

Example 2: The approximate value of π is given by $\pi = 3.14\ldots$ Therefore we have

$$a^{3.14} = a^{314/100} = \left(\sqrt[100]{a}\right)^{314}$$

is an approximation to the number a^π. Better and better approximations for a^π can be obtained by incorporating more and more decimal places in the decimal representation $\pi = 3.14\,15\,92\ldots$

The logarithm: Let a be a fixed, positive real number $a \neq 1$. For each given positive real number y the equation

$$\boxed{y = a^x}$$

has a unique real solution x, which is denoted by

$$\boxed{x = \log_a y}$$

and is called the *logarithm of y to base a.*[10]

Laws for logarithms: For all positive real numbers c, d and all real numbers x one has:

$$\boxed{\begin{array}{lll} \log_a(cd) = \log_a c + \log_a d\,, & \log_a\left(\dfrac{c}{d}\right) = \log_a c - \log_a d\,, & \\ \log_a c^x = x\log_a c\,, & \log_a a = 1\,, & \log_a 1 = 0\,. \end{array}}$$

From the relation $\log(cd) = \log c + \log d$ one sees that the logarithm has the fundamental property that multiplication of two numbers corresponds to the *addition* of the logarithms of those numbers.

[9] Limits of sequences of real numbers will be considered in 1.2.

[10] The word logarithm has a Greek root and means 'ratio number'.

Historical remark: In his monograph *Arithmetica integra* (Collected arithmetic), Michael Stifel noted in 1544 the the comparison of

$$1 \quad a \quad a^2 \quad a^3 \quad a^4 \quad \ldots$$

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots$$

allows the reduction of the multiplication of the numbers in the first row to the addition of the powers in the second row. This is precisely the basic idea of calculations with logarithms. Stifel remarks on this: "One could write an entire book on the properties of these wonderful numbers, but I have to be modest and close my eyes to this at this point." In the year 1614 the Scotch nobleman John Neper (or Napier) published the first incomplete tables of logarithms (with a base proportional to 1/e). These tables were improved bit for bit. After discussions with Henry Briggs, Neper agreed to use the basis 10 for all logarithms. In 1617 Briggs published a table of logarithms up to 14 decimal places (to base $a = 10$). The appearance of these tables was of great help to Kepler in the completion of his famous "Rudolfian tables" in 1624 (cf. 0.1.12). He propagated the advantages of this powerful new method of calculation with ardent zeal.

In our modern times with the widespread use of computers these tables are no longer of importance and represent a historical episode.

Natural logarithms: Logarithms $\log_{\mathrm{e}} y$ to base e are referred to as *natural logarithms* (logarithmus naturalis) and are denoted $\ln y$. If $a > 0$ is an arbitrary base, then one has the relation

$$\boxed{a^x = \mathrm{e}^{x \ln a}}$$

for all real numbers x. If one knows the natural logarithm, then one can find the logarithm to an arbitrary base by means of the formula

$$\boxed{\log_a y = \frac{\ln y}{\ln a}.}$$

Example 3: For $a = 10$ one has $\ln a = 2.302585\ldots$ and $\dfrac{1}{\ln a} = 0.434294\ldots$

In 1.12.1 we will give applications of the function $y = \mathrm{e}^x$ with the help of differential equations to radioactive decay and growth process. These examples show that the Euler number $\mathrm{e} = 2.718283\ldots$ is the natural base for the exponential function. The inverse of $y = \mathrm{e}^x$ gives $x = \ln y$. This motivates the nomenclature 'natural logarithm'.

0.1.10 Elementary algebraic formulas

0.1.10.1 The geometric and arithmetic series

Summation symbol and product symbol: We define

$$\sum_{k=0}^{n} a_k := a_0 + a_1 + a_2 + \ldots + a_n$$

and

$$\prod_{k=0}^{n} a_k := a_0 a_1 a_2 \ldots a_n \,.$$

The finite geometric series:

$$a + aq + aq^2 + \ldots + aq^n = a\frac{1 - q^{n+1}}{1 - q}, \qquad n = 1, 2, \ldots \tag{0.14}$$

This formula is valid for all real or complex numbers a and q with $q \neq 1$. The geometric series (0.14) is characterized by the fact that the *quotient* of two successive terms is a constant. With the help of the summation symbol one can write (0.14) in the form

$$\sum_{k=0}^{n} aq^k = a\frac{1 - q^{n+1}}{1 - q}, \qquad q \neq 1, \quad n = 1, 2, \ldots$$

Example 1: $1 + q + q^2 = \dfrac{1 - q^3}{1 - q} \qquad (q \neq 1)$.

The arithmetic series:

$$a + (a + d) + (a + 2d) + \ldots + (a + nd) = \frac{n+1}{2}(a + (a + dn)). \tag{0.15}$$

The arithmetic series (0.15) is characterized by the property that the *difference* of two successive terms is a constant. In words:

The sum of an arithmetic series is equal to the sum of the first and the last term multiplied by half the total number of terms.

With the help of the summation symbol, the formula (0.15) can be written:

$$\sum_{k=0}^{n} (a + kd) = \frac{n+1}{2}(a + (a + nd)).$$

Arithmetic series can be found in ancient texts of Babylonian and Egyptian times (around 2000 BC). Geometric series and the formula for the sum are found in Euclid's *Elements* (around 300 BC).

Example 2: It is reported that the teacher of the young Gauss (1777–1855) wanted a relaxing day by giving his students the assignment of adding the numbers 1 to 40. Just after assigning this, the little boy Gauss (who was to become one of the greatest mathematicians of all times) came to the teachers desk with his slate and the result of 820. It apparently was immediately clear to the youngster that instead of the original series $1 + 2 + \ldots + 40$ one should rather consider

$$\begin{array}{cccc} 1 & 2 & 3 \ldots & 40 \\ 40 & 39 & 38 \ldots & 1. \end{array}$$

Here we have 40 pairs of numbers (those in columns above) whose sum is 41. Consequently, the sum of the first series is half of the sum of these pairs, i.e., $20 \cdot 41 = 820$.

This is an example for a moment of inspiration in mathematics. A problem who initially seems to be quite complicated is reduced by some elegant trick to a different, easier problem which is quickly solved.

0.1.10.2 Calculations with the summation and product symbols

Summation symbol: The following manipulations are often applied:

1. $\sum_{k=0}^{n} a_k = \sum_{j=0}^{n} a_j$, (change of summation index).

2. $\sum_{k=0}^{n} a_k = \sum_{j=N}^{n+N} a_{j-N}$, (shift of the summation index; $j = k + N$).

3. $\sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k = \sum_{k=0}^{n} (a_k + b_k)$, (rule for addition).

4. $\Big(\sum_{j=1}^{m} a_j\Big)\Big(\sum_{k=1}^{n} b_k\Big) = \sum_{j=1}^{m}\sum_{k=1}^{n} a_j b_k$, (distributive law).

5. $\sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk} = \sum_{k=1}^{n}\sum_{j=1}^{m} a_{jk}$, (commutative law).

Product symbol: Analogously to the summations symbol one has the following properties of the product symbol:

1. $\prod_{k=0}^{n} a_k = \prod_{j=0}^{n} a_j$.

2. $\prod_{k=0}^{n} a_k = \prod_{j=N}^{n+N} a_{j-N}$.

3. $\prod_{k=0}^{n} a_k \prod_{k=0}^{n} b_k = \prod_{k=0}^{n} a_k b_k$.

4. $\prod_{j=1}^{m}\prod_{k=1}^{n} a_{jk} = \prod_{k=1}^{n}\prod_{j=1}^{m} a_{jk}$.

0.1.10.3 The binomial formula

Three classical binomial formulas:

$(a+b)^2 = a^2 + 2ab + b^2$,	(first binomial formula),
$(a-b)^2 = a^2 - 2ab + b^2$,	(second binomial formula),
$(a-b)(a+b) = a^2 - b^2$,	(third binomial formula).

These formulas are valid for all real or complex numbers a and b. The second binomial formula is actually a consequence of the first, by replacing b by $-b$.

The general third binomial formula: One has

$$\sum_{k=0}^{n} a^{n-k}b^k = a^n + a^{n-1}b + \ldots + ab^{n-1} + b^n = \frac{a^{n+1} - b^{n+1}}{a - b}$$

for all $n = 1, 2, \ldots$ and all real or complex numbers a and b with $a \neq b$.

Binomial coefficients: For all $k = 1, 2, \ldots$ and all real numbers α we set

$$\binom{\alpha}{k} := \frac{\alpha}{1} \cdot \frac{(\alpha - 1)}{2} \cdot \frac{(\alpha - 2)}{3} \cdot \ldots \cdot \frac{(\alpha - k + 1)}{k}.$$

Furthermore let

$$\binom{\alpha}{0} := 1.$$

Example 1: $\binom{3}{2} = \frac{3 \cdot 2}{1 \cdot 2} = 3, \quad \binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10.$

The general first binomial formula (binomial theorem):

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{n-1} ab^{n-1} + b^n. \qquad (0.16)$$

This fundamental formula of elementary mathematics is valid for all $n = 1, 2, \ldots$ and all real or complex a and b. With the help of the summation symbol, (0.16) can be written:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k. \qquad (0.17)$$

The general second binomial formula:

$$(a - b)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a^{n-k}b^k.$$

This formula follows immediately from (0.17) upon replacing b by $-b$.

Pascal triangle: In Table 0.10 each coefficient is obtained as the sum of the two coefficients lying above the given one. This gives a convenient way to obtain the coefficients for the general binomial formulas.

Example 2:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$
$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

The Pascal triangle is named after Blaise Pascal (1623–1662), who at the age of 20 built the first addition machine. The modern computer language *Pascal* is named in his honor. One can also find the Pascal triangles for $n = 1, \ldots, 8$ in the Chinese monograph *The precious mirror of four elements* by Chu Shih–Chieh, written in 1303.

Table 0.10. Pascal's triangle.

	Coefficients of the binomial formulas										
$n = 0$						1					
$n = 1$					1		1				
$n = 2$				1		2		1			
$n = 3$			1		3		3		1		
$n = 4$		1		4		6		4		1	
$n = 5$	1		5		10		10		5		1

Newton's binomial series for real exponents: The 24-year old Isaac Newton (1643–1727) found by intuitive reasoning the general formula for the series:

$$\boxed{(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \ldots = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k\,.} \tag{0.18}$$

For $\alpha = 1, 2, \ldots$, the infinite series (0.18) is actually *finite* an is nothing but the binomial formula.

Theorem of Euler (1774): The binomial series converges for all *real exponents* α and all complex numbers x with $|x| < 1$.

It had been attempted for a long time to prove the convergence of this series. It wasn't until Euler was 67 that he succeeded, more than one hundred years after Newton's discovery of the series.

The polynomial theorem: This theorem generalizes the binomial theorem to more than two summands. Special cases are:

$$\begin{aligned}(a+b+c)^2 &= a^2+b^2+c^2+2ab+2ac+2bc\,,\\ (a+b+c)^3 &= a^3+b^3+c^3+3a^2b+3a^2c+3b^2c\\ &\quad +6abc+3ab^2+3ac^2+3bc^2\,.\end{aligned}$$

The general form of this theorem for arbitrary real or complex non-vanishing numbers $a_1, \ldots, a_N$ and natural numbers $n = 1, 2, \ldots$ is:

$$\boxed{(a_1+a_2+\ldots+a_N)^n = \sum_{m_1+\ldots+m_N=n} \frac{n!}{m_1!m_2!\cdots m_N!}\, a_1^{m_1}a_2^{m_2}\cdots a_N^{m_N}\,.}$$

The summation here is over all N-tuples $(m_1, m_2, \ldots, m_N)$ of natural numbers running from 0 to n and whose sum is n. Moreover $n! = 1 \cdot 2 \cdot \ldots \cdot n$.

Properties of binomial coefficients: For natural numbers n, k with $0 \le k \le n$ and real or complex numbers α, β one has:

(i) *symmetry law*

$$\boxed{\binom{n}{k} = \binom{n}{n-k} = \frac{n!}{k!(n-k)!}\,.}$$

(ii) *addition law*[11]

$$\boxed{\binom{\alpha}{k} + \binom{\alpha}{k+1} = \binom{\alpha+1}{k+1}\,,} \tag{0.19}$$

$$\binom{\alpha}{0} + \binom{\alpha+1}{1} + \ldots + \binom{\alpha+k}{k} = \binom{\alpha+k+1}{k}\,,$$
$$\binom{\alpha}{0}\binom{\beta}{k} + \binom{\alpha}{1}\binom{\beta}{k-1} + \ldots + \binom{\alpha}{k}\binom{\beta}{0} = \binom{\alpha+\beta}{k}\,.$$

Example 3: If we set $\alpha = \beta = k = n$ in the last equation, then from the symmetry law we get the relation:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}\,.$$

From the binomial theorem for $a = b = 1$ and $a = -b = 1$ we get:

$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n\,,$$
$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0\,.$$

0.1.10.4 Sums of powers and Bernoulli numbers

Sums of natural numbers:

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}\,,$$
$$\sum_{k=1}^{n} 2k = 2 + 4 + \ldots + 2n = n(n+1)\,,$$
$$\sum_{k=1}^{n} (2k-1) = 1 + 3 + \ldots + (2n-1) = n^2\,.$$

Sums of squares:

$$\boxed{\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}\,,}$$

$$\sum_{k=1}^{n} (2k-1)^2 = 1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}\,.$$

[11] The Pascal triangle is based on the formula (0.19).

Sums of third and fourth powers:

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4},$$

$$\sum_{k=1}^{n} k^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Bernoulli numbers: Jacob Bernoulli (1645–1705) ran across these numbers as he attempted calculating an empirical formula for the sums of powers

$$S_n^p := 1^p + 2^p + \ldots + n^p$$

of natural numbers. He found for $n = 1, 2, \ldots$ and for the exponents $p = 1, 2, \ldots$ the general formula:

$$\boxed{S_n^p = \frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + \frac{B_2}{2}\binom{p}{1}n^{p-1} + \frac{B_3}{3}\binom{p}{2}n^{p-2} + \ldots + \frac{B_p}{p}\binom{p}{p-1}n.}$$

He also noticed that the sum of the coefficients always turns out to equal 1, i.e, we have

$$\frac{1}{p+1} + \frac{1}{2} + \frac{B_2}{2}\binom{p}{1} + \frac{B_3}{3}\binom{p}{2} + \ldots + \frac{B_p}{p}\binom{p}{p-1} = 1.$$

From this one gets for $p = 2, 3, \ldots$ successively the Bernoulli numbers $B_2, B_3, \ldots$ One also sets $B_0 := 1$ and $B_1 := -1/2$ (see Table 0.11). For odd numbers $n \geq 3$ one has $B_n = 0$. The recursion formula can also be written in the form

$$\boxed{\sum_{k=0}^{n}\binom{p+1}{k}B_k = 0.}$$

Symbolically, this equation is

$$(1+B)^{p+1} - B_{p+1} = 0,$$

if one agrees to replace B^n by B_n after multiplying out the expression on the left.

Table 0.11. Bernoulli numbers B_k ($B_3 = B_5 = B_7 = \ldots = 0$).

k	B_k	k	B_k	k	B_k	k	B_k
0	1	4	$-\frac{1}{30}$	10	$\frac{5}{66}$	16	$-\frac{3617}{510}$
1	$-\frac{1}{2}$	6	$\frac{1}{42}$	12	$-\frac{691}{2730}$	18	$\frac{43\,867}{798}$
2	$\frac{1}{6}$	8	$-\frac{1}{30}$	14	$\frac{7}{6}$	20	$-\frac{174\,611}{330}$

Example:

$$S_n^1 = \frac{1}{2}n^2 + \frac{1}{2}n\,,$$
$$S_n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\,,$$
$$S_n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\,,$$
$$S_n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n\,.$$

In addition one has:

$$\boxed{\frac{S_n^p}{p!} = \frac{B_0(n+1)^{p+1}}{0!(p+1)!} + \frac{B_1(n+1)^p}{1!p!} + \frac{B_2(n+1)^{p-1}}{2!(p-1)!} + \ldots + \frac{B_p(n+1)}{p!1!}\,.}$$

Bernoulli numbers and infinite series: For all complex numbers x with $0 < |x| < 2\pi$, one has:

$$\boxed{\frac{x}{\mathrm{e}^x - 1} = \frac{B_0}{0!} + \frac{B_1}{1!}x + \frac{B_2}{2!}x^2 + \ldots = \sum_{k=0}^{\infty} \frac{B_k}{k!}x^k\,.}$$

Bernoulli numbers also appear in the power series expansion of the functions

$$\tan x\,,\ \cot x\,,\ \tanh x\,,\ \coth x\,,\ \frac{1}{\sin x}\,,\ \frac{1}{\sinh x}\,,$$
$$\ln|\tan x|\,,\ \ln|\sin x|\,,\ \ln\cos x$$

(cf. 0.7.2).

Bernoulli numbers also play an important role in the summation of the inverses of powers of natural numbers. Euler discovered in 1734 the famous formula

$$\boxed{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\,.}$$

More generally, Euler discovered for $k = 1, 2, \ldots$ the values[12]:

$$\boxed{1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(2\pi)^{2k}}{2(2k)!}|B_{2k}|\,.}$$

Even earlier, the brothers Johann and Jakob Bernoulli had tried for a long time to determine the value of these series.

[12] For this, Euler used the product formula

$$\sin \pi x = \pi x \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2}\right)\,,$$

which he had discovered and which holds for all complex numbers x; this is in fact a generalization of the fundamental theorem of algebra (cf. 2.1.6) to the sine function

0.1.10.5 The Euler numbers

Defining relations: For all complex numbers x with $|x| < \frac{\pi}{2}$, the infinite series

$$\boxed{\frac{1}{\cosh x} = 1 + \frac{E_1}{1!}x + \frac{E_2}{2!}x^2 + \ldots = \sum_{k=0}^{\infty} \frac{E_k}{k!}x^k}$$

converges. The coefficients E_k which occur in this series are called *Euler numbers* (cf. Table 0.12). One has $E_0 = 1$ and for odd n, $E_n = 0$. The Euler numbers satisfy the symbolic equation

$$\boxed{(E+1)^n + (E-1)^n = 0\,, \qquad n = 1, 2, \ldots\,,}$$

in which one agrees to replace E^n by E_n after the multiplication has been carried out. This gives a convenient recursion formula for the E_n. The relation between the Euler and the Bernoulli numbers is, again in symbolic form, given by:

$$E_{2n} = \frac{4^{2n+1}}{2n+1}\left(B_n - \frac{1}{4}\right)^{2n+1}\,, \qquad n = 1, 2, \ldots$$

Table 0.12. The Euler numbers E_k ($E_1 = E_3 = E_5 = \ldots = 0$).

k	E_k	k	E_k	k	E_k
0	1	6	−61	12	2,702,765
2	−1	8	1,385	14	−199,360,981
4	5	10	−50,521		

Euler numbers and infinite series: The Euler numbers occur in the power series expansion of the functions

$$\frac{1}{\cosh x}\,, \quad \frac{1}{\cos x}$$

(cf. 0.7.2). For $k = 1, 2, \ldots$ one has in addition the formula

$$\boxed{1 - \frac{1}{3^{2k+1}} + \frac{1}{5^{2k+1}} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} = \frac{\pi^{2k+1}}{2^{2k+2}(2k)!}|E_{2k}|\,.}$$

0.1.11 Important inequalities

The rules for manipulations with inequalities can be found in section 1.1.5

The triangle inequality[13]:

$$\boxed{\big||z| - |w|\big| \le |z - w| \le |z| + |w| \quad \text{for all } z, w \in \mathbb{C}\,.}$$

[13]The statement 'for all $a \in \mathbb{R}$' means that the formula is valid for all real numbers a. The statement 'for all $z \in \mathbb{C}$' means that the statement is valid for all complex numbers. Note that each real number is also a complex number. The absolute value $|z|$ of a real or complex number is introduced in 1.1.2.1.

In addition one has for n complex summands $x_1, \ldots, x_n$ the triangle inequality

$$\boxed{\left|\sum_{k=1}^{n} x_k\right| \le \sum_{k=1}^{n} |x_k|\,.}$$

The Bernoulli inequality: For all real numbers $x \ge -1$ and $n = 1, 2, \ldots$ one has

$$\boxed{(1+x)^n \ge 1 + nx\,.}$$

The binomial inequality:

$$\boxed{|ab| \le \frac{1}{2}\left(a^2 + b^2\right) \quad \text{for all } a, b \in \mathbb{R}\,.}$$

The inequality for means: For all positive real numbers c and d one has:

$$\boxed{\frac{2}{\dfrac{1}{c} + \dfrac{1}{d}} \le \sqrt{cd} \le \frac{c+d}{2} \le \sqrt{\frac{c^2 + d^2}{2}}\,.}$$

The means which appear here are called, from left to right, harmonic mean, geometric mean, arithmetic mean and quadratic mean. All these means lie in between the two values $\min\{c, d\}$ and $\max\{c, d\}$, which justifies the term mean.[14]

Inequality for general means: For positive real numbers $x_1, \ldots, x_n$ one has:

$$\boxed{\min\{x_1, \ldots, x_n\} \le h \le g \le m \le s \le \max\{x_1, \ldots, x_n\}\,.}$$

In this formula we have used the notations:

$$m := \frac{x_1 + x_2 + \ldots + x_n}{n} = \frac{1}{n}\sum_{k=1}^{n} x_k, \qquad \text{(arithmetic mean or mean value)},$$

$$g := (x_1 x_2 \ldots x_n)^{1/n} = \left(\prod_{k=1}^{n} x_k\right)^{1/n}, \qquad \text{(geometric mean)},$$

$$h := \frac{n}{\dfrac{1}{x_1} + \ldots + \dfrac{1}{x_n}}, \qquad \text{(harmonic mean)}$$

and

$$s := \left(\frac{1}{n}\sum_{k=1}^{n} x_k^2\right)^{1/2}, \qquad \text{(quadratic mean)}.$$

[14]The terms $\min\{c, d\}$ (resp. $\max\{c, d\}$) denote the smallest (resp. the largest) of the two numbers c and d.

The Young inequality: One has

$$\boxed{|ab| \le \frac{|a|^p}{p} + \frac{|b|^q}{q} \quad \text{for all } a, b \in \mathbb{C}} \tag{0.20}$$

and all real exponents p and q which satisfy $p, q > 1$ and

$$\frac{1}{p} + \frac{1}{q} = 1 \,.$$

In the special case $p = q = 2$ the Young inequality is nothing but the binomial inequality. If $n = 2, 3, \ldots$, then the general Young inequality is valid:

$$\boxed{\left|\prod_{k=1}^{n} x_k\right| \le \sum_{k=1}^{n} \frac{|x_k|^{p_k}}{p_k} \quad \text{for all } x_k \in \mathbb{C}} \tag{0.21}$$

and all real exponents $p_k > 1$ with $\displaystyle\sum_{k=1}^{n} \frac{1}{p_k} = 1$.

The Schwarz inequality:

$$\boxed{\left|\sum_{k=1}^{n} x_k y_k\right| \le \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} |y_k|^2\right)^{1/2} \quad \text{for all } x_k, y_k \in \mathbb{C} \,.}$$

The Hölder inequality[15]**:** One has

$$\boxed{\big|(x|y)\big| \le \|x\|_p \|y\|_q \quad \text{for all } x, y \in \mathbb{C}^N}$$

and all real exponents $p, q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. The notations used are defined as follows:

$$(x|y) := \sum_{k=1}^{N} \overline{x}_k y_k \qquad \text{and} \qquad \|x\|_p := \left(\sum_{k=1}^{N} |x_k|^p\right)^{1/p}$$

as well as

$$\|x\|_\infty := \max_{1 \le k \le N} |x_k| \,.$$

The notation $\overline{x}_k$ denotes the complex conjugate number to x_k (cf. 1.1.2).

The Minkowski inequality:

$$\boxed{\|x + y\|_p \le \|x\|_p + \|y\|_p \quad \text{for all } x, y \in \mathbb{C}^N \,, \; 1 \le p \le \infty \,.}$$

Jensen's inequality:

$$\boxed{\|x\|_p \le \|x\|_r \quad \text{for all } x \in \mathbb{C}^N \,, \; 0 < r < p \le \infty \,.}$$

[15] The statement 'for all $x \in \mathbb{C}^N$' means 'for all N-tuples $(x_1, \ldots, x_N)$ of complex numbers x_k'.

Integral inequalities: The following inequalities hold, provided the integral on the right hand side exists (and is therefore finite)[16]. In addition, the real coefficients $p, q > 1$ should satisfy the condition $\frac{1}{p} + \frac{1}{q} = 1$. Then:

(i) *triangle inequality*

$$\left| \int_G f \mathrm{d}x \right| \leq \int_G |f(x)| \mathrm{d}x \,.$$

(ii) *Hölder inequality*

$$\left| \int_G f(x)g(x)\mathrm{d}x \right| \leq \left(\int_G |f(x)|^p \mathrm{d}x \right)^{1/p} \left(\int_G |g(x)|^q \mathrm{d}x \right)^{1/q} .$$

In the special case $p = q = 2$ this reduces to the Schwarz inequality.

(iii) *Minkowski inequality* $(1 \leq r < \infty)$

$$\left(\int_G \left| f(x) + g(x) \right|^r \mathrm{d}x \right)^{1/r} \leq \left(\int_G \left| f(x) \right|^r \mathrm{d}x \right)^{1/r} + \left(\int_G \left| g(x) \right|^r \mathrm{d}x \right)^{1/r} .$$

(iv) *Jensen's inequality* $(0 < p < r < \infty)$

$$\left(\int_G (\left| f(x) \right|^p \mathrm{d}x \right)^{1/p} \leq \left(\int_G \left| f(x) \right|^r \mathrm{d}x \right)^{1/r} .$$

The Jensen convexity inequality: Let $m = 1, 2, \ldots$ If the real valued function $F : \mathbb{R}^N \to \mathbb{R}$ is convex, then

$$F\left(\sum_{k=1}^{m} \lambda_k x_k \right) \leq \sum_{k=1}^{m} \lambda_k F(x_k)$$

for all $x_k \in \mathbb{R}^N$ and all non-negative real coefficients λ_k with $\sum_{k=1}^{m} \lambda_k = 1$ (cf. 1.4.5.5).

The Jensen convexity inequality for integrals:

$$F\left(\frac{\int_G p(x)g(x)\mathrm{d}x}{\int_G p(x)\mathrm{d}x} \right) \leq \frac{\int_G p(x)F(g(x))\mathrm{d}x}{\int_G p(x)\mathrm{d}x} . \tag{0.22}$$

Here it is assumed that:

[16]These formulas hold under very general assumptions. One can use the classical one-dimensional integral (Riemann integral) $\int_G f \mathrm{d}x = \int_a^b f \mathrm{d}x$, the several variable classical integral or the modern Lebesgue integral. The values of the function $f(x)$ may be real or complex.

(i) The real valued function $F\colon \mathbb{R} \to \mathbb{R}$ is convex.

(ii) The function $p\colon G \to \mathbb{R}$ is non-negative and is integrable on the open set G in $\mathbb{R}^N$ with $\int_G p\mathrm{d}x > 0$.

(iii) The function $g\colon G \to \mathbb{R}$ has the property that all integrals in (0.22) exist[17].

For example, one may choose $p(x) \equiv 1$.

The fundamental convexity inequality: Let $n = 1, 2, \ldots$ For all non-negative real numbers x_k and λ_k with $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$ one has

$$\boxed{f^{-1}\left(\sum_{k=1}^{n} \lambda_k f(x_k)\right) \leq g^{-1}\left(\sum_{k=1}^{n} \lambda_k g(x_k)\right),} \tag{0.23}$$

provided the following assumptions are fulfilled:

(i) The functions $f, g\colon\ [0,\infty[\to [0,\infty[$ are increasing and surjective. We denote by $f^{-1}, g^{-1}\colon\ [0,\infty[\to [0,\infty[$ the inverse functions to f and g.

(ii) The composition $y = g\left(f^{-1}(x)\right)$ of functions is *convex* on the interval $[0,\infty[$.

Except for the triangle inequality one gets all the inequalities above from (0.23). The idea behind all of these is the fruitful notion of *convexity.*

Example 1: If we choose $f(x) := \ln x$ and $g(x) := x$, then we have $f^{-1}(x) = \mathrm{e}^x$ and $g^{-1}(x) = x$. From (0.23) we get the inequality for the weighted mean

$$\boxed{\prod_{k=1}^{n} x_k^{\lambda_k} \leq \sum_{k=1}^{n} \lambda_k x_k\,,} \tag{0.24}$$

which is valid for all non-negative real numbers x_k and λ_k which satisfy $\sum\limits_{k=1}^{n} \lambda_k = 1$. This inequality is equivalent to the Young inequality (0.21).

In the special case $\lambda_k = 1/n$ for all k, the inequality (0.24) is just the inequality $g \leq m$ between the geometric mean g and the arithmetic mean m.

The duality inequality:

$$\boxed{(x|y) \leq F(x) + F^*(y) \quad \text{for all } x, y \in \mathbb{R}^N.} \tag{0.25}$$

Here, the function $F\colon\ \mathbb{R}^N \to \mathbb{R}$ is given, and the *dual* function $F^*\colon\ \mathbb{R}^N \to \mathbb{R}$ is given by the relation

$$F^*(y) := \sup_{x\in\mathbb{R}^N} (x|y) - F(x).$$

[17] If $G :=]a,b[$ is an open bounded interval, then is it sufficient for example that p and g are continuous on $[a,b]$ (or more generally, almost everywhere continuous and bounded). In this case we have

$$\int_G \ldots \mathrm{d}x = \int_a^b \ldots \mathrm{d}x\,.$$

If G is a bounded, open (non-empty) set in $\mathbb{R}^N$, then it is sufficient that p and g are continuous on the closure $\overline{G}$ (or more generally almost everywhere continuous and bounded).

Example 2: Let $N = 1, p > 1$ and $F(x) := \frac{|x|^p}{p}$ for all $x \in \mathbb{R}$. Then one has

$$F^*(y) = \frac{|y|^q}{q} \quad \text{for all } y \in \mathbb{R}\,,$$

where q is determined from the equation $\frac{1}{p} + \frac{1}{q} = 1$. In this special case, (0.25) is nothing but the Young inequality $xy \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}$.

Standard literature: A large collection of further inequalities can be found in the standard references [19] and [15].

0.1.12 Application to the motion of the planets – a triumph of mathematics in space

One can not have a pure understanding of what one has until one has a complete understanding of what others had before oneself.

Johann Wolfgang von Goethe (1749–1832)

The results of the previous sections are correctly considered today to belong to elementary mathematics. Actually it was the result of centuries of toil and thought – always in interaction with the resolution of important questions put to man by nature – before these realizations, today considered to be elementary, could be attained. As an example of this we consider here in more detail planetary motion.

Conic sections were already investigated intensively in ancient times. To describe the location of the planets in the heavens, the ancient astronomers used the idea of Appolonius von Perga (roughly 260–190 BC) of *epicycles.* According to this theory, the planets move along a small circular orbit, which in turn moves along a larger circular orbit (cf. Figure 0.21(a)).

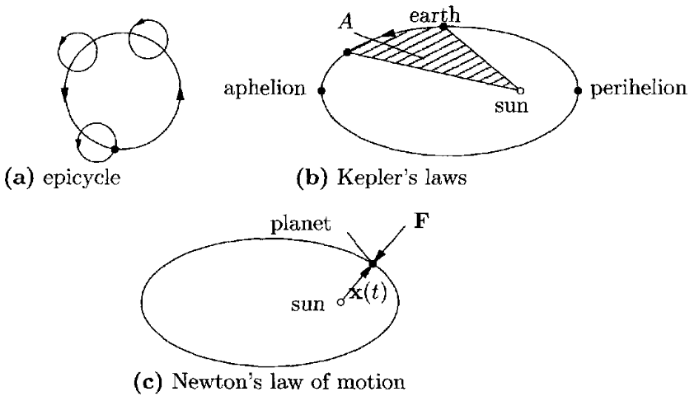


Figure 0.21. Historical occurrences of conic sections.

This theory gave a relatively accurate description of the apparent complicated annual motion of the planets in the sky. The theory of epicycles is a very vivid example for how the attempt to fit theory with observation can lead to a totally *wrong* model.

Copernicus' view of the world: In 1543, the year of death of Nicolaus Copernicus (born in 1473 in the old Polish Hansa city Toruń), his epochal work *De revolutionibus orbium coelestium* (On the motion of the heavenly orbits) appeared. In this work he broke with the tradition of the view of the world of the ancients, shaped by Ptolemy, according to which the earth was the center of the universe. On the contrary, Copernicus created the idea that the earth orbits the sun, while keeping the idea of *circular* orbits.

The three Kepler laws: Based on extensive observations of the Danish astronomer Tycho Brahe (1564–1601), Johannes Kepler (1571–1630, born in the city of Weil in Germany) found after extensive calculation the following *three laws for planetary motion* (Figure 0.21(b)):

1. *The planets move in elliptical orbits, with the sun at one of the focal points of the ellipse.*
2. *The motion sweeps out equal areas in equal times (denoted A in Figure 0.21(b)).*
3. *The ratio of the square of the orbital period T and the third power of the long axis a of the ellipse is a constant for all planets*:

$$\boxed{\frac{T^2}{a^3} = \text{const}\,.}$$

The first two of the laws were published by Kepler in 1609 in his monograph *Astronomia nova* (New Astronomy). Ten years later the third law appeared in his thesis *Harmonices mundi* (World Harmonies)[18].

In 1624, Kepler finished the enormous work involved in completing the "Rudolfian tables", which the German Emperor Rudolf II had commissioned him with in 1601. These tables were used by astronomers for the next 200 years. With the help of these tables it was possible to precisely predict the motion of the planets and solar and lunar eclipses for all times past and future. In these days of computer computational power it is impossible to imagine what an achievement this was, particularly since for use in astronomy one needs very precise results, not just rough approximations. Kepler even had to work without tables for logarithms. The first table of logarithms was published by the Scotch nobleman Neper in 1614. Kepler immediately realized the computational power afforded by these mathematical tool, reducing multiplications to additions. In fact, Kepler's paper on this was of great help in spreading the popularity of logarithms.

Newtonian mechanics: Exactly one hundred years after the death of Copernicus, Isaac Newton – one of the true geniuses of human kind – was born in 1643 as the son of a leaseholder in a small village on the east coast of England. Lagrange wrote about him: "He is the luckiest of all; the system of the universe can only be discovered once". At the age of 26, Newton became Professor at the famous Trinity College in Cambridge (England). Already at the age of 23 he used the third of Kepler's laws to estimate the power of gravitational attraction and found that this must be proportional to the inverse of the square of the distance. In 1687 his famous book *Philosophiae naturalis principia mathematica* (Mathematical Principles of Science) appeared. In this book, he founded classical mechanics and derived and applied his famous *law of motion*

$$\boxed{\text{force} = \text{mass} \times \text{acceleration.}}$$

[18]Kepler discovered the third law on May 18, 1618, five days before the window incident in Prague, which began the thirty years' war.

At the same time he created the theory of differential and integral calculus. Newton's law written in modern notion is the differential equation for the motion of the planets

$$\boxed{m\mathbf{x}''(t) = \mathbf{F}(\mathbf{x}(t))\,.} \tag{0.26}$$

The vector $\mathbf{x}(t)$ describes the position of the planets[19] at the time t (Figure 0.21(c)). The second derivative with respect to time, $\mathbf{x}''(t)$, corresponds to the vector of acceleration of the planet at time t, and the positive constant m is the mass of the planet. The gravitational attraction of the sun according to Newton has the form

$$\boxed{\mathbf{F}(\mathbf{x}) = -\frac{\mathrm{G}mM}{|\mathbf{x}|^2}\mathbf{e}}$$

with the unit vector

$$\mathbf{e} = \frac{\mathbf{x}}{|\mathbf{x}|}\,.$$

The negative sign of $\mathbf{F}$ corresponds to the fact that gravitational force points in the direction $-\mathbf{x}(t)$, that is, from the planet toward the sun. Furthermore, M denotes the mass of the sun, G is a universal constant of natural, called the *gravitational constant*:

$$\mathrm{G} = 6.6726 \cdot 10^{-11}\,\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}\,.$$

Newton found solutions of (0.26) which are ellipses

$$r = \frac{p}{1 - \varepsilon\cos\varphi}$$

(in polar coordinates) with the numerical eccentricity ε and the half-parameter p determined according to the following equations:

$$\varepsilon = \sqrt{1 + \frac{2ED^2}{\mathrm{G}^2m^3M^2}}\,, \qquad p = \frac{D^2}{\mathrm{G}^2m^3M^2}\,.$$

The energy E and the angular momentum D are determined from the position and the velocity of the planet at some fixed time. The orbital motion $\varphi = \varphi(t)$ is obtained by solving the equation

$$t = \frac{m}{D}\int_0^{\varphi} r^2(\varphi)\mathrm{d}\varphi$$

for the angle φ.

Gauss rediscovers Ceres: In the new years night of 1801 a tiny star of the magnitude 8 was discovered at the observatory in Palermo, which moved relatively quickly and then vanished again. This amounted to an incredible challenge for the astronomers of the day. Only 9 degrees of the orbit were known. The methods used up until then for celestial calculations failed. The 24-year old Gauss however succeeded in surmounting the difficulties of mastering an equation of the eighth degree, by developing totally new methods, which he published in 1809 in his work *Theoria motus corporum coelestium in sectionibus conicis Solem ambientium*[20].

[19] Vector calculus will be described in detail in 1.8.

[20] A translation of this title is *The theory of the motion of the planets, which move in conic sections around the sun.*

According to Gauss' calculations, Ceres could be observed again in the new years night 1802. Ceres was the first of the asteroids to be observed. It is estimated that there are approximately 50,000 such asteroids moving in a belt between Mars and Jupiter, whose total mass is just a few thousandths that of the earth. The diameter of Ceres is 768 km. It is the largest known asteroid.

The discovery of Neptune: During a night in March, 1781, Wilhelm Herschel discovered a new planet, which was later named Uranus and whose orbital period around the sun is 84 years (cf. Table 0.13). Two young astronomers, John Adams (1819–1892) in Cambridge and Jean Leverrier (1811–1877) in Paris, determined independently of each other the orbit of Uranus and concluded from the observed perturbation in Uranus' orbit the existence of a new planet, which according to Leverrier had been observed by Gottfried Galle in 1846 at the Berlin Observatory and received the name Neptune. This was a triumph of Newtonian mechanics and at the same time one of practical calculations in the theory of celestial motions.

From the observed perturbations in the motion of Neptune one later concluded the existence of a further, tiny planet very far from the sun, which was discovered in 1930 and was named Pluto after the Roman God of the underworld (cf. Table 0.13).

Table 0.13. A model of the solar system scaled to $1\,\mathrm{m} \,\widehat{=}\, 10^6\,\mathrm{km}$.

Planet	***Distance from the sun***	***Orbital period***	***Numerical orbital eccentricity*** ε	***Planet's diameter***	***Comparative size***
Sun	–	–	–	1.4 m	–
Mercury	58 m	88 days	0.206	5 mm	pea
Venus	108 m	255 days	0.007	12 mm	cherry
Earth	149 m	1 year	0.017	13 mm	cherry
Mars	229 m	2 years	0.093	7 mm	pea
Jupiter	778 m	12 years	0.048	143 mm	coconut
Saturn	1400 m	30 years	0.056	121 mm	coconut
Uranus	2900 m	84 years	0.047	50 mm	apple
Neptune	4500 m	165 years	0.009	53 mm	apple
Pluto	5900 m	249 years	0.249	10 mm	cherry

The perihelion motion of Mercury: The calculation of the orbits of the planets is quite complicated by virtue of the fact that not only the gravitational force of the sun, but also of the other planets must be accounted for. This in done in the context of mathematical *perturbation theory*, which in general considers the behavior of solutions under small perturbations of the (coefficients of the) equations. In spite of incredibly precise calculations, the orbit of the planet nearest to the sun, Mercury, had a rotation of the long axi of the ellipse describing its motion by 43 arc seconds a century, which was inexplicable. This discrepancy wasn't explained until the advent of Einstein's general theory of relativity in 1916.

The background microwave radiation of the big bang: There is a solution to the equations of the general theory of relativity which describes an expanding universe. The starting point of this expansion is referred to as the *big bang.* In 1965 the American physicists Penzias and Wilson at the Bell Laboratory in New Jersey discovered an extremely weak (microwave), completely isotropic (the same in all directions) radiation, which is now viewed to be a relict of and experimental evidence for the big bang from 15 billion years ago. This was a scientific sensation. Both scientist were awarded the Nobel prize for this discovery. Since the radiation can be viewed as a photon gas at the temperature of 3 degrees Kelvin (above absolute zero), one also speaks of the 3K radiation. The complete isotropy of this radiation on the other hand was for some time quite difficult to explain; it is an apparent contradiction to the formation of galaxies in the universe. In 1992, the satellite COBE., designed by George Smoot, after extensive preparations over several years, finally observed a detailed anisotropy in the background microwave radiation. This gives us a view back in time at the distribution of matter in the universe at the very young age of 300,000 years after the big bang and makes the formation of galaxies at about 10 billion years ago understandable.[21].

Astrophysics, differential equations, numerics, fast computers and the death of the sun: Our source of life, the sun, formed together with the planets about 5 billion years ago by attraction and compression of dark matter. Modern mathematics is in a position to describe the life and death of the sun. One uses a model for the sun which consists of a complicated system of differential equations, the derivation of which was the work of decades of astronomers. It is impossible to give exact solutions to this complicated system of differential equations. However, modern methods in numerics provides effective ways of calculating approximations to solutions with the computational power of supercomputers. The chair of Roland Bulirsch at the Technical University in Munich has carried through these calculations. This has been made vividly imaginable by motion pictures describing the solutions found in this way; these show how the sun at an age of about 11 billion years will start to expand to the orbit of Venus, at which time all life on the planet Earth will long have ceased to exist from the incredible heat caused by this expansion. Somewhat later the sun will start to collapse and will become a brown dwarf from which no more light can escape.

0.2 Elementary functions and their graphical representation

Basic idea: A real-valued function[22]

$$\boxed{y = f(x)}$$

assigns, in a unique fashion, a real number y to the real number x. One must differentiate in thought between the function f as an assignment and the value $f(x)$ of the function at the number x.

(i) The set of all x for which the assignment is defined is called the *domain* $D(f)$ of the function f.

(ii) The set of image points y for all $x \in D(f)$[23] is called the *range* $R(f)$ of the function

[21] The fascinating story of modern cosmology and of the COBE-project is described in the book [28].

[22] Real-valued functions are special *maps.* The definition and properties of general maps are discussed in 4.3.3. For simplicity, real-valued functions are also briefly referred to as real functions.

[23] The symbol $x \in D(f)$ indicates that x is an element of the set $D(f)$.

f.

(iii) The set of all point pairs $(x, f(x))$ is called the *graph* $G(f)$ of the function f.

Functions can be defined by a *table of values* or by a *graphical representation.*

Example: For the function $y = 2x + 1$, the table of values is

x	0	1	2	3	4
y	1	3	5	7	9

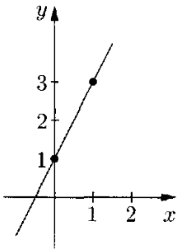


Figure 0.22.

The graphical representation of $y = 2x + 1$, the graph of f is the plane of points (x, y), is the line through the two points $(0, 1)$ and $(1, 3)$.

Increasing and decreasing functions: A function f is said to be (strictly) *increasing* if

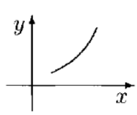
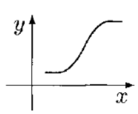
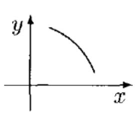
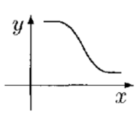
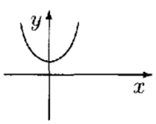
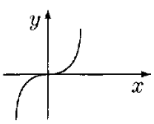
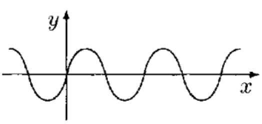
$$\boxed{x < u \text{ implies } f(x) < f(u)\,.} \tag{0.27}$$

A function f is said to be *non-decreasing*, *decreasing* or *non-increasing*, if in (0.27) the symbol '$f(x) < f(u)$' is replaced by, in order

$$f(x) \le f(u)\,, \qquad f(x) > f(u)\,, \qquad f(x) \ge f(u)$$

(see Table 0.14).

Table 0.14. Properties of functions.

Increasing	***Non-decreasing***	***Decreasing***	***Non-increasing***
			
Even	***Odd***	***Periodic***	
			

Basic idea of the inverse function: We consider the function

$$\boxed{y = x^2\,, \qquad x \ge 0\,.} \tag{0.28}$$

The equation (0.28) has for each $y \ge 0$ exactly one solution $x \ge 0$, which one denotes by $\sqrt{y}$:

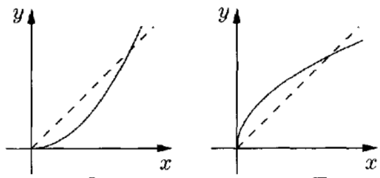
$$\boxed{x = \sqrt{y}\,.}$$

Exchanging formally x with y, we get the *square root* function

$$\boxed{y = \sqrt{x}\,.} \tag{0.29}$$

The graph of the inverse function (0.29) is obtained from the graph of the original function (0.28) by reflecting the graph on the diagonal (Figure 0.23).

This construction can be carried out for arbitrary continuous, increasing functions (cf. 1.4.4). As we will see in the next sections, one get in this manner many important functions (for example $y = \ln x$, $y = \arcsin x$, $y = \arccos x$ etc.).



(a) $y = x^2$ (b) $y = \sqrt{x}$

Figure 0.23. Power functions.

Graphical representation of functions with Mathematica: The software package Mathematica contains a built-in series of important mathematical functions. These can be displayed by tables of values or by plotting the graphs.

0.2.1 Transformation of functions

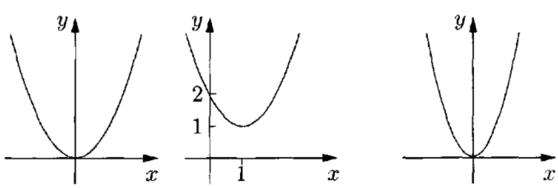
It suffices to know certain standard forms of functions. From these one can get graphical representations of other interesting functions by the processes of translation, dilation and reflection.

Translation: The graph of the function

$$\boxed{y = f(x-a) + b}$$

is obtained from the graph of $y = f(x)$ by the translation in which each point (x, y) is shifted to $(x + a, y + b)$.

Example 1: The graph of $y = (x-1)^2 + 1$ is obtained from the graph of $y = x^2$ by the translation in which the point $(0,0)$ is translated to the point $(1,1)$ (Figure 0.24).



(a) $y = x^2$ (c) $y = (x-1)^2 + 1$ (b) $y = 2x^2$

Figure 0.24. Translation and dilation of a graph.

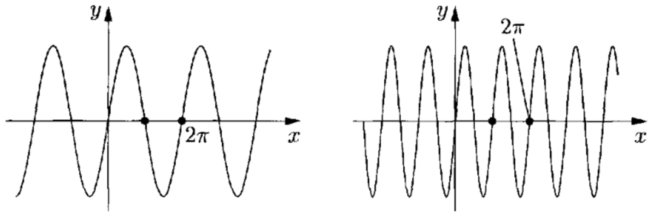
Dilation along axi: The graph of the function

$$\boxed{y = b\, f\left(\frac{x}{a}\right)}$$

with fixed $a > 0$ and $b > 0$ is obtained from the graph of $y = f(x)$ by stretching the x-axis by a factor of a and stretching the y-axis by a factor of b.

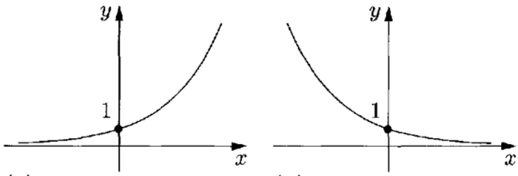
Example 2: From $y = x^2$ one gets $y = 2x^2$ by stretching the y-axis by a factor of 2 (Figure 0.24).

Example 3: From $y = \sin x$ one gets $y = \sin 2x$ by dilating the x-axis by a factor of $\frac{1}{2}$ (Figure 0.25).



(a) $y = \sin x$ (b) $y = \sin 2x$

Figure 0.25. Sinusoidal waves.



(a) $y = \mathrm{e}^x$ (b) $y = \mathrm{e}^{-x}$

Figure 0.26. Exponential functions.

Reflection: The graphs of

$$\boxed{y = f(-x)} \quad \text{resp.} \quad \boxed{y = -f(x)}$$

are obtained from the graph of $y = f(x)$ by reflection on the y-axis (resp. on the x-axis).

Example 4: The graph of $y = \mathrm{e}^{-x}$ results by reflecting the graph of $y = \mathrm{e}^x$ on the y-axis (Figure 0.26).

Even and odd functions: A function $y = f(x)$ is said to be *even* (resp. odd), if

$$f(-x) = f(x) \qquad (\text{resp. } f(-x) = -f(x))$$

for all $x \in D(f)$ (Table 0.14).

The graph of an even (resp. odd) function is invariant under reflection of the x-axis (resp. reflection of both axi) on the origin.

Example 5: The function $y = x^2$ is even, while $y = x^3$ is odd.

Periodic functions: The function f has by definition a *period* p, if

$$\boxed{f(x+p) = f(x) \quad \text{for all } x \in \mathbb{R},}$$

i.e., if the relation is satisfied for all real numbers x. The graph of a periodic function is invariant under translations of the x-axis by p.

Example 6: The function $y = \sin x$ has a period of 2π (Figure 0.25).

0.2.2 Linear functions

The linear function

$$\boxed{y = mx + b}$$

has a graph which is a line with slope m and which has y-intercept b (see Figure 0.10 in 0.1.7.1).

0.2.3 Quadratic functions

The simplest quadratic function

$$\boxed{y = ax^2} \qquad (0.30)$$

for $a \neq 0$ has a graph which is a parabola (Figure 0.27). A general quadratic function

$$\boxed{y = ax^2 + 2bx + c} \qquad (0.31)$$

can be put in the form

$$y = a\left(x + \frac{b}{a}\right)^2 - \frac{D}{a} \qquad (0.32)$$

with the discriminant $D := b^2 - ac$ by means of *quadratic completion.* Thus (0.31) results from (0.30) by a translation which moves the apex $(0,0)$ to $\left(-\frac{b}{a}, -\frac{D}{a}\right)$.

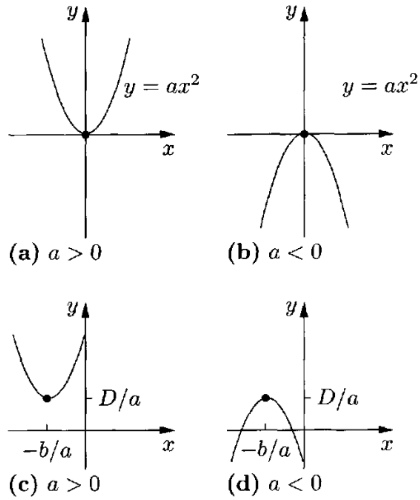


Figure 0.27.

Quadratic equations: The equation

$$\boxed{ax^2 + 2bx + c = 0}$$

has for real coefficients a, b and c with $a > 0$ the solutions

$$\boxed{x_\pm = \frac{-b \pm \sqrt{D}}{a} = \frac{-b \pm \sqrt{b^2 - ac}}{a}\,.}$$

Case 1: $D > 0$. There are two different real zeros x_+ and x_-, which correspond to two different points of intersection of the parabola (0.31) with the x-axis (Figure 0.28(a)).

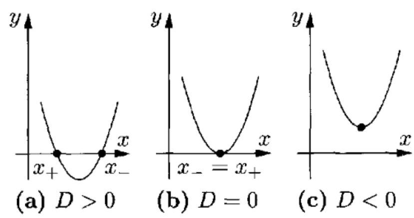


Figure 0.28.

Case 2: $D = 0$. There is one real zero $x_+ = x_-$. The parabola (0.31) is tangent to the x-axis (Figure 0.28(b)).

Case 3: $D < 0$. There are two *complex* zeros

$$x_\pm = \frac{-b \pm \mathrm{i}\sqrt{-D}}{a} = \frac{-b \pm \mathrm{i}\sqrt{ac - b^2}}{a},$$

where i is the imaginary unit with $\mathrm{i}^2 = -1$ (cf. 1.1.2). In this case the x-axis is not intersected by the (real) parabola (0.31) (Figure 0.28(c)).

Example 1: The equation $x^2 - 6x + 8 = 0$ has the two zeros

$$x_\pm = 3 \pm \sqrt{3^2 - 8} = 3 \pm 1\,,$$

that is $x_+ = 4$ and $x_- = 2$.

Example 2: The equation $x^2 - 2x + 1 = 0$ has the zero

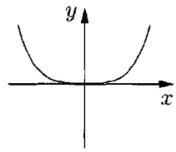
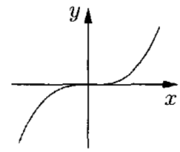
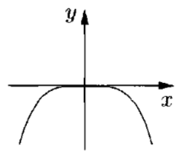
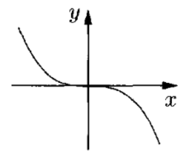
$$x_\pm = 1 \pm \sqrt{1 - 1} = 1\,.$$

Example 3: For $x^2 + 2x + 2 = 0$ we get the zeros

$$x_\pm = -1 \pm \sqrt{1 - 2} = -1 \pm \mathrm{i}\,.$$

0.2.4 The power function

Table 0.15. The power function $y = ax^n$.

$n \geq 2$:	*Even*	*Odd*
$a > 0$		
$a < 0$		

Let $n = 2, 3, \ldots$ The function

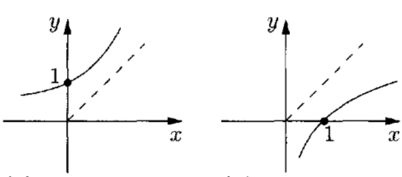
$$\boxed{y = ax^n}$$

for even n is shaped similarly as $y = ax^2$ and for odd n similarly as $y = ax^3$ (Table 0.15).

0.2.5 The Euler e-function

The shortest path between two real points is through the complex domain. *Jacques Hadamard* (1865–1963)

In order to recognize deep connections among different parts of mathematics, it is important to consider the functions e^x, $\sin x$ and $\cos x$ also for complex arguments x.



(a) $y = e^x$ (b) $y = \ln x$

Figure 0.29.

Complex numbers of the form $x = a + bi$ with real numbers a and b are discussed in detail in 1.1.2. One just has to note that the imaginary unit i satisfies the relation

$$\boxed{i^2 = -1.}$$

Every real number is at the same time a complex number.

Definition: For all complex numbers x, the infinite series[24]

$$\boxed{e^x := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}} \qquad (0.33)$$

converges.

In this way the exponential function $y = e^x$ is defined for all complex arguments x,

[24] Infinite series are considered in detail in section 1.10.

which turns out to be the most important single function in all of mathematics. For real x this function was introduced by Newton at the age of 33 in 1676 (Figure 0.29(a)).

Addition theorem: For all complex numbers x and z one has the fundamental formula:

$$\boxed{e^{x+z} = e^x e^z\,.}$$

Euler made the very surprising discovery about 75 years after Newton that the e-function and the trigonometric functions (for complex arguments) are closely related (see the Euler formula (0.35) in 0.2.8.). Therefore one refers to the exponential function $y = e^x$ as the Euler e-function. For $x = 1$ we get

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

In addition the Euler limit formula holds[25]

$$\boxed{e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n}$$

for all real numbers x. One has $e = 2.71828183$.

Increasing property: The function $y = e^x$ is strictly increasing and continuous for all real arguments.

Behavior at infinity[26]**:**

$$\boxed{\lim_{x \to +\infty} e^x = +\infty\,, \qquad \lim_{x \to -\infty} e^x = 0\,.}$$

For negative arguments of large absolute value the graph of $y = e^x$ approaches the x-axis asymptotically (Figure 0.29(a)). The limiting relation

$$\boxed{\lim_{x \to +\infty} \frac{e^x}{x^n} = +\infty\,, \qquad n = 1, 2, \ldots\,,}$$

states that the exponential function for large arguments *grows faster than every power function.*

The complexity of computer algorithms: If a computer algorithm depends on a natural number N (if, for example, N is the number of equations) and the needed computation time behaves like e^N, then the computation time explodes for large N, making the algorithm practically useless for large N. Investigations of this kind are done in the context of the modern *complexity theory*. Especially many algorithms used in computer algebra have a high complexity.

Derivative: The function $y = e^x$ is infinitely often differentiable for all real or complex number x, and the derivative is[27]

$$\boxed{\frac{de^x}{dx} = e^x\,.}$$

Periodicity in the complex domain: The Euler e-function has the complex period of $2\pi i$, that is, for all complex numbers x one has:

$$\boxed{e^{x+2\pi i} = e^x\,.}$$

[25] Limits of sequences of numbers are introduced in 1.2.
[26] Limits of functions are investigated in 1.3.
[27] The notion of derivative of real or complex functions, one of the most fundamental notions of analysis, is found in 1.4.1 (resp. 1.14.3).

If one restricts oneself to real arguments x, then this periodicity is invisible (see Figure 0.29(a)).

Non-vanishing of the e-function: For all complex numbers x, we have $e^x \neq 0$.[28]

0.2.6 The logarithm

The inverse of the e-function: Since the e-function is strictly increasing and continuous for all real arguments, the equation

$$\boxed{y = e^x}$$

has a unique real number x as solution for all $y > 0$, which is denoted

$$\boxed{x = \ln y}$$

and is called the *natural logarithm* (logarithmus naturalis). Formally exchanging x and y, we get the function

$$\boxed{y = \ln x\,,}$$

which is the inverse function of the function $y = e^x$. The graph of $y = \ln x$ is obtained from the graph of $y = e^x$ by reflection on the diagonal (Figure 0.29(b)).

From the addition theorem $e^{u+v} = e^u e^v$ the fundamental property of the logarithm follows:[29]

$$\boxed{\ln(xy) = \ln x + \ln y}$$

for all positive real numbers x and y.

Logarithm laws: See section 0.1.9.

Limit relations:

$$\boxed{\lim_{x\to+0} \ln x = -\infty\,, \qquad \lim_{x\to+\infty} \ln x = +\infty\,.}$$

For every real number $\alpha > 0$ one has

$$\boxed{\lim_{x\to+0} x^\alpha \ln x = 0\,.}$$

It follows that the function $y = \ln x$ approaches minus infinity extremely slowly near $x = 0$.

Derivative: For all real numbers $x > 0$ one has

$$\boxed{\frac{d\ln x}{dx} = \frac{1}{x}\,.}$$

[28] More precisely, the map $x \longmapsto e^x$ is a surjective map from the complex plane $\mathbb{C}$ onto $\mathbb{C} \setminus \{0\}$.

[29] If we set $x := e^u$ and $y := e^v$, then we get $xy = e^{u+v}$. This yields $u = \ln x$, $v = \ln y$ and $u + v = \ln(xy)$.

0.2.7 The general exponential function

Definition: For every positive real numbers a and every real number x we set

$$\boxed{a^x := e^{x \ln a}.}$$

In this way the general exponential function a^x is reduced to the e-function (Figure 0.30).

Power laws: See section 0.1.9.

General logarithm: Let a be a fixed positive real number $a \neq 1$. For every positive real number y, the equation

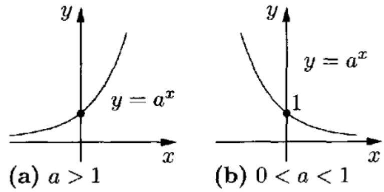


Figure 0.30. The general exponential.

$$y = a^x$$

has a unique real solution x, which we denote by $x = \log_a y$. Formally exchanging x and y, we get the inverse function to $y = a^x$:

$$y = \log_a x.$$

For this one has the relation

$$\boxed{\log_a y = \frac{\ln y}{\ln a}}$$

(cf. 0.1.9). One has $\ln a > 0$ for $a > 1$ and $\ln a < 0$ for $0 < a < 1$.

Two important functional equations: Let $a > 0$.

(i) The only continuous function[30] $f: \mathbb{R} \longrightarrow \mathbb{R}$, which satisfies the relation

$$\boxed{f(x+y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}}$$

together with the normalization $f(1) = a$, is the exponential function $f(x) = a^x$.

(ii) The only continuous function $g:]0, \infty[\longrightarrow \mathbb{R}$, which satisfies the condition

$$\boxed{g(xy) = g(x) + g(y) \quad \text{for all } x, y \in]0, \infty[}$$

together with the normalization $g(a) = 1$, is the logarithm $g(x) = \log_a x$.

Both of these statements show that the exponential and logarithm are very naturally and useful functions and that the mathematicians of the past certainly would have had to run across these functions sooner or later.

0.2.8 Sine and cosine

Analytical definition: From a modern perspective it is convenient to define the two functions $y = \sin x$ and $y = \cos x$ by means of their infinite series expansions

$$\boxed{\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}. \end{aligned}} \tag{0.34}$$

[30]The notion of continuity will be introduced in 1.3.1.2.

These two series converge for all complex numbers[31] x.

The Euler formula (1749): For all complex numbers x the following fundamental formula is valid:

$$\boxed{e^{\pm ix} = \cos x \pm i \sin x\,.} \tag{0.35}$$

This formula dominates the entire theory of trigonometric functions. The realtion (0.35) follows immediately from the power series expansions (0.33) and (0.34) for e^{ix}, $\cos x$ and $\sin x$, when one takes note of the fact that $i^2 = -1$. In 1.3.3 one can find important applications of this formula to the theory of vibrations. From (0.35) we get

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}\,, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}\,.} \tag{0.36}$$

These formulas, together with the addition theorem $e^{u+v} = e^u e^v$, easily yield the following fundamental addition theorems for sine and cosine.

Addition theorems: For all complex numbers x and y one has:

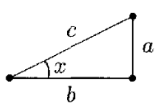
$$\boxed{\begin{aligned} \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y\,, \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y\,. \end{aligned}} \tag{0.37}$$

Evenness and oddness: For all complex numbers x one has:

$$\boxed{\sin(-x) = -\sin x\,, \qquad \cos(-x) = \cos x\,.}$$

Geometric interpretation on a right triangle: We consider a right triangle with an angle x measured in radians (cf. 0.1.2). Then $\sin x$ and $\cos x$ are given as the ratios of the sides as shown in Table 0.16.

Table 0.16. Interpretation of trigonometric functions in terms of a right triangle.

Right triangle	***Sine***	***Cosine***
	$\sin x = \frac{a}{c}$	$\cos x = \frac{b}{c}$
$0 < x < \frac{\pi}{2}$	(length of opposite side a divided by the hypotenuse c)	(length of adjacent side b divided by the hypotenuse c)

[31] Compare the remarks made at the beginning of 0.2.5 about complex numbers.

The symbol '$\sin x$' is read 'sine of x' and the symbol '$\cos x$' is read 'cosine of x'. The latin word sinus means bulge. In older literature one also uses the functions

secant: $\sec x := \frac{1}{\cos x}$, *cosecant:* $\operatorname{cosec} x := \frac{1}{\sin x}$.

Geometric interpretation on the unit circle: Using the unit circle, the quantities $\sin x$ and $\cos x$ are just the lengths of the segments shown in Figure 0.31(a)-(d). From this one see immediately that $\sin x$ and $\cos x$ have the same values after a rotation of 2π. This is the geometric interpretation of the 2π-periodicity of the $\sin x$ and $\cos x$:

$$\boxed{\sin(x + 2\pi) = \sin x\,, \qquad \cos(x + 2\pi) = \cos x\,.} \tag{0.38}$$

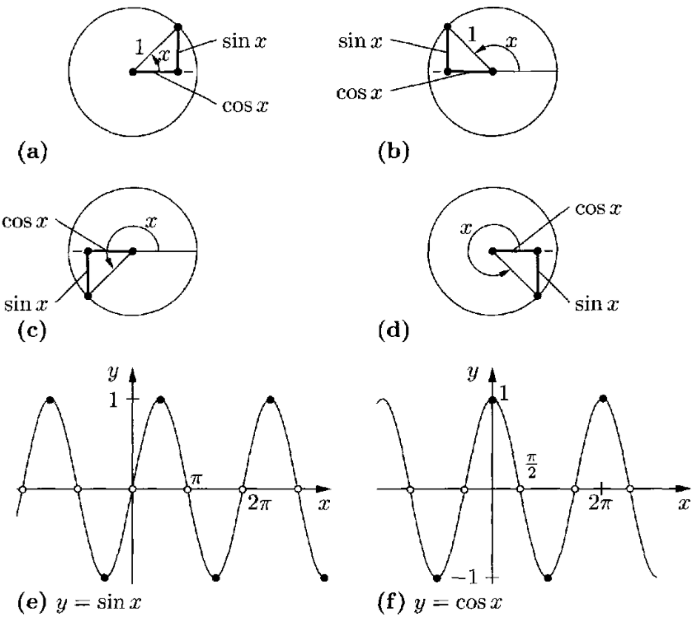


Figure 0.31. Trigonometric functions and the unit circle.

These relations hold for all complex arguments x. Looking at the unit circle again, one sees the following symmetries

$$\boxed{\sin(\pi - x) = \sin x\,, \qquad \cos(\pi - x) = -\cos x} \tag{0.39}$$

for $0 \le x \le \pi/2$. In fact these relations hold for all complex numbers x. Finally one gets from Figure 0.31(a) and the theorem of Pythagoras the relation

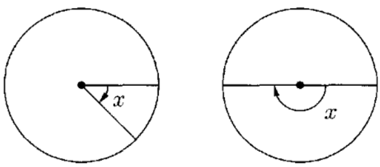
$$\boxed{\cos^2 x + \sin^2 x = 1\,,} \tag{0.40}$$

which holds not only for real angles x, but also for all complex arguments x. In the same way we get from the theorem of Pythogoras the values for $\sin x$ and $\cos x$ listed in Table 0.17 (cf. 3.2.1.2).

The validity of (0.38), (0.39) and (0.40) for all complex numbers follows easily from the addition theorem (0.37) and the relations $\sin 0 = \sin 2\pi = 0$ and $\cos 0 = \cos 2\pi = 1$.

Table 0.17. Exact values of the sine and cosine functions for important angles.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	***(radians)***
	0	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	***(degrees)***
sin x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	



(a) $x = -\frac{\pi}{4}$ **(b)** $x = -\pi$

Figure 0.32. Negative angles.

Negative angles: Applying the geometric interpretation of the unit circle one gets the graphical representation of the functions $y = \sin x$ and $y = \cos x$ as shown in Figure 0.31(e),(f). Here negative angles $x < 0$ were introduced as in Figure 0.32, by measuring positive angles in counter-clockwise direction (*positive mathematical direction*) and negative angles in clockwise direction (*negative mathematical direction*).

Zeros: From Figure 0.31(e),(f) it follows that:

(i) The function $y = \sin x$ has zeros at the points $x = k\pi$, where k is an arbitrary integer; in other words the set of zeros is given by $x = 0, \pm\pi, \pm 2\pi, \ldots$

(ii) The function $y = \cos x$ has zeros at the points $x = k\pi + \frac{\pi}{2}$, where k is an arbitrary integer.

(iii) Both functions $y = \sin x$ and $y = \cos x$ have only real zeros in the complex plane. These zeros are those described in (i) and (ii).

The law of translation: It is suffient to know the values of $\sin x$ for all angles x with $0 \le x \le \frac{\pi}{2}$. All other values can be obtained by the following formulas, which in turn are consequences of the addition theorems:

$$\sin\left(\frac{\pi}{2} + x\right) = \cos x\,, \qquad \sin(\pi + x) = -\sin x\,, \qquad \sin\left(\frac{3\pi}{2} + x\right) = -\cos x\,,$$
$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x\,, \qquad \cos(\pi + x) = -\cos x\,, \qquad \cos\left(\frac{3\pi}{2} + x\right) = \sin x\,.$$

De Moivre's formula for multiples of a given angle[32]: Let $n = 2, 3, \ldots$ Then for

[32]This formula, found by de Moivre (1667–1754), inspired Euler to the discovery of his famous formula

$$e^{ix} = \cos x + i \sin x\,.$$

Today it is more convenient to work the other way around: de Moivre's formula (0.41) is a consequence of the Euler formula, using

$$\cos nx + i \sin nx = e^{inx} = \left(e^{ix}\right)^n = (\cos x + i \sin x)^n$$

and the binomial formula (cf. 0.1.10.3).

all complex numbers x, one has

$$\boxed{\cos nx + \mathrm{i}\sin nx = \sum_{k=0}^{n} \mathrm{i}^k \binom{n}{k} \cos^{n-k} x \sin^k x\,.} \tag{0.41}$$

Seperating here the real and imaginary part of the complex numbers, one gets

$$\cos nx = \cos^n x - \binom{n}{2}\cos^{n-2} x \sin^2 x + \binom{n}{4}\cos^{n-4} x \sin^4 x - \ldots \tag{0.42}$$

$$\sin nx = \binom{n}{1}\cos^{n-1} x \sin x - \binom{n}{3}\cos^{n-3} x \sin^3 x + \binom{n}{5}\cos^{n-5} x \sin^5 x - \ldots$$

For $n = 2, 3, 4$ we get the following special cases:

$$\begin{array}{|ll|}\hline \sin 2x = 2\sin x\cos x\,, & \cos 2x = \cos^2 x - \sin^2 x\,,\\ \sin 3x = 3\sin x - 4\sin^3 x\,, & \cos 3x = 4\cos^3 x - 3\cos x\,,\\ \sin 4x = 8\cos^3 x\sin x - 4\cos x\sin x\,, & \cos 4x = 8\cos^4 x - 8\cos^2 x + 1\,.\\ \hline\end{array}$$

The formula for half-angles: For all complex numbers x one has:

$$\sin^2\frac{x}{2} = \frac{1}{2}(1-\cos x)\,, \qquad \cos^2\frac{x}{2} = \frac{1}{2}(1+\cos x)\,,$$

$$\sin\frac{x}{2} = \begin{cases} \sqrt{\frac{1}{2}(1-\cos x)}\,, & 0 \le x \le \pi\,,\\ -\sqrt{\frac{1}{2}(1-\cos x)}\,, & \pi \le x \le 2\pi\,,\end{cases}$$

$$\cos\frac{x}{2} = \begin{cases} \sqrt{\frac{1}{2}(1+\cos x)}\,, & -\pi \le x \le \pi\,,\\ -\sqrt{\frac{1}{2}(1+\cos x)}\,, & \pi \le x \le 3\pi\,.\end{cases}$$

Formulas for sums: For all complex numbers x and y one has:

$$\boxed{\begin{aligned} \sin x \pm \sin y &= 2\sin\frac{x\pm y}{2}\,\cos\frac{x\mp y}{2}\,,\\ \cos x + \cos y &= 2\cos\frac{x+y}{2}\,\cos\frac{x-y}{2}\,,\\ \cos x - \cos y &= 2\sin\frac{x+y}{2}\,\sin\frac{y-x}{2}\,,\\ \cos x \pm \sin x &= \sqrt{2}\sin\left(\frac{\pi}{4}\pm x\right)\,.\end{aligned}}$$

Formulas for products of two factors:

$$\sin x\sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)\,,$$

$$\cos x\cos y = \frac{1}{2}\left(\cos(x-y) + \cos(x+y)\right)\,,$$

$$\sin x\cos y = \frac{1}{2}\left(\sin(x-y) + \sin(x+y)\right)\,.$$

Formulas for products of three factors:

$$\sin x \sin y \sin z = \frac{1}{4}\left(\sin(x+y-z) + \sin(y+z-x) + \sin(z+x-y) - \sin(x+y+z)\right),$$

$$\sin x \cos y \cos z = \frac{1}{4}\left(\sin(x+y-z) - \sin(y+z-x) + \sin(z+x-y) + \sin(x+y+z)\right),$$

$$\sin x \sin y \cos z = \frac{1}{4}\left(-\cos(x+y-z) + \cos(y+z-x) + \cos(z+x-y) - \cos(x+y+z)\right),$$

$$\cos x \cos y \cos z = \frac{1}{4}\left(\cos(x+y-z) + \cos(y+z-x) + \cos(z+x-y) + \cos(x+y+z)\right).$$

Formulas for powers:

$$\sin^2 x = \frac{1}{2}(1-\cos 2x), \qquad \cos^2 x = \frac{1}{2}(1+\cos 2x),$$

$$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x), \qquad \cos^3 x = \frac{1}{4}(3\cos x + \cos 3x),$$

$$\sin^4 x = \frac{1}{8}(\cos 4x - 4\cos 2x + 3), \qquad \cos^4 x = \frac{1}{4}(\cos 4x + 4\cos 2x + 3).$$

More general formulas for $\sin^n x$ and $\cos^n x$ follow from de Moivre's formula (0.42).

Addition theorems for three summands:

$$\sin(x+y+z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z,$$

$$\cos(x+y+z) = \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z.$$

All of these formulas are verified by expressing $\cos x$ and $\sin x$ as linear combinations of $e^{\pm ix}$ according to (0.36). Then it only remains to verify some elementary algebraic identities. One can also apply the addition theorem (0.37).

The Euler product formula[33]**:** For all complex numbers one has:

$$\boxed{\sin \pi x = \pi x \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2}\right).}$$

One can read off of this formula immediately exactly where the sine has zeros: $\sin \pi x$ has zeros at $x = 0, \pm 1, \pm 2, \ldots$ These zeros are in addition *simple* (see 1.14.6.3).

Partial fraction decomposition: For all complex numbers x different from $0, \pm 1, \pm 2, \ldots$ one has

$$\boxed{\frac{\cos \pi x}{\sin \pi x} = \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{x-k} + \frac{1}{x+k}\right).}$$

[33] Infinite products are considered in 1.10.6.

Derivatives: For all complex numbers x one has:

$$\boxed{\frac{\mathrm{d}\sin x}{\mathrm{d}x} = \cos x\,, \qquad \frac{\mathrm{d}\cos x}{\mathrm{d}x} = -\sin x\,.}$$

Parametrization of the unit circle with the aid of trigonometric functions: See section 0.1.7.2.

Applications of trigonometric functions in plane trigonometry (land surveying) and spherical trigonometry (navigation and air traffic): See section 3.2.

Historical remarks: Ever since ancient times, the developement of trigonometry has been inseparably connected with technological developments in surveying and navigation, construction and use of calenders and the science of astronomy. Trigonometry had a heyday in the hands of the arabians in the 8th century. In 1260 the book *Treatise on the complete quadrilaterial* was written by at-Tusi, the most important islamic mathematician in the area of trigonometry. This book was the starting point of an independent branch of mathematics concerned with trigonometry. The most important European mathematician of the fifteenth century was Regiomontanus (1436–1476), whose name was in reality Johannes Müller. His most important work[34] *De triangulis omnimodis libri quinque* didn't appear until 1533, long after his death. This treatise contains a complete presentation of plane and spherical trigonometry, and founded the modern branch of mathematics referred to as trigonometry. Unfortunately all formulas in that book were expressed awkwardly in words.[35] Since Regiomontanus didn't have decimal numbers at his disposal[36], he used in the sense of Table 0.16 the formula

$$a = c\sin x \qquad \text{with} \quad c = 10,000,000\,.$$

His values for a correspond to an accuracy of 7 decimal places for $\sin x$. Euler (1707–1783) was the first to use $c = 1$.

At the end of the sixteenth century, Vieta (1540–1603) calculated, in his monograph *Canon*, a table of trigonometric functions, which proceeds from arc minute to arc minute.

Just like tables for logarithms, tables for values of trigonometric functions are obsolete in the day of computers.

0.2.9 Tangent and cotangent

Analytic definition: For all complex numbers x not equal to one of the values $\frac{\pi}{2} + k\pi$ with $k \in \mathbb{Z}$, we set[37]

$$\boxed{\tan x := \frac{\sin x}{\cos x}\,.}$$

We further define for all complex numbers x not equal to one of the values $k\pi$ with $k \in \mathbb{Z}$ the function

$$\boxed{\cot x := \frac{\cos x}{\sin x}\,.}$$

[34] Translated into English the title means "Five books about all kinds of triangles".

[35] The use of formulas goes back to Vieta *In artem analyticam isagoge*, which appeared in 1591.

[36] Decimal numbers were introduced in 1585 by Stevin in his book *La disme* (The decimal system). This lead to the unification of measurements in continental Europe, based on the decimal system.

[37] We denote by the symbol $\mathbb{Z}$ the set of all integers $k = 0, \pm 1, \pm 2, \ldots$

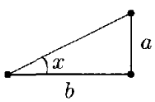
Translation property: For all complex numbers x with $x \neq k\pi$, $k \in \mathbb{Z}$, one has:

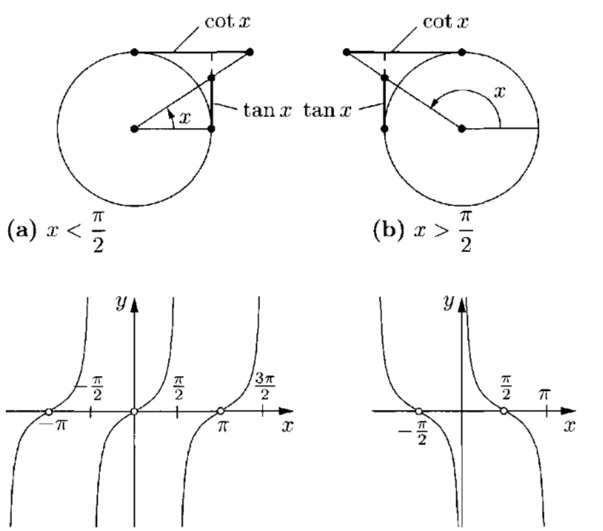
$$\boxed{\cot x = \tan\left(\frac{\pi}{2} - x\right).}$$

Because of this, all properties of the function cotangent follow directly from those of tangent.

Geometric interpretation in a right triangle: We consider a right triangle with the angle x measured in radians (cf. 0.1.2). Then the values of $\tan x$ and $\cot x$ are given by the ratios of sides as shown in Table 0.18.

Table 0.18. Interpretation of trigonometric functions in terms of a right triangle.

Right triangle	***Tangent***	***Cotangent***
	$\tan x = \frac{a}{b}$	$\cot x = \frac{b}{a}$
$0 < x < \frac{\pi}{2}$	(length of opposite side a divided by length of adjacent side b)	(length of adjacent side b divided by length of opposite side a)



(c) $y = \tan x$ (period π) **(d)** $y = \cot x$ (period π)

Figure 0.33. Geometrical interpretation of the tangent and cotangent functions.

Geometric interpretation on the unit circle: Using the unit circle, the values of $\tan x$ and $\cot x$ are the lengths of the segments shown in Figure 0.33(a),(b). One gets from this the special values listed in Table 0.19.

Table 0.19. Exact values of tan and cot for important angles.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	***(radians)***
	0	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	***(degrees)***
tan x	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	–	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	
cot x	–	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	–	

Zeros and poles: The function $y = \tan x$ has for complex arguments x exactly the zeros $k\pi$ with $k \in \mathbb{Z}$ and precisely the poles $k\pi + \frac{\pi}{2}$ with $k \in \mathbb{Z}$. All of these zeros and poles are simple (Figure 0.33(c)).

The function $\cot x$ has for complex arguments x exactly the poles $k\pi$ with $k \in \mathbb{Z}$ and exactly the zeros $k\pi + \dfrac{\pi}{2}$ with $k \in \mathbb{Z}$. Again, all of these zeros and poles are simple[38] (Figure 0.33(d)).

Partial fraction decomposition: For all complex numbers x with $x \notin \mathbb{Z}$ one has:

$$\cot \pi x = \frac{1}{x} + \sum_{k=1}^{\infty} \left(\frac{1}{x-k} + \frac{1}{x+k} \right).$$

Derivative: For all complex numbers x with $x \neq \frac{\pi}{2} + k\pi$ and $k \in \mathbb{Z}$ one has:

$$\frac{\mathrm{d} \tan x}{\mathrm{d}x} = \frac{1}{\cos^2 x}.$$

For all complex numbers x with $x \neq k\pi$ and $k \in \mathbb{Z}$ one has:

$$\frac{\mathrm{d} \cot x}{\mathrm{d}x} = -\frac{1}{\sin^2 x}.$$

Power series: For all complex numbers x with $|x| < \frac{\pi}{2}$ one has:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots = \sum_{k=1}^{\infty} 4^k(4^k - 1)\frac{|B_{2k}|x^{2k-1}}{(2k)!}.$$

For all complex numbers x with $0 < |x| < \pi$ one has:

$$\begin{aligned} \cot x &= \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \ldots \\ &= \frac{1}{x} - \sum_{k=1}^{\infty} \frac{4^k |B_{2k}| x^{2k-1}}{(2k)!}. \end{aligned}$$

Here B_{2k} denote the Bernoulli numbers.

Convention: The following formulas hold for all complex arguments x and y with the exception of those arguments for which the function has a pole.

[38] The notion of simple zero or pole is defined in 1.14.6.3

Periodicity:

$$\boxed{\tan(x+\pi) = \tan x\,, \qquad \cot(x+\pi) = \cot x\,.}$$

Oddness:

$$\boxed{\tan(-x) = -\tan x\,, \qquad \cot(-x) = -\cot x\,.}$$

Addition theorems:

$$\boxed{\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}\,, \qquad \cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}\,;}$$

$$\tan\left(\frac{\pi}{2} \pm x\right) = \mp \cot x\,, \qquad \tan(\pi \pm x) = \pm \tan x\,, \qquad \tan\left(\frac{3\pi}{2} \pm x\right) = \mp \cot x\,,$$
$$\cot\left(\frac{\pi}{2} \pm x\right) = \mp \tan x\,, \qquad \cot(\pi \pm x) = \pm \cot x\,, \qquad \cot\left(\frac{3\pi}{2} \pm x\right) = \mp \tan x\,.$$

Multiples of arguments:

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{2}{\cot x - \tan x}\,, \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x} = \frac{\cot x - \tan x}{2}\,,$$
$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}\,, \qquad \cot 3x = \frac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}\,,$$
$$\tan 4x = \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}\,, \qquad \cot 4x = \frac{\cot^4 x - 6\cot^2 x + 1}{4\cot^3 x - 4\cot x}\,.$$

Half-arguments:

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}\,,$$
$$\cot\frac{x}{2} = \frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}\,.$$

Sums:

$$\tan x \pm \tan y = \frac{\sin(x\pm y)}{\cos x \cos y}\,, \qquad \cot x \pm \cot y = \pm\frac{\sin(x \pm y)}{\sin x \sin y}\,,$$
$$\tan x + \cot y = \frac{\cos(x-y)}{\cos x \sin y}\,, \qquad \cot x - \tan y = \frac{\cos(x+y)}{\sin x \cos y}\,.$$

Products:

$$\tan x \tan y = \frac{\tan x + \tan y}{\cot x + \cot y} = -\frac{\tan x - \tan y}{\cot x - \cot y}\,,$$
$$\cot x \cot y = \frac{\cot x + \cot y}{\tan x + \tan y} = -\frac{\cot x - \cot y}{\tan x - \tan y}\,,$$
$$\tan x \cot y = \frac{\tan x + \cot y}{\cot x + \tan y} = -\frac{\tan x - \cot y}{\cot x - \tan y}\,.$$

Squares:

$\sin^2 x$	–	$1-\cos^2 x$	$\dfrac{\tan^2 x}{1+\tan^2 x}$	$\dfrac{1}{1+\cot^2 x}$
$\cos^2 x$	$1-\sin^2 x$	–	$\dfrac{1}{1+\tan^2 x}$	$\dfrac{\cot^2 x}{1+\cot^2 x}$
$\tan^2 x$	$\dfrac{\sin^2 x}{1-\sin^2 x}$	$\dfrac{1-\cos^2 x}{\cos^2 x}$	–	$\dfrac{1}{\cot^2 x}$
$\cot^2 x$	$\dfrac{1-\sin^2 x}{\sin^2 x}$	$\dfrac{\cos^2 x}{1-\cos^2 x}$	$\dfrac{1}{\tan^2 x}$	–

0.2.10 The hyperbolic functions sinh x and cosh x

Sinus hyperbolicus and cosinus hyperbolicus (hyperbolic sine and cosine): For all complex numbers x we define the functions

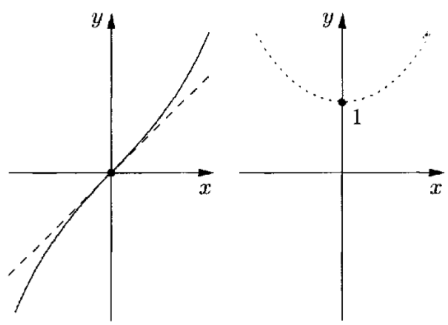
$$\boxed{\sinh x := \frac{e^x - e^{-x}}{2}, \quad \cosh x := \frac{e^x + e^{-x}}{2}.}$$

The function 'sinh' is read 'sinch', 'cosh' is read 'cosh'. For real arguments x the graph is drawn in Figure 0.34.

Relation to the trigonometric functions: For all complex numbers x one has:

$$\boxed{\sinh \mathrm{i}x = \mathrm{i}\sin x, \quad \cosh \mathrm{i}x = \cos x.}$$

Because of this relation every formula about the trigonometric functions sine and cosine gives rise to a formula about the hyperbolic functions cosh and sinh. For example $\cos^2 \mathrm{i}x + \sin^2 \mathrm{i}x = 1$ for all complex numbers x implies the following formula:



(a) $y = \sinh x$ (b) $y = \cosh x$

Figure 0.34. Hyperbolic functions.

$$\boxed{\cosh^2 x - \sinh^2 x = 1.}$$

The terminology hyperbolic function arises from the fact that these functions $x = a\cosh t$, $y = b\sinh t$, $t \in \mathbb{R}$ are the parameterization of a hyperbola (cf. 0.1.7.4).

The following formulas hold for all complex numbers x and y.

Evenness and oddness:

$$\sinh(-x) = -\sinh x, \quad \cosh(-x) = \cosh x.$$

Periodicity in the complex domain:

$$\sinh(x + 2\pi\mathrm{i}) = \sinh x\,, \qquad \cosh(x + 2\pi\mathrm{i}) = \cosh x\,.$$

Power series:

$$\boxed{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\,, \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots}$$

Derivative:

$$\boxed{\frac{\mathrm{d}\sinh x}{\mathrm{d}x} = \cosh x\,, \qquad \frac{\mathrm{d}\cosh x}{\mathrm{d}x} = \sinh x\,.}$$

Addition theorems:

$$\boxed{\begin{aligned}\sinh(x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y\,,\\ \cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y\,.\end{aligned}}$$

Doubled arguments:

$$\begin{aligned}\sinh 2x &= 2\sinh x \cosh x\,,\\ \cosh 2x &= \sinh^2 x + \cosh^2 x\,.\end{aligned}$$

Half-arguments:

$$\begin{aligned}\sinh\frac{x}{2} &= \sqrt{\frac{1}{2}(\cosh x - 1)} \qquad \text{for} \quad x \geq 0\,,\\ \sinh\frac{x}{2} &= -\sqrt{\frac{1}{2}(\cosh x - 1)} \qquad \text{for} \quad x < 0\,,\\ \cosh\frac{x}{2} &= \sqrt{\frac{1}{2}(\cosh x + 1)} \qquad \text{for} \quad x \in \mathbb{R}\,.\end{aligned}$$

Formula of de Moivre:

$$\boxed{(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx\,, \qquad n = 1, 2, \ldots}$$

Sums:

$$\begin{aligned}\sinh x \pm \sinh y &= 2\sinh\frac{1}{2}(x \pm y)\cosh\frac{1}{2}(x \mp y)\,,\\ \cosh x + \cosh y &= 2\cosh\frac{1}{2}(x + y)\cosh\frac{1}{2}(x - y)\,,\\ \cosh x - \cosh y &= 2\sinh\frac{1}{2}(x + y)\sinh\frac{1}{2}(x - y)\,.\end{aligned}$$

0.2.11 The hyperbolic functions $\tanh x$ and $\coth x$

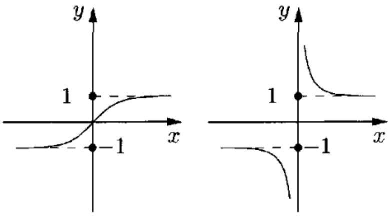
Tangens hyperbolicus and cotangens hyperbolicus (hyperbolic tangent and cotangent): For all complex numbers $x \neq \left(k\pi + \frac{\pi}{2}\right)\mathrm{i}$ with $k \in \mathbb{Z}$ we define a function

$$\boxed{\tanh x := \frac{\sinh x}{\cosh x}\,.}$$

For all complex numbers $x \neq k\pi\mathrm{i}$ with $k \in \mathbb{Z}$ we define the function

$$\boxed{\coth x := \frac{\cosh x}{\sinh x}.}$$

The graphical representation of these two functions for real arguments x is given in Figure 0.35.



(a) $y = \tanh x$ **(b)** $y = \coth x$

Figure 0.35. Hyperbolic functions.

The following formulas hold for all complex arguments x and y for which the functions do not have poles[39]

Relationship with the trigonometric functions:

$$\boxed{\tanh x = -\mathrm{i} \tan \mathrm{i}x, \qquad \coth x = \mathrm{i} \cot \mathrm{i}x.}$$

Table 0.20. Zeros and poles of the hyperbolic functions and trigonometric functions (all zeros and poles are simple).

Function	***Period***	***Zeros ($k \in \mathbb{Z}$)***	***Poles ($k \in \mathbb{Z}$)***	***Parity***
$\sinh x$	$2\pi\mathrm{i}$	$\pi k\mathrm{i}$	–	odd
$\cosh x$	$2\pi\mathrm{i}$	$\left(\pi k + \frac{\pi}{2}\right)\mathrm{i}$	–	even
$\tanh x$	$\pi\mathrm{i}$	$\pi k\mathrm{i}$	$\left(\pi k + \frac{\pi}{2}\right)\mathrm{i}$	odd
$\coth x$	$\pi\mathrm{i}$	$\left(\pi k + \frac{\pi}{2}\right)\mathrm{i}$	$\pi k\mathrm{i}$	odd
$\sin x$	2π	πk	–	odd
$\cos x$	2π	$\pi k + \frac{\pi}{2}$	–	even
$\tan x$	π	πk	$\pi k + \frac{\pi}{2}$	odd
$\cot x$	π	$\pi k + \frac{\pi}{2}$	πk	odd

Derivative:

$$\boxed{\frac{\mathrm{d}\tanh x}{\mathrm{d}x} = \frac{1}{\cosh^2 x}, \qquad \frac{\mathrm{d}\coth x}{\mathrm{d}x} = -\frac{1}{\sinh^2 x}.}$$

[39] In older liturature also the following functions are used (hyperbolic secant and cosecant):

$$\mathrm{cosech}x, := \frac{1}{\sinh x} \qquad \text{(cosecans hyperbolicus)},$$
$$\mathrm{sech}x, := \frac{1}{\cosh x} \qquad \text{(secans hyperbolicus)}.$$

Addition theorems:

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}, \qquad \coth(x \pm y) = \frac{1 \pm \coth x \coth y}{\coth x \pm \coth y}.$$

Doubled arguments:

$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}, \qquad \coth 2x = \frac{1 + \coth^2 x}{2\coth x}.$$

Half-arguments:

$$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1},$$
$$\coth \frac{x}{2} = \frac{\sinh x}{\cosh x - 1} = \frac{\cosh x + 1}{\sinh x}.$$

Sums:

$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y}.$$

Squares:

$\sinh^2 x$	–	$\cosh^2 x - 1$	$\frac{\tanh^2 x}{1 - \tanh^2 x}$	$\frac{1}{\coth^2 x - 1}$
$\cosh^2 x$	$\sinh^2 x + 1$	–	$\frac{1}{1 - \tanh^2 x}$	$\frac{\coth^2 x}{\coth^2 x - 1}$
$\tanh^2 x$	$\frac{\sinh^2 x}{\sinh^2 x + 1}$	$\frac{\cosh^2 x - 1}{\cosh^2 x}$	–	$\frac{1}{\coth^2 x}$
$\coth^2 x$	$\frac{\sinh^2 x + 1}{\sinh^2 x}$	$\frac{\cosh^2 x}{\cosh^2 x - 1}$	$\frac{1}{\tanh^2 x}$	–

Power series expansion: See section 0.7.2.

0.2.12 The inverse trigonometric functions

The function arcsine: The equation

$$y = \sin x, \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2},$$

has for every real number y with $-1 \le y \le 1$ exactly one solution, which we denote by $x = \arcsin y$. Formally exchanging x and y, we get the function

$$y = \arcsin x, \qquad -1 \le x \le 1.$$

The graph of this function is obtained from that of $y = \sin x$ by reflection on the diagonal[40] (see Table 0.21).

[40] In older literature the principal branch and other branches of the function $y = \arcsin x$ are used. This distinction can however lead to erroneous interpretation of (many-valued) formulas. In order to avoid that, we will use in this book only the one-to-one inverse function, which corresponds to the older principal branch (see Tables 0.21 and 0.22). The notation $y = \arcsin x$ means: y is the size of the angle y (measured in radians), whose sine has the value x (latin: arcus cuius sinus est x). Instead of $\arcsin x$, $\arccos x$, $\arctan x$ and $\text{arccot}\, x$ one speaks of the functions arcsine, arccosine, arctangent and arccotangent (of x).

Table 0.21. Inverse trigonometric functions – graphs.

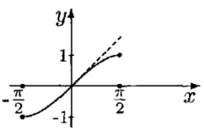
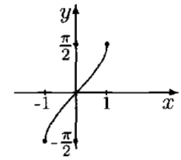
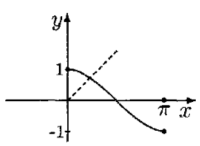
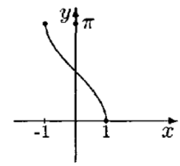
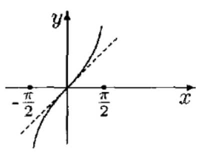
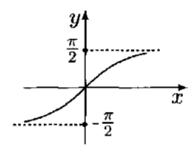
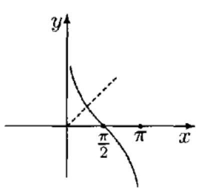
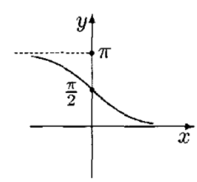
Original function	***Inverse function***
 $y = \sin x$	 $y = \arcsin x$
 $y = \cos x$	 $y = \arccos x$
 $y = \tan x$	 $y = \arctan x$
 $y = \cot x$	 $y = \text{arccot} x$

Table 0.22. Inverse trigonometric functions – formulas.

Equation	***Bounds on y***	***Solutions x (k ∈ Z)***
$y = \sin x$	$-1 \leq y \leq 1$	$x = \arcsin y + 2k\pi\,, \quad x = \pi - \arcsin y + 2\pi k\,,$
$y = \cos x$	$-1 \leq y \leq 1$	$x = \pm \arccos y + 2k\pi\,,$
$y = \tan x$	$-\infty < y < \infty$	$x = \arctan y + k\pi\,,$
$y = \cot x$	$-\infty < y < \infty$	$x = \text{arccot}\, y + k\pi\,.$

Transformation formulas: For all real numbers x with $-1 < x < 1$ one has:

$$\arcsin x = -\arcsin(-x) = \frac{\pi}{2} - \arccos x = \arctan \frac{x}{\sqrt{1-x^2}}\,.$$

For all real numbers x one has:

$$\arctan x = -\arctan(-x) = \frac{\pi}{2} - \operatorname{arccot} x = \arcsin \frac{x}{\sqrt{1+x^2}} .$$

Derivative: For all real numbers x with $-1 < x < 1$ one has:

$$\boxed{\frac{\mathrm{d} \arcsin x}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}} , \qquad \frac{\mathrm{d} \arccos x}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}} .}$$

For all real numbers x one has:

$$\boxed{\frac{\mathrm{d} \arctan x}{\mathrm{d}x} = \frac{1}{1+x^2} , \qquad \frac{\mathrm{d} \operatorname{arccot} x}{\mathrm{d}x} = -\frac{1}{1+x^2} .}$$

Power series: See section 0.7.2.

0.2.13 The inverse hyperbolic functions

Arcsinh: The equation

$$y = \sinh x , \qquad -\infty < x < \infty ,$$

has, for every real number y, exactly one solution, which is denoted by $x = \operatorname{arsinh} y$. Formally exchanging x and y, we get the function

$$y = \operatorname{arsinh} x , \qquad -\infty < x < \infty .$$

The graph of this function is obtained from the graph of the function $y = \sinh x$ by a reflection on the diagonal[41] (see Table 0.23).

Derivative:

$$\begin{aligned}
\frac{\mathrm{d} \operatorname{arsinh} x}{\mathrm{d}x} &= \frac{1}{\sqrt{1+x^2}} , && -\infty < x < \infty , \\
\frac{\mathrm{d} \operatorname{arcosh} x}{\mathrm{d}x} &= \frac{1}{\sqrt{1-x^2}} , && x > 1 , \\
\frac{\mathrm{d} \operatorname{artanh} x}{\mathrm{d}x} &= \frac{1}{1-x^2} , && |x| > 1 , \\
\frac{\mathrm{d} \operatorname{arcoth} x}{\mathrm{d}x} &= \frac{1}{1-x^2} , && |x| < 1 .
\end{aligned}$$

Power series: See section 0.7.2.

[41]The Latin names for the inverse hyperbolic functions are area sinus hyperbolicus, area cosinus hyperbolicus, area tangens hyperbolicus and area cotangens hyperbolicus (of x). The notation used here is from the fact that these functions give values which are the *arguments* of the hyperbolic functions.

Table 0.23. Inverse hyperbolic functions – graphs.

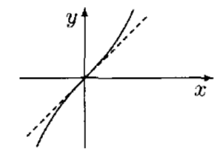
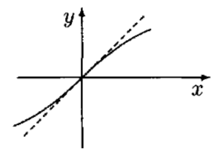
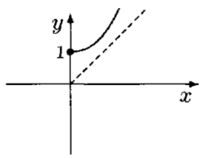
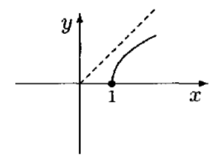
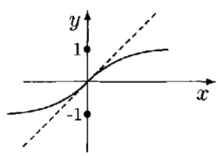
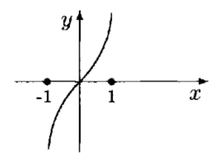
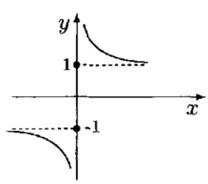
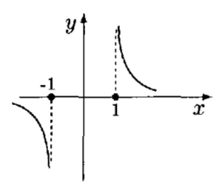
Original function	***Inverse function***
 $y = \sinh x$	 $y = \operatorname{arsinh} x$
 $y = \cosh x$	 $y = \operatorname{arcosh} x$
 $y = \tanh x$	 $y = \operatorname{artanh} x$
 $y = \coth x$	 $y = \operatorname{arcoth} x$

Table 0.24. Inverse hyperbolic functions – formulas.

Equation	***Bounds on y***	***Solution x***
$y = \sinh x$	$-\infty < y < \infty$	$x = \operatorname{arsinh} y = \ln\left(y + \sqrt{y^2+1}\right)$,
$y = \cosh x$	$y \geq 1$	$x = \pm\operatorname{arcosh} y = \pm\ln\left(y + \sqrt{y^2-1}\right)$,
$y = \tanh x$	$-1 < y < 1$	$x = \operatorname{artanh} y = \dfrac{1}{2}\ln\dfrac{1+y}{1-y}$,
$y = \coth x$	$y > 1\,,\ y < -1$	$x = \operatorname{arcoth} y = \dfrac{1}{2}\ln\dfrac{y+1}{y-1}$.

Transformation formulas:

$$\operatorname{arsinh} x = (\operatorname{sgn} x)\operatorname{arcosh}\sqrt{1+x^2} = \operatorname{artanh}\frac{x}{\sqrt{1+x^2}}\,, \qquad -\infty < x < \infty\,,$$

$$\operatorname{arcosh} x = \operatorname{arsinh}\sqrt{x^2-1}\,, \qquad x \geq 1\,,$$

$$\operatorname{arcoth} x = \operatorname{artanh}\frac{1}{x}\,, \qquad -1 < x < 1\,.$$

0.2.14 Polynomials

A (real) *polynomial of degree* n is a function of the form

$$\boxed{y = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0\,.} \tag{0.43}$$

Here n can take any of the values $n = 0, 1, 2, \ldots$, and all coefficients a_k are real numbers with $a_n \neq 0$.

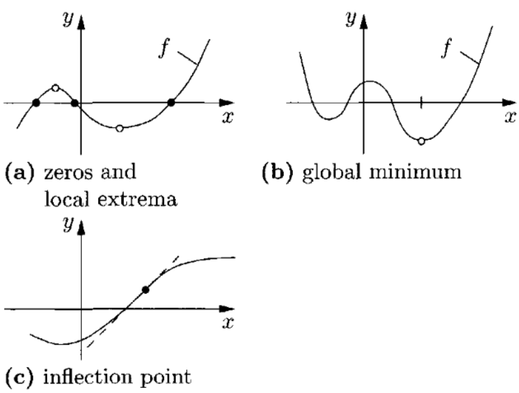
Smoothness: The function $y = f(x)$ in (0.43) is continuous and infinitely often differentiable in every point $x \in \mathbb{R}$. The first derivative is:

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1\,.$$

Behavior at infinity: The function $y = f(x)$ in (0.43) behaves for $x \to \pm\infty$ in the same way as the function $y = ax^n$, i.e., for $n \geq 1$ one has[42]:

$$\lim_{x\to+\infty} f(x) = \begin{cases} +\infty & \text{for} \quad a_n > 0\,, \\ -\infty & \text{for} \quad a_n < 0\,, \end{cases}$$

$$\lim_{x\to-\infty} f(x) = \begin{cases} +\infty & \text{for} \quad a_n > 0 \quad \text{and } n \text{ even} \\ & \text{or for } a_n < 0 \text{ and } n \text{ odd}\,, \\ -\infty & \text{for} \quad a_n > 0 \quad \text{and } n \text{ odd} \\ & \text{or for } a_n < 0 \text{ and } n \text{ even}\,. \end{cases}$$



(a) zeros and local extrema

(b) global minimum

(c) inflection point

Figure 0.36. Local properties of polynomials.

Zeros: If n is odd, then the graph of $y = f(x)$ intersects the x-axis at least once (Figure 0.36(a)). This point of intersection corresponds to a solution of the equation $f(x) = 0$.

Global minimum: If n is even and $a_n > 0$, then $y = f(x)$ has a global minimum, i.e., there is a point a with $f(a) \leq f(x)$ for all $x \in \mathbb{R}$ (Figure 0.36(b)).

If n is even and $a_n < 0$, then $y = f(x)$ has a global maximum.

Local extrema: Let $n \geq 2$. Then the function $y = f(x)$ has

[42]The meaning of the limit symbol 'lim' will be explained in 1.3.1.1 .

at most $n-1$ local extrema, which are alternately local minima and local maxima.

Inflection points: Let $n \geq 3$. Then the graph of $y = f(x)$ has at most $n-2$ inflection points (Figure 0.36(c)).

0.2.15 Rational functions

0.2.15.1 Special rational functions

Let $b > 0$ be a fixed real number. The function

$$\boxed{y = \frac{b}{x}, \qquad x \in \mathbb{R}, \quad x \neq 0,}$$

represents a equilateral hyperboloid, which has the x- and the y-axis as asymptotes. The vertices are $S_\pm = \left(\pm\sqrt{b}, \pm\sqrt{b}\right)$ (Figure 0.37).

Behavior at infinity: $\lim\limits_{x\to\pm\infty} \dfrac{b}{x} = 0$.

Pole at the point $x = 0$: $\lim\limits_{x\to\pm 0} \dfrac{b}{x} = \pm\infty$.

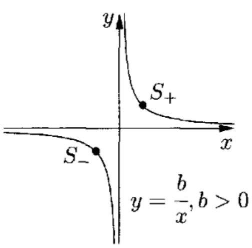
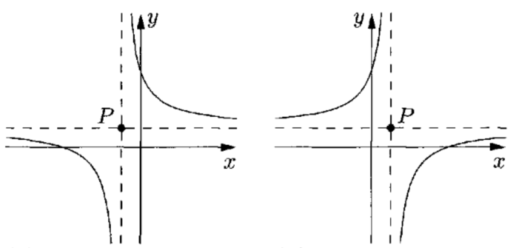


Figure 0.37.



(a) $\Delta < 0$ (b) $\Delta > 0$

Figure 0.38.

0.2.15.2 Rational function with linear numerators and denominators

Let the real numbers a, b, c and d be given with $c \neq 0$ and $\Delta := ad - bc \neq 0$. The function

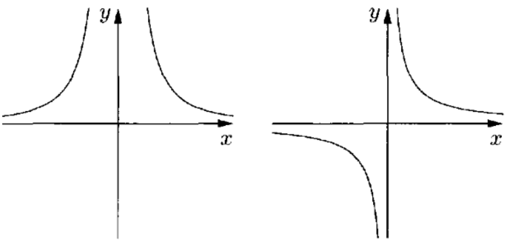
$$\boxed{y = \frac{ax+b}{cx+d}, \qquad x \in \mathbb{R},\ x \neq -\frac{d}{c}} \tag{0.44}$$

is transformed by the change of coordinates $x = u - \dfrac{d}{c}$, $y = w + \dfrac{a}{c}$ to the simpler form

$$w = -\frac{\Delta}{c^2 u}.$$

Thus, the general equation (0.44) results for the normalized form $y = -\dfrac{\Delta}{c^2 x}$ by a simple change of coordinates, which maps the point $(0, 0)$ to the point $P = \left(-\dfrac{d}{c}, \dfrac{a}{c}\right)$ (Figure 0.38).

0.2.15.3 Special rational function with a denominator of nth degree



(a) $y = \frac{b}{x^n}$, n even (b) $y = \frac{b}{x^n}$, n odd

Figure 0.39.

Let $b > 0$ be given and $n = 1, 2, \ldots$ The function

$$y = \frac{b}{x^n}, \qquad x \in \mathbb{R},\ x \neq 0,$$

is displayed in Figure 0.39.

0.2.15.4 Rational functions with quadratic denominator

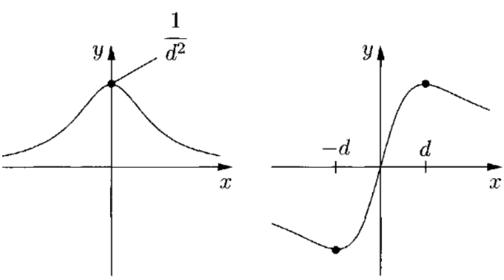
Special case 1: Let $d > 0$ be given. The functions

$$y = \frac{1}{x^2 + d^2}, \qquad x \in \mathbb{R},$$

and

$$y = \frac{x}{x^2 + d^2}, \qquad x \in \mathbb{R},$$

are pictured in Figure 0.40.



(a) $y = \frac{1}{x^2 + d^2}$ (b) $y = \frac{x}{x^2 + d^2}$

Figure 0.40.

Special case 2: Let two real numbers $x_\pm$ be given with $x_- < x_+$. The function $y = f(x)$ given by

$$y = \frac{1}{(x - x_+)(x - x_-)} \qquad (0.45)$$

can be put in the form

$$y = \frac{1}{x_+ - x_-}\left(\frac{1}{x - x_+} - \frac{1}{x - x_-}\right).$$

This is a special case of the so-call partial fraction decomposition (cf. 2.1.7). One has:

$$\lim_{x \to x_+ \pm 0} f(x) = \pm\infty, \quad \lim_{x \to x_- \pm 0} f(x) = \mp\infty, \quad \lim_{x \to \pm\infty} f(x) = 0.$$

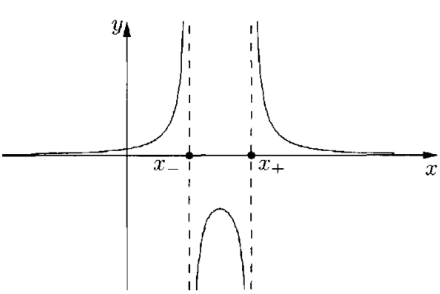


Figure 0.41.

Thus the poles of the function are at the points x_+ and x_- (see Figure 0.41).

Special case 3: The function

$$y = \frac{x - 1}{x^2 - 1} \qquad (0.46)$$

is initially not defined at the point $x = 1$. However, if one uses the decomposition $x^2 - 1 = (x - 1)(x + 1)$, then we get

$$y = \frac{1}{x + 1}, \qquad x \in \mathbb{R},\ x \neq -1.$$

One says that the function (0.46) has a *apparent singularity* at the point $x = 1$.

General case: Let real numbers a, b, c and d be given with $a^2 + b^2 \neq 0$. The behavior of the function $y = f(x)$ defined by

$$\boxed{y = \frac{ax+b}{x^2+2cx+d}} \tag{0.47}$$

depends in an essential way on the sign of the *discriminant* $D := c^2 - d$. Independently of this, one always has

$$\boxed{\lim_{x\to\pm\infty} f(x) = 0\,.}$$

Case 1: $D > 0$. Then one has

$$x^2 + 2cx + d = (x - x_+)(x - x_-)$$

with $x_\pm = -c \pm \sqrt{D}$. This yields the partial fraction decomposition

$$f(x) = \frac{A}{x - x_+} + \frac{B}{x - x_-}\,.$$

The constants A and B are determined by calculating the limits:

$$A = \lim_{x\to x_+}(x - x_+)f(x) = \frac{ax_+ + b}{x_+ - x_-}\,, \qquad B = \lim_{x\to x_-}(x - x_-)f(x) = \frac{ax_- + b}{x_- - x_+}\,.$$

There are poles at the points $x_\pm$.

Case 2: $D = 0$. In this case one has $x_+ = x_-$. We thus get

$$f(x) = \frac{ax+b}{(x - x_+)^2}\,.$$

This yields

$$\lim_{x\to x_+\pm 0} f(x) = \begin{cases} +\infty\,, & \text{if} \quad ax_+ + b > 0\,, \\ -\infty\,, & \text{if} \quad ax_+ + b < 0\,, \end{cases}$$

i.e., the point x_+ is a pole.

Case 3: $D < 0$. In this case we have $x^2 + 2cx + d > 0$ for all $x \in \mathbb{R}$. Consequently the function $y = f(x)$ in (0.47) is continuous and infinitely often differentiable for all points $x \in \mathbb{R}$, in other words, f is smooth.

0.2.15.5 The general rational function

A (real) *rational function* is a function $y = f(x)$ of the form

$$\boxed{y = \frac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}\,,}$$

where there are polynomials in both the numerator and the denominator (cf. 0.2.14).

Behavior at infinity: We set $c := a_n/b_m$. Then we have:

$$\boxed{\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} cx^{n-m}\,.}$$

From this we can discuss all possible cases.

Case 1: $c > 0$.

$$\lim_{x\to+\infty} f(x) = \begin{cases} c & \text{for } n = m\,, \\ +\infty & \text{for } n > m\,, \\ 0 & \text{for } n < m\,. \end{cases}$$

$$\lim_{x\to-\infty} f(x) = \begin{cases} c & \text{for } n = m\,, \\ +\infty & \text{for } n > m \text{ and } n-m \text{ even}\,, \\ -\infty & \text{for } n > m \text{ and } n-m \text{ odd}\,, \\ 0 & \text{for } n < m\,. \end{cases}$$

Case 2: $c < 0$. Here one must replace $\pm\infty$ by $\mp\infty$.

Partial fraction decomposition: The precise structure of rational functions is given by the partial fraction decomposition (cf. 2.1.7).

0.3 Mathematics and computers – a revolution in mathematics

One can say that we live in the age of mathematics, and that our culture has been 'mathematized'. This is proved beyond a doubt by the widespread use of computers.

Arthur Jaffe
(Harvard University, Cambridge, USA)

In solving mathematical problems one utilizes (at least) four important techniques:

(i) the use of numerical algorithms;

(ii) the algorithmic treatment of analytical, algebraic and geometric problems;

(iii) reference to tables and collections of formulas;

(iv) the graphical representation of situations.

Modern software programs can carry out all four of these effectively on computers:

(a) For the solution of standard problems of mathematics we suggest the system Mathematica.

(b) For more complicated problems of scientific calculations a combination of Maple and Matlab often leads to success.

(c) Many software packages also contain the program library Imsl math/stat/sfun library (International Mathematical and Statistical Library).

To solve a given mathematical problem, one should first check whether the problem is amenable to the procedures available in Mathamatica. This is the case for example for many of the problems considered in this book. Only after this has been checked with negative result should one resort to (b) or (c).

There is a long list of literature on this topic at the beginning of the bibliography. The handbooks listed there for using Mathematica are skillfully written and didacticly apt, addressed to a large audience. Experience shows that one requires a certain amount of time to get used to such programs, before they can be applied efficiently. It is worthwhile to invest this time; the gains are potentially great.

Modern software systems, which are continually being refined, are already able to do a great amount of the routine work for the user, freeing him for other activities. However, this can not replace a thorough occupation with the basics of mathematics. In this connection the following picture is helpful. At a construction site one sees today huge cranes, which do an enormous amount of work for humans. But still it is the humans which decide what is to be built and how the building should be designed. For this human qualities like phantasy and originality are required, something one can not expect (or want!) a machine to possess.

0.4 Tables of mathematical statistics and standard procedures for practitioners

The goal of this section is to give a large audience of potential readers an acquaintance with the basics and practical application of mathematical statistics. To meet this goal, we assume on the part of the reader almost nothing in the way of mathematical background. A discussion of the fundamentals of mathematical statistics can be found in 6.3.

Mathematical statistics on the computer: Elementary standard procedures can be done with Mathematica. More specialized statistical packages which are wide spread are **SPSS** and **SAS.**

0.4.1 The most important empirical data for sequences of measurements (trials)

Many measurements in technology, science or medicine have the characteristic property that the results of measurements vary from trial to trial. One says, that the measurements have a component of randomness. The quantity one wishes to measure, X, is called a *random variable.*

Example 1: The height X of a person is random, i.e., X is a random variable.

Sequence of measurements: If we measure a random quantity X, then we get measurements

$$\boxed{x_1\,,\ \ldots\,,\ x_n\,.}$$

Example 2: The Tables 0.25 and 0.26 show the result of measuring the height of 8 men in cm.

Table 0.25

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{x}$	Δx
168	170	172	175	176	177	180	182	175	4,8

Table 0.26

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{x}$	Δx
174	174	174	174	176	176	176	176	175	1,07

Empirical mean and empirical standard deviation: Two basic characteristics of a sequence of measurements $x_1, \ldots, x_n$ are the empirical mean

$$\boxed{\overline{x} := \frac{1}{n}(x_1 + x_2 + \ldots + x_n)}$$

and the empirical standard deviation Δx. The square of this quantity is given by the formula[43]

$$\boxed{(\Delta x)^2 := \frac{1}{n-1}\left[(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \ldots + (x_n - \overline{x})^2\right] .}$$

Example 3: For the values in Table 0.25 one gets

$$\overline{x} = \frac{1}{8}\big(168 + 170 + 172 + 175 + 176 + 177 + 180 + 182\big) = 175\,.$$

One says that the average height of the men is 175 cm. The same average is obtained from the values in Table 0.26. A glance at the tables however shows that the variation in the values of Table 0.25 is much higher than that of Table 0.26. For Table 0.26 we get

$$\begin{aligned}(\Delta x)^2 &= \frac{1}{7}\left[(174-175)^2 + (174-175)^2 + \ldots + (176-175)^2\right]\\ &= \frac{1}{7}\big[1+1+1+1+1+1+1+1\big] = \frac{8}{7}\,,\end{aligned}$$

in other words, $\Delta x = 1.07$. On the other hand, the values for Table 0.25 yield for the standard deviation from the equation

$$\begin{aligned}(\Delta x)^2 &= \frac{1}{7}\left[(168-175)^2 + (170-175)^2 + \ldots + (182-175)^2\right]\\ &= \frac{1}{7}\big[49 + 25 + 9 + 1 + 4 + 25 + 49\big] = 23\end{aligned}$$

the value $\Delta x = 4.8$.

Rule of thumb:

> The smaller the empirical standard deviation is, the smaller is the variation of the measurements from the mean $\overline{x}$.

In the limiting case $\Delta x = 0$, all the measurements coincide with the mean $\overline{x}$.

The distribution of the measurements – the histogram: To get a general idea of the distribution of the measurements, one uses, especially for larger sets of measurements, a graphical representation called the *histogram*.

(i) One divides the measurements into several classes $K_1, K_2, \ldots, K_s$. These are neighboring intervals.

(ii) Let m_r denote the number of measurements which belong to the class K_r.

(iii) If n measurements have been made $x_1, \ldots, x_n$, then the quantity $\dfrac{m_r}{n}$ is called the relative frequency of the measurements with respect to the class K_r.

[43]The appearance of the denominator $n-1$ instead of that probably expected by many readers, namely n, can be justified by estimation theory. In fact the quantity Δx is a *expectation faithful estimation* for the theoretical standard deviation ΔX of the random variable X (cf. 6.3.2). For large n the difference between n and $n-1$ is negligible.

The quantity $(\Delta x)^2$ is called the *variance*.

(iv) One graphs the classes K_j, with a column of height $\frac{m_r}{n}$ over each K_r.

Example 4: In Table 0.27 the measurements for the heights of 100 men in centimeters are listed. The histogram constructed from these data is displayed in Figure 0.42.

Table 0.27

Class K_r	***Measure-ments***	***Frequency*** m_r	***Relative frequency*** $\frac{m_r}{100}$
K_1	$150 \le x < 165$	2	0,02
K_2	$165 \le x < 170$	18	0,18
K_3	$170 \le x < 175$	30	0,30
K_4	$175 \le x < 180$	32	0,32
K_5	$180 \le x < 185$	16	0,16
K_6	$185 \le x < 200$	2	0,02

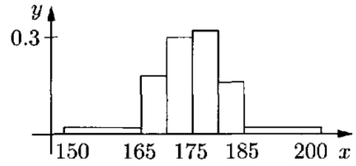


Figure 0.42

0.4.2 The theoretical distribution function

The sequences of trials of a random variable X generally vary from trial to trial. For example, the measurements of heights of persons leads to different results if one measures all men in a house, a city or a state. The notion of theoretical distribution function is necessary in order to build up a theory of random variables.

Definition: The *theoretical distribution function* Φ of a random variable X is defined by the following prescription:

$$\boxed{\Phi(x) := P\,(X < x)\ .}$$

This means that the value $\Phi(x)$ is equal to the *probability* that the random variable is less than the given number x.

The normal distribution: Many measured quantities follow a *normal* distribution. To explain this, we consider a Gauss bell curve

$$\boxed{\varphi(x) := \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-(x-\mu)^2/2\sigma^2}\ .} \tag{0.48}$$

Such a curve has a maximum at the point $x = \mu$. The smaller the positive value σ is, the more it is more concentrated at the point $x = \mu$. One calls μ the *mean* and σ the *standard deviation* of the normal distribution (Figure 0.43(a)).

> The area of the hatched surface in Figure 0.43(b) is equal to the probability that the random variable lies in the interval $[a, b]$.

The distribution function Φ of the normal distribution (0.48) is displayed in Figure 0.43(d). The value $\Phi(a)$ in Figure 0.43(d) is equal to the area of the surface under the bell curve, which lies to the left of a. The difference

$$\boxed{\Phi(b) - \Phi(a)}$$

is equal to the area of the hatched surface in Figure 0.43(b).

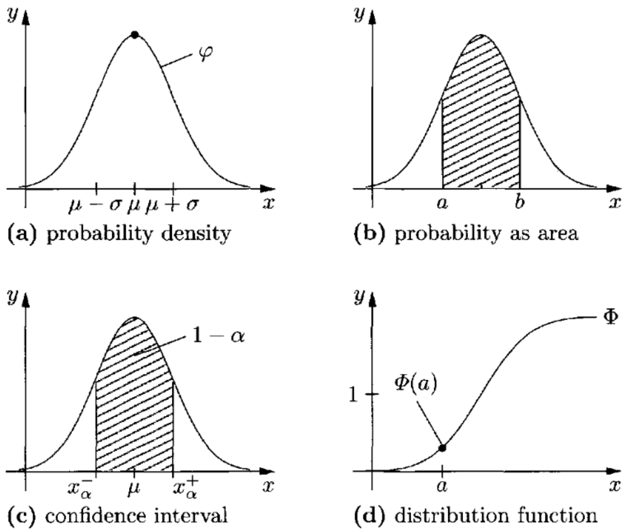


Figure 0.43. Properties of the Gaussian normal distribution.

Confidence interval: This notion is of central importance for mathematical statistics. The α-confidence interval $[x_\alpha^-, x_\alpha^+]$ of the random variable X is defined in such a way that the probability that all measurements of X lie in the interval is $1 - \alpha$, i.e., the probability that x satisfies the inequality

$$x_\alpha^- \leq x \leq x_\alpha^+$$

is $1 - \alpha$. In Figure 0.43(c) the endpoints of the interval x_α^+ and x_α^- are chosen in such a way that they are symmetric around the mean μ and the area of the hatched surface is equal to $1 - \alpha$. One has

$$x_\alpha^+ = \mu + \sigma z_\alpha\,, \qquad x_\alpha^- = \mu - \sigma z_\alpha\,.$$

The value of z_α for the important cases for many applications, namely $\alpha = 0.01\,, 0.05\,, 0.1$, are listed in Table 0.28.

Table 0.28

α	0.01	0.05	0.1
z_α	2.6	2.0	1.6

The random variable X lies with the probability $1 - \alpha$ in the α-confidence interval.

Example: Let $\mu = 10$ and $\sigma = 2$. For $\alpha = 0.01$ we get

$$x_\alpha^+ = 10 + 2 \cdot 2.6 = 15.2\,, \qquad x_\alpha^- = 10 - 2 \cdot 2.6 = 4.8\,.$$

It follows that with a probability of $1 - \alpha = 0.99$, the measured value x lies between 4.8 and 15.2. Intuitively this means the following.

(a) If n is a large number and we take a total of n measurements of X, then there are approximately $(1 - \alpha)n = 0.99n$ measured values in between 4.8 and 15.2.

(b) If we measure X for example 1000 times, then approximately 990 of the values lie between 4.8 and 15.2.

0.4.3 Checking for a normal distribution

Many test procedures in applications are based on the assumption that a random variable X follows a normal distribution. We describe here a simple graphical procedure to test *whether* X is normally distributed.

(i) We draw a line in the (z, y)-coordinate plane, which contains the pairs of points $(z, \Phi(z))$, which we take from Table 0.29 (Figure 0.44). Note that the value on the y-axis in the present case has an irregular scale.

(ii) For given measured values $x_1, \ldots, x_n$ of X we form the quantities

$$z_j := \frac{x_j - \overline{x}}{\Delta x}, \qquad j = 1, \ldots, n.$$

(iii) We calculate the numbers

$$\Phi_*(z_j) = \frac{1}{n}\,(\text{number of measured values } z_k\text{, which are smaller than } z_j\ )$$

and plot the points $\big(z_j, \Phi_*(z_j)\big)$ in Figure 0.44.

If these points lie approximately on the line drawn in (i), then X is approximately normally distributed.

Example: The open circles in Figure 0.44 represent measurements which are approximately normally distributed.

Table 0.29. Sample values of the normal distribution function.

z	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5
$\Phi(z)$	0.01	0.02	0.07	0.16	0.31	0.5	0.69	0.84	0.93	0.98	0.99

A more precise table of the values of Φ is given in Table 0.29. The diagram of Figure 0.44 can also be obtained as so-called probability paper.

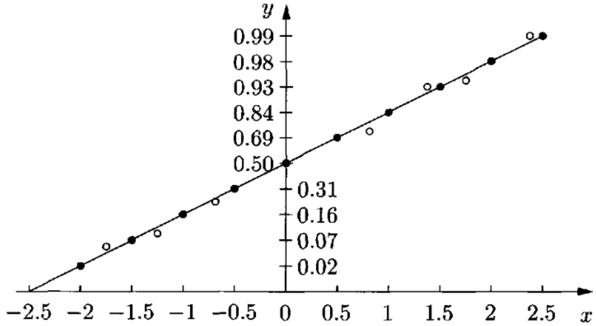


Figure 0.44. Checking data for approximate normal distribution.

The χ^2-fit test for normal distributions: This test, which is much more significant than the intuitive method just explained with the probability paper, is described in 6.3.4.5.

0.4.4 The statistical evaluation of a sequence of measurements

We assume that X is a normally distributed random variable, whose normal distribution (0.48) has mean μ and standard deviation σ.

The confidence limit for the mean μ:

(i) We take n measurements of the quantity X and get the measurements $x_1, \ldots, x_n$.

(ii) We choose a small number α as the probability of being in error and determine the value of $t_{\alpha,m}$ with $m = n - 1$ from Table in 0.4.6.3.

Then the unknown mean μ for the normal distribution satisfies the inequality:

$$\boxed{\overline{x} - t_{\alpha,m}\frac{\Delta x}{\sqrt{n}} \leq \mu \leq \overline{x} + t_{\alpha,m}\frac{\Delta x}{\sqrt{n}}\,.}$$

This statement has a probability of α of being in error.

Example 1: For the measurements of heights listed in Table 0.25 one has $n = 8$, $\overline{x} = 175$, $\Delta x = 4.8$. If we choose $\alpha = 0.01$, then we get from 0.4.6.3 for $m = 7$ the value $t_{\alpha,m} = 3.5$. If we assume that the height is a normally distributed random variable, then we have, with error probability $\alpha = 0.01$, for the mean:

$$169 \leq \mu \leq 181\,.$$

The confidence interval for the standard deviation σ: With probability α of error, the standard deviation σ satisfies the inequality:

$$\boxed{\frac{(n-1)\,(\Delta x)^2}{b} \leq \sigma^2 \leq \frac{(n-1)\,(\Delta x)^2}{a}\,.}$$

The values $a := \chi^2_{1-\alpha/2}$ and $b := \chi^2_{\alpha/2}$ are extracted from the table in 0.4.6.4 with $m = n - 1$ degrees of freedom.

Example 2: We consider again the height measurements listed in Table 0.25. For $\alpha = 0.01$ and $m = 7$ we get $a = 1.24$ and $b = 20.3$ from 0.4.6.4. Consequently we get with error probability $\alpha = 0.01$ the estimate

$$2.8 \leq \sigma \leq 11.40\,.$$

It is not surprising that these estimates are quite rough. This is because of the small number of measurements.

A more detailed justification is given in 6.3.3.

0.4.5 The statistical comparison of two sequences of measurements

Let two sequences of trials of random variables X and Y be given,

$$\boxed{x_1, \ldots, x_{n_1} \qquad \text{and} \qquad y_1, \ldots, y_{n_2}.} \tag{0.49}$$

Two basic questions are:

(i) Is there a dependence between the two sequences of measurements?

(ii) Is there a significant difference between the two random variables?

To investigate (i) one uses correlation coefficients. An answer to (ii) is provided by the F-test, the t-test and the Wilcoxon-test. This will be considered in the sequel.

0.4.5.1 Empirical correlation coefficients

The empirical correlation coefficient of the two sequences of measurements (0.49) with $n_1 = n_2 = n$ is given by the number

$$\varrho = \frac{(x_1 - \overline{x})(y_1 - \overline{y}) + (x_2 - \overline{x})(y_2 - \overline{y}) + \ldots + (x_n - \overline{x})(y_n - \overline{y})}{(n-1)\Delta x \Delta y}.$$

One has $-1 \le \varrho \le 1$. For $\varrho = 0$ there is no dependency between the two sequences.

> The dependency between the two sequences is stronger the larger the quantity ϱ^2 is.

Regression line: If one plots the pairs (x_j, y_j) of measurements in a Cartesian coordinate system, then the so-called *regression line*

$$\boxed{y = \overline{y} + \varrho \frac{\Delta y}{\Delta x}(x - \overline{x})}$$

is the line closest to the set of plotted points (Figure 0.45), i.e., this line is a solution of the minimum problem

$$\sum_{j=1}^{n} (y_j - a - bx_j)^2 \stackrel{!}{=} \min, \qquad a, b \text{ real},$$

and the minimal value is equal to $(\Delta y)^2 (1 - \varrho^2)$. The fit of the regression line on the measurements is thus optimal for $\varrho^2 = 1$.

Table 0.30

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{x}$	Δx
168	170	172	175	176	177	180	182	175	5
y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	$\overline{y}$	Δy
157	160	163	165	167	167	168	173	165	5

Example: For the two sequences of measurements listed in Table 0.30 one gets the correlation coefficient

$$\varrho = 0.96$$

with the regression line

$$y = \overline{y} + 0.96(x - \overline{x}). \qquad (0.50)$$

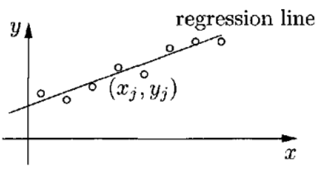


Figure 0.45.

Here there is a strong dependency between the two sequences of measurements. The measurements are approximated quite well by the regression line (0.50).

0.4.5.2 The comparison of two means with the t-test

The t-test is often used in applications. With this test one can verify whether the means of two sequences of trials differ essentially from one another.

(i) We consider the two sequences $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ of two random variables X and Y, which we assume are normally distributed.

In addition one assumes that the standard deviation of X and Y are the same. This assumption can be checked with the help of the F-test 0.4.5.3.

(ii) We calculate the number

$$t = \frac{\overline{x} - \overline{y}}{\sqrt{(n_1 - 1)(\Delta x)^2 + (n_2 - 1)(\Delta y)^2}} \sqrt{\frac{n_1 n_2 (n_1 + n_2 - 2)}{n_1 + n_2}} .$$

(iii) For a given α and $m = n_1 + n_2 - 1$ we determine the value $t_{\alpha,m}$ from the table in 0.4.6.3.

Case 1: Assume

$$\boxed{|t| > t_{\alpha,m} .}$$

In this case the means of X and Y differ, i.e., the differences between the measured empirical means $\overline{x}$ and $\overline{y}$ are not random, but have some explanation. One also says in this case that there is a significant difference between the random variables X and Y.

Case 2: One has

$$\boxed{|t| < t_{\alpha,m} .}$$

One may assume that the means of X and Y do not differ significantly.

The statements both have a probability of error of α. This means the following. If one applies this test in 100 different situations, then there is a probability that this will lead to a false conclusion in $100 \cdot \alpha$ cases.

Example 1: For $\alpha = 0.01$ there might be in 100 tests one case in which the test leads to a wrong conclusion.

Example 2: Two medicines A and B are being given to patients which have the same illness. The random variable is the number of days X (resp. Y) for the medicine A (resp. B) to be administered until the illness has been cured. Table 0.31 lists some measurements. For example, the mean duration until cure is 20 days under the use of medicine A.

Table 0.31

medicine A :	$\overline{x} = 20$	$\Delta x = 5$	$n_1 = 15$ patients
medicine B :	$\overline{y} = 26$	$\Delta y = 4$	$n_2 = 15$ patients.

We get

$$t = \frac{26 - 20}{\sqrt{14 \cdot 25 + 14 \cdot 16}} \cdot \sqrt{\frac{15 \cdot 15\,(30 - 2)}{15 + 15}} = 3.6 .$$

We find for the value of $\alpha = 0.01$ and $m = 15 + 15 - 1 = 29$ the value $t_{\alpha,m} = 2.8$ from the table in 0.4.6.3.

Because of $t > t_{\alpha,m}$ there is a significant difference between the two medicines, i.e., medicine A is better than B.

0.4.5.3 The F-test

This test verifies whether the standard deviations of two normally distributed random variables differ from one another.

(i) We consider the two sequences of measurements $x_1, \ldots, x_{n_1}$ and $y_1, \ldots, y_{n_2}$ of the two random variables X and Y, both of which we assume follow a normal distribution.

(ii) We form the quotients

$$F := \begin{cases} \left(\dfrac{\Delta x}{\Delta y}\right)^2, & \text{if } \Delta x > \Delta y, \\ \left(\dfrac{\Delta y}{\Delta x}\right)^2, & \text{if } \Delta x \le \Delta y. \end{cases}$$

(iii) We look up the bold-faced value $F_{0.01;m_1m_2}$ in 0.4.6.5 for $m_1 := n_1 - 1$ and $m_2 := n_2 - 1$.

Case 1: One has

$$\boxed{F > F_{0.01;m_1m_2}\,.}$$

In this case the standard deviations of X and Y are *not* the same, i.e., the difference between the measured empirical standard deviations Δx and Δy is not a random variation, but has some deeper meaning.

Case 2: One has

$$\boxed{F \le F_{0.01;m_1m_2}\,.}$$

One may assume that the standard deviations of X and Y are essentially the same.

These statements both have a probability of error of 0.02. This means the following. Carrying out this test in 100 different situations, it is likely that in two of these situations the test leads to a wrong conclusion.

Example: We consider again the situation giving rise to the data of Table 0.31. One has $F = (\Delta x/\Delta y)^2 = 1.6$. From the table in 0.4.6.5 with $m_1 = m_2 = 14$ we find $F_{0.01;m_1m_2} = 3.7$. Because of $F < F_{0.01;m_1m_2}$ we can be assured that X and Y have the same standard deviation.

0.4.5.4 The Wilcoxon-test

The t-test can only be applied to normally distributed quantities. The Wilcoxon-test is much more general and can be applied for example to check whether two sequences of trials come from random variables with different distributions, i.e., whether these quantities differ from each other in an essential way. This test is described in 6.3.4.5.

0.4.6 Tables of mathematical statistics

0.4.6.1 Interpolation of tables

Linear interpolation: Each table consists of entries and table values. In Table 0.32, x denotes the entries and $f(x)$ the values.

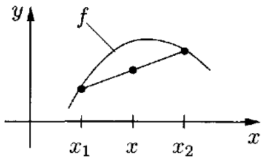


Figure 0.46.

Table 0.32

x	$f(x)$
1	0.52
2	0.60
3	0.64

First basic problem: Interpolation of table values $f(x)$ for known entries x: If the entry x is not in the table, one can use the method of linear interpolation, which is illustrated in Figure 0.46. Here the graph of $y = f(x)$ is replaced by the secant between two of the pairs $(x_1, f(x_1))$ and $(x_2, f(x_2))$. The approximate value $f_*(x)$ for $f(x)$ is then derived from the linear interpolation formula:

$$\boxed{f_*(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)\,.} \tag{0.51}$$

Example 1: Let $x = 1.5$. In Table 0.32 one finds the nearest entries

$$x_1 = 1 \qquad \text{and} \qquad x_2 = 2$$

with the values $f(x_1) = 0.52$ and $f(x_2) = 0.60$. From the interpolation formula (0.51) it follows

$$\begin{aligned} f_*(x) &= 0.52 + \frac{0.60 - 0.52}{1}(1.5 - 1) \\ &= 0.52 + 0.08 \cdot 0.5 = 0.56\,. \end{aligned}$$

Second basic problem: interpolation of the entry x for a known value $f(x)$: To determine x from $f(x)$, one uses the formula:

$$\boxed{x = x_1 + \frac{f(x) - f(x_1)}{f(x_2) - f(x_1)}(x_2 - x_1)\,.} \tag{0.52}$$

Example 2: Suppose that the value $f(x) = 0.62$ is given. The nearest table values in Table 0.32 are $f(x_1) = 0.60$ and $f(x_2) = 0.64$ with $x_1 = 2$ and $x_2 = 3$. Applying (0.52) we get

$$x = 2 + \frac{0.62 - 0.60}{0.64 - 0.60}(3 - 2) = 2 + \frac{0.02}{0.04} = 2.5\,.$$

Higher precision with Mathematica: Linear interpolation is a method of producing an approximation. For the needs of mathematical statistics this is quite sufficent. One should not be led to believe that more digits after the decimal point is an increase in accuracy in an endeavor like statistics which by its very nature is not precise.

In physics and technology, however, one often requires a higher precision. For this the method of quadratic interpolation is often applied. In the day of wide-spread computers one can use computer software programs to get very precise values for special functions (for example with Mathematica).

0.4.6.2 Normal distribution

Table 0.33. The density function

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}z^2\right)$$ *of the normalized, centered normal distribution.*

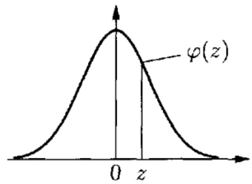


Figure 0.47.

z	0	1	2	3	4	5	6	7	8	9
0.0	3 989^{-4}	3 989	3 989	3 988	3 986	3 984	3 982	3 980	3 977	3 973
0.1	3 970^{-4}	3 965	3 961	3 956	3 951	3 945	3 939	3 932	3 925	3 918
0.2	3 910^{-4}	3 902	3 894	3 885	3 876	3 867	3 857	3 847	3 836	3 825
0.3	3 814^{-4}	3 802	3 790	3 778	3 765	3 752	3 739	3 725	3 712	3 697
0.4	3 683^{-4}	3 668	3 653	3 637	3 621	3 605	3 589	3 572	3 555	3 538
0.5	3 521^{-4}	3 503	3 485	3 467	3 448	3 429	3 410	3 391	3 372	3 352
0.6	3 332^{-4}	3 312	3 292	3 271	3 251	3 230	3 209	3 187	3 166	3 144
0.7	3 123^{-4}	3 101	3 079	3 056	3 034	3 011	2 989	2 966	2 943	2 920
0.8	2 897^{-4}	2 874	2 850	2 827	2 803	2 780	2 756	2 732	2 709	2 685
0.9	2 661^{-4}	2 637	2 613	2 589	2 565	2 541	2 516	2 492	2 468	2 444
1.0	2 420^{-4}	2 396	2 371	2 347	2 323	2 299	2 275	2 251	2 227	2 203
1.1	2 179^{-4}	2 155	2 131	2 107	2 083	2 059	2 036	2 012	1 989	1 965
1.2	1 942^{-4}	1 919	1 895	1 872	1 849	1 826	1 804	1 781	1 758	1 736
1.3	1 714^{-4}	1 691	1 669	1 647	1 626	1 604	1 582	1 561	1 539	1 518
1.4	1 497^{-4}	1 476	1 456	1 435	1 415	1 394	1 374	1 354	1 334	1 315
1.5	1 295^{-4}	1 276	1 257	1 238	1 219	1 200	1 182	1 163	1 145	1 127
1.6	1 109^{-4}	1 092	1 074	1 057	1 040	1 023	1 006	9893^{-5}	9 728	9 566
1.7	9 405^{-5}	9 246	9 089	8 933	8 780	8 628	8 478	8 329	8 183	8 038
1.8	7 895^{-5}	7 754	7 614	7 477	7 341	7 206	7 074	6 943	6 814	6 687
1.9	6 562^{-5}	6 438	6 316	6 195	6 077	5 960	5 844	5 730	5 618	5 508
2.0	5 399^{-5}	5 292	5 186	5 082	4 980	4 879	4 780	4 682	4 586	4 491
2.1	4 398^{-5}	4 307	4 217	4 128	4 041	3 955	3 871	3 788	3 706	3 626
2.2	3 547^{-5}	3 470	3 394	3 319	3 246	3 174	3 103	3 034	2 965	2 898
2.3	2 833^{-5}	2 768	2 705	2 643	2 582	2 522	2 463	2 406	2 349	2 294
2.4	2 239^{-5}	2 186	2 134	2 083	2 033	1 984	1 936	1 888	1 842	1 797
2.5	1 753^{-5}	1 709	1 667	1 625	1 585	1 545	1 506	1 468	1 431	1 394
2.6	1 358^{-5}	1 323	1 289	1 256	1 223	1 191	1 160	1 130	1 100	1 071
2.7	1 042^{-5}	1 014	9 871^{-6}	9 606	9 347	9 094	8 846	8 605	8 370	8 140
2.8	7 915^{-6}	7 697	7 483	7 274	7 071	6 873	6 679	6 491	6 307	6 127
2.9	5 953^{-6}	5 782	5 616	5 454	5 296	5 143	4 993	4 847	4 705	4 567
3.0	4 432^{-6}	4 301	4 173	4 049	3 928	3 810	3 695	3 584	3 475	3 370
3.1	3 267^{-6}	3 167	3 070	2 975	2 884	2 794	2 707	2 623	2 541	2 461
3.2	2 384^{-6}	2 309	2 236	2 165	2 096	2 029	1 964	1 901	1 840	1 780
3.3	1 723^{-6}	1 667	1 612	1 560	1 508	1 459	1 411	1 364	1 319	1 275
3.4	1 232^{-6}	1 191	1 151	1 112	1 075	1 038	1 003	9 689^{-7}	9 358	9 037
3.5	8 727^{-7}	8 426	8 135	7 853	7 581	7 317	7 061	6 814	6 575	6 343
3.6	6 119^{-7}	5 902	5 693	5 490	5 294	5 105	4 921	4 744	4 573	4 408
3.7	4 248^{-7}	4 093	3 944	3 800	3 661	3 526	3 396	3 271	3 149	3 032
3.8	2 919^{-7}	2 810	2 705	2 604	2 506	2 411	2 320	2 232	2 147	2 065
3.9	1 987^{-7}	1 910	1 837	1 766	1 698	1 633	1 569	1 508	1 449	1 393
4.0	1 338^{-7}	1 286	1 235	1 186	1 140	1 094	1 051	1 009	9 687^{-8}	9 299
4.1	8 926^{-8}	8 567	8 222	7 890	7 570	7 263	6 967	6 683	6 410	6 147
4.2	5 894^{-8}	5 652	5 418	5 194	4 979	4 772	4 573	4 382	4 199	4 023
4.3	3 854^{-8}	3 691	3 535	3 386	3 242	3 104	2 972	2 845	2 723	2 606
4.4	2 494^{-8}	2 387	2 284	2 185	2 090	1 999	1 912	1 829	1 749	1 672
4.5	1 598^{-8}	1 528	1 461	1 396	1 334	1 275	1 218	1 164	1 112	1 062
4.6	1 014^{-8}	9 684^{-9}	9 248	8 830	8 430	8 047	7 681	7 331	6 996	6 676
4.7	6 370^{-9}	6 077	5 797	5 530	5 274	5 030	4 796	4 573	4 360	4 156
4.8	3 961^{-9}	3 775	3 598	3 428	3 267	3 112	2 965	2 824	2 960	2 561
4.9	2 439^{-9}	2 322	2 211	2 105	2 003	1 907	1 814	1 727	1 643	1 563

Remark: 3989^{-4} is to be understood to mean $3989 \cdot 10^{-4}$.

Table 0.34. Probability integral $\Phi_0(z) = \int_0^z \varphi(x)\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_0^z \exp\left(-\frac{1}{2}x^2\right)\mathrm{d}x$ *of the normalized, centered normal distribution.*

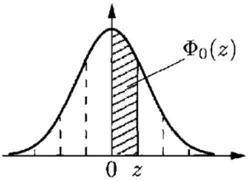


Figure 0.48.

The distribution function $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{1}{2}x^2\right)\mathrm{d}x$ is related to $\Phi_0(z)$ by the relation $\Phi(z) = \frac{1}{2} + \Phi_0(z)$; moreover, $\Phi_0(-z) = -\Phi_0(z)$.

z	0	1	2	3	4	5	6	7	8	9
0.0	0.0 000	040	080	120	160	199	239	279	319	359
0.1	398	438	478	517	557	596	636	675	714	753
0.2	793	832	871	910	948	987	·026	·064	·103	·141
0.3	0.1 179	217	255	293	331	368	406	443	480	517
0.4	554	591	628	664	700	736	772	808	844	879
0.5	915	950	985	·019	·054	·088	·123	·157	·190	·224
0.6	0.2 257	291	324	357	389	422	454	486	517	549
0.7	580	611	642	673	703	734	764	794	823	852
0.8	881	910	939	967	995	·023	·051	·078	·106	·133
0.9	0.3 159	186	212	238	264	289	315	340	365	389
1.0	413	438	461	485	508	531	554	577	599	621
1.1	643	665	686	708	729	749	770	790	810	830
1.2	849	869	888	907	925	944	962	980	997	·015
1.3	0.4 032	049	066	082	099	115	131	147	162	177
1.4	192	207	222	236	251	265	279	292	306	319
1.5	332	345	357	370	382	394	406	418	429	441
1.6	452	463	474	484	495	505	515	525	535	545
1.7	554	564	573	582	591	599	608	616	625	633
1.8	641	649	656	664	671	678	686	693	699	706
1.9	713	719	726	732	738	744	750	756	761	767
2.0	772	778	783	788	793	798	803	808	812	817
2.1	821	826	830	834	838	842	846	850	854	857
2.2	860	864	867	871	874	877	880	883	886	889
	966	*474*	*906*	*263*	*545*	*755*	*894*	*962*	*962*	*893*
2.3	892	895	898	900	903	906	908	911	913	915
	759	*559*	*296*	*969*	*581*	*133*	*625*	*060*	*437*	*758*
2.4	918	920	922	924	926	928	930	932	934	936
	025	*237*	*397*	*506*	*564*	*572*	*531*	*443*	*309*	*128*
2.5	937	939	941	942	944	946	947	949	950	952
	903	*634*	*323*	*969*	*574*	*139*	*664*	*151*	*600*	*012*
2.6	953	954	956	957	958	959	960	962	963	964
	388	*729*	*035*	*308*	*547*	*754*	*930*	*074*	*189*	*274*
2.7	965	966	967	968	969	970	971	971	972	973
	330	*358*	*359*	*333*	*280*	*202*	*099*	*972*	*821*	*646*
2.8	974	975	975	976	977	978	978	979	980	980
	449	*229*	*988*	*726*	*443*	*140*	*818*	*476*	*116*	*738*
2.9	981	981	982	983	983	984	984	985	985	986
	342	*929*	*498*	*052*	*589*	*111*	*618*	*110*	*588*	*051*

Remarks: 0.4 860 is to be interpreted here to mean 0.4 860 966.
966

A dot in front of an entry indicates a jump of one in decimal place.
For example in the line $z = 0.5$, the entry ·019 means .2019.

Table 0.34. (continued)

z	0	1	2	3	4	5	6	7	8	9
3.0	0.4 986 *501*	986 *938*	987 *361*	987 *772*	988 *171*	988 *558*	988 *933*	989 *297*	989 *650*	989 *992*
3.1	990 *324*	990 *646*	990 *957*	991 *260*	991 *553*	991 *836*	992 *112*	992 *378*	992 *636*	992 *886*
3.2	993 *129*	993 *363*	993 *590*	993 *810*	994 *024*	994 *230*	994 *429*	994 *623*	994 *810*	994 *991*
3.3	995 *166*	995 *335*	995 *499*	995 *658*	995 *811*	995 *959*	996 *103*	996 *242*	996 *376*	996 *505*
3.4	996 *631*	996 *752*	996 *869*	996 *982*	997 *091*	997 *197*	997 *299*	997 *398*	997 *493*	997 *585*
3.5	997 *674*	997 *759*	997 *842*	997 *922*	997 *999*	998 *074*	998 *146*	998 *215*	998 *282*	998 *347*
3.6	998 *409*	998 *469*	998 *527*	998 *583*	998 *637*	998 *689*	998 *739*	998 *787*	998 *834*	998 *879*
3.7	998 *922*	998 *964*	999 *004*	999 *043*	999 *080*	999 *116*	999 *150*	999 *184*	999 *216*	999 *247*
3.8	999 *276*	999 *305*	999 *333*	999 *359*	999 *385*	999 *409*	999 *433*	999 *456*	999 *478*	999 *499*
3.9	999 *519*	999 *539*	999 *557*	999 *575*	999 *593*	999 *609*	999 *625*	999 *641*	999 *655*	999 *670*
4.0	999 *683*	999 *696*	999 *709*	999 *721*	999 *733*	999 *744*	999 *755*	999 *765*	999 *775*	999 *784*
4.1	999 *793*	999 *802*	999 *811*	999 *819*	999 *826*	999 *834*	999 *841*	999 *848*	999 *854*	999 *861*
4.2	999 *867*	999 *872*	999 *878*	999 *883*	999 *888*	999 *893*	999 *898*	999 *902*	999 *907*	999 *911*
4.3	999 *915*	999 *918*	999 *922*	999 *925*	999 *929*	999 *932*	999 *935*	999 *938*	999 *941*	999 *943*
4.4	999 *946*	999 *948*	999 *951*	999 *953*	999 *955*	999 *957*	999 *959*	999 *961*	999 *963*	999 *964*
4.5	999 *966*	999 *968*	999 *969*	999 *971*	999 *972*	999 *973*	999 *974*	999 *976*	999 *977*	999 *978*
5.0	999 *997*									

0.4.6.3 Values $t_{\alpha, m}$ of the Student t-distribution

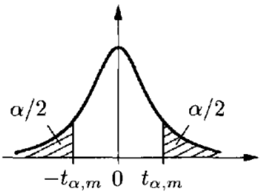


Figure 0.49.

m \ α	0.10	0.05	0.025	0.020	0.010	0.005	0.003	0.002	0.001
1	6.314	12.706	25.452	31.821	63.657	127.3	212.2	318.3	636.6
2	2.920	4.303	6.205	6.965	9.925	14.089	18.216	22.327	31.600
3	2.353	3.182	4.177	4.541	5.841	7.453	8.891	10.214	12.922
4	2.132	2.776	3.495	3.747	4.604	5.597	6.435	7.173	8.610
5	2.015	2.571	3.163	3.365	4.032	4.773	5.376	5.893	6.869
6	1.943	2.447	2.969	3.143	3.707	4.317	4.800	5.208	5.959
7	1.895	2.365	2.841	2.998	3.499	4.029	4.442	4.785	5.408
8	1.860	2.306	2.752	2.896	3.355	3.833	4.199	4.501	5.041
9	1.833	2.262	2.685	2.821	3.250	3.690	4.024	4.297	4.781
10	1.812	2.228	2.634	2.764	3.169	3.581	3.892	4.144	4.587
12	1.782	2.179	2.560	2.681	3.055	3.428	3.706	3.930	4.318
14	1.761	2.145	2.510	2.624	2.977	3.326	3.583	3.787	4.140
16	1.746	2.120	2.473	2.583	2.921	3.252	3.494	3.686	4.015
18	1.734	2.101	2.445	2.552	2.878	3.193	3.428	3.610	3.922
20	1.725	2.086	2.423	2.528	2.845	3.153	3.376	3.552	3.849
22	1.717	2.074	2.405	2.508	2.819	3.119	3.335	3.505	3.792
24	1.711	2.064	2.391	2.492	2.797	3.092	3.302	3.467	3.745
26	1.706	2.056	2.379	2.479	2.779	3.067	3.274	3.435	3.704
28	1.701	2.048	2.369	2.467	2.763	3.047	3.250	3.408	3.674
30	1.697	2.042	2.360	2.457	2.750	3.030	3.230	3.386	3.646
∞	1.645	1.960	2.241	2.326	2.576	2.807	2.968	3.090	3.291

0.4.6.4 Values χ^2_α of the χ^2-distribution

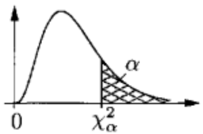


Figure 0.50.

Number m of degrees of freedom	probability α															
	0.99	**0.98**	**0.95**	**0.90**	**0.80**	**0.70**	**0.50**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.005**	**0.002**	**0.001**
1	0.000 16	0.000 6	0.003 9	0.016	0.064	0.148	0.455	1.07	1.64	2.7	3.8	5.4	6.6	7.9	9.5	10.83
2	0.020	0.040	0.103	0.211	0.446	0.713	1.386	2.41	3.22	4.6	6.0	7.8	9.2	10.6	12.4	13.8
3	0.115	0.185	0.352	0.584	1.005	1.424	2.366	3.67	4.64	6.3	7.8	9.8	11.3	12.8	14.8	16.3
4	0.30	0.43	0.71	1.06	1.65	2.19	3.36	4.9	6.0	7.8	9.5	11.7	13.3	14.9	16.9	18.5
5	0.55	0.75	1.14	1.61	2.34	3.00	4.35	6.1	7.3	9.2	11.1	13.4	15.1	16.8	18.9	20.5
6	0.87	1.13	1.63	2.20	3.07	3.83	5.35	7.2	8.6	10.6	12.6	15.0	16.8	18.5	20.7	22.5
7	1.24	1.56	2.17	2.83	3.82	4.67	6.35	8.4	9.8	12.0	14.1	16.6	18.5	20.3	22.6	24.3
8	1.65	2.03	2.73	3.49	4.59	5.53	7.34	9.5	11.0	13.4	15.5	18.2	20.1	22.0	24.3	26.1
9	2.09	2.53	3.32	4.17	5.38	6.39	8.34	10.7	12.2	14.7	16.9	19.7	21.7	23.6	26.1	27.9
10	2.56	3.06	3.94	4.86	6.18	7.27	9.34	11.8	13.4	16.0	18.3	21.2	23.2	25.2	27.7	29.6
11	3.1	3.6	4.6	5.6	7.0	8.1	10.3	12.9	14.6	17.3	19.7	22.6	24.7	26.8	29.4	31.3
12	3.6	4.2	5.2	6.3	7.8	9.0	11.3	14.0	15.8	18.5	21.0	24.1	26.2	28.3	30.9	32.9
13	4.1	4.8	5.9	7.0	8.6	9.9	12.3	15.1	17.0	19.8	22.4	25.5	27.7	29.8	32.5	34.5
14	4.7	5.4	6.6	7.8	9.5	10.8	13.3	16.2	18.2	21.1	23.7	26.9	29.1	31.3	34.0	36.1
15	5.2	6.0	7.3	8.5	10.3	11.7	14.3	17.3	19.3	22.3	25.0	28.3	30.6	32.8	35.6	37.7
16	5.8	6.6	8.0	9.3	11.2	12.6	15.3	18.4	20.5	23.5	26.3	29.6	32.0	34.3	37.1	39.3
17	6.4	7.3	8.7	10.1	12.0	13.5	16.3	19.5	21.6	24.8	27.6	31.0	33.4	35.7	38.6	40.8
18	7.0	7.9	9.4	10.9	12.9	14.4	17.3	20.6	22.8	26.0	28.9	32.3	34.8	37.2	40.1	42.3
19	7.6	8.6	10.1	11.7	13.7	15.4	18.3	21.7	23.9	27.2	30.1	33.7	36.2	38.6	41.6	43.8
20	8.3	9.2	10.9	12.4	14.6	16.3	19.3	22.8	25.0	28.4	31.4	35.0	37.6	40.0	43.0	45.3
21	8.9	9.9	11.6	13.2	15.4	17.2	20.3	23.9	26.2	29.6	32.7	36.3	38.9	41.4	44.5	46.8
22	9.5	10.6	12.3	14.0	16.3	18.1	21.3	24.9	27.3	30.8	33.9	37.7	40.3	42.8	45.9	48.3
23	10.2	11.3	13.1	14.8	17.2	19.0	22.3	26.0	28.4	32.0	35.2	39.0	41.6	44.2	47.3	49.7
24	10.9	12.0	13.8	15.7	18.1	19.9	23.3	27.1	29.6	33.2	36.4	40.3	43.0	45.6	48.7	51.2
25	11.5	12.7	14.6	16.5	18.9	20.9	24.3	28.2	30.7	34.4	37.7	41.6	44.3	46.9	50.1	52.6
26	12.2	13.4	15.4	17.3	19.8	21.8	25.3	29.2	31.8	35.6	38.9	42.9	45.6	48.3	51.6	54.1
27	12.9	14.1	16.2	18.1	20.7	22.7	26.3	30.3	32.9	36.7	40.1	44.1	47.0	49.6	52.9	55.5
28	13.6	14.8	16.9	18.9	21.6	23.6	27.3	31.4	34.0	37.9	41.3	45.4	48.3	51.0	54.4	56.9
29	14.3	15.6	17.7	19.8	22.5	24.6	28.3	32.5	35.1	39.1	42.6	46.7	49.6	52.3	55.7	58.3
30	15.0	16.3	18.5	20.6	23.4	25.5	29.3	33.5	36.3	40.3	43.8	48.0	50.9	53.7	57.1	59.7

0.4.6.5 Values $F_{0.05;m_1m_2}$ and values $F_{0.01;m_1m_2}$ (in boldface) of the F-distribution

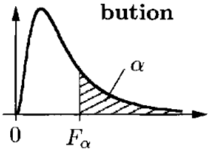


Figure 0.51.

m_2	m_1 = 1	2	3	4	5	6	7	8	9	10	11	12
1	161	200	216	225	230	234	237	239	241	242	243	244
	4 052	**4 999**	**5 403**	**5 625**	**5 764**	**5 859**	**5 928**	**5 981**	**6 022**	**6 056**	**6 083**	**6 106**
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.39	19.40	19.41
	98.50	**99.00**	**99.17**	**99.25**	**99.30**	**99.33**	**99.36**	**99.37**	**99.39**	**99.40**	**99.41**	**99.42**
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74
	34.12	**30.82**	**29.46**	**28.71**	**28.24**	**27.91**	**27.67**	**27.49**	**27.34**	**27.23**	**27.13**	**27.05**
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91
	21.20	**18.00**	**16.69**	**15.98**	**15.52**	**15.21**	**14.98**	**14.80**	**14.66**	**14.55**	**14.45**	**14.37**
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68
	16.26	**13.27**	**12.06**	**11.39**	**10.97**	**10.67**	**10.46**	**10.29**	**10.16**	**10.05**	**9.96**	**9.89**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00
	13.74	**10.92**	**9.78**	**9.15**	**8.75**	**8.47**	**8.26**	**8.10**	**7.98**	**7.87**	**7.79**	**7.72**
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57
	12.25	**9.55**	**8.45**	**7.85**	**7.46**	**7.19**	**7.00**	**6.84**	**6.72**	**6.62**	**6.54**	**6.47**
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28
	11.26	**8.65**	**7.59**	**7.01**	**6.63**	**6.37**	**6.18**	**6.03**	**5.91**	**5.81**	**5.73**	**5.67**
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07
	10.56	**8.02**	**6.99**	**6.42**	**6.06**	**5.80**	**5.61**	**5.47**	**5.35**	**5.26**	**5.18**	**5.11**
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91
	10.04	**7.56**	**6.55**	**5.99**	**5.64**	**5.39**	**5.20**	**5.06**	**4.94**	**4.85**	**4.77**	**4.71**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79
	9.65	**7.21**	**6.22**	**5.67**	**5.32**	**5.07**	**4.89**	**4.74**	**4.63**	**4.54**	**4.46**	**4.40**
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69
	9.33	**6.93**	**5.95**	**5.41**	**5.06**	**4.82**	**4.64**	**4.50**	**4.39**	**4.30**	**4.22**	**4.16**
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60
	9.07	**6.70**	**5.74**	**5.21**	**4.86**	**4.62**	**4.44**	**4.30**	**4.19**	**4.10**	**4.02**	**3.96**
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53
	8.86	**6.51**	**5.56**	**5.04**	**4.70**	**4.46**	**4.28**	**4.14**	**4.03**	**3.94**	**3.86**	**3.80**
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48
	8.68	**6.36**	**5.42**	**4.89**	**4.56**	**4.32**	**4.14**	**4.00**	**3.89**	**3.80**	**3.73**	**3.67**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42
	8.53	**6.23**	**5.29**	**4.77**	**4.44**	**4.20**	**4.03**	**3.89**	**3.78**	**3.69**	**3.62**	**3.55**
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38
	8.40	**6.11**	**5.18**	**4.67**	**4.34**	**4.10**	**3.93**	**3.79**	**3.68**	**3.59**	**3.52**	**3.46**
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34
	8.29	**6.01**	**5.09**	**4.58**	**4.25**	**4.01**	**3.84**	**3.71**	**3.60**	**3.51**	**3.43**	**3.37**
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31
	8.18	**5.93**	**5.01**	**4.50**	**4.17**	**3.94**	**3.77**	**3.63**	**3.52**	**3.43**	**3.36**	**3.30**
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28
	8.10	**5.85**	**4.94**	**4.43**	**4.10**	**3.87**	**3.70**	**3.56**	**3.46**	**3.37**	**3.29**	**3.23**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25
	8.02	**5.78**	**4.87**	**4.37**	**4.04**	**3.81**	**3.64**	**3.51**	**3.40**	**3.31**	**3.24**	**3.17**
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23
	7.95	**5.72**	**4.82**	**4.31**	**3.99**	**3.76**	**3.59**	**3.45**	**3.35**	**3.26**	**3.18**	**3.12**
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.24	2.20
	7.88	**5.66**	**4.76**	**4.26**	**3.94**	**3.71**	**3.54**	**3.41**	**3.30**	**3.21**	**3.14**	**3.07**

m_1: 14	16	20	24	30	40	50	75	100	200	500	∞	m_2
245	246	248	249	250	251	252	253	253	254	254	254	1
6 143	**6 169**	**6 209**	**6 235**	**6 261**	**6 287**	**6 302**	**6 323**	**6 334**	**6 352**	**6 361**	**6 366**	
19.42	19.43	19.44	19.45	19.46	19.47	19.48	19.48	19.49	19.49	19.50	19.50	2
99.43	**99.44**	**99.45**	**99.46**	**99.47**	**99.47**	**99.48**	**99.49**	**99.49**	**99.49**	**99.50**	**99.50**	
8.71	8.69	8.66	8.64	8.62	8.59	8.58	8.57	8.55	8.54	8.53	8.53	3
26.92	**26.83**	**26.69**	**26.60**	**26.50**	**26.41**	**26.35**	**26.27**	**26.23**	**26.18**	**26.14**	**26.12**	
5.87	5.84	5.80	5.77	5.75	5.72	5.70	5.68	5.66	5.65	5.64	5.63	4
14.25	**14.15**	**14.02**	**13.93**	**13.84**	**13.74**	**13.69**	**13.61**	**13.57**	**13.52**	**13.48**	**13.46**	
4.64	4.60	4.56	4.53	4.50	4.46	4.44	4.42	4.41	4.39	4.37	4.36	5
9.77	**9.68**	**9.55**	**9.47**	**9.38**	**9.29**	**9.24**	**9.17**	**9.13**	**9.08**	**9.04**	**9.02**	
3.96	3.92	3.87	3.84	3.81	3.77	3.75	3.72	3.71	3.69	3.68	3.67	6
7.60	**7.52**	**7.39**	**7.31**	**7.23**	**7.14**	**7.09**	**7.02**	**6.99**	**6.93**	**6.90**	**6.88**	
3.53	3.49	3.44	3.41	3.38	3.34	3.32	3.29	3.27	3.25	3.24	3.23	7
6.36	**6.27**	**6.16**	**6.07**	**5.99**	**5.91**	**5.86**	**5.78**	**5.75**	**5.70**	**5.67**	**5.65**	
3.24	3.20	3.15	3.12	3.08	3.05	3.02	3.00	2.97	2.95	2.94	2.93	8
5.56	**5.48**	**5.36**	**5.28**	**5.20**	**5.12**	**5.07**	**5.00**	**4.96**	**4.91**	**4.88**	**4.86**	
3.03	2.99	2.93	2.90	2.86	2.83	2.80	2.77	2.76	2.73	2.72	2.71	9
5.00	**4.92**	**4.81**	**4.73**	**4.65**	**4.57**	**4.52**	**4.45**	**4.42**	**4.36**	**4.33**	**4.31**	
2.86	2.83	2.77	2.74	2.70	2.66	2.64	2.61	2.59	2.56	2.55	2.54	10
4.60	**4.52**	**4.41**	**4.33**	**4.25**	**4.17**	**4.12**	**4.05**	**4.01**	**3.96**	**3.93**	**3.91**	
2.74	2.70	2.65	2.61	2.57	2.53	2.51	2.47	2.46	2.43	2.42	2.40	11
4.29	**4.21**	**4.10**	**4.02**	**3.94**	**3.86**	**3.81**	**3.74**	**3.71**	**3.66**	**3.62**	**3.60**	
2.64	2.60	2.54	2.51	2.47	2.43	2.40	2.36	2.35	2.32	2.31	2.30	12
4.05	**3.97**	**3.86**	**3.78**	**3.70**	**3.62**	**3.57**	**3.49**	**3.47**	**3.41**	**3.38**	**3.36**	
2.55	2.51	2.46	2.42	2.38	2.34	2.31	2.28	2.26	2.23	2.22	2.21	13
3.86	**3.78**	**3.66**	**3.59**	**3.51**	**3.43**	**3.38**	**3.30**	**3.27**	**3.22**	**3.19**	**3.17**	
2.48	2.44	2.39	2.35	2.31	2.27	2.24	2.21	2.19	2.16	2.14	2.13	14
3.70	**3.62**	**3.51**	**3.43**	**3.35**	**3.27**	**3.22**	**3.14**	**3.11**	**3.06**	**3.03**	**3.00**	
2.42	2.38	2.33	2.29	2.25	2.20	2.18	2.15	2.12	2.10	2.08	2.07	15
3.56	**3.49**	**3.37**	**3.29**	**3.21**	**3.13**	**3.08**	**3.00**	**2.98**	**2.92**	**2.89**	**2.87**	
2.37	2.33	2.28	2.24	2.19	2.15	2.12	2.09	2.07	2.04	2.02	2.01	16
3.45	**3.37**	**3.26**	**3.18**	**3.10**	**3.02**	**2.97**	**2.86**	**2.86**	**2.81**	**2.78**	**2.75**	
2.33	2.29	2.23	2.19	2.15	2.10	2.08	2.04	2.02	1.99	1.97	1.96	17
3.35	**3.27**	**2.16**	**3.08**	**3.00**	**2.92**	**2.87**	**2.79**	**2.76**	**2.71**	**2.68**	**2.65**	
2.29	2.25	2.19	2.15	2.11	2.06	2.04	2.00	1.98	1.95	1.93	1.92	18
3.27	**3.19**	**3.08**	**3.00**	**2.92**	**2.84**	**2.78**	**2.71**	**2.68**	**2.62**	**2.59**	**2.57**	
2.26	2.21	2.15	2.11	2.07	2.03	2.00	1.96	1.94	1.91	1.90	1.88	19
3.19	**3.12**	**3.00**	**2.92**	**2.84**	**2.76**	**2.71**	**2.63**	**2.60**	**2.55**	**2.51**	**2.49**	
2.22	2.18	2.12	2.08	2.04	1.99	1.97	1.92	1.91	1.88	1.86	1.84	20
3.13	**3.05**	**2.94**	**2.86**	**2.78**	**2.69**	**2.64**	**2.56**	**2.54**	**2.48**	**2.44**	**2.42**	
2.20	2.16	2.10	2.05	2.01	1.96	1.94	1.89	1.88	1.84	1.82	1.81	21
3.07	**2.99**	**2.88**	**2.80**	**2.72**	**2.64**	**2.58**	**2.51**	**2.48**	**2.42**	**2.38**	**2.36**	
2.17	2.13	2.07	2.03	1.98	1.94	1.91	1.87	1.85	1.81	1.80	1.78	22
3.02	**2.94**	**2.83**	**2.75**	**2.67**	**2.58**	**2.53**	**2.46**	**2.42**	**2.36**	**2.33**	**2.31**	
2.15	2.11	2.05	2.00	1.96	1.91	1.88	1.84	1.82	1.79	1.77	1.76	23
2.97	**2.89**	**2.78**	**2.70**	**2.62**	**2.54**	**2.48**	**2.41**	**2.37**	**2.32**	**2.28**	**2.26**	

m_2	m_1											
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22	2.18
	7.82	**5.61**	**4.72**	**4.22**	**3.90**	**3.67**	**3.50**	**3.36**	**3.26**	**3.17**	**3.09**	**3.03**
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16
	7.77	**5.57**	**4.68**	**4.18**	**3.86**	**3.63**	**3.46**	**3.32**	**3.22**	**3.13**	**3.06**	**2.99**
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15
	7.72	**5.53**	**4.64**	**4.14**	**3.82**	**3.59**	**3.42**	**3.29**	**3.18**	**3.09**	**3.02**	**2.96**
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.16	2.13
	7.68	**5.49**	**4.60**	**4.11**	**3.78**	**3.56**	**3.39**	**3.26**	**3.15**	**3.06**	**2.99**	**2.93**
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12
	7.64	**5.45**	**4.57**	**4.07**	**3.76**	**3.53**	**3.36**	**3.23**	**3.12**	**3.03**	**2.96**	**2.90**
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.14	2.10
	7.60	**5.42**	**4.54**	**4.04**	**3.73**	**3.50**	**3.33**	**3.20**	**3.09**	**3.00**	**2.93**	**2.87**
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09
	7.56	**5.39**	**4.51**	**4.02**	**3.70**	**3.47**	**3.30**	**3.17**	**3.07**	**2.98**	**2.90**	**2.84**
32	4.15	3.29	2.90	2.67	2.51	2.40	2.31	2.24	2.19	2.14	2.10	2.07
	7.50	**5.34**	**4.46**	**3.97**	**3.65**	**3.43**	**3.25**	**3.13**	**3.02**	**2.93**	**2.86**	**2.80**
34	4.13	3.28	2.88	2.65	2.49	2.38	2.29	2.23	2.17	2.12	2.08	2.05
	7.44	**5.29**	**4.42**	**3.93**	**3.61**	**3.39**	**3.22**	**3.09**	**2.98**	**2.89**	**2.82**	**2.76**
36	4.11	3.26	2.87	2.63	2.48	2.36	2.28	2.21	2.15	2.11	2.07	2.03
	7.40	**5.25**	**4.38**	**3.89**	**3.57**	**3.35**	**3.18**	**3.05**	**2.95**	**2.86**	**2.79**	**2.72**
38	4.10	3.24	2.85	2.62	2.46	2.35	2.26	2.19	2.14	2.09	2.05	2.02
	7.35	**5.21**	**4.34**	**3.86**	**3.54**	**3.32**	**3.15**	**3.02**	**2.91**	**2.82**	**2.75**	**2.69**
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00
	7.31	**5.18**	**4.31**	**3.83**	**3.51**	**3.29**	**3.12**	**2.99**	**2.89**	**2.80**	**2.73**	**2.66**
42	4.07	3.22	2.83	2.59	2.44	2.32	2.24	2.17	2.11	2.06	2.03	1.99
	7.28	**5.15**	**4.29**	**3.80**	**3.49**	**3.27**	**3.10**	**2.97**	**2.86**	**2.78**	**2.70**	**2.64**
44	4.06	3.21	2.82	2.58	2.43	2.31	2.23	2.16	2.10	2.05	2.01	1.98
	7.25	**5.12**	**4.26**	**3.78**	**3.47**	**3.24**	**3.08**	**2.95**	**2.84**	**2.75**	**2.68**	**2.62**
46	4.05	3.20	2.81	2.57	2.42	2.30	2.22	2.15	2.09	2.04	2.00	1.97
	7.22	**5.10**	**4.24**	**3.76**	**3.44**	**3.22**	**3.06**	**2.93**	**2.82**	**2.73**	**2.66**	**2.60**
48	4.04	3.19	2.80	2.57	2.41	2.30	2.21	2.14	2.08	2.03	1.99	1.96
	7.20	**5.08**	**4.22**	**3.74**	**3.43**	**3.20**	**3.04**	**2.91**	**2.80**	**2.72**	**2.64**	**2.58**
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.99	1.95
	7.17	**5.06**	**4.20**	**3.72**	**3.41**	**3.19**	**3.02**	**2.89**	**2.79**	**2.70**	**2.63**	**2.56**
55	4.02	3.16	2.78	2.54	2.38	2.27	2.18	2.11	2.06	2.01	1.97	1.93
	7.12	**5.01**	**4.16**	**3.68**	**3.37**	**3.15**	**2.98**	**2.85**	**2.75**	**2.66**	**2.59**	**2.53**
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92
	7.08	**4.98**	**4.13**	**3.65**	**3.34**	**3.12**	**2.95**	**2.82**	**2.72**	**2.63**	**2.56**	**2.50**
65	3.99	3.14	2.75	2.51	2.36	2.24	2.15	2.08	2.03	1.98	1.94	1.90
	7.04	**4.95**	**4.10**	**3.62**	**3.31**	**3.09**	**2.93**	**2.80**	**2.69**	**2.61**	**2.53**	**2.47**
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02	1.97	1.93	1.89
	7.01	**4.92**	**4.08**	**3.60**	**3.29**	**3.07**	**2.91**	**2.78**	**2.67**	**2.59**	**2.51**	**2.45**
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.91	1.88
	6.96	**4.88**	**4.04**	**3.56**	**3.26**	**3.04**	**2.87**	**2.74**	**2.64**	**2.55**	**2.48**	**2.42**
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.89	1.85
	6.90	**4.82**	**3.98**	**3.51**	**3.21**	**2.99**	**2.82**	**2.69**	**2.59**	**2.50**	**2.43**	**2.37**
125	3.92	3.07	2.68	2.44	2.29	2.17	2.08	2.01	1.96	1.91	1.87	1.83
	6.84	**4.78**	**3.94**	**3.47**	**3.17**	**2.95**	**2.79**	**2.66**	**2.55**	**2.50**	**2.40**	**2.33**
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94	1.89	1.85	1.82
	6.81	**4.75**	**3.92**	**3.45**	**3.14**	**2.92**	**2.76**	**2.63**	**2.53**	**2.44**	**2.37**	**2.31**
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80
	6.76	**4.71**	**3.88**	**3.41**	**3.11**	**2.89**	**2.73**	**2.60**	**2.50**	**2.41**	**2.34**	**2.27**
400	3.86	3.02	2.62	2.39	2.23	2.12	2.03	1.96	1.90	1.85	1.81	1.78
	6.70	**4.66**	**3.83**	**3.36**	**3.06**	**2.85**	**2.69**	**2.55**	**2.46**	**2.37**	**2.29**	**2.23**
1000	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89	1.84	1.80	1.76
	6.66	**4.63**	**3.80**	**3.34**	**3.04**	**2.82**	**2.66**	**2.53**	**2.43**	**2.34**	**2.27**	**2.20**
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75
	6.63	**4.61**	**3.78**	**3.32**	**3.02**	**2.80**	**2.64**	**2.51**	**2.41**	**2.32**	**2.25**	**2.18**

m_1												m_2
14	**16**	**20**	**24**	**30**	**40**	**50**	**75**	**100**	**200**	**500**	∞	
2.13	2.09	2.03	1.98	1.94	1.89	1.86	1.82	1.80	1.77	1.75	1.73	24
2.93	**2.85**	**2.74**	**2.66**	**2.58**	**2.49**	**2.44**	**2.36**	**2.33**	**2.27**	**2.24**	**2.21**	
2.11	2.07	2.01	1.96	1.92	1.87	1.84	1.80	1.78	1.75	1.73	1.71	25
2.89	**2.81**	**2.70**	**2.62**	**2.54**	**2.45**	**2.40**	**2.32**	**2.29**	**2.23**	**2.19**	**2.17**	
2.10	2.05	1.99	1.95	1.90	1.85	1.82	1.78	1.76	1.73	1.70	1.69	26
2.86	**2.78**	**2.66**	**2.58**	**2.50**	**2.42**	**2.36**	**2.28**	**2.25**	**2.19**	**2.16**	**2.13**	
2.08	2.04	1.97	1.93	1.88	1.84	1.81	1.76	1.74	1.71	1.68	1.67	27
2.82	**2.75**	**2.63**	**2.55**	**2.47**	**2.38**	**2.33**	**2.25**	**2.22**	**2.16**	**2.12**	**2.10**	
2.06	2.02	1.96	1.91	1.87	1.82	1.79	1.75	1.73	1.69	1.67	1.65	28
2.80	**2.71**	**2.60**	**2.52**	**2.44**	**2.35**	**2.30**	**2.22**	**2.19**	**2.13**	**2.09**	**2.06**	
2.05	2.01	1.94	1.90	1.85	1.80	1.77	1.73	1.71	1.67	1.65	1.64	29
2.77	**2.69**	**2.57**	**2.49**	**2.41**	**2.33**	**2.27**	**2.19**	**2.16**	**2.10**	**2.06**	**2.03**	
2.04	1.99	1.93	1.89	1.84	1.79	1.76	1.72	1.70	1.66	1.64	1.62	30
2.74	**2.66**	**2.55**	**2.47**	**2.38**	**2.30**	**2.25**	**2.16**	**2.13**	**2.07**	**2.03**	**2.01**	
2.01	1.97	1.91	1.86	1.82	1.77	1.74	1.69	1.67	1.63	1.61	1.59	32
2.70	**2.62**	**2.50**	**2.42**	**2.34**	**2.25**	**2.20**	**2.12**	**2.08**	**2.02**	**1.98**	**1.96**	
1.99	1.95	1.89	1.84	1.80	1.75	1.71	1.67	1.65	1.61	1.59	1.57	34
2.66	**2.58**	**2.46**	**2.38**	**2.30**	**2.21**	**2.16**	**2.08**	**2.04**	**1.98**	**1.94**	**1.91**	
1.98	1.93	1.87	1.82	1.78	1.73	1.69	1.65	1.62	1.59	1.56	1.55	36
2.62	**2.54**	**2.43**	**2.35**	**2.26**	**2.17**	**2.12**	**2.04**	**2.00**	**1.94**	**1.90**	**1.87**	
1.96	1.92	1.85	1.81	1.76	1.71	1.68	1.63	1.61	1.57	1.54	1.53	38
2.59	**2.51**	**2.40**	**2.32**	**2.23**	**2.14**	**2.09**	**2.00**	**1.97**	**1.90**	**1.86**	**1.84**	
1.95	1.90	1.84	1.79	1.74	1.69	1.66	1.61	1.59	1.55	1.53	1.51	40
2.56	**2.48**	**2.37**	**2.29**	**2.20**	**2.11**	**2.06**	**1.97**	**1.94**	**1.87**	**1.83**	**1.80**	
1.93	1.89	1.83	1.78	1.73	1.68	1.65	1.60	1.57	1.53	1.51	1.49	42
2.54	**2.46**	**2.34**	**2.26**	**2.18**	**2.09**	**2.03**	**1.94**	**1.91**	**1.85**	**1.80**	**1.78**	
1.92	1.88	1.81	1.77	1.72	1.67	1.63	1.58	1.56	1.52	1.49	1.48	44
2.52	**2.44**	**2.32**	**2.24**	**2.15**	**2.06**	**2.01**	**1.92**	**1.89**	**1.82**	**1.78**	**1.75**	
1.91	1.87	1.80	1.76	1.71	1.65	1.62	1.57	1.55	1.51	1.48	1.46	46
2.50	**2.42**	**2.30**	**2.22**	**2.13**	**2.04**	**1.99**	**1.90**	**1.86**	**1.80**	**1.75**	**1.73**	
1.90	1.86	1.79	1.75	1.70	1.64	1.61	1.56	1.54	1.49	1.47	1.45	48
2.48	**2.40**	**2.28**	**2.20**	**2.12**	**2.03**	**1.97**	**1.88**	**1.84**	**1.78**	**1.73**	**1.70**	
1.89	1.85	1.78	1.74	1.69	1.63	1.60	1.55	1.52	1.48	1.46	1.44	50
2.46	**2.38**	**2.26**	**2.18**	**2.10**	**2.00**	**1.95**	**1.86**	**1.82**	**1.76**	**1.71**	**1.68**	
1.88	1.83	1.76	1.72	1.67	1.61	1.58	1.52	1.50	1.46	1.43	1.41	55
2.43	**2.34**	**2.23**	**2.15**	**2.06**	**1.96**	**1.91**	**1.82**	**1.78**	**1.71**	**1.67**	**1.64**	
1.86	1.82	1.75	1.70	1.65	1.59	1.56	1.50	1.48	1.44	1.41	1.39	60
2.39	**2.31**	**2.20**	**2.12**	**2.03**	**1.94**	**1.88**	**1.79**	**1.75**	**1.68**	**1.63**	**1.60**	
1.85	1.80	1.73	1.69	1.63	1.58	1.54	1.49	1.46	1.42	1.39	1.37	65
2.37	**2.29**	**2.18**	**2.09**	**2.00**	**1.90**	**1.85**	**1.76**	**1.72**	**1.65**	**1.60**	**1.56**	
1.84	1.79	1.72	1.67	1.62	1.57	1.53	1.47	1.45	1.40	1.37	1.35	70
2.35	**2.27**	**2.15**	**2.07**	**1.98**	**1.88**	**1.83**	**1.74**	**1.70**	**1.62**	**1.57**	**1.53**	
1.82	1.77	1.70	1.65	1.60	1.54	1.51	1.45	1.43	1.38	1.35	1.32	80
2.31	**2.23**	**2.12**	**2.03**	**1.94**	**1.85**	**1.79**	**1.70**	**1.66**	**1.58**	**1.53**	**1.49**	
1.79	1.75	1.68	1.63	1.57	1.52	1.48	1.42	1.39	1.34	1.31	1.28	100
2.26	**2.19**	**2.06**	**1.98**	**1.89**	**1.79**	**1.73**	**1.64**	**1.60**	**1.52**	**1.47**	**1.43**	
1.77	1.72	1.65	1.60	1.55	1.49	1.45	1.39	1.36	1.31	1.27	1.25	125
2.23	**2.15**	**2.03**	**1.94**	**1.85**	**1.75**	**1.69**	**1.59**	**1.55**	**1.47**	**1.41**	**1.37**	
1.76	1.71	1.64	1.59	1.53	1.48	1.44	1.37	1.34	1.29	1.25	1.22	150
2.20	**2.12**	**2.00**	**1.91**	**1.83**	**1.72**	**1.66**	**1.56**	**1.52**	**1.43**	**1.38**	**1.33**	
1.74	1.69	1.62	1.57	1.52	1.46	1.41	1.35	1.32	1.26	1.22	1.19	200
2.17	**2.09**	**1.97**	**1.88**	**1.79**	**1.69**	**1.63**	**1.53**	**1.48**	**1.39**	**1.33**	**1.28**	
1.72	1.67	1.60	1.54	1.49	1.42	1.38	1.32	1.28	1.22	1.16	1.13	400
2.12	**2.04**	**1.92**	**1.84**	**1.74**	**1.64**	**1.57**	**1.47**	**1.42**	**1.32**	**1.24**	**1.19**	
1.70	1.65	1.58	1.53	1.47	1.41	1.36	1.30	1.26	1.19	1.13	1.08	1000
2.09	**2.02**	**1.89**	**1.81**	**1.71**	**1.61**	**1.54**	**1.44**	**1.38**	**1.28**	**1.19**	**1.11**	
1.69	1.64	1.57	1.52	1.46	1.39	1.35	1.28	1.24	1.17	1.11	1.00	∞
2.08	**2.00**	**1.88**	**1.79**	**1.70**	**1.59**	**1.52**	**1.41**	**1.36**	**1.25**	**1.15**	**1.00**	

0.4.6.6 The Fischer Z-distribution

Remark on the table: The table contains the values of z_0, for which the probability: that the Fischer random variable Z with (r_1, r_2) degrees of freedom is not smaller than z_0; is equal to 0.01, in other words,

$$P(Z \geq z_0) = \int_{z_0}^{\infty} f(z)\,\mathrm{d}z = 0.01.$$

Here $f(z)$ is given by the formula

$$f(z) = \frac{2 r_1^{\frac{r_1}{2}} r_2^{\frac{r_2}{2}}}{B\left(\frac{r_1}{2}, \frac{r_2}{2}\right)} \frac{\mathrm{e}^{r_1 z}}{(r_1 \mathrm{e}^{2z} + r_2)^{\frac{r_1+r_2}{2}}}.$$

r_2	r_1									
	1	2	3	4	5	6	8	12	24	∞
1	4.1535	4.2585	4.2974	4.3175	4.3297	4.3379	4.3482	4.3585	4.3689	4.3794
2	2.2950	2.2976	2.2984	2.2988	2.2991	2.2992	2.2994	2.2997	2.2999	2.3001
3	1.7649	1.7140	1.6915	1.6786	1.6703	1.6645	1.6569	1.6489	1.6404	1.6314
4	1.5270	1.4452	1.4075	1.3856	1.3711	1.3609	1.3473	1.3327	1.3170	1.3000
5	1.3943	1.2929	1.2449	1.2164	1.1974	1.1838	1.1656	1.1457	1.1239	1.0997
6	1.3103	1.1955	1.1401	1.1068	1.0843	1.0680	1.0460	1.0218	0.9948	0.9643
7	1.2526	1.1281	1.0682	1.0300	1.0048	0.9864	0.9614	0.9335	0.9020	0.8658
8	1.2106	1.0787	1.0135	0.9734	0.9459	0.9259	0.8983	0.8673	0.8319	0.7904
9	1.1786	1.0411	0.9724	0.9299	0.9006	0.8791	0.8494	0.8157	0.7769	0.7305
10	1.1535	1.0114	0.9399	0.8954	0.8646	0.8419	0.8104	0.7744	0.7324	0.6816
11	1.1333	0.9874	0.9136	0.8674	0.8354	0.8116	0.7785	0.7405	0.6958	0.6408
12	1.1166	0.9677	0.8919	0.8443	0.8111	0.7864	0.7520	0.7122	0.6649	0.6061
13	1.1027	0.9511	0.8737	0.8248	0.7907	0.7652	0.7295	0.6882	0.6386	0.5761
14	1.0909	0.9370	0.8581	0.8082	0.7732	0.7471	0.7103	0.6675	0.6159	0.5500
15	1.0807	0.9249	0.8448	0.7939	0.7582	0.7314	0.6937	0.6496	0.5961	0.5269
16	1.0719	0.9144	0.8331	0.7814	0.7450	0.7177	0.6791	0.6339	0.5786	0.5064
17	1.0641	0.9051	0.8229	0.7705	0.7335	0.7057	0.6663	0.6199	0.5630	0.4879
18	1.0572	0.8970	0.8138	0.7607	0.7232	0.6950	0.6549	0.6075	0.5491	0.4712
19	1.0511	0.8897	0.8057	0.7521	0.7140	0.6854	0.6447	0.5964	0.5366	0.4560
20	1.0457	0.8831	0.7985	0.7443	0.7058	0.6768	0.6355	0.5864	0.5253	0.4421
21	1.0408	0.8772	0.7920	0.7372	0.6984	0.6690	0.6272	0.5773	0.5150	0.4294
22	1.0363	0.8719	0.7860	0.7309	0.6916	0.6620	0.6196	0.5691	0.5056	0.4176
23	1.0322	0.8670	0.7806	0.7251	0.6855	0.6555	0.6127	0.5615	0.4969	0.4068
24	1.0285	0.8626	0.7757	0.7197	0.6799	0.6496	0.6064	0.5545	0.4890	0.3967
25	1.0251	0.8585	0.7712	0.7148	0.6747	0.6442	0.6006	0.5481	0.4816	0.3872
26	1.0220	0.8548	0.7670	0.7103	0.6699	0.6392	0.5952	0.5422	0.4748	0.3784
27	1.0191	0.8513	0.7631	0.7062	0.6655	0.6346	0.5902	0.5367	0.4685	0.3701
28	1.0164	0.8481	0.7595	0.7023	0.6614	0.6303	0.5856	0.5316	0.4626	0.3624
29	1.0139	0.8451	0.7562	0.6987	0.6576	0.6263	0.5813	0.5269	0.4570	0.3550
30	1.0116	0.8423	0.7531	0.6954	0.6540	0.6226	0.5773	0.5224	0.4519	0.3481
40	0.9949	0.8223	0.7307	0.6712	0.6283	0.5956	0.5481	0.4901	0.4138	0.2922
60	0.9784	0.8025	0.7086	0.6472	0.6028	0.5687	0.5189	0.4574	0.3746	0.2352
120	0.9622	0.7829	0.6867	0.6234	0.5774	0.5419	0.4897	0.4243	0.3339	0.1612
∞	0.9462	0.7636	0.6651	0.5999	0.5522	0.5152	0.4604	0.3908	0.2913	0.0000

0.4.6.7 Critical numbers for the Wilcoxon test

$\alpha = 0.05$

	n_2											
	4	5	6	7	8	9	10	11	12	13	14	n_1
	–	–	–	–	8.0	9.0	10.0	10.0	11.0	12.0	13.0	2
	–	7.5	8.0	9.5	10.0	11.5	12.0	13.5	14.0	15.5	16.0	3
	8.0	9.0	10.0	11.0	12.0	13.0	15.0	16.0	17.0	18.0	19.0	4
	9.0	10.5	12.0	12.5	14.0	15.5	17.0	18.5	19.0	20.5	22.0	5
			13.0	15.0	16.0	17.0	19.0	20.0	22.0	23.0	25.0	6
15	47.5			16.5	18.0	19.5	21.0	22.5	24.0	25.5	27.0	7
14	46.0	48.0			19.0	21.0	23.0	25.0	26.0	28.0	29.0	8
13	43.5	45.0	47.5			22.5	25.0	26.5	28.0	30.5	32.0	9
12	41.0	43.0	45.0	47.0			27.0	29.0	30.0	32.0	34.0	10
11	38.5	40.0	42.5	44.0	46.5			30.5	33.0	34.5	37.0	11
10	36.0	38.0	40.0	42.0	43.0	45.0			35.0	37.0	39.0	12
9	33.5	35.0	37.5	39.0	40.5	42.0	44.5			38.5	41.0	13
8	31.0	33.0	34.0	36.0	38.0	39.0	41.0	42.0			43.0	14
7	28.5	30.0	31.5	33.0	34.5	36.0	37.5	39.0	40.5			
6	26.0	27.0	29.0	30.0	32.0	33.0	34.0	36.0	37.0	38.0	39.0	
5	23.5	24.0	25.5	27.0	28.5	30.0	30.5	32.0	33.5	35.0	35.5	
4	20.0	21.0	23.0	24.0	25.0	26.0	27.0	28.0	29.0	30.0	32.0	
3	17.5	18.0	19.5	20.0	21.5	22.0	23.5	24.0	25.5	26.0	27.5	
2	14.0	15.0	15.0	16.0	17.0	18.0	18.0	19.0	20.0	21.0	22.0	
n_1	15	16	17	18	19	20	21	22	23	24	25	
	n_2											

$\alpha = 0.01$

	n_2											
	4	5	6	7	8	9	10	11	12	13	14	n_1
	–	–	–	–	–	13.5	15.0	16.5	17.0	18.5	20.0	3
	–	–	12.0	14.0	15.0	17.0	18.0	20.0	21.0	22.0	24.0	4
	–	12.5	14.0	15.5	18.0	19.5	21.0	22.5	24.0	25.5	28.0	5
			16.0	18.0	20.0	22.0	24.0	26.0	27.0	29.0	31.0	6
15	61.5			20.5	22.0	24.5	26.0	28.5	30.0	32.5	34.0	7
14	59.0	62.0			25.0	27.0	29.0	31.0	33.0	35.0	38.0	8
13	55.5	58.0	61.5			29.5	32.0	33.5	36.0	38.5	41.0	9
12	53.0	55.0	58.0	61.0			34.0	36.0	39.0	41.0	44.0	10
11	49.5	52.0	54.5	57.0	59.5			39.5	42.0	44.5	47.0	11
10	46.0	49.0	51.0	53.0	56.0	58.0			44.0	47.0	50.0	12
9	42.5	45.0	47.5	50.0	52.5	54.0	56.5			50.5	53.0	13
8	40.0	42.0	44.0	46.0	48.0	50.0	52.0	54.0			56.0	14
7	36.5	38.0	40.5	42.0	44.5	46.0	48.5	50.0	51.5			
6	33.0	35.0	36.0	38.0	40.0	42.0	44.0	45.0	47.0	49.0	51.0	
5	29.5	31.0	32.5	34.0	35.5	37.0	38.5	41.0	42.5	44.0	45.5	
4	25.0	27.0	28.0	30.0	31.0	32.0	34.0	35.0	37.0	38.0	40.0	
3	20.5	22.0	23.5	25.0	25.5	27.0	28.5	29.0	30.5	32.0	32.5	
2	–	–	–	–	19.0	20.0	21.0	22.0	23.0	24.0	25.0	
n_1	15	16	17	18	19	20	21	22	23	24	25	
	n_2											

0.4.6.8 The Kolmogorow–Smirnow λ-distribution

Remark on the table:

The tables on probability theory and mathematical statistics are taken in part from [17] and [27].

λ	$Q(\lambda)$	λ	$Q(\lambda)$	λ	$Q(\lambda)$	λ	$Q(\lambda)$	λ	$Q(\lambda)$	λ	$Q(\lambda)$
0.32	0.0000	0.66	0.2236	1.00	0.7300	1.34	0.9449	1.68	0.9929	2.00	0.9993
0.33	0.0001	0.67	0.2396	1.01	0.7406	1.35	0.9478	1.69	0.9934	2.01	0.9994
0.34	0.0002	0.68	0.2558	1.02	0.7508	1.36	0.9505	1.70	0.9938	2.02	0.9994
0.35	0.0003	0.69	0.2722	1.03	0.7608	1.37	0.9531	1.71	0.9942	2.03	0.9995
0.36	0.0005	0.70	0.2888	1.04	0.7704	1.38	0.9556	1.72	0.9946	2.04	0.9995
0.37	0.0008	0.71	0.3055	1.05	0.7798	1.39	0.9580	1.73	0.9950	2.05	0.9996
0.38	0.0013	0.72	0.3223	1.06	0.7889	1.40	0.9603	1.74	0.9953	2.06	0.9996
0.39	0.0019	0.73	0.3391	1.07	0.7976	1.41	0.9625	1.75	0.9956	2.07	0.9996
0.40	0.0028	0.74	0.3560	1.08	0.8061	1.42	0.9646	1.76	0.9959	2.08	0.9996
0.41	0.0040	0.75	0.3728	1.09	0.8143	1.43	0.9665	1.77	0.9962	2.09	0.9997
0.42	0.0055	0.76	0.3896	1.10	0.8223	1.44	0.9684	1.78	0.9965	2.10	0.9997
0.43	0.0074	0.77	0.4064	1.11	0.8299	1.45	0.9702	1.79	0.9967	2.11	0.9997
0.44	0.0097	0.78	0.4230	1.12	0.8374	1.46	0.9718	1.80	0.9969	2.12	0.9997
0.45	0.0126	0.79	0.4395	1.13	0.8445	1.47	0.9734	1.81	0.9971	2.13	0.9998
0.46	0.0160	0.80	0.4559	1.14	0.8514	1.48	0.9750	1.82	0.9973	2.14	0.9998
0.47	0.0200	0.81	0.4720	1.15	0.8580	1.49	0.9764	1.83	0.9975	2.15	0.9998
0.48	0.0247	0.82	0.4880	1.16	0.8644	1.50	0.9778	1.84	0.9977	2.16	0.9998
0.49	0.0300	0.83	0.5038	1.17	0.8706	1.51	0.9791	1.85	0.9979	2.17	0.9998
0.50	0.0361	0.84	0.5194	1.18	0.8765	1.52	0.9803	1.86	0.9980	2.18	0.9999
0.51	0.0428	0.85	0.5347	1.19	0.8823	1.53	0.9815	1.87	0.9981	2.19	0.9999
0.52	0.0503	0.86	0.5497	1.20	0.8877	1.54	0.9826	1.88	0.9983	2.20	0.9999
0.53	0.0585	0.87	0.5645	1.21	0.8930	1.55	0.9836	1.89	0.9984	2.21	0.9999
0.54	0.0675	0.88	0.5791	1.22	0.8981	1.56	0.9846	1.90	0.9985	2.22	0.9999
0.55	0.0772	0.89	0.5933	1.23	0.9030	1.57	0.9855	1.91	0.9986	2.23	0.9999
0.56	0.0876	0.90	0.6073	1.24	0.9076	1.58	0.9864	1.92	0.9987	2.24	0.9999
0.57	0.0987	0.91	0.6209	1.25	0.9121	1.59	0.9873	1.93	0.9988	2.25	0.9999
0.58	0.1104	0.92	0.6343	1.26	0.9164	1.60	0.9880	1.94	0.9989	2.26	0.9999
0.59	0.1228	0.93	0.6473	1.27	0.9206	1.61	0.9888	1.95	0.9990	2.27	0.9999
0.60	0.1357	0.94	0.6601	1.28	0.9245	1.62	0.9895	1.96	0.9991	2.28	0.9999
0.61	0.1492	0.95	0.6725	1.29	0.9283	1.63	0.9902	1.97	0.9991	2.29	0.9999
0.62	0.1632	0.96	0.6846	1.30	0.9319	1.64	0.9908	1.98	0.9992	2.30	0.9999
0.63	0.1778	0.97	0.6964	1.31	0.9354	1.65	0.9914	1.99	0.9993	2.31	1.0000
0.64	0.1927	0.98	0.7079	1.32	0.9387	1.66	0.9919				
0.65	0.2080	0.99	0.7191	1.33	0.9418	1.67	0.9924				

0.4.6.9 The Poisson distribution

$$P(X = r) = \frac{\lambda^r}{r!}e^{-\lambda}$$

	λ							
r	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0	0.904 837	0.818 731	0.740 818	0.670 320	0.606 531	0.548 812	0.496 585	0.449 329
1	0.090 484	0.163 746	0.222 245	0.268 128	0.303 265	0.329 287	0.347 610	0.359 463
2	0.004 524	0.016 375	0.033 337	0.053 626	0.075 816	0.098 786	0.121 663	0.143 785
3	0.000 151	0.001 092	0.003 334	0.007 150	0.012 636	0.019 757	0.028 388	0.038 343
4	0.000 004	0.000 055	0.000 250	0.000 715	0.001 580	0.002 964	0.004 968	0.007 669
5	–	0.000 002	0.000 015	0.000 057	0.000 158	0.000 356	0.000 696	0.001 227
6	–	–	0.000 001	0.000 004	0.000 013	0.000 036	0.000 081	0.000 164
7	–	–	–	–	0.000 001	0.000 003	0.000 008	0.000 019
8	–	–	–	–	–	–	0.000 001	0.000 002

	λ							
r	0.9	1.0	1.5	2.0	2.5	3.0	3.5	4.0
0	0.406 570	0.367 879	0.223 130	0.135 335	0.082 085	0.049 787	0.030 197	0.018 316
1	0.365 913	0.367 879	0.334 695	0.270 671	0.205 212	0.149 361	0.105 691	0.073 263
2	0.164 661	0.183 940	0.251 021	0.270 671	0.256 516	0.224 042	0.184 959	0.146 525
3	0.049 398	0.061 313	0.125 510	0.180 447	0.213 763	0.224 042	0.215 785	0.195 367
4	0.011 115	0.015 328	0.047 067	0.090 224	0.133 602	0.168 031	0.188 812	0.195 367
5	0.002 001	0.003 066	0.014 120	0.036 089	0.066 801	0.100 819	0.132 169	0.156 293
6	0.000 300	0.000 511	0.003 530	0.012 030	0.027 834	0.050 409	0.077 098	0.104 196
7	0.000 039	0.000 073	0.000 756	0.003 437	0.009 941	0.021 604	0.038 549	0.059 540
8	0.000 004	0.000 009	0.000 142	0.000 859	0.003 106	0.008 102	0.016 865	0.029 770
9	–	0.000 001	0.000 024	0.000 191	0.000 863	0.002 701	0.006 559	0.013 231
10	–	–	0.000 004	0.000 038	0.000 216	0.000 810	0.002 296	0.005 292
11	–	–	–	0.000 007	0.000 049	0.000 221	0.000 730	0.001 925
12	–	–	–	0.000 001	0.000 010	0.000 055	0.000 213	0.000 642
13	–	–	–	–	0.000 002	0.000 013	0.000 057	0.000 197
14	–	–	–	–	–	0.000 003	0.000 014	0.000 056
15	–	–	–	–	–	0.000 001	0.000 003	0.000 015
16	–	–	–	–	–	–	0.000 001	0.000 004
17	–	–	–	–	–	–	–	0.000 001

	λ						
r	4.5	5.0	6.0	7.0	8.0	9.0	10.0
0	0.011 109	0.006 738	0.002 479	0.000 912	0.000 335	0.000 123	0.000 045
1	0.049 990	0.033 690	0.014 873	0.006 383	0.002 684	0.001 111	0.000 454
2	0.112 479	0.083 224	0.044 618	0.022 341	0.010 735	0.004 998	0.002 270
3	0.168 718	0.140 374	0.089 235	0.052 129	0.028 626	0.014 994	0.007 567
4	0.189 808	0.175 467	0.133 853	0.091 226	0.057 252	0.033 737	0.018 917
5	0.170 827	0.175 467	0.160 623	0.127 717	0.091 604	0.060 727	0.037 833
6	0.128 120	0.146 223	0.160 623	0.149 003	0.122 138	0.091 090	0.063 055
7	0.082 363	0.104 445	0.137 677	0.149 003	0.139 587	0.117 116	0.090 079
8	0.046 329	0.065 278	0.103 258	0.130 377	0.139 587	0.131 756	0.112 599
9	0.023 165	0.036 266	0.068 838	0.101 405	0.124 077	0.131 756	0.125 110
10	0.010 424	0.018 133	0.041 303	0.070 983	0.099 262	0.118 580	0.125 110
11	0.004 264	0.008 242	0.022 529	0.045 171	0.072 190	0.097 020	0.113 736
12	0.001 599	0.003 434	0.011 264	0.026 350	0.048 127	0.072 765	0.094 780
13	0.000 554	0.001 321	0.005 199	0.014 188	0.029 616	0.050 376	0.072 908
14	0.000 178	0.000 472	0.002 228	0.007 094	0.016 924	0.032 384	0.052 077
15	0.000 053	0.000 157	0.000 891	0.003 311	0.009 026	0.019 431	0.034 718
16	0.000 015	0.000 049	0.000 334	0.001 448	0.004 513	0.010 930	0.021 699
17	0.000 004	0.000 014	0.000 118	0.000 596	0.002 124	0.005 786	0.012 764
18	0.000 001	0.000 004	0.000 039	0.000 232	0.000 944	0.002 893	0.007 091
19	–	0.000 001	0.000 012	0.000 085	0.000 397	0.001 370	0.003 732
20	–	–	0.000 004	0.000 030	0.000 159	0.000 617	0.001 866
21	–	–	0.000 001	0.000 010	0.000 061	0.000 264	0.000 889
22	–	–	–	0.000 003	0.000 022	0.000 108	0.000 404
23	–	–	–	0.000 001	0.000 008	0.000 042	0.000 176
24	–	–	–	–	0.000 003	0.000 016	0.000 073
25	–	–	–	–	0.000 001	0.000 006	0.000 029
26	–	–	–	–	–	0.000 002	0.000 011
27	–	–	–	–	–	0.000 001	0.000 004
28	–	–	–	–	–	–	0.000 001
29	–	–	–	–	–	–	0.000 001

0.5 Tables of values of special functions

Remark on the following tables:
Some of these tables are taken from [21].

0.5.1 The gamma functions $\Gamma(x)$ and $1/\Gamma(x)$

Remark on this table: See also section 1.14.16.

x	$\Gamma(x)$	$1/\Gamma(x)$	x	$\Gamma(x)$	$1/\Gamma(x)$	x	$\Gamma(x)$	$1/\Gamma(x)$
1.00	1.000 00	1.000 0	1.40	0.887 26	1.127 0	1.70	0.908 64	1.100 5
1.01	0.994 33	005 7	1.41	886 76	127 7	1.71	910 57	098 2
1.02	988 84	011 3	1.42	886 36	128 2	1.72	912 58	095 8
1.03	983 55	016 7	1.43	886 04	128 6	1.73	914 67	093 3
1.04	978 44	022 0	1.44	885 81	128 9	1.74	916 83	090 7
1.05	973 50	027 2	1.45	885 66	129 1	1.75	919 06	088 1
1.06	968 74	032 3	1.46	885 60	129 1	1.76	921 37	085 4
1.07	964 15	037 2	1.47	885 63	129 1	1.77	923 76	082 5
1.08	959 73	042 0	1.48	885 75	129 1	1.78	926 23	079 6
1.09	955 46	046 6	1.49	885 95	128 8	1.79	928 77	076 7
1.10	0.951 35	1.051 1	1.50	0.886 23	1.128 4	1.80	0.931 38	1.073 7
1.11	947 40	055 5	1.51	886 59	127 9	1.81	934 08	070 6
1.12	943 59	059 8	1.52	887 04	127 3	1.82	936 85	067 4
1.13	939 93	063 9	1.53	887 57	126 7	1.83	939 69	064 2
1.14	936 42	067 9	1.54	888 18	125 9	1.84	942 61	060 9
1.15	933 04	071 8	1.55	888 87	125 0	1.85	945 61	057 5
1.16	929 80	075 5	1.56	889 64	124 0	1.86	948 69	054 1
1.17	926 70	079 1	1.57	890 49	123 0	1.87	951 84	050 6
1.18	923 73	082 6	1.58	891 42	121 8	1.88	955 07	047 1
1.19	920 89	085 9	1.59	892 43	120 5	1.89	958 38	043 5
1.20	0.918 17	1.089 1	1.60	0.893 52	1.119 1	1.90	0.961 77	1.039 8
1.21	915 58	092 2	1.61	894 68	117 7	1.91	965 23	036 0
1.22	913 11	095 2	1.62	895 92	116 1	1.92	968 77	032 2
1.23	910 75	098 0	1.63	897 24	114 5	1.93	972 40	028 4
1.24	908 52	100 7	1.64	898 64	112 8	1.94	976 10	024 5
1.25	906 40	103 2	1.65	900 12	110 9	1.95	979 88	020 6
1.26	904 40	105 7	1.66	901 67	109 1	1.96	983 74	016 5
1.27	902 50	108 0	1.67	903 30	107 1	1.97	987 68	012 5
1.28	900 72	110 2	1.68	905 00	104 9	1.98	991 71	008 3
1.29	899 04	112 3	1.69	906 78	102 8	1.99	995 81	004 2
1.30	0.897 47	1.114 2						
1.31	896 00	116 1						
1.32	894 64	117 8						
1.33	893 38	119 4						
1.34	892 22	120 8						
1.35	891 15	122 2						
1.36	890 18	123 4						
1.37	889 31	124 4						
1.38	888 54	125 4						
1.39	887 85	126 3						

If x is a natural number n with $n \geq 1$, then

$$\Gamma(n) = (n-1)!,$$

so that, for example, $\Gamma(2) = 1$.
To calculate $\Gamma(x)$ for x which is less than 1 but not an integer, one can use the formula

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}.$$

If $x > 2$, then for the calculation the formula

$$\Gamma(x) = (x-1) \cdot \Gamma(x-1)$$

can be used.

Examples:

1. $\Gamma(-0.2) = \dfrac{\Gamma(0.8)}{-0.2} = -\dfrac{\Gamma(1.8)}{0.2 \cdot 0.8} = -\dfrac{0.931\,38}{0.16} = -5.821\,13.$

2. $\Gamma(3.2) = 2.2 \cdot \Gamma(2.2) = 2.2 \cdot 1.2 \cdot 2.\Gamma(1.2) = 2.2 \cdot 1.2 \cdot 0.918\,17 = 2.423\,97.$

0.5.2 Cylinder functions (also known as Bessel functions)

Remark: See also section 1.14.22.

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$	$I_0(x)$	$I_1(x)$	$K_0(x)$	$K_1(x)$
0.0	+1.0000	+0.0000	$-\infty$	$-\infty$	1.000	0.000	∞	∞
0.1	+0.9975	+0.0499	−1.5342	−6.4590	1.003	0.0501	2.4271	9.8538
0.2	+0.9900	+0.0995	−1.0811	−3.3238	1.010	0.1005	1.7527	4.7760
0.3	+0.9776	+0.1483	−0.8073	−2.2931	1.023	0.1517	1.3725	3.0560
0.4	+0.9604	+0.1960	−0.6060	−1.7809	1.040	0.2040	1.1145	2.1844
0.5	+0.9385	+0.2423	−0.4445	−1.4715	1.063	0.2579	0.9244	1.6564
0.6	+0.9120	+0.2867	−0.3085	−1.2604	1.092	0.3137	0.7775	1.3028
0.7	+0.8812	+0.3290	−0.1907	−1.1032	1.126	0.3719	0.6605	1.0503
0.8	+0.8463	+0.3688	−0.0868	−0.9781	1.167	0.4329	0.5653	0.8618
0.9	+0.8075	+0.4059	+0.0056	−0.8731	1.213	0.4971	0.4867	0.7165
1.0	+0.7652	+0.4401	+0.0883	−0.7812	1.266	0.5652	0.4210	0.6019
1.1	+0.7196	+0.4709	+0.1622	−0.6981	1.326	0.6375	0.3656	0.5098
1.2	+0.6711	+0.4983	+0.2281	−0.6211	1.394	0.7147	0.3185	0.4346
1.3	+0.6201	+0.5220	+0.2865	−0.5485	1.469	0.7973	0.2782	0.3725
1.4	+0.5669	+0.5419	+0.3379	−0.4791	1.553	0.8861	0.2437	0.3208
1.5	+0.5118	+0.5579	+0.3824	−0.4123	1.647	0.9817	0.2138	0.2774
1.6	+0.4554	+0.5699	+0.4204	−0.3476	1.750	1.085	0.1880	0.2406
1.7	+0.3980	+0.5778	+0.4520	−0.2847	1.864	1.196	0.1655	0.2094
1.8	+0.3400	+0.5815	+0.4774	−0.2237	1.990	1.317	0.1459	0.1826
1.9	+0.2818	+0.5812	+0.4968	−0.1644	2.128	1.448	0.1288	0.1597
2.0	+0.2239	+0.5767	+0.5104	−0.1070	2.280	1.591	0.1139	0.1399
2.1	+0.1666	+0.5683	+0.5183	−0.0517	2.446	1.745	0.1008	0.1227
2.2	+0.1104	+0.5560	+0.5208	+0.0015	2.629	1.914	0.08927	0.1079
2.3	+0.0555	+0.5399	+0.5181	+0.0523	2.830	2.098	0.07914	0.09498
2.4	+0.0025	+0.5202	+0.5104	+0.1005	3.049	2.298	0.07022	0.08372
2.5	−0.0484	+0.4971	+0.4981	+0.1459	3.290	2.517	0.06235	0.07389
2.6	−0.0968	+0.4708	+0.4813	+0.1884	3.553	2.755	0.05540	0.06528
2.7	−0.1424	+0.4416	+0.4605	+0.2276	3.842	3.016	0.04926	0.05774
2.8	−0.1850	+0.4097	+0.4359	+0.2635	4.157	3.301	0.04382	0.05111
2.9	−0.2243	+0.3754	+0.4079	+0.2959	4.503	3.613	0.03901	0.04529
3.0	−0.2601	+0.3391	+0.3769	+0.3247	4.881	3.953	0.03474	0.04016
3.1	−0.2921	+0.3009	+0.3431	+0.3496	5.294	4.326	0.03095	0.03563
3.2	−0.3202	+0.2613	+0.3070	+0.3707	5.747	4.734	0.02759	0.03164
3.3	−0.3443	+0.2207	+0.2691	+0.3879	6.243	5.181	0.02461	0.02812
3.4	−0.3643	+0.1792	+0.2296	+0.4010	6.785	5.670	0.02196	0.02500
3.5	−0.3801	+0.1374	+0.1890	+0.4102	7.378	6.206	0.01960	0.02224
3.6	−0.3918	+0.0955	+0.1477	+0.4154	8.028	6.793	0.01750	0.01979
3.7	−0.3992	+0.0538	+0.1061	+0.4167	8.739	7.436	0.01563	0.01763
3.8	−0.4026	+0.0128	+0.0645	+0.4141	9.517	8.140	0.01397	0.01571
3.9	−0.4018	−0.0272	+0.0234	+0.4078	10.37	8.913	0.01248	0.01400
4.0	−0.3971	−0.0660	−0.0169	+0.3979	11.30	9.759	0.01116	0.01248
4.1	−0.3887	−0.1033	−0.0561	+0.3846	12.32	10.69	0.009980	0.01114
4.2	−0.3766	−0.1386	−0.0938	+0.3680	13.44	11.71	0.008927	0.009938
4.3	−0.3610	−0.1719	−0.1296	+0.3484	14.67	12.82	0.007988	0.008872
4.4	−0.3423	−0.2028	−0.1633	+0.3260	16.01	14.05	0.007149	0.007923
4.5	−0.3205	−0.2311	−0.1947	+0.3010	17.48	15.39	0.006400	0.007078
4.6	−0.2961	−0.2566	−0.2235	+0.2737	19.09	16.86	0.005730	0.006325
4.7	−0.2693	−0.2791	−0.2494	+0.2445	20.86	18.48	0.005132	0.005654
4.8	−0.2404	−0.2985	−0.2723	+0.2136	22.79	20.25	0.004597	0.005055
4.9	−0.2097	−0.3147	−0.2921	+0.1812	24.91	22.20	0.004119	0.004521

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$	$I_0(x)$	$I_1(x)$	$K_0(x)$	$K_1(x)$
5.0	−0.1776	−0.3276	−0.3085	+0.1479	27.24	24.34	$3691 \cdot 10^{-6}$	$4045 \cdot 10^{-6}$
5.1	−0.1443	−0.3371	−0.3216	+0.1137	29.79	26.68	$3308 \cdot 10^{-6}$	$3619 \cdot 10^{-6}$
5.2	−0.1103	−0.3432	−0.3313	+0.0792	32.58	29.25	$2966 \cdot 10^{-6}$	$3239 \cdot 10^{-6}$
5.3	−0.0758	−0.3460	−0.3374	+0.0445	35.65	32.08	$2659 \cdot 10^{-6}$	$2900 \cdot 10^{-6}$
5.4	−0.0412	−0.3453	−0.3402	+0.0101	39.01	35.18	$2385 \cdot 10^{-6}$	$2597 \cdot 10^{-6}$
5.5	−0.0068	−0.3414	−0.3395	−0.0238	42.69	38.59	$2139 \cdot 10^{-6}$	$2326 \cdot 10^{-6}$
5.6	+0.0270	−0.3343	−0.3354	−0.0568	46.74	42.33	$1918 \cdot 10^{-6}$	$2083 \cdot 10^{-6}$
5.7	+0.0599	−0.3241	−0.3282	−0.0887	51.17	46.44	$1721 \cdot 10^{-6}$	$1866 \cdot 10^{-6}$
5.8	+0.0917	−0.3110	−0.3177	−0.1192	56.04	50.95	$1544 \cdot 10^{-6}$	$1673 \cdot 10^{-6}$
5.9	+0.1220	−0.2951	−0.3044	−0.1481	61.38	55.90	$1386 \cdot 10^{-6}$	$1499 \cdot 10^{-6}$
6.0	+0.1506	−0.2767	−0.2882	−0.1750	67.23	61.34	$1244 \cdot 10^{-6}$	$1344 \cdot 10^{-6}$
6.1	+0.1773	−0.2559	−0.2694	−0.1998	73.66	67.32	$1117 \cdot 10^{-6}$	$1205 \cdot 10^{-6}$
6.2	+0.2017	−0.2329	−0.2483	−0.2223	80.72	73.89	$1003 \cdot 10^{-6}$	$1081 \cdot 10^{-6}$
6.3	+0.2238	−0.2081	−0.2251	−0.2422	88.46	81.10	$9001 \cdot 10^{-7}$	$9691 \cdot 10^{-7}$
6.4	+0.2433	−0.1816	−0.1999	−0.2596	96.96	89.03	$8083 \cdot 10^{-7}$	$8693 \cdot 10^{-7}$
6.5	+0.2601	−0.1538	−0.1732	−0.2741	106.3	97.74	$7259 \cdot 10^{-7}$	$7799 \cdot 10^{-7}$
6.6	+0.2740	−0.1250	−0.1452	−0.2857	116.5	107.3	$6520 \cdot 10^{-7}$	$6998 \cdot 10^{-7}$
6.7	+0.2851	−0.0953	−0.1162	−0.2945	127.8	117.8	$5857 \cdot 10^{-7}$	$6280 \cdot 10^{-7}$
6.8	+0.2931	−0.0652	−0.0864	−0.3002	140.1	129.4	$5262 \cdot 10^{-7}$	$5636 \cdot 10^{-7}$
6.9	+0.2981	−0.0349	−0.0563	−0.3029	153.7	142.1	$4728 \cdot 10^{-7}$	$5059 \cdot 10^{-7}$
7.0	+0.3001	−0.0047	−0.0259	−0.3027	168.6	156.0	$4248 \cdot 10^{-7}$	$4542 \cdot 10^{-7}$
7.1	+0.2991	+0.0252	+0.0042	−0.2995	185.0	171.4	$3817 \cdot 10^{-7}$	$4078 \cdot 10^{-7}$
7.2	+0.2951	+0.0543	+0.0339	−0.2934	202.9	188.3	$3431 \cdot 10^{-7}$	$3662 \cdot 10^{-7}$
7.3	+0.2882	+0.0826	+0.0628	−0.2846	222.7	206.8	$3084 \cdot 10^{-7}$	$3288 \cdot 10^{-7}$
7.4	+0.2786	+0.1096	+0.0907	−0.2731	244.3	227.2	$2772 \cdot 10^{-7}$	$2953 \cdot 10^{-7}$
7.5	+0.2663	+0.1352	+0.1173	−0.2591	268.2	249.6	$2492 \cdot 10^{-7}$	$2653 \cdot 10^{-7}$
7.6	+0.2516	+0.1592	+0.1424	−0.2428	294.3	274.2	$2240 \cdot 10^{-7}$	$2383 \cdot 10^{-7}$
7.7	+0.2346	+0.1813	+0.1658	−0.2243	323.1	301.3	$2014 \cdot 10^{-7}$	$2141 \cdot 10^{-7}$
7.8	+0.2154	+0.2014	+0.1872	−0.2039	354.7	331.1	$1811 \cdot 10^{-7}$	$1924 \cdot 10^{-7}$
7.9	+0.1944	+0.2192	+0.2065	−0.1817	389.4	363.9	$1629 \cdot 10^{-7}$	$1729 \cdot 10^{-7}$
8.0	+0.1717	+0.2346	+0.2235	−0.1581	427.6	399.9	$1465 \cdot 10^{-7}$	$1554 \cdot 10^{-7}$
8.1	+0.1475	+0.2476	+0.2381	−0.1331	469.5	439.5	$1317 \cdot 10^{-7}$	$1396 \cdot 10^{-7}$
8.2	+0.1222	+0.2580	+0.2501	−0.1072	515.6	483.0	$1185 \cdot 10^{-7}$	$1255 \cdot 10^{-7}$
8.3	+0.0960	+0.2657	+0.2595	−0.0806	566.3	531.0	$1066 \cdot 10^{-7}$	$1128 \cdot 10^{-7}$
8.4	+0.0692	+0.2708	+0.2662	−0.0535	621.9	583.7	$9588 \cdot 10^{-8}$	$1014 \cdot 10^{-7}$
8.5	+0.0419	+0.2731	+0.2702	−0.0262	683.2	641.6	$8626 \cdot 10^{-8}$	$9120 \cdot 10^{-8}$
8.6	+0.0146	+0.2728	+0.2715	+0.0011	750.5	705.4	$7761 \cdot 10^{-8}$	$8200 \cdot 10^{-8}$
8.7	−0.0125	+0.2697	+0.2700	+0.0280	824.4	775.5	$6983 \cdot 10^{-8}$	$7374 \cdot 10^{-8}$
8.8	−0.0392	+0.2641	+0.2659	+0.0544	905.8	852.7	$6283 \cdot 10^{-8}$	$6631 \cdot 10^{-8}$
8.9	−0.0653	+0.2559	+0.2592	+0.0799	995.2	937.5	$5654 \cdot 10^{-8}$	$5964 \cdot 10^{-8}$
9.0	−0.0903	+0.2453	+0.2499	+0.1043	1094.0	1031.0	$5088 \cdot 10^{-8}$	$5364 \cdot 10^{-8}$
9.1	−0.1142	+0.2324	+0.2383	+0.1275	1202.0	1134.0	$4579 \cdot 10^{-8}$	$4825 \cdot 10^{-8}$
9.2	−0.1367	+0.2174	+0.2245	+0.1491	1321.0	1247.0	$4121 \cdot 10^{-8}$	$4340 \cdot 10^{-8}$
9.3	−0.1577	+0.2004	+0.2086	+0.1691	1451.0	1371.0	$3710 \cdot 10^{-8}$	$3904 \cdot 10^{-8}$
9.4	−0.1768	+0.1816	+0.1907	+0.1871	1595.0	1508.0	$3339 \cdot 10^{-8}$	$3512 \cdot 10^{-8}$
9.5	−0.1939	+0.1613	+0.1712	+0.2032	1753.0	1685.0	$3006 \cdot 10^{-8}$	$3160 \cdot 10^{-8}$
9.6	−0.2090	+0.1395	+0.1502	+0.2171	1927.0	1824.0	$2706 \cdot 10^{-8}$	$2843 \cdot 10^{-8}$
9.7	−0.2218	+0.1166	+0.2179	+0.2287	2119.0	2006.0	$2436 \cdot 10^{-8}$	$2559 \cdot 10^{-8}$
9.8	−0.2323	+0.0928	+0.1045	+0.2379	2329.0	2207.0	$2193 \cdot 10^{-8}$	$2302 \cdot 10^{-8}$
9.9	−0.2403	+0.0684	+0.0804	+0.2447	2561.0	2428.0	$1975 \cdot 10^{-8}$	$2072 \cdot 10^{-8}$
10.0	−0.2459	+0.0435	+0.0557	+0.2490	2816.0	2671.0	$1778 \cdot 10^{-8}$	$1865 \cdot 10^{-8}$

Some values of Bessel functions of higher order p, for integral arguments

For $p = 0.5$, 1.5 and 2.5 see the table *Spherical cylinder functions* below.

p	$J_p(1)$	$J_p(2)$	$J_p(3)$	$J_p(4)$	$J_p(5)$
0	+0. 765 2	+0. 223 9	−0. 260 1	−0. 397 1	−0. 177 6
1.0	+0. 440 1	+0. 576 7	+0. 339 1	−0. 066 04	−0. 327 6
2.0	+0. 114 9	+0. 352 8	+0. 486 1	+0. 364 1	+0. 046 57
3.0	+0. 019 56	+0. 128 9	+0. 309 1	+0. 430 2	+0. 364 8
3.5	+0. 718 6·10^{-2}	+0. 068 52	+0. 210 1	+0. 365 8	+0. 410 0
4.0	+0. 247 7·10^{-2}	+0. 034 00	+0. 132 0	+0. 281 1	+0. 391 2
4.5	+0. 807·10^{-3}	+0. 015 89	+0. 077 60	+0. 199 3	+0. 333 7
5.0	+0. 249 8·10^{-3}	+0. 704 0·10^{-2}	+0. 043 03	+0. 132 1	+0. 261 1
5.5	+0. 74·10^{-4}	+0. 297 3·10^{-2}	+0. 022 66	+0. 082 61	+0. 190 6
6.0	+0. 209 4·10^{-4}	+0. 120 2·10^{-2}	+0. 011 39	+0. 049 09	+0. 131 0
6.5	+0. 6·10^{-5}	+0. 467·10^{-3}	+0. 549 3·10^{-2}	+0. 027 87	+0. 085 58
7.0	+0. 1502·10^{-5}	+0. 174 9·10^{-3}	+0. 254 7·10^{-2}	+0. 015 18	+0. 053 38
8.0	+0. 942 2·10^{-7}	+0. 221 8·10^{-4}	+0. 493 4·10^{-3}	+0. 402 9·10^{-2}	+0. 018 41
9.0	+0. 524 9·10^{-8}	+0. 249 2·10^{-5}	+0. 844 0·10^{-4}	+0. 938 6·10^{-3}	+0. 552 0·10^{-2}
10.0	+0. 263 1·10^{-9}	+0. 251 5·10^{-6}	+0. 129 3·10^{-4}	+0. 195 0·10^{-3}	+0. 146 8·10^{-2}

p	$J_p(6)$	$J_p(7)$	$J_p(8)$	$J_p(9)$	$J_p(10)$
0	+0. 150 6	+0. 300 1	+0. 171 7	−0. 090 33	−0. 245 9
1.0	−0. 276 7	−0. 468 3·10^{-2}	+0. 234 6	+0. 245 3	+0. 043 47
2.0	−0. 242 9	−0. 301 4	−0. 113 0	+0. 144 8	+0. 254 6
3.0	+0. 114 8	−0. 167 6	−0. 291 1	−0. 180 9	+0. 058 38
3.5	+0. 267 1	−0. 340 3·10^{-2}	−0. 232 6	−0. 268 3	−0. 099 65
4.0	+0. 357 6	+0. 157 8	−0. 105 4	−0. 265 5	−0. 219 6
4.5	+0. 384 6	+0. 280 0	+0. 047 12	−0. 183 9	−0. 266 4
5.0	+0. 362 1	+0. 347 9	+0. 185 8	−0. 055 04	−0. 234 1
5.5	+0. 309 8	+0. 363 4	+0. 285 6	+0. 084 39	−0. 140 1
6.0	+0. 245 8	+0. 339 2	+0. 337 6	+0. 204 3	−0. 014 46
6.5	+0. 183 3	+0. 291 1	+0. 345 6	+0. 287 0	+0. 112 3
7.0	+0. 129 6	+0. 233 6	+0. 320 6	+0. 327 5	+0. 216 7
8.0	+0. 056 53	+0. 128 0	+0. 223 5	+0. 305 1	+0. 317 9
9.0	+0. 021 17	+0. 058 92	+0. 126 3	+0. 214 9	+0. 291 9
10.0	+0. 696 4·10^{-2}	+0. 023 54	+0. 060 77	+0. 124 7	+0. 207 5

p	$J_p(11)$	$J_p(12)$	$J_p(13)$	$J_p(14)$	$J_p(15)$
0	−0. 171 2	+0. 047 69	+0. 206 9	+0. 171 1	−0. 014 22
1.0	−0. 176 8	−0. 223 4	−0. 070 32	+0. 133 4	+0. 205 1
2.0	+0. 139 0	−0. 084 93	−0. 217 7	−0. 152 0	+0. 041 57
3.0	+0. 227 3	+0. 195 1	+0. 332 0·10^{-2}	−0. 176 8	−0. 194 0
3.5	+0. 129 4	+0. 234 8	+0. 140 7	−0. 062 45	−0. 199 1
4.0	−0. 015 04	+0. 182 5	+0. 219 3	+0. 076 24	−0. 119 2
4.5	−0. 151 9	+0. 064 57	+0. 213 4	+0. 183 0	+0. 798 4·10^{-}[illegible]
5.0	−0. 238 3	−0. 073 47	+0. 131 6	+0. 220 4	+0. 130 5
5.5	−0. 253 8	−0. 186 4	+0. 705 5·10^{-2}	+0. 180 1	+0. 203 9
6.0	−0. 201 6	−0. 243 7	−0. 118 0	+0. 081 17	+0. 206 1
6.5	−0. 101 8	−0. 235 4	−0. 207 5	−0. 041 51	+0. 141 5
7.0	+0. 018 38	−0. 170 3	−0. 240 6	−0. 150 8	+0. 034 46
8.0	+0. 225 0	+0. 045 10	−0. 141 0	−0. 232 0	−0. 174 0
9.0	+0. 308 9	+0. 230 4	+0. 066 98	−0. 114 3	−0. 220 0
10.0	+0. 280 4	+0. 300 5	+0. 233 8	+0. 085 01	−0. 090 07

p	$J_p(16)$	$J_p(17)$	$J_p(18)$	$J_p(19)$	$J_p(20)$
0	-0.1749	-0.1699	-0.01336	+0.1466	+0.1670
1.0	+0.09040	-0.09767	-0.1880	-0.1057	+0.06683
2.0	+0.1862	+0.1584	$-0.7533\cdot10^{-2}$	-0.1578	-0.1603
3.0	-0.04385	+0.1349	+0.1863	+0.07249	-0.09890
3.5	-0.1585	+0.01461	+0.1651	+0.1649	+0.02152
4.0	-0.2026	-0.1107	+0.06964	+0.1806	+0.1307
4.5	-0.1619	-0.1875	-0.05501	+0.1165	+0.1801
5.0	-0.05747	-0.1870	-0.1554	$+0.3572\cdot10^{-2}$	+0.1512
5.5	+0.06743	-0.1139	-0.1926	-0.1097	+0.05953
6.0	+0.1667	$+0.7153\cdot10^{-3}$	-0.1560	-0.1788	-0.05509
6.5	+0.2083	+0.1138	-0.06273	-0.1800	-0.1474
7.0	+0.1825	+0.1875	+0.05140	-0.1165	-0.1842
8.0	$-0.7021\cdot10^{-2}$	+0.1537	+0.1959	+0.09294	-0.07387
9.0	-0.1895	-0.04286	+0.1228	+0.1947	+0.1251
10.0	-0.2062	-0.1991	-0.07317	+0.09155	+0.1865

Spherical cylinder functions (Bessel functions) $J_{\pm(n+1/2)}$

x	$J_{1/2}$	$J_{3/2}$	$J_{5/2}$	$J_{-1/2}$	$J_{-3/2}$	$J_{-5/2}$
0	0.0000	0.0000	0.0000	$+\infty$	$-\infty$	$+\infty$
1	+0.6714	+0.2403	+0.0495	+0.4311	-1.1025	+2.8764
2	+0.5130	+0.4913	+0.2239	-0.2348	-0.3956	+0.8282
3	+0.0650	+0.4777	+0.4127	-0.4560	+0.0870	+0.3690
4	-0.3019	+0.1853	+0.4409	-0.2608	+0.3671	-0.0146
5	-0.3422	-0.1697	+0.2404	+0.1012	+0.3219	-0.2944
6	-0.0910	-0.3279	-0.0730	+0.3128	+0.0389	-0.3322
7	+0.1981	-0.1991	-0.2834	+0.2274	-0.2306	-0.1285
8	+0.2791	+0.0759	-0.2506	-0.0410	-0.2740	+0.1438
9	+0.1096	+0.2545	-0.0248	-0.2423	-0.0827	+0.2699
10	-0.1373	+0.1980	+0.1967	-0.2117	+0.1584	+0.1642
11	-0.2406	-0.0229	+0.2343	+0.0011	+0.2405	-0.0666
12	-0.1236	-0.2047	+0.0724	+0.1944	+0.1074	-0.2212
13	+0.0930	-0.1937	-0.1377	+0.2008	-0.1084	-0.1758
14	+0.2112	-0.0141	-0.2143	+0.0292	-0.2133	+0.0166
15	+0.1340	+0.1654	-0.1009	-0.1565	-0.1235	+0.1812
16	-0.0574	+0.1874	+0.0926	-0.1910	+0.0694	+0.1780
17	-0.1860	+0.0423	+0.1935	-0.0532	+0.1892	+0.0199
18	-0.1412	-0.1320	+0.1192	+0.1242	+0.1343	-0.1466
19	+0.0274	-0.1795	-0.0558	+0.1810	-0.0370	-0.1751
20	+0.1629	-0.0647	-0.1726	+0.0728	-0.1665	-0.0478
21	+0.1457	+0.1023	-0.1311	-0.0954	-0.1411	+0.1155
22	-0.0015	+0.1700	+0.0247	-0.1701	+0.0092	+0.1688
23	-0.1408	+0.0825	+0.1516	-0.0886	+0.1446	+0.0698
24	-0.1475	-0.0752	+0.1381	+0.0691	+0.1446	-0.0872
25	-0.0211	-0.1590	+0.0020	+0.1582	+0.0148	-0.1599
26	+0.1193	-0.0966	-0.1305	+0.1012	-0.1232	-0.0870
27	+0.1469	+0.0503	-0.1413	-0.0449	-0.1452	+0.0610
28	+0.0408	+0.1466	-0.0251	-0.1451	-0.0357	+0.1490
29	-0.0983	+0.1074	+0.1094	-0.1108	+0.1021	+0.1003
30	-0.1439	-0.0273	+0.1412	+0.0225	+0.1432	-0.0368
31	-0.0579	-0.1330	+0.0450	+0.1311	+0.0537	-0.1363
32	+0.0778	-0.1152	-0.0886	+0.1177	-0.0814	-0.1100
33	+0.1389	+0.0061	-0.1383	-0.0018	-0.1388	+0.0145
34	+0.0724	+0.1182	-0.0620	-0.1161	-0.0690	+0.1222
35	-0.0578	+0.1202	+0.0680	-0.1219	+0.0612	+0.1166
36	-0.1319	+0.0134	+0.1330	-0.0170	+0.1324	+0.0060
37	-0.0844	-0.1027	+0.0761	+0.1004	+0.0817	-0.1070
38	+0.0384	-0.1226	-0.0480	+0.1236	-0.0416	-0.1203
39	+0.1231	-0.0309	-0.1255	+0.0341	-0.1240	-0.0245
40	+0.0940	+0.0865	-0.0875	-0.0841	-0.0919	+0.0910

The n^{th} zero of some Bessel functions

n	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$
1	2.405	3.832	5.135	6.379	7.588	8.771
2	5.520	7.016	8.417	9.760	11.064	12.339
3	8.654	10.173	11.620	13.015	14.373	15.700
4	11.792	13.323	14.796	16.224	17.616	18.980
5	14.931	16.470	17.960	19.410	20.827	22.218
6	18.071	19.616	21.117	22.583	24.018	25.430
7	21.212	22.760	24.270	25.749	27.200	28.627
8	24.353	25.903	27.421	28.909	30.371	31.812
9	27.494	29.047	30.569	32.065	33.537	34.989

0.5.3 Spherical functions (Legendre polynomials)

Remark: See also section 1.13.2.13.

$x = P_1(x)$	$P_2(x)$	$P_3(x)$	$P_4(x)$	$P_5(x)$	$P_6(x)$	$P_7(x)$
0.00	−0. 5000	0. 0000	0. 3750	0. 0000	−0. 3125	0. 0000
0.05	−0. 4962	−0. 0747	0. 3657	0. 0927	−0. 2962	−0. 1069
0.10	−0. 4850	−0. 1475	0. 3379	0. 1788	−0. 2488	−0. 1995
0.15	−0. 4662	−0. 2166	0. 2928	0. 2523	−0. 1746	−0. 2649
0.20	−0. 4400	−0. 2800	0. 2320	0. 3075	−0. 0806	−0. 2935
0.25	−0. 4062	−0. 3359	0. 1577	0. 3397	+0. 0243	−0. 2799
0.30	−0. 3650	−0. 3825	+0. 0729	0. 3454	0. 1292	−0. 2241
0.35	−0. 3162	−0. 4178	−0. 0187	0. 3225	0. 2225	−0. 1318
0.40	−0. 2600	−0. 4400	−0. 1130	0. 2706	0. 2926	−0. 0146
0.45	−0. 1962	−0. 4472	−0. 2050	0. 1917	0. 3290	+0. 1106
0.50	−0. 1250	−0. 4375	−0. 2891	+0. 0898	0. 3232	0. 2231
0.55	−0. 0462	−0. 4091	−0. 3590	−0. 0282	0. 2708	0. 3007
0.60	+0. 0400	−0. 3600	−0. 4080	−0. 1526	0. 1721	0. 3226
0.65	0. 1338	−0. 2884	−0. 4284	−0. 2705	+0. 0347	0. 2737
0.70	0. 2350	−0. 1925	−0. 4121	−0. 3652	−0. 1253	+0. 1502
0.75	0. 3438	−0. 0703	−0. 3501	−0. 4164	−0. 2808	−0. 0342
0.80	0. 4600	+0. 0800	−0. 2330	−0. 3995	−0. 3918	−0. 2397
0.85	0. 5838	0. 2603	−0. 0506	−0. 2857	−0. 4030	−0. 3913
0.90	0. 7150	0. 4725	+0. 2079	−0. 0411	−0. 2412	−0. 3678
0.95	0. 8538	0. 7184	0. 5541	+0. 3727	+0. 1875	+0. 0112

One has: $P_n(1) = 1$ for all $n = 1, 2, \ldots$

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x),$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5),$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \qquad P_7(x) = \frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

0.5.4 Elliptic integrals

Remark: See also section 1.14.19.

a) Elliptic integrals of the first kind $F(k, \varphi)$, $k = \sin\alpha$.

	$\alpha = 0°$	10°	20°	30°	40°
φ=0°	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.1745	0.1746	0.1746	0.1748	0.1749
20	0.3491	0.3493	0.3499	0.3508	0.3520
30	0.5236	0.5243	0.5263	0.5294	0.5334
40	0.6981	0.6997	0.7043	0.7116	0.7213
50	0.8727	0.8756	0.8842	0.8982	0.9173
60	1.0472	1.0519	1.0660	1.0896	1.1226
70	1.2217	1.2286	1.2495	1.2853	1.3372
80	1.3963	1.4056	1.4344	1.4846	1.5597
90	1.5708	1.5828	1.6200	1.6858	1.7868

	$\alpha = 50°$	60°	70°	80°	90°
φ=0°	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.1751	0.1752	0.1753	0.1754	0.1754
20	0.3533	0.3545	0.3555	0.3561	0.3564
30	0.5379	0.5422	0.5459	0.5484	0.5493
40	0.7323	0.7436	0.7535	0.7604	0.7629
50	0.9401	0.9647	0.9876	1.0044	1.0107
60	1.1643	1.2126	1.2619	1.3014	1.3170
70	1.4068	1.4944	1.5959	1.6918	1.7354
80	1.6660	1.8125	2.0119	2.2653	2.4362
90	1.9356	2.1565	2.5046	3.1534	∞

b) Elliptic integrals of the second kind $E(k, \varphi)$, $k = \sin\alpha$.

	$\alpha = 0°$	10°	20°	30°	40°
φ=0°	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.1745	0.1745	0.1744	0.1743	0.1742
20	0.3491	0.3489	0.3483	0.3473	0.3462
30	0.5236	0.5229	0.5209	0.5179	0.5141
40	0.6981	0.3966	0.6921	0.6851	0.6763
50	0.8727	0.8698	0.8614	0.8483	0.8317
60	1.0472	1.0426	1.0290	1.0076	0.9801
70	1.2217	1.2149	1.1949	1.1632	1.1221
80	1.3963	1.3870	1.3597	1.3161	1.2590
90	1.5708	1.5589	1.5238	1.4675	1.3931

	$\alpha = 50°$	60°	70°	80°	90°
φ=0°	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.1740	0.1739	0.1738	1.1737	0.1736
20	0.3450	0.3438	0.3429	0.3422	0.3420
30	0.5100	0.5061	0.5029	0.5007	0.5000
40	0.6667	0.6575	0.6497	0.6446	0.6428
50	0.8134	0.7954	0.7801	0.7697	0.7660
60	0.9493	0.9184	0.8914	0.8728	0.8660
70	1.0750	1.0266	0.9830	0.9514	0.9397
80	1.1926	1.1225	1.0565	1.0054	0.9848
90	1.3055	1.2111	1.1184	1.0401	1.0000

c) Complete elliptic integrals K and E, $k = \sin\alpha$; for $\alpha = 90°$, we set K$= \infty$, E$= 1$.

$\alpha°$	K	E	$\alpha°$	K	E	$\alpha°$	K	E
0	1.5708	1.5708	30	1.6858	1.4675	60	2.1565	1.2111
1	1.5709	1.5707	31	1.6941	1.4608	61	2.1842	1.2015
2	1.5713	1.5703	32	1.7028	1.4539	62	2.2132	1.1920
3	1.5719	1.5697	33	1.7119	1.4469	63	2.2435	1.1826
4	1.5727	1.5689	34	1.7214	1.4397	64	2.2754	1.1732
5	1.5738	1.5678	35	1.7312	1.4323	65	2.3088	1.1638
6	1.5751	1.5665	36	1.7415	1.4248	66	2.3439	1.1545
7	1.5767	1.5649	37	1.7522	1.4171	67	2.3809	1.1453
8	1.5785	1.5632	38	1.7633	1.4092	68	2.4198	1.1362
9	1.5805	1.5611	39	1.7748	1.4013	69	2.4610	1.1272
10	1.5828	1.5589	40	1.7868	1.3931	70	2.5046	1.1184
11	1.5854	1.5564	41	1.7992	1.3849	71	2.5507	1.1096
12	1.5882	1.5537	42	1.8122	1.3765	72	2.5998	1.1011
13	1.5913	1.5507	43	1.8256	1.3680	73	2.6521	1.0927
14	1.5946	1.5476	44	1.8396	1.3594	74	2.7081	1.0844
15	1.5981	1.5442	45	1.8541	1.3506	75	2.7681	1.0764
16	1.6020	1.5405	46	1.8691	1.3418	76	2.8327	1.0686
17	1.6061	1.5367	47	1.8848	1.3329	77	2.9026	1.0611
18	1.6105	1.5326	48	1.9011	1.3238	78	2.9786	1.0538
19	1.6151	1.5283	49	1.9180	1.3147	79	3.0617	1.0468
20	1.6200	1.5238	50	1.9356	1.3055	80	3.1534	1.0401
21	1.6252	1.5191	51	1.9539	1.2963	81	3.2553	1.0338
22	1.6307	1.5141	52	1.9729	1.2870	82	3.3699	1.0278
23	1.6365	1.5090	53	1.9927	1.2776	83	3.5004	1.0223
24	1.6426	1.5037	54	2.0133	1.2681	84	3.6519	1.0172
25	1.6490	1.4981	55	2.0347	1.2587	85	3.8317	1.0127
26	1.6557	1.4924	56	2.0571	1.2492	86	4.0528	1.0086
27	1.6627	1.4864	57	2.0804	1.2397	87	4.3387	1.0053
28	1.6701	1.4803	58	2.1047	1.2301	88	4.7427	1.0026
29	1.6777	1.4740	59	2.1300	1.2206	89	5.4349	1.0008

$$F(k,\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}} = \int_0^{\sin\varphi} \frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{1-k^2x^2}},$$

$$E(k,\varphi) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\psi}\,\mathrm{d}\psi = \int_0^{\sin\varphi} \sqrt{\frac{1-k^2x^2}{1-x^2}}\,\mathrm{d}x,$$

$$\mathrm{K} = F\Big(k,\frac{\pi}{2}\Big) = \int_0^{\pi/2} \frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}} = \int_0^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}\sqrt{1-k^2x^2}},$$

$$\mathrm{E} = E\Big(k,\frac{\pi}{2}\Big) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\psi}\,\mathrm{d}\psi = \int_0^{1} \sqrt{\frac{1-k^2x^2}{1-x^2}}\,\mathrm{d}x.$$

$$4(n+1)^2\int \mathrm{E}x^n\mathrm{d}x - (2n+3)(2n+5)\int \mathrm{E}x^{n+1}\mathrm{d}x$$
$$= (2n+3)^2\int \mathrm{K}x^{n+1}\mathrm{d}x - 4(n+1)^2\int \mathrm{K}x^n\mathrm{d}x = 2x^{n+1}(\mathrm{E} - (2n+3)(1-x)\mathrm{K}).$$

0.5.5 Integral trigonometric and exponential functions

Definition: $\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \operatorname{si}(x) = \operatorname{Si}(x) - \frac{\pi}{2} = -\int_x^\infty \frac{\sin t}{t}\,dt,$

$$\operatorname{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}\,dt, \qquad \operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt,$$

$$\operatorname{li}(x) = \int_0^x \frac{dt}{\ln t}, \qquad \operatorname{li}(x) = \operatorname{Ei}(\ln x).$$

x	Si(x)	Ci(x)	Ei(x)	x	Si(x)	Ci(x)	Ei(x)
0.00	0.0000	$-\infty$	$-\infty$	0.40	0.3965	−0.3788	0.1048
0.01	0.0100	−4.0280	−4.0179	0.41	0.4062	−0.3561	0.1418
0.02	0.0200	−3.3349	−3.3147	0.42	0.4159	−0.3341	0.1783
0.03	0.0300	−2.9296	−2.8991	0.43	0.4256	−0.3126	0.2143
0.04	0.0400	−2.6421	−2.6013	0.44	0.4353	−0.2918	0.2498
0.05	0.0500	−2.4191	−2.3679	0.45	0.4450	−0.2715	0.2849
0.06	0.0600	−2.2371	−2.1753	0.46	0.4546	−0.2517	0.3195
0.07	0.0700	−2.0833	−2.0108	0.47	0.4643	−0.2325	0.3537
0.08	0.0800	−1.9501	−1.8669	0.48	0.4739	−0.2138	0.3876
0.09	0.0900	−1.8328	−1.7387	0.49	0.4835	−0.1956	0.4211
0.10	0.0999	−1.7279	−1.6228	0.50	0.4931	−0.1778	0.4542
0.11	0.1099	−1.6331	−1.5170	0.51	0.5027	−0.1605	0.4870
0.12	0.1199	−1.5466	−1.4193	0.52	0.5123	−0.1436	0.5195
0.13	0.1299	−1.4672	−1.3287	0.53	0.5218	−0.1271	0.5517
0.14	0.1399	−1.3938	−1.2438	0.54	0.5313	−0.1110	0.5836
0.15	0.1498	−1.3255	−1.1641	0.55	0.5408	−0.0953	0.6153
0.16	0.1598	−1.2618	−1.0887	0.56	0.5503	−0.0800	0.6467
0.17	0.1697	−1.2020	−1.0172	0.57	0.5598	−0.0650	0.6778
0.18	0.1797	−1.1457	−0.9491	0.58	0.5693	−0.0504	0.7087
0.19	0.1896	−1.0925	−0.8841	0.59	0.5787	−0.0362	0.7394
0.20	0.1996	−1.0422	−0.8218	0.60	0.5881	−0.0223	0.7699
0.21	0.2095	−0.9944	−0.7619	0.61	0.5975	−0.0087	0.8002
0.22	0.2194	−0.9490	−0.7042	0.62	0.6069	+0.0046	0.8302
0.23	0.2293	−0.9057	−0.6485	0.63	0.6163	0.0176	0.8601
0.24	0.2392	−0.8643	−0.5947	0.64	0.6256	0.0303	0.8898
0.25	0.2491	−0.8247	−0.5425	0.65	0.6349	0.0427	0.9194
0.26	0.2590	−0.7867	−0.4919	0.66	0.6442	0.0548	0.9488
0.27	0.2689	−0.7503	−0.4427	0.67	0.6535	0.0666	0.9780
0.28	0.2788	−0.7153	−0.3949	0.68	0.6628	0.0782	1.0071
0.29	0.2886	−0.6816	−0.3482	0.69	0.6720	0.0895	1.0361
0.30	0.2985	−0.6492	−0.3027	0.70	0.6812	0.1005	1.0649
0.31	0.3083	−0.6179	−0.2582	0.71	0.6904	0.1113	1.0936
0.32	0.3182	−0.5877	−0.2147	0.72	0.6996	0.1219	1.1222
0.33	0.3280	−0.5585	−0.1721	0.73	0.7087	0.1322	1.1507
0.34	0.3378	−0.5304	−0.1304	0.74	0.7179	0.1423	1.1791
0.35	0.3476	−0.5031	−0.0894	0.75	0.7270	0.1522	1.2073
0.36	0.3574	−0.4767	−0.0493	0.76	0.7360	0.1618	1.2355
0.37	0.3672	−0.4511	−0.0098	0.77	0.7451	0.1712	1.2636
0.38	0.3770	−0.4263	+0.0290	0.78	0.7541	0.1805	1.2916
0.39	0.3867	−0.4022	0.0672	0.79	0.7631	0.1895	1.3195

x	Si(x)	Ci(x)	Ei(x)	x	Si(x)	Ci(x)	Ei(x)
0.80	0.7721	0.1983	1.3474	2.6	1.8004	0.2533	7.5761
0.81	0.7811	0.2069	1.3752	2.7	1.8182	0.2201	8.1103
0.82	0.7900	0.2153	1.4029	2.8	1.8321	0.1865	8.6793
0.83	0.7989	0.2235	1.4306	2.9	2.8422	0.1529	9.2860
0.84	0.8078	0.2316	1.4582	3.0	1.8487	0.1196	9.9338
0.85	0.8166	0.2394	1.4857	3.1	1.8517	0.08699	10.6263
0.86	0.8254	0.2471	1.5132	3.2	1.8514	0.05526	11.3673
0.87	0.8342	0.2546	1.5407	3.3	1.8481	+0.02468	12.1610
0.88	0.8430	0.2619	1.5681	3.4	1.8419	−0.00452	13.0121
0.89	0.8518	0.2691	1.5955	3.5	1.8331	−0.03213	13.9254
0.90	0.8605	0.2761	1.6228	3.6	1.8219	−0.05797	14.9063
0.91	0.8692	0.2829	1.6501	3.7	1.8086	−0.0819	15.9606
0.92	0.8778	0.2896	1.6774	3.8	1.7934	−0.1038	17.0948
0.93	0.8865	0.2961	1.7047	3.9	1.7765	−0.1235	18.3157
0.94	0.8951	0.3024	1.7319	4.0	1.7582	−0.1410	19.6309
0.95	0.9036	0.3086	1.7591	4.1	1.7387	−0.1562	21.0485
0.96	0.9122	0.3147	1.7864	4.2	1.7184	−0.1690	22.5774
0.97	0.9207	0.3206	1.8136	4.3	1.6973	−0.1795	24.2274
0.98	0.9292	0.3263	1.8407	4.4	1.6758	−0.1877	26.0090
0.99	0.9377	0.3319	1.8679	4.5	1.6541	−0.1935	27.9337
1.0	0.9461	0.3374	1.8951	4.6	1.6325	−0.1970	30.0141
1.1	1.0287	0.3849	2.1674	4.7	1.6110	−0.1984	32.2639
1.2	1.1080	0.4205	2.4421	4.8	1.5900	−0.1976	34.6979
1.3	1.1840	0.4457	2.7214	4.9	1.5696	−0.1948	37.3325
1.4	1.2562	0.4620	3.0072	5.0	1.5499	−0.1900	40.1853
1.5	1.3247	0.4704	3.3013	6	1.4247	−0.0681	85.9898
1.6	1.3892	0.4717	3.6053	7	1.4546	+0.0767	191.505
1.7	1.4496	0.4670	3.9210	8	1.5742	+0.1224	440.380
1.8	1.5058	0.4568	4.2499	9	1.6650	+0.05535	1037.88
1.9	1.5578	0.4419	4.5937	10	1.6583	−0.04546	2492.23
2.0	1.6054	0.4230	4.9542	11	1.5783	−0.08956	6071.41
2.1	1.6487	0.4005	5.3332	12	1.5050	−0.04978	14959.5
2.2	1.6876	0.3751	5.7326	13	1.4994	+0.02676	37197.7
2.3	1.7222	0.3472	6.1544	14	1.5562	+0.06940	93192.5
2.4	1.7525	0.3173	6.6007	15	1.6182	+0.04628	234956.0
2.5	1.7785	0.2859	7.0738				

x	Si(x)	Ci(x)	x	Si(x)	Ci(x)
20	1.5482	+0.04442	120	1.5640	+0.00478
25	1.5315	−0.00685	140	1.5722	+0.00701
30	1.5668	−0.03303	160	1.5769	+0.00141
35	1.5969	−0.01148	180	1.5741	−0.00443
40	1.5870	+0.01902	200	1.5684	−0.00438
45	1.5587	+0.01863	300	1.5709	−0.00333
50	1.5516	−0.00563	400	1.5721	−0.00212
55	1.5707	−0.01817	500	1.5726	−0.00093
60	1.5867	−0.00481	600	1.5725	+0.00008
65	1.5792	+0.01285	700	1.5720	+0.00078
70	1.5616	+0.01092	800	1.5714	+0.00112
80	1.5723	−0.01240	10^3	1.5702	+0.00083
90	1.5757	+0.00999	10^4	1.5709	−0.00003
100	1.5622	−0.00515	10^5	1.5708	+0.00000
110	1.5799	−0.00032	∞	$\pi/2$	+0.00000

0.5.6 Fresnel integrals

Remark: See also section 0.10.1.

x	$C(x)$	$S(x)$	x	$C(x)$	$S(x)$	x	$C(x)$	$S(x)$
0.0	0	0	8.5	0.6129	0.5755	21.0	0.5738	0.5459
0.1	0.2521	0.0084	9.0	0.5608	0.6172	21.5	0.5423	0.5748
0.2	0.3554	0.0238	9.5	0.4969	0.6286	22.0	0.5012	0.5849
0.3	0.4331	0.0434	10.0	0.4370	0.6084	22.5	0.4607	0.5742
0.4	0.4966	0.0665	10.5	0.3951	0.5632	23.0	0.4307	0.5458
0.5	0.5502	0.0924	11.0	0.3804	0.5048	23.5	0.4181	0.5068
0.6	0.5962	0.1205	11.5	0.3951	0.4478	24.0	0.4256	0.4670
0.7	0.6356	0.1504	12.0	0.4346	0.4058	24.5	0.4511	0.4361
0.8	0.6693	0.1818	12.5	0.4881	0.3882	25.0	0.4879	0.4212
0.9	0.6979	0.2143	13.0	0.5425	0.3983	25.5	0.5269	0.4258
1.0	0.7217	0.2476	13.5	0.5846	0.4325	26.0	0.5586	0.4483
1.5	0.7791	0.4155	14.0	0.6047	0.4818	26.5	0.5755	0.4829
2.0	0.7533	0.5628	14.5	0.5989	0.5337	27.0	0.5738	0.5211
2.5	0.6710	0.6658	15.0	0.5693	0.5758	27.5	0.5541	0.5534
3.0	0.5610	0.7117	15.5	0.5240	0.5982	28.0	0.5217	0.5721
3.5	0.4520	0.7002	16.0	0.4743	0.5961	28.5	0.4846	0.5731
4.0	0.3682	0.6421	16.5	0.4323	0.5709	29.0	0.4518	0.5562
4.5	0.3252	0.5565	17.0	0.4080	0.5293	29.5	0.4314	0.5260
5.0	0.3285	0.4659	17.5	0.4066	0.4818	30.0	0.4279	0.4900
5.5	0.3724	0.3918	18.0	0.4278	0.4400	30.5	0.4420	0.4570
6.0	0.4433	0.3499	18.5	0.4660	0.4139	31.0	0.4700	0.4350
6.5	0.5222	0.3471	19.0	0.5113	0.4093	31.5	0.5048	0.4291
7.0	0.5901	0.3812	19.5	0.5528	0.4269	32.0	0.5379	9.4406
7.5	0.6318	0.4415	20.0	0.5804	0.4616	32.5	0.5613	0.4663
8.0	0.6393	0.5120	20.5	0.5878	0.5049	33.0	0.5694	0.4999

0.5.7 The function $\int_0^x e^{t^2}\,dt$

x	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0100	0.0200	0.0300	0.0400	0.0500	0.0601	0.0701	0.0802	0.0902
0.1	0.1003	0.1104	0.1206	0.1307	0.1409	0.1511	0.1614	0.1717	0.1820	0.1923
0.2	0.2027	0.2131	0.2236	0.2341	0.2447	0.2553	0.2660	0.2767	0.2875	0.2983
0.3	0.3092	0.3202	0.3313	0.3424	0.3536	0.3648	0.3762	0.3876	0.3991	0.4107
0.4	0.4224	0.4342	0.4461	0.4580	0.4701	0.4823	0.4946	0.5070	0.5196	0.5322
0.5	0.5450	0.5579	0.5709	0.5841	0.5974	0.6109	0.6245	0.6382	0.6522	0.6662
0.6	0.6805	0.6949	0.7095	0.7243	0.7393	0.7544	0.7698	0.7853	0.8011	0.8171
0.7	0.8333	0.8497	0.8664	0.8833	0.9005	0.9179	0.9356	0.9536	0.9718	0.9903
0.8	1.0091	1.0282	1.0477	1.0674	1.0875	1.1079	1.1287	1.1498	1.1713	1.1932
0.9	1.2155	1.2382	1.2613	1.2848	1.3088	1.3332	1.3581	1.3835	1.4093	1.4357
1.0	1.463	1.490	1.518	1.547	1.576	1.606	1.636	1.667	1.699	1.731
1.1	1.765	1.799	1.833	1.869	1.905	1.942	1.980	2.019	2.059	2.099
1.2	2.141	2.184	2.228	2.272	2.318	2.365	2.414	2.463	2.514	2.566
1.3	2.620	2.675	2.731	2.789	2.848	2.909	2.972	3.037	3.103	3.171
1.4	3.241	3.313	3.387	3.463	3.542	3.622	3.705	3.791	3.879	3.970
1.5	4.063	4.159	4.259	4.361	4.467	4.575	4.688	4.803	4.923	5.046
1.6	5.174	5.305	5.441	5.581	5.726	5.876	6.030	6.190	6.356	6.527
1.7	6.704	6.887	7.076	7.272	7.475	7.685	7.903	8.128	8.362	8.604
1.8	8.85	9.11	9.38	9.66	9.95	10.25	10.57	10.89	11.23	11.58
1.9	11.94	12.32	12.70	13.11	13.54	13.98	14.43	14.91	15.40	15.92

0.5.8 Changing from degrees to radians

Arclength of the unit circle

Angle	*Arc*	*Angle*	*Arc*	*Angle*	*Arc*
1″	0.000 005	1°	0.017 453	31°	0.541 052
2	0.000 010	2	0.034 907	32	0.558 505
3	0.000 015	3	0.052 360	33	0.575 959
4	0.000 019	4	0.069 813	34	0.593 412
5	0.000 024	5	0.087 266	35	0.610 865
6	0.000 029	6	0.104 720	36	0.628 319
7	0.000 034	7	0.122 173	37	0.645 772
8	0.000 039	8	0.139 626	38	0.663 225
9	0.000 044	9	0.157 080	39	0.680 678
10	0.000 048	10	0.174 533	40	0.698 132
20	0.000 097	11	0.191 986	45	0.785 398
30	0.000 145	12	0.209 440	50	0.872 665
40	0.000 194	13	0.226 893	55	0.959 931
50	0.000 242	14	0.244 346	60	1.047 198
		15	0.261 799	65	1.134 464
1′	0.000 291	16	0.279 253	70	1.221 730
2	0.000 582	17	0.296 706	75	1.308 997
3	0.000 873	18	0.314 159	80	1.396 263
4	0.001 164	19	0.331 613	85	1.483 530
5	0.001 454	20	0.349 066	90	1.570 796
6	0.001 745	21	0.366 519	100	1.745 329
7	0.002 036	22	0.383 972	120	2.094 395
8	0.002 327	23	0.401 426	150	2.617 994
9	0.002 618	24	0.418 879	180	3.141 593
10	0.002 909	25	0.436 332	200	3.490 659
20	0.005 818	26	0.453 786	250	4.363 323
30	0.008 727	27	0.471 239	270	4.712 389
40	0.011 636	28	0.488 692	300	5.235 988
50	0.014 544	29	0.506 145	360	6.283 185
		30	0.523 599	400	6.981 317

Examples:

1) 52° 37′ 23″

50°			= 0.872 665
2°			= 0.034 907
	30′		= 0.008 727
	7′		= 0.002 036
		20″	= 0.000 097
		3″	= 0.000 015
			0.918 447
52°	37′	23″	= 0.91845 rad

2) 5.645 radians (arclength)

5.235 988	= 300°			
0.409 012				
0.401 426	= 23°			
0.007 586				
0.005 818	=	20′		
0.001 768				
0.001 745	=	6′		
0.000 023	=		5″	
5.645 rad	= 323°	26′	5″	

The radian is the plane angle for which the quotient of the length of the corresponding circular arc and its radius is equal to 1 (abbreviated rad).

0.6 Table of prime numbers ≤ 4000

The prime number *twins*, i.e., two consecutive odd numbers which are prime, are indicated by boldface (starting at 41–43). It is known that there are infinitely many such twins.

2	3	5	7	11	13	17	19	23	29
31	37	**41**	**43**	47	53	**59**	**61**	67	**71**
73	79	83	89	97	**101**	**103**	**107**	**109**	113
127	131	**137**	**139**	**149**	**151**	157	163	167	173
179	181	**191**	**193**	**197**	**199**	211	223	**227**	**229**
233	239	241	251	257	263	**269**	**271**	277	**281**
283	293	307	**311**	**313**	317	331	337	**347**	**349**
353	359	367	373	379	383	389	397	401	409
419	**421**	**431**	**433**	439	443	449	457	**461**	**463**
467	479	487	491	499	503	509	**521**	**523**	541
547	557	563	569	571	577	587	593	**599**	**601**
607	613	**617**	**619**	631	**641**	**643**	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	**809**
811	**821**	**823**	**827**	**829**	839	853	**857**	**859**	863
877	**881**	**883**	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013
1019	**1021**	**1031**	**1033**	1039	**1049**	**1051**	**1061**	**1063**	1069
1087	**1091**	**1093**	1097	1103	1109	1117	1123	1129	**1151**
1153	1163	1171	1181	1187	1193	1201	1213	1217	1223
1229	**1231**	1237	1249	1259	**1277**	**1279**	1283	**1289**	**1291**
1297	**1301**	**1303**	1307	**1319**	**1321**	1327	1361	1367	1373
1381	1399	1409	1423	**1427**	**1429**	1433	1439	1447	**1451**
1453	1459	1471	**1481**	**1483**	**1487**	**1489**	1493	1499	1511
1523	1531	1543	1549	1553	1559	1567	1571	1579	1583
1597	1601	**1607**	**1609**	1613	**1619**	**1621**	1627	1637	1657
1663	**1667**	**1669**	1693	**1697**	**1699**	1709	**1721**	**1723**	1733
1741	1747	1753	1759	1777	1783	**1787**	**1789**	1801	1811
1823	1831	1847	1861	1867	**1871**	**1873**	**1877**	**1879**	1889
1901	1907	1913	**1931**	**1933**	**1949**	**1951**	1973	1979	1987
1993	**1997**	**1999**	2003	2011	2017	**2027**	**2029**	2039	2053
2063	2069	**2081**	**2083**	**2087**	**2089**	2099	**2111**	**2113**	**2129**
2131	2137	**2141**	**2143**	2153	2161	2179	2203	2207	2213
2221	**2237**	**2239**	2243	2251	**2267**	**2269**	2273	2281	2287
2293	2297	**2309**	**2311**	2333	**2339**	**2341**	2347	2351	2357
2371	2377	**2381**	**2383**	2389	2393	2399	2411	2417	2423
2437	2441	2447	2459	2467	2473	2477	2503	2521	2531
2539	2543	**2549**	**2551**	2557	2579	**2591**	**2593**	2609	2617
2621	2633	2647	**2657**	**2659**	2663	2671	2677	2683	**2687**
2689	2693	2699	2707	**2711**	**2713**	2719	**2729**	**2731**	2741
2749	2753	2767	2777	**2789**	**2791**	2797	**2801**	**2803**	2819
2833	2837	2843	2851	2857	2861	2879	2887	2897	2903
2909	2917	2927	2939	2953	2957	2963	**2969**	**2971**	**2999**
3001	3011	3019	3023	3037	3041	3049	3061	3067	3079
3083	3089	3109	**3119**	**3121**	3137	3163	**3167**	**3169**	3181
3187	3191	3203	3209	3217	3221	3229	**3251**	**3253**	**3257**
3259	3271	**3299**	**3301**	3307	3313	3319	3323	**3329**	**3331**
3343	3347	**3359**	**3361**	**3371**	**3373**	**3389**	**3391**	3407	3413
3433	3449	3457	**3461**	**3463**	**3467**	**3469**	3491	3499	3511
3517	**3527**	**3529**	3533	**3539**	**3541**	3547	**3557**	**3559**	3571
3581	**3583**	3593	3607	3613	3617	3623	3631	3637	3643
3659	**3671**	**3673**	3677	3691	3697	3701	3709	3719	3727
3733	3739	3761	**3767**	**3769**	3779	3793	3797	3803	**3821**
3823	3833	3847	**3851**	**3853**	3863	3877	3881	3889	3907
3911	**3917**	**3919**	3923	**3929**	**3931**	3943	3947	3967	3989

0.7 Formulas for series and products

For infinite series and infinite products the notion of convergence is fundamental (cf. 1.10.1 and 1.10.6).

0.7.1 Special series

One gets important series by inserting special values in the power series listed in 0.7.2 or in Fourier series listed in 0.7.4.

0.7.1.1 The Leibniz series and related series

$$\boxed{1 - \frac{1}{3} + \frac{1}{5} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \qquad \text{(Leibniz, 1676)},}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2\,,$$

$$\ln\left(1 - \frac{1}{2^2}\right) + \ln\left(1 - \frac{1}{3^2}\right) + \ldots = \sum_{k=2}^{\infty} \ln\left(1 - \frac{1}{k^2}\right) = -\ln 2\,,$$

$$\boxed{2 + \frac{1}{2!} + \frac{1}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} = \mathrm{e} \qquad \text{(Euler number)},}$$

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{\mathrm{e}}\,,$$

$$\boxed{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2 \qquad \text{(geometric series)},}$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = \frac{2}{3} \qquad \text{(alternating geometric series)},$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1\,,$$

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots = \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\,,$$

$$\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \ldots = \sum_{n=2}^{\infty} \frac{1}{(n-1)(n+1)} = \frac{3}{4}\,,$$

$$\frac{1}{3\cdot 5} + \frac{1}{7\cdot 9} + \frac{1}{11\cdot 13} + \ldots = \sum_{n=1}^{\infty} \frac{1}{(4n-1)(4n+1)} = \frac{1}{2} - \frac{\pi}{8}\,,$$

$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}\,,$$

$$\frac{1}{1\cdot 2\cdot 3\cdots k}+\frac{1}{2\cdot 3\cdots (k+1)}+\dots = \sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+k-1)}=\frac{1}{(k-1)(k-1)!}\,,\qquad k=2,3,\dots,$$

$$\sum_{n=p+1}^{\infty}\frac{1}{n^2-p^2}=\frac{1}{2p}\left(1+\frac{1}{2}+\dots+\frac{1}{2p}\right),\qquad p=1,2,\dots,\qquad \text{(Jakob Bernoulli, 1689).}$$

0.7.1.2 Special values of the Riemannian ζ-function and related series

The series

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\dots=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

converges for all real numbers $s>1$ and more generally for all complex numbers s with $\operatorname{Re} s>1$. This function is of fundamental importance in the mathematical discipline of number theory, in particular with the distribution of prime numbers (see section 2.7.3). It is called the *Riemann ζ-function* and was studied by Euler and particularly by Riemann in 1859.

The formula of L. Euler (1734)[44] :

$$\zeta(2k)=1+\frac{1}{2^{2k}}+\frac{1}{3^{2k}}+\dots=\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}\,,\qquad k=1,2,\dots$$

Special cases:

$$\zeta(2)=1+\frac{1}{2^2}+\frac{1}{3^2}+\dots=\frac{\pi^2}{6}\,,$$

$$\zeta(4)=\frac{\pi^4}{90}\,,\qquad \zeta(6)=\frac{\pi^6}{945}\,,\qquad \zeta(8)=\frac{\pi^8}{9\,450}\,,$$

$$1-\frac{1}{2^{2k}}+\frac{1}{3^{2k}}-\frac{1}{4^{2k}}+\dots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{2k}}=\frac{\pi^{2k}\left(2^{2k}-1\right)}{(2k)!}\,|B_{2k}|\,,\qquad k=1,2,\dots$$

Special cases:

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}\,,$$

$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\dots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}=\frac{7\pi^4}{720}\,,$$

[44]The Bernoulli numbers B_k and the Euler numbers E_k can be found in sections 0.1.10.4 and 0.1.10.5.

$$1+\frac{1}{3^{2k}}+\frac{1}{5^{2k}}+\ldots=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k}}=\frac{\pi^{2k}\left(2^{2k-1}\right)}{2(2k)!}|B_{2k}|, \quad k=1,2,\ldots$$

Special cases:

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8},$$
$$1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{96}.$$

$$1-\frac{1}{3^{2k+1}}+\frac{1}{5^{2k+1}}-\ldots=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+1}}=\frac{\pi^{2k+1}}{2^{2k+2}(2k)!}|E_{2k}|, \quad k=0,1,2,\ldots$$

Special cases: For $k=0$ one gets the Leibniz series $1-\frac{1}{3}+\frac{1}{5}-\ldots$; for $k=1$:

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\ldots=\frac{\pi^3}{32}.$$

0.7.1.3 The Euler–McLaurin summation formula

The asymptotic formula of Euler (1734):

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln(n+1)\right)=\mathrm{C}. \tag{0.53}$$

The Euler constant C has the value $\mathrm{C}=0.57721\,56649\,01532\ldots$, which had already been calculated by Euler. The asympotic formula (0.53) is a special case of the Euler–McLaurin summation formula (0.54).

Bernoulli polynomials:

$$B_n(x):=\sum_{k=0}^{\infty}\binom{n}{k}B_k x^{n-k}.$$

Modified Bernoulli polynomials:[45]

$$C_n(x):=B_n\big(x-[x]\big).$$

The Euler–McLaurin summation formula: For $n=1,2,\ldots$ one has

$$f(0)+f(1)+\ldots+f(n)=\int_0^n f(x)\,\mathrm{d}x+\frac{f(0)+f(n)}{2}+S_n \tag{0.54}$$

with[46]

$$S_n:=\frac{B_2}{2!}f'+\frac{B_4}{4!}f^{(3)}+\ldots+\frac{B_{2p}}{(2p)!}f^{(2p-1)}\bigg|_1^n+R_p, \qquad p=2,3,\ldots,$$

[45] We denote by $[x]$ the largest integer n smaller than or equal to x: (Gauss bracket). The function C_n coincides in the interval $[0,1[$ with B_n and is extended periodically with period 1.

[46] The symbol $g|_1^n$ means $g(n)-g(1)$.

and the remainder term

$$R_p = \frac{1}{(2p+1)!}\int_0^n f^{(2p+1)}(x)C_{2p+1}(x)\,\mathrm{d}x\,.$$

Here it is assumed that the function $f\colon [0,n] \to \mathbb{R}$ is sufficiently smooth, i.e., has continuous derivatives up to order $2p+1$ on the interval $[0,n]$.

0.7.1.4 Infinite partial fraction decomposition

The following series converge for all complex numbers x with the exception of those values for which the denominator vanishes[47]:

$$\cot \pi x = \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{x-k} + \frac{1}{x+k}\right),$$

$$\tan \pi x = -\sum_{k=1}^{\infty}\frac{1}{x-\left(k-\frac{1}{2}\right)} + \frac{1}{x+\left(k-\frac{1}{2}\right)},$$

$$\frac{\pi}{\sin \pi x} = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{(-1)^k 2x}{x^2-k^2},$$

$$\left(\frac{\pi}{\sin \pi x}\right)^2 = \sum_{k=-\infty}^{\infty}\frac{1}{(x-k)^2},$$

$$\left(\frac{\pi}{\cos \pi x}\right)^2 = \sum_{k=-\infty}^{\infty}\frac{1}{\left(x-k+\frac{1}{2}\right)^2}.$$

0.7.2 Power series

Comments on the power series table: The power series listed in the following table converge for all complex numbers x for which the stated inequalities hold. The properties of power series will be considered in more detail in 1.10.3.

The given first term in the series may be used as an approximation for $|x|$ sufficiently small.

Example: One has

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5\,040} + \ldots = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

If $|x|$ is small, then, approximately

$$\sin x = x\,.$$

Successively improved approximations are obtained by

$$\sin x = x - \frac{x^3}{6}, \qquad \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} \qquad \text{etc.}$$

[47] These series are special cases of the theorem of Mittag-Leffler (cf. 1.14.6.4).
A sum $\sum_{k=-\infty}^{\infty} \ldots$ stands for the sum of the two infinite sums: $\sum_{k=0}^{\infty} \ldots + \sum_{k=-\infty}^{-1} \ldots$

For the frequently appearing factorials one can use the following table:

n	0	1	2	3	4	5	6	7	8	9	10
$n!$	1	1	2	6	24	120	720	5040	40,320	362,880	3,628,800

In the expansions of

$$\frac{x}{e^x - 1}\,,\ \tan x\,,\ \cot x\,,\ \frac{1}{\sin x} \equiv \operatorname{cosec} x\,,\ \tanh x\,,\ \coth x\,,\ \frac{1}{\sinh x} \equiv \operatorname{cosec} x$$

and

$$\frac{1}{\cosh x} \equiv \operatorname{sech} x\,,\ \frac{1}{\cos x} \equiv \sec x\,,\ \ln\cos x\,,\ \ln|x| - \ln|\sin x|\,,$$

respectively, the Bernoulli numbers B_k resp. Euler numbers E_k appear (see sections 0.1.10.4 and 0.1.10.5).

Function	***Power series expansion***	***Domain of convergence*** $(x \in \mathbb{C})$
	geometric series	
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k$	$\lvert x\rvert < 1$
$\frac{1}{1+x}$	$1 - x + x^2 - x^3 + \ldots = \sum_{k=0}^{\infty} (-1)^k x^k$	$\lvert x\rvert < 1$
	The binomial series of Newton	
$(1+x)^\alpha$	$1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \ldots = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$ (α is an arbitrary real number[48])	$\lvert x\rvert < 1$ $(x = \pm 1,\ \alpha > 0)$
$(a+x)^\alpha$	$a^\alpha\left(1 + \frac{x}{a}\right)^\alpha = a^\alpha + \alpha a^{\alpha-1}x + a^{\alpha-2}\binom{\alpha}{2}x^2 + \ldots$ $= \sum_{k=0}^{\infty} a^{\alpha-k}\binom{\alpha}{k}x^k$ (a is a positive real number)	$\lvert x\rvert < a$ ($x = \pm a$ for $\alpha > 0$)
$(a+x)^n$	$a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \ldots + \binom{n}{1}ax^{n-1} + x^n$ ($n = 1, 2, \ldots$; a and x are arbitrary complex numbers)	$\lvert x\rvert < \infty$
$(a+x)^{-n}$	$(a+x)^{-n} := \frac{1}{(a+x)^n}$	
$(a+x)^{1/n}$	$(a+x)^{1/n} := \sqrt[n]{a+x}$	
$(a+x)^{-1/n}$	$(a+x)^{-1/n} := \frac{1}{\sqrt[n]{a+x}}$	

[48] The generalized binomial coefficients are defined by $\binom{\alpha}{1} = \alpha$, $\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{1\cdot 2}$, $\binom{\alpha}{3} = \frac{\alpha(\alpha-1)(\alpha-2)}{1\cdot 2\cdot 3}$ etc.

Special cases of the binomial series for integral exponents
(a a complex number with $a \neq 0$)

$\frac{1}{(a \pm x)^n}$	$\frac{1}{a^n} \mp \frac{nx}{a^{n+1}} + \frac{n(n+1)x^2}{2a^{n+2}} \mp \dots$ $= \frac{1}{a^n} + \sum_{k=1}^{\infty} \frac{n(n+1)\dots(n-k+1)}{k!a^{n+k}} (\mp x)^k$	$\lvert x\rvert < \lvert a\rvert$
$\frac{1}{a \pm x}$	$\frac{1}{a} \mp \frac{x}{a^2} + \frac{x^2}{a^3} \mp \frac{x^3}{a^4} + \dots = \sum_{k=0}^{\infty} \frac{(\mp x)^k}{a^{k+1}}$	$\lvert x\rvert < \lvert a\rvert$
$(a \pm x)^2$	$a^2 \pm 2ax + x^2$	$\lvert x\rvert < \infty$
$\frac{1}{(a \pm x)^2}$	$\frac{1}{a^2} \mp \frac{2x}{a^3} + \frac{3x^2}{a^4} \mp \frac{4x^3}{a^5} + \dots = \sum_{k=0}^{\infty} \frac{(k+1)(\mp x)^k}{a^{2+k}}$	$\lvert x\rvert < \lvert a\rvert$
$(a \pm x)^3$	$a^3 \pm 3a^2x + 3ax^2 \pm x^3$	$\lvert x\rvert < \infty$
$\frac{1}{(a \pm x)^3}$	$\frac{1}{a^3} \mp \frac{3x}{a^4} + \frac{6x^2}{a^5} \mp \frac{10x^3}{a^6} + \dots$ $= \sum_{k=0}^{\infty} \frac{(k+1)(k+2)(\mp x)^k}{2a^{3+k}}$	$\lvert x\rvert < \lvert a\rvert$

Special cases of the binomial series for rational exponents
(b a positive real number)

$\sqrt{b \pm x}$	$\sqrt{b} \pm \frac{x}{2\sqrt{b}} - \frac{x^2}{8b\sqrt{b}} \pm \frac{x^3}{16b^2\sqrt{b}} - \dots$ $= \sqrt{b} \pm \frac{x}{2\sqrt{b}} + \sum_{k=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2k-3)(-1)^{k+1}(\pm x)^k}{(2 \cdot 4 \cdot 6 \dots 2k)b^{k-1}\sqrt{b}}$	$\lvert x\rvert < b$
$\frac{1}{\sqrt{b \pm x}}$	$\frac{1}{\sqrt{b}} \mp \frac{x}{2b\sqrt{b}} + \frac{3x^2}{8b^2\sqrt{b}} \mp \frac{15x^3}{48b^3\sqrt{b}} + \dots$ $= \frac{1}{\sqrt{b}} \mp \frac{x}{2b\sqrt{b}} + \sum_{k=2}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2k-1)(-1)^k(\pm x)^k}{(2 \cdot 4 \cdot 6 \dots 2k)b^k\sqrt{b}}$	$\lvert x\rvert < b$
$\sqrt[3]{b \pm x}$	$\sqrt[3]{b} \pm \frac{x}{3\sqrt[3]{b^2}} - \frac{x^2}{9b\sqrt[3]{b^2}} \pm \frac{5x^3}{81b^2\sqrt[3]{b^2}} - \dots$ $= \sqrt[3]{b} \pm \frac{x}{3\sqrt[3]{b^2}} + \sum_{k=2}^{\infty} \frac{2 \cdot 5 \cdot 8 \dots (3k-4)(-1)^{k+1}(\pm x)^k}{(3 \cdot 6 \cdot 9 \dots 3k)b^{k-1}\sqrt[3]{b^2}}$	$\lvert x\rvert < b$
$\frac{1}{\sqrt[3]{b \pm x}}$	$\frac{1}{\sqrt[3]{b}} \mp \frac{x}{3b\sqrt[3]{b}} + \frac{2x^2}{9b^2\sqrt[3]{b}} \mp \frac{14x^3}{81b^3\sqrt[3]{b}} + \dots$ $= \frac{1}{\sqrt[3]{b}} + \sum_{k=1}^{\infty} \frac{4 \cdot 7 \cdot 10 \dots (3k-2)(-1)^k(\pm x)^k}{(3 \cdot 6 \cdot 9 \dots 3k)b^k\sqrt[3]{b}}$	$\lvert x\rvert < b$

Hypergeometric series (generalized binomial series) of Gauss		
$F(\alpha,\beta,\gamma,x)$	$1+\frac{\alpha\beta}{\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2\gamma(\gamma+1)}x^2+\ldots$ $=1+\sum_{k=1}^{\infty}\frac{\alpha(\alpha+1)\ldots(\alpha+k-1)\beta(\beta+1)\ldots(\beta+k-1)}{k!\gamma(\gamma+1)\ldots(\gamma+k-1)}x^k$	$\|x\|<1$

Special cases of the hypergeometric series		
$(1+x)^\alpha$	$=F(-\alpha,1,1,-x)$	
$\arcsin x$	$=xF\left(\frac{1}{2},\frac{1}{2},\frac{3}{2},x^2\right)$	
$\ln(1+x)$	$=xF(1,1,2,-x)$	
e^x	$=\lim_{\beta\to+\infty}F\left(1,\beta,1,\frac{x}{\beta}\right)$	
$P_n(x)$	$=F\left(n+1,-n,1,\frac{1-x}{2}\right)$, $n=0,1,2,\ldots$	
(Legendre polynomials, see page 123 below)		
$Q_n(x)$	$=\frac{\sqrt{\pi}\Gamma(n+1)}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)}\cdot\frac{1}{x^{n+1}}F\left(\frac{n+1}{2},\frac{n+2}{2},\frac{2n+3}{2},\frac{1}{x^2}\right)$	$\|x\|>1$
(Legendre functions, see page 123 below)		

Exponential function		
e^x	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots=\sum_{k=0}^{\infty}\frac{x^k}{k!}$	$\|x\|<\infty$
e^{bx}	$1+bx+\frac{(bx)^2}{2!}+\frac{(bx)^3}{3!}+\ldots=\sum_{k=0}^{\infty}\frac{(bx)^k}{k!}$ (b is a complex number)	$\|x\|<\infty$
a^x	$a^x=e^{bx}$ with $b=\ln a$ (a real and positive)	
$\frac{x}{e^x-1}$	$1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^4}{7\,200}+\ldots=\sum_{k=0}^{\infty}\frac{B_k}{k!}x^k$	$\|x\|<2\pi$

Trigonometric functions and hyperbolic functions

$\sin \mathrm{i}x = \mathrm{i}\sinh x$, $\cos \mathrm{i}x = \cosh x$, $\sinh \mathrm{i}x = \mathrm{i}\sin x$, $\cosh \mathrm{i}x = \cos x$
(for all complex numbers x)

$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$	$\lvert x\rvert < \infty$
$\sinh x$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$	$\lvert x\rvert < \infty$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$	$\lvert x\rvert < \infty$
$\cosh x$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$	$\lvert x\rvert < \infty$
$\tan x$	$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots = \sum_{k=1}^{\infty} 4^k\left(4^k - 1\right)\frac{\lvert B_{2k}\rvert x^{2k-1}}{(2k)!}$	$\lvert x\rvert < \frac{\pi}{2}$
$\tanh x$	$x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \ldots = \sum_{k=1}^{\infty} 4^k\left(4^k - 1\right)\frac{B_{2k} x^{2k-1}}{(2k)!}$	$\lvert x\rvert < \frac{\pi}{2}$
$\frac{1}{x} - \cot x$	$\frac{x}{3} + \frac{x^3}{45} + \frac{2x^5}{945} + \frac{x^7}{4\,725} + \ldots = \sum_{k=1}^{\infty} \frac{4^k \lvert B_{2k}\rvert x^{2k-1}}{(2k)!}$	$0 < \lvert x\rvert < \pi$
$\coth x - \frac{1}{x}$	$\frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4\,725} + \ldots = \sum_{k=1}^{\infty} \frac{4^k B_{2k} x^{2k-1}}{(2k)!}$	$0 < \lvert x\rvert < \pi$
$\frac{1}{\cos x}$	$1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \ldots = \sum_{k=0}^{\infty} \frac{\lvert E_k\rvert x^k}{k!}$	$\lvert x\rvert < \frac{\pi}{2}$
$\frac{1}{\cosh x}$	$1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \ldots = \sum_{k=0}^{\infty} \frac{E_k x^k}{k!}$	$\lvert x\rvert < \frac{\pi}{2}$
$\frac{1}{\sin x} - \frac{1}{x}$	$\frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15\,120} + \frac{127x^7}{604\,800} + \ldots$ $= \sum_{k=1}^{\infty} \frac{2\left(2^{2k-1} - 1\right)}{(2k)!}\lvert B_{2k}\rvert x^{2k-1}$	$0 < \lvert x\rvert < \pi$
$\frac{1}{x} - \frac{1}{\sinh x}$	$\frac{x}{6} - \frac{7x^3}{360} + \frac{31x^5}{15\,120} - \frac{127x^7}{604\,800} + \ldots$ $= \sum_{k=1}^{\infty} \frac{2\left(2^{2k-1} - 1\right)}{(2k)!} B_{2k} x^{2k-1}$	$0 < \lvert x\rvert < \pi$

Inverse trigonometric functions and inverse hyperbolic functions		
$\arctan x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$	$\lvert x\rvert < 1$ (or $x = \pm 1$)
$\frac{\pi}{4} = \arctan 1$	$1 - \frac{1}{3} + \frac{1}{5} - \ldots$ (Leibniz series)	
$\operatorname{artanh} x$	$x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$	$\lvert x\rvert < 1$
$\frac{\pi}{2} - \operatorname{arccot} x$	$\frac{\pi}{2} - \operatorname{arccot} x = \arctan x$	
$\arctan \frac{1}{x}$	$\frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \ldots$	$0 < x < 1$
$\arctan \frac{1}{x}$	$-\frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \ldots$	$-1 < x < 0$
$\operatorname{arcoth} \frac{1}{x}$	$x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$	$0 < \lvert x\rvert < 1$
$\arcsin x$	$x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{15x^7}{336} + \cdots$ $= x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \ldots (2k-1)x^{2k+1}}{2 \cdot 4 \cdot 6 \cdots 2k(2k+1)}$	$\lvert x\rvert < 1$
$\frac{\pi}{2} - \arccos x$	$\frac{\pi}{2} - \arccos x = \arcsin x$	
$\operatorname{arsinh} x$	$x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{15x^7}{336} + \ldots$ $= x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)(-1)^k x^{2k+1}}{2 \cdot 4 \cdot 6 \cdots (2k)(2k+1)}$	$\lvert x\rvert < 1$

Logarithmic functions		
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$	$\lvert x\rvert < 1$ (and $x = 1$)
$\ln 2$	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$	
$-\ln(1-x)$	$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots = \sum_{k=1}^{\infty} \frac{x^k}{k}$	$\lvert x\rvert < 1$ (and $x = -1$)
$\ln \frac{1+x}{1-x} = 2 \operatorname{artanh} x$	$2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \ldots = \sum_{k=1}^{\infty} \frac{2x^{2k+1}}{2k+1}$	$\lvert x\rvert < 1$

$\ln\lvert x\rvert$ $-\ln\lvert\sin x\rvert$	$\frac{x^2}{6}+\frac{x^4}{180}+\frac{x^6}{2\,835}+\ldots=\sum_{k=1}^{\infty}\frac{2^{2k-1}}{k(2k)!}\lvert B_{2k}\rvert x^{2k}$	$0<\lvert x\rvert<\pi$
$-\ln\cos x$	$\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{45}+\frac{17x^8}{2\,520}+\ldots$ $=\sum_{k=1}^{\infty}\frac{2^{2k-1}\left(4^k-1\right)}{k(2k)!}\lvert B_{2k}\rvert x^{2k}$	$\lvert x\rvert<\frac{\pi}{2}$
$\ln\lvert\tan x\rvert$ $-\ln\lvert x\rvert$	$\frac{x^2}{3}+\frac{7x^4}{90}+\frac{62x^6}{2\,835}+\ldots$ $=\sum_{k=1}^{\infty}\frac{4^k\left(2^{2k-1}-1\right)}{k(2k)!}\lvert B_{2k}\rvert x^{2k}$	$0<\lvert x\rvert<\frac{\pi}{2}$

Complete elliptic integrals		
$K(k)$	$\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}}=\frac{\pi}{2}\left(1+\frac{k^2}{4}+\frac{9k^4}{64}+\ldots\right)$ $=\frac{\pi}{2}\left(1+\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right)^2 k^{2n}\right)$	$\lvert k\rvert<1$
$E(k)$	$\int_0^{\pi/2}\sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\varphi=\frac{\pi}{2}\left(1-\frac{k^2}{4}+\frac{9k^4}{192}-\ldots\right)$ $=\frac{\pi}{2}\left(1+\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right)^2\frac{(-1)^n k^{2n}}{2n-1}\right)$	$\lvert k\rvert<1$

The Euler gamma function (generalized factorial)		$x\in\mathbb{C}$
	$\Gamma(x+1)=x!\,,\quad \Gamma(x+1)=x\Gamma(x)$	$x\neq 0,-1,-2,\ldots$
	$\Gamma(x)=\int_0^{\infty}\mathrm{e}^{-t}t^{x-1}\mathrm{d}t$	$\mathrm{Re}\,x>0$
$\ln\Gamma(x+1)$	$-\mathrm{C}x+\frac{\zeta(2)x^2}{2}-\frac{\zeta(3)x^3}{3}+\ldots=-\mathrm{C}x+\sum_{k=2}^{\infty}(-1)^k\frac{\zeta(k)x^k}{k}$ $=\frac{1}{2}\ln\frac{\pi x}{\sin\pi x}-\frac{1}{2}\ln\frac{1+x}{1-x}+(1-\mathrm{C})x$ $+\sum_{k=1}^{\infty}\frac{\left(1-\zeta(2k+1)\right)x^{2k+1}}{2k+1}$ (Legendre series[49])	$\lvert x\rvert<1$
$\Gamma(x+1)$	$\sqrt{\frac{\pi x}{\sin\pi x}\cdot\frac{1-x}{1+x}}\exp\left((1-\mathrm{C})x+\sum_{k=1}^{\infty}\frac{\left(1-\zeta(2k+1)\right)x^{2k+1}}{2k+1}\right)$	$\lvert x\rvert<1$

[49] Here C denotes the Euler constant, and ζ is the Riemannian ζ-function.