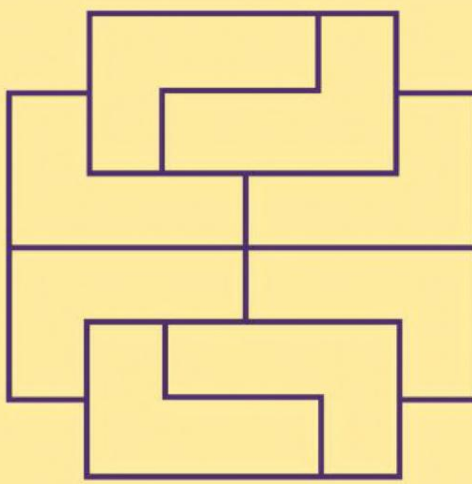
PROBLEM BOOKS IN MATHEMATICS

Arthur Engel

Problem-Solving Strategies



Springer

Arthur Engel

Problem-Solving Strategies

With 223 Figures

Springer

Angel Engel
Institut für Didaktik der Mathematik
Johann Wolfgang Goethe–Universität Frankfurt am Main
Senckenberganlage 9–11
60054 Frankfurt am Main 11
Germany

Series Editor:
Paul R. Halmos
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053
USA

Mathematics Subject Classification (1991): 00A07

Library of Congress Cataloging-in-Publication Data

Engel, Arthur.
Problem-solving strategies/Arthur Engel.
p. cm. — (Problem books in mathematics)
Includes index.
ISBN 0-387-98219-1 (softcover: alk. paper)
1. Problem solving. I. Title. II. Series.
QA63.E54 1997
510′.76—dc21 97-10090

ISBN 0–387–98219–1 Springer-Verlag New York Berlin Heidelburg SPIN 10557554

Contents

Abbreviations and Notations

Abbreviations

ARO	Allrussian Mathematical Olympiad
ATMO	Austrian Mathematical Olympiad
AuMO	Australian Mathematical Olympiad
AUO	Allunion Mathematical Olympiad
BrMO	British Mathematical Olympiad
BWM	German National Olympiad
BMO	Balkan Mathematical Olympiad
ChNO	Chinese National Olympiad
HMO	Hungarian Mathematical Olympiad (Kűrschak Competition)
IIM	International Intellectual Marathon (Mathematics/Physics Competition)
IMO	International Mathematical Olympiad
LMO	Leningrad Mathematical Olympiad
MMO	Moskov Mathematical Olympiad
PAMO	Polish-Austrian Mathematical Olympiad

PMO	Polish Mathematical Olympiad
RO	Russian Olympiad (ARO from 1994 on)
SPMO	St. Petersburg Mathematical Olympiad
TT	Tournament of the Towns
USO	US Olympiad

Notations for Numerical Sets

$\mathbb{N}$ or $\mathbb{Z}^+$	the positive integers (natural numbers), i.e., $\{1,2,3,\dots\}$
$\mathbb{N}_0$	the nonnegative integers, $\{0,1,2,\dots\}$
$\mathbb{Z}$	the integers
$\mathbb{Q}$	the rational numbers
$\mathbb{Q}^+$	the positive rational numbers
$\mathbb{Q}_0^+$	the nonnegative rational numbers
$\mathbb{R}$	the real numbers
$\mathbb{R}^+$	the positive real numbers
$\mathbb{C}$	the complex numbers
$\mathbb{Z}_n$	the integers modulo n
$1\,..\,n$	the integers $1,\ 2,\ \dots,\ n$

Notations from Sets, Logic, and Geometry

$\Longleftrightarrow$	iff, if and only if
$\Longrightarrow$	implies
$A \subset B$	A is a subset of B
$A \setminus B$	A without B
$A \cap B$	the intersection of A and B
$A \cup B$	the union of A and B
$a \in A$	the element a belongs to the set A
$\lvert AB\rvert$	also AB, the distance between the points A and B
box	parallelepiped, solid bounded by three pairs of parallel planes

1

The Invariance Principle

We present our first *Higher Problem-Solving Strategy*. It is extremely useful in solving certain types of difficult problems, which are easily recognizable. We will teach it by solving problems which use this strategy. In fact, **problem solving can be learned only by solving problems.** But it must be supported by strategies provided by the trainer.

Our first strategy is the *search for invariants*, and it is called the **Invariance Principle**. The principle is applicable to algorithms (games, transformations). Some task is repeatedly performed. **What stays the same? What remains invariant?** Here is a saying easy to remember:

If there is repetition, look for what does not change!

In algorithms there is a starting state S and a sequence of legal steps (moves, transformations). One looks for answers to the following questions:

1. Can a given end state be reached?

2. Find all reachable end states.

3. Is there convergence to an end state?

4. Find all periods with or without tails, if any.

Since the Invariance Principle is a *heuristic principle*, it is best learned by experience, which we will gain by solving the key examples **E1** to **E10**.

E1. *Starting with a point $S = (a, b)$ of the plane with $0 < b < a$, we generate a sequence of points (x_n, y_n) according to the rule*

$$x_0 = a, \quad y_0 = b, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}.$$

Here it is easy to find an *invariant*. From $x_{n+1}y_{n+1} = x_n y_n$, for all n we deduce $x_n y_n = ab$ for all n. This is the *invariant* we are looking for. Initially, we have $y_0 < x_0$. This relation also remains invariant. Indeed, suppose $y_n < x_n$ for some n. Then x_{n+1} is the midpoint of the segment with endpoints y_n, x_n. Moreover, $y_{n+1} < x_{n+1}$ since the harmonic mean is strictly less than the arithmetic mean. Thus,

$$0 < x_{n+1} - y_{n+1} = \frac{x_n - y_n}{x_n + y_n} \cdot \frac{x_n - y_n}{2} < \frac{x_n - y_n}{2}$$

for all n. So we have $\lim x_n = \lim y_n = x$ with $x^2 = ab$ or $x = \sqrt{ab}$.

Here the invariant helped us very much, but its recognition was not yet the solution, although the completion of the solution was trivial.

E2. *Suppose the positive integer n is odd. First Al writes the numbers $1, 2, \ldots, 2n$ on the blackboard. Then he picks any two numbers a, b, erases them, and writes, instead, $|a - b|$. Prove that an odd number will remain at the end.*

Solution. Suppose S is the sum of all the numbers still on the blackboard. Initially this sum is $S = 1+2+\cdots+2n = n(2n+1)$, an odd number. Each step reduces S by $2\min(a, b)$, which is an even number. So the parity of S is an *invariant*. During the whole reduction process we have $S \equiv 1 \bmod 2$. Initially the parity is odd. So, it will also be odd at the end.

E3. *A circle is divided into six sectors. Then the numbers $1, 0, 1, 0, 0, 0$ are written into the sectors (counterclockwise, say). You may increase two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?*

Solution. Suppose $a_1, \ldots, a_6$ are the numbers currently on the sectors. Then $I = a_1 - a_2 + a_3 - a_4 + a_5 - a_6$ is an *invariant*. Initially $I = 2$. The goal $I = 0$ cannot be reached.

E4. *In the Parliament of Sikinia, each member has* **at most three enemies.** *Prove that the house can be separated into two houses, so that each member has* **at most one enemy** *in his own house.*

Solution. Initially, we separate the members in any way into the two houses. Let H be the total sum of all the enemies each member has in his own house. Now suppose A has at least two enemies in his own house. Then he has at most one enemy in the other house. If A switches houses, the number H will decrease. This decrease cannot go on forever. At some time, H reaches its absolute minimum. Then we have reached the required distribution.

Here we have a new idea. We construct a positive integral function which decreases at each step of the algorithm. So we know that our algorithm will terminate. There is no strictly decreasing infinite sequence of positive integers. H is not strictly an invariant, but decreases monotonically until it becomes constant. Here, the monotonicity relation is the invariant.

E5. *Suppose not all four integers a, b, c, d are equal. Start with (a, b, c, d) and repeatedly replace (a, b, c, d) by $(a - b, b - c, c - d, d - a)$. Then at least one number of the quadruple will eventually become arbitrarily large.*

Solution. Let $P_n = (a_n, b_n, c_n, d_n)$ be the quadruple after n iterations. Then we have $a_n + b_n + c_n + d_n = 0$ for $n \geq 1$. We do not see yet how to use this invariant. But geometric interpretation is mostly helpful. A very important function for the point P_n in 4-space is the square of its distance from the origin $(0, 0, 0, 0)$, which is $a_n^2 + b_n^2 + c_n^2 + d_n^2$. If we could prove that it has no upper bound, we would be finished.

We try to find a relation between P_{n+1} and P_n:

$$\begin{aligned} a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2 &= (a_n - b_n)^2 + (b_n - c_n)^2 + (c_n - d_n)^2 + (d_n - a_n)^2 \\ &= 2(a_n^2 + b_n^2 + c_n^2 + d_n^2) \\ &\quad - 2a_nb_n - 2b_nc_n - 2c_nd_n - 2d_na_n. \end{aligned}$$

Now we can use $a_n + b_n + c_n + d_n = 0$ or rather its square:

$$0 = (a_n+b_n+c_n+d_n)^2 = (a_n+c_n)^2+(b_n+d_n)^2+2a_nb_n+2a_nd_n+2b_nc_n+2c_nd_n. \tag{1}$$

Adding (1) and (2), for $a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + d_{n+1}^2$, we get

$$2(a_n^2 + b_n^2 + c_n^2 + d_n^2) + (a_n + c_n)^2 + (b_n + d_n)^2 \geq 2(a_n^2 + b_n^2 + c_n^2 + d_n^2).$$

From this invariant inequality relationship we conclude that, for $n \geq 2$,

$$a_n^2 + b_n^2 + c_n^2 + d_n^2 \geq 2^{n-1}(a_1^2 + b_1^2 + c_1^2 + d_1^2). \tag{2}$$

The distance of the points P_n from the origin increases without bound, which means that at least one component must become arbitrarily large. Can you always have equality in (2)?

Here we learned that the distance from the origin is a very important function. Each time you have a sequence of points you should consider it.

E6. *An algorithm is defined as follows:*

Start: (x_0, y_0) *with* $0 < x_0 < y_0$.

Step: $x_{n+1} = \dfrac{x_n + y_n}{2}, \qquad y_{n+1} = \sqrt{x_{n+1}y_n}.$

Figure 1.1 and the arithmetic mean-geometric mean inequality show that

$$x_n < y_n \Rightarrow x_{n+1} < y_{n+1}, \quad y_{n+1} - x_{n+1} < \frac{y_n - x_n}{4}$$

for all n. Find the common limit $\lim x_n = \lim y_n = x = y$.

Here, invariants can help. But there are no systematic methods to find invariants, just *heuristics*. These are methods which often work, but not always. Two of these heuristics tell us to look for the change in x_n/y_n or $y_n - x_n$ when going from n to $n+1$.

(a)
$$\frac{x_{n+1}}{y_{n+1}} = \frac{x_{n+1}}{\sqrt{x_{n+1} y_n}} = \sqrt{\frac{x_{n+1}}{y_n}} = \sqrt{\frac{1 + x_n/y_n}{2}}. \tag{1}$$

This reminds us of the half-angle relation

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1 + \cos\alpha}{2}}.$$

Since we always have $0 < x_n/y_n < 1$, we may set $x_n/y_n = \cos\alpha_n$. Then (1) becomes

$$\cos\alpha_{n+1} = \cos\frac{\alpha_n}{2} \Rightarrow \alpha_n = \frac{\alpha_0}{2^n} \Rightarrow 2^n \alpha_n = \alpha_0,$$

which is equivalent to

$$2^n \arccos\frac{x_n}{y_n} = \arccos\frac{x_0}{y_0}. \tag{2}$$

This is an *invariant!*

(b) To avoid square roots, we consider $y_n^2 - x_n^2$ instead of $y_n - x_n$ and get

$$y_{n+1}^2 - x_{n+1}^2 = \frac{y_n^2 - x_n^2}{4} \Rightarrow 2\sqrt{y_{n+1}^2 - x_{n+1}^2} = \sqrt{y_n^2 - x_n^2}$$

or

$$2^n \sqrt{y_n^2 - x_n^2} = \sqrt{y_0^2 - x_0^2}, \tag{3}$$

which is a second *invariant*.

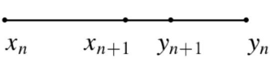


Fig. 1.1

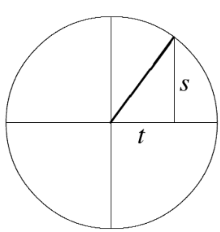


Fig. 1.2. $\arccos t = \arcsin s,\ s = \sqrt{1 - t^2}$.

From Fig. 1.2 and (2), (3), we get

$$\arccos\frac{x_0}{y_0} = 2^n \arccos\frac{x_n}{y_n} = 2^n \arcsin\frac{\sqrt{y_n^2 - x_n^2}}{y_n} = 2^n \arcsin\frac{\sqrt{y_0^2 - x_0^2}}{2^n y_n}.$$

The right-hand side converges to $\sqrt{y_0^2 - x_0^2}/y$ for $n \to \infty$. Finally, we get

$$x = y = \frac{\sqrt{y_0^2 - x_0^2}}{\arccos(x_0/y_0)}. \tag{4}$$

It would be pretty hopeless to solve this problem without invariants. By the way, this is a hard problem by any competition standard.

E7. *Each of the numbers $a_1, \ldots, a_n$ is 1 or -1, and we have*

$$S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0.$$

Prove that $4 \mid n$.

Solution. This is a number theoretic problem, but it can also be solved by invariance. If we replace any a_i by $-a_i$, then S does not change mod 4 since four cyclically adjacent terms change their sign. Indeed, if two of these terms are positive and two negative, nothing changes. If one or three have the same sign, S changes by ± 4. Finally, if all four are of the same sign, then S changes by ± 8.

Initially, we have $S = 0$ which implies $S \equiv 0 \bmod 4$. Now, step-by-step, we change each negative sign into a positive sign. This does not change S mod 4. At the end, we still have $S \equiv 0 \bmod 4$, but also S$= n$, i.e, $4|n$.

E8. *$2n$ ambassadors are invited to a banquet. Every ambassador has at most $n - 1$ enemies. Prove that the ambassadors can be seated around a round table, so that nobody sits next to an enemy.*

Solution. First, we seat the ambassadors in any way. Let H be the number of neighboring hostile couples. We must find an algorithm which reduces this number whenever $H > 0$. Let (A, B) be a hostile couple with B sitting to the right of A (Fig. 1.3). We must separate them so as to cause as little disturbance as possible. This will be achieved if we reverse some arc BA' getting Fig. 1.4. H will be reduced if (A, A') and (B, B') in Fig. 1.4 are friendly couples. It remains to be shown that such a couple always exists with B' sitting to the right of A'. We start in A and go around the table counterclockwise. We will encounter at least n friends of A. To their right, there are at least n seats. They cannot all be occupied by enemies of B since B has at most $n - 1$ enemies. Thus, there is a friend A' of A with right neighbor B', a friend of B.

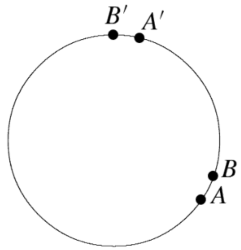


Fig. 1.3. Invert arc $A'B$.

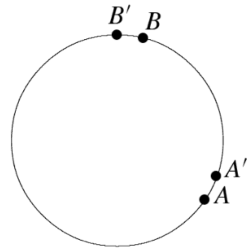


Fig. 1.4

Remark. This problem is similar to **E4**, but considerably harder. It is the following theorem in graph theory: *Let G be a linear graph with n vertices. Then G has a Hamiltonian path if the sum of the degrees of any two vertices is equal to or larger than* $n - 1$. In our special case, we have proved that there is even a Hamiltonian circuit.

E9. *To each vertex of a pentagon, we assign an integer* x_i *with sum* $s = \sum x_i > 0$. *If* x, y, z *are the numbers assigned to three successive vertices and if* $y < 0$, *then we replace* (x, y, z) *by* $(x + y, -y, y + z)$. *This step is repeated as long as there is a* $y < 0$. *Decide if the algorithm always stops.* (Most difficult problem of IMO 1986.)

Solution. The algorithm always stops. The key to the proof is (as in Examples 4 and 8) to find an integer-valued, nonnegative function $f(x_1, \ldots, x_5)$ of the vertex labels whose value decreases when the given operation is performed. All but one of the eleven students who solved the problem found the same function

$$f(x_1, x_2, x_3, x_4, x_5) = \sum_{i=1}^{5} (x_i - x_{i+2})^2, \quad x_6 = x_1, \quad x_7 = x_2.$$

Suppose $y = x_4 < 0$. Then $f_{new} - f_{old} = 2sx_4 < 0$, since $s > 0$. If the algorithm does not stop, we can find an infinite decreasing sequence $f_0 > f_1 > f_2 > \cdots$ of nonnegative integers. Such a sequence does not exist.

Bernard Chazelle (Princeton) asked: How many steps are needed until stop? He considered the infinite multiset S of all sums defined by $s(i, j) = x_i + \cdots + x_{j-1}$ with $1 \le i \le 5$ and $j > i$. A multiset is a set which can have equal elements. In this set, all elements but one either remain invariant or are switched with others. Only $s(4, 5) = x_4$ changes to $-x_4$. Thus, exactly one negative element of S changes to positive at each step. There are only finitely many negative elements in S, since $s > 0$. The number of steps until stop is equal to the number of negative elements of S. We see that the x_i need not be integers.

Remark. It is interesting to find a formula with the computer, which, for input a, b, c, d, e, gives the number of steps until stop. This can be done without much effort if $s = 1$. For instance, the input $(n, n, 1 - 4n, n, n)$ gives the step number $f(n) = 20n - 10$.

E10. Shrinking squares. An empirical exploration. *Start with a sequence* $S = (a, b, c, d)$ *of positive integers and find the derived sequence* $S_1 = T(S) = (|a-b|, |b-c|, |c-d|, |d-a|)$. *Does the sequence* S, S_1, $S_2 = T(S_1)$, $S_3 = T(S_2)$, ... *always end up with* $(0, 0, 0, 0)$?

Let us collect material for solution hints:

$$(0, 3, 10, 13) \mapsto (3, 7, 3, 13) \mapsto (4, 4, 10, 10) \mapsto (0, 6, 0, 6) \mapsto (6, 6, 6, 6) \mapsto (0, 0, 0, 0),$$

$$(8, 17, 3, 107) \mapsto (9, 14, 104, 99) \mapsto (5, 90, 5, 90) \mapsto (85, 85, 85, 85) \mapsto (0, 0, 0, 0),$$

$$(91, 108, 95, 294) \mapsto (17, 13, 99, 203) \mapsto (4, 86, 104, 186) \mapsto (82, 18, 82, 182) \mapsto (64, 64, 100, 100) \mapsto (0, 36, 0, 36) \mapsto (36, 36, 36, 36) \mapsto (0, 0, 0, 0).$$

Observations:

1. Let $\max S$ be the maximal element of S. Then $\max S_{i+1} \leq \max S_i$, and $\max S_{i+4} < \max S_i$ as long as $\max S_i > 0$. Verify these observations. This gives a proof of our conjecture.

2. S and tS have the same life expectancy.

3. After four steps at most, all four terms of the sequence become even. Indeed, it is sufficient to calculate modulo 2. Because of cyclic symmetry, we need to test just six sequences $0001 \mapsto 0011 \mapsto 0101 \mapsto 1111 \mapsto 0000$ and $1110 \mapsto 0011$. Thus, we have proved our conjecture. After four steps at most, each term is divisible by 2, after 8 steps at most, by $2^2, \ldots,$ after $4k$ steps at most, by 2^k. As soon as $\max S < 2^k$, all terms must be 0.

In observation 1, we used another strategy, the **Extremal Principle: Pick the maximal element!** Chapter 3 is devoted to this principle.

In observation 3, we used **symmetry.** You should always think of this strategy, although we did not devote a chapter to this idea.

Generalizations:

(a) Start with four real numbers, e.g.,

$\sqrt{2}$	π	$\sqrt{3}$	e
$\pi - \sqrt{2}$	$\pi - \sqrt{3}$	$e - \sqrt{3}$	$e - \sqrt{2}$
$\sqrt{3} - \sqrt{2}$	$\pi - e$	$\sqrt{3} - \sqrt{2}$	$\pi - e$
$\pi - e - \sqrt{3} + \sqrt{2}$	$\pi - e - \sqrt{3} + \sqrt{2}$	$\pi - e - \sqrt{3} + \sqrt{2}$	$\pi - e - \sqrt{3} + \sqrt{2}$
0	0	0	0.

Some more trials suggest that, even for all nonnegative real quadruples, we always end up with $(0, 0, 0, 0)$. But with $t > 1$ and $S = (1, t, t^2, t^3)$ we have

$$T(S) = [t - 1, (t - 1)t, (t - 1)t^2, (t - 1)(t^2 + t + 1)].$$

If $t^3 = t^2 + t + 1$, i.e., $t = 1.8392867552\ldots$, then the process never stops because of the second observation. This t is unique up to a transformation $f(t) = at + b$.

(b) Start with $S = (a_0, a_1, \ldots, a_{n-1})$, a_i nonnegative integers. For $n = 2$, we reach $(0, 0)$ after 2 steps at most. For $n = 3$, we get, for 011, a pure cycle of length 3: $011 \mapsto 101 \mapsto 110 \mapsto 011$. For $n = 5$ we get $00011 \mapsto 00101 \mapsto 01111 \mapsto 10001 \mapsto 10010 \mapsto 10111 \mapsto 11000 \mapsto 01001 \mapsto 11011 \mapsto 01100 \mapsto 10100 \mapsto 11101 \mapsto 00110 \mapsto 01010 \mapsto 11110 \mapsto 00011$, which has a pure cycle of length 15.

1. Find the periods for $n = 6$ $(n = 7)$ starting with 000011 (0000011).
2. Prove that, for $n = 8$, the algorithm stops starting with 00000011.
3. Prove that, for $n = 2^r$, we always reach $(0, 0, \ldots, 0)$, and, for $n \neq 2^r$, we get (up to some exceptions) a cycle containing just two numbers: 0 and evenly often some number $a > 0$. Because of observation 2, we may assume that $a = 1$. Then $| a - b | = a + b \bmod 2$, and we do our calculations in GF(2), i.e., the finite field with two elements 0 and 1.
4. Let $n \neq 2^r$ and $c(n)$ be the cycle length. Prove that $c(2n) = 2c(n)$ (up to some exceptions).
5. Prove that, for odd n, $S = (0, 0, \ldots, 1, 1)$ always lies on a cycle.
6. *Algebraization.* To the sequence $(a_0, \ldots, a_{n-1})$, we assign the polynomial $p(x) = a_{n-1} + \cdots + a_0 x^{n-1}$ with coefficients from GF(2), and $x^n = 1$. The polynomial $(1 + x)p(x)$ belongs to $T(S)$. Use this algebraization if you can.
7. The following table was generated by means of a computer. Guess as many properties of $c(n)$ as you can, and prove those you can.

n	3	5	7	9	11	13	15	17	19	21	23	25
$c(n)$	3	15	7	63	341	819	15	255	9709	63	2047	25575

n	27	29	31	33	35	37	39	41	43
$c(n)$	13797	47507	31	1023	4095	3233097	4095	41943	5461

Problems

1. Start with the positive integers $1, \ldots, 4n - 1$. In one move you may replace any two integers by their difference. Prove that an even integer will be left after $4n - 2$ steps.

2. Start with the set $\{3, 4, 12\}$. In each step you may choose two of the numbers a, b and replace them by $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you reach the goal (a) or (b) in finitely many steps:

   (a) $\{4, 6, 12\}$, (b) $\{x, y, z\}$ with $|x - 4|, |y - 6|, |z - 12|$ each less than $1/\sqrt{3}$?

3. Assume an 8×8 chessboard with the usual coloring. You may repaint all squares (a) of a row or column (b) of a 2×2 square. The goal is to attain just one black square. Can you reach the goal?

4. We start with the state (a, b) where a, b are positive integers. To this initial state we apply the following algorithm:

   while $a > 0$, **do if** $a < b$ **then** $(a, b) \leftarrow (2a, b - a)$ **else** $(a, b) \leftarrow (a - b, 2b)$.

   For which starting positions does the algorithm stop? In how many steps does it stop, if it stops? What can you tell about periods and tails?

   The same questions, when a, b are positive reals.

5. Around a circle, 5 ones and 4 zeros are arranged in any order. Then between any two equal digits, you write 0 and between different digits 1. Finally, the original digits are wiped out. If this process is repeated indefinitely, you can never get 9 zeros. Generalize!

6. There are a white, b black, and c red chips on a table. In one step, you may choose two chips of different colors and replace them by a chip of the third color. If just one chip will remain at the end, its color will not depend on the evolution of the game. When can this final state be reached?

7. There are a white, b black, and c red chips on a table. In one step, you may choose two chips of different colors and replace each one by a chip of the third color. Find conditions for all chips to become of the same color. Suppose you have initially 13 white 15 black and 17 red chips. Can all chips become of the same color? What states can be reached from these numbers?

8. There is a positive integer in each square of a rectangular table. In each move, you may double each number in a row or subtract 1 from each number of a column. Prove that you can reach a table of zeros by a sequence of these permitted moves.

9. Each of the numbers 1 to 10^6 is repeatedly replaced by its digital sum until we reach 10^6 one-digit numbers. Will these have more 1's or 2's?

10. The vertices of an n-gon are labeled by real numbers $x_1, \ldots, x_n$. Let a, b, c, d be four successive labels. If $(a - d)(b - c) < 0$, then we may switch b with c. Decide if this switching operation can be performed infinitely often.

11. In Fig. 1.5, you may switch the signs of all numbers of a row, column, or a parallel to one of the diagonals. In particular, you may switch the sign of each corner square. Prove that at least one -1 will remain in the table.

1	1	1	1
1	1	1	1
1	1	1	1
1	-1	1	1

Fig. 1.5

12. There is a row of 1000 integers. There is a second row below, which is constructed as follows. Under each number a of the first row, there is a positive integer $f(a)$ such that $f(a)$ equals the number of occurrences of a in the first row. In the same way, we get the 3rd row from the 2nd row, and so on. Prove that, finally, one of the rows is identical to the next row.
13. There is an integer in each square of an 8×8 chessboard. In one move, you may choose any 4×4 or 3×3 square and add 1 to each integer of the chosen square. Can you always get a table with each entry divisible by (a) 2, (b) 3?
14. We strike the first digit of the number 7^{1996}, and then add it to the remaining number. This is repeated until a number with 10 digits remains. Prove that this number has two equal digits.
15. There is a checker at point $(1, 1)$ of the lattice (x, y) with x, y positive integers. It moves as follows. At any move it may double one coordinate, or it may subtract the smaller coordinate from the larger . Which points of the lattice can the checker reach?
16. Each term in a sequence $1, 0, 1, 0, 1, 0, \ldots$ starting with the seventh is the sum of the last 6 terms mod 10. Prove that the sequence $\ldots, 0, 1, 0, 1, 0, 1, \ldots$ never occurs.
17. Starting with any 35 integers, you may select 23 of them and add 1 to each. By repeating this step, one can make all 35 integers equal. Prove this. Now replace 35 and 23 by m and n, respectively. What condition must m and n satisfy to make the equalization still possible?
18. The integers $1, \ldots, 2n$ are arranged in any order on $2n$ places numbered $1, \ldots, 2n$. Now we add its place number to each integer. Prove that there are two among the sums which have the same remainder mod $2n$.
19. The n holes of a socket are arranged along a circle at equal (unit) distances and numbered $1, \ldots, n$. For what n can the prongs of a plug fitting the socket be numbered such that at least one prong in each plug-in goes into a hole of the same number (good numbering)?
20. A game for computing $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$.
    We start with $x = a$, $y = b$, $u = a$, $v = b$ and move as follows:
    if $x < y$ then, set $y \leftarrow y - x$ and $v \leftarrow v + u$
    if $x > y$, then set $x \leftarrow x - y$ and $u \leftarrow u + v$
    The game ends with $x = y = \gcd(a, b)$ and $(u + v)/2 = \operatorname{lcm}(a, b)$. Show this.
21. Three integers a, b, c are written on a blackboard. Then one of the integers is erased and replaced by the sum of the other two diminished by 1. This operation is repeated many times with the final result 17, 1967, 1983. Could the initial numbers be (a) 2, 2, 2 (b) 3, 3, 3?
22. There is a chip on each dot in Fig. 1.6. In one move, you may simultaneously move any two chips by one place in opposite directions. The goal is to get all chips into one dot. When can this goal be reached?

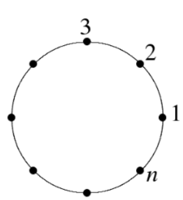


Fig. 1.6

23. Start with n pairwise different integers $x_1, x_2, \dots, x_n$, $(n > 2)$ and repeat the following step:

$$T : (x_1, \dots, x_n) \mapsto \left(\frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, \dots, \frac{x_n + x_1}{2} \right).$$

Show that $T, T^2, \dots$ finally leads to nonintegral components.

24. Start with an $m \times n$ table of integers. In one step, you may change the sign of all numbers in any row or column. Show that you can achieve a nonnegative sum of any row or column. (Construct an integral function which increases at each step, but is bounded above. Then it must become constant at some step, reaching its maximum.)

25. Assume a convex $2m$-gon $A_1, \dots, A_{2m}$. In its interior we choose a point P, which does not lie on any diagonal. Show that P lies inside an even number of triangles with vertices among $A_1, \dots, A_{2m}$.

26. Three automata I, H, T print pairs of positive integers on tickets. For input (a, b), I and H give $(a + 1, b + 1)$ and $(a/2, b/2)$, respectively. H accepts only even a, b. T needs two pairs (a, b) and (b, c) as input and yields output (a, c). Starting with $(5, 19)$ can you reach the ticket (a) $(1, 50)$ (b) $(1, 100)$? Initially, we have (a, b), $a < b$. For what n is $(1, n)$ reachable?

27. Three automata I, R, S print pairs of positive integers on tickets. For entry (x, y), the automata I, R, S give tickets $(x - y, y)$, $(x + y, y)$, (y, x), respectively, as outputs. Initially, we have the ticket $(1, 2)$. With these automata, can I get the tickets (a) $(19, 79)$ (b) $(819, 357)$? Find an invariant. What pairs (p, q) can I get starting with (a, b)? Via which pair should I best go?

28. n numbers are written on a blackboard. In one step you may erase any two of the numbers, say a and b, and write, instead $(a + b)/4$. Repeating this step $n - 1$ times, there is one number left. Prove that, initially, if there were n ones on the board, at the end, a number, which is not less than $1/n$ will remain.

29. The following operation is performed with a nonconvex non-self-intersecting polygon P. Let A, B be two nonneighboring vertices. Suppose P lies on the same side of AB. Reflect one part of the polygon connecting A with B at the midpoint O of AB. Prove that the polygon becomes convex after finitely many such reflections.

30. Solve the equation $(x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = x$.

31. Let $a_1, a_2, \dots, a_n$ be a permutation of $1, 2, \dots, n$. If n is odd, then the product $P = (a_1 - 1)(a_2 - 2) \dots (a_n - n)$ is even. Prove this.

32. Many handshakes are exchanged at a big international congress. We call a person an *odd person* if he has exchanged an odd number of handshakes. Otherwise he will be called an *even person.* Show that, at any moment, there is an even number of odd persons.

33. Start with two points on a line labeled $0,\ 1$ in that order. In one move you may add or delete two *neighboring* points $(0,\ 0)$ or $(1,\ 1)$. Your goal is to reach a single pair of points labeled $(1,\ 0)$ in that order. Can you reach this goal?

34. Is it possible to transform $f(x) = x^2 + 4x + 3$ into $g(x) = x^2 + 10x + 9$ by a sequence of transformations of the form

$$f(x) \mapsto x^2 f(1/x + 1) \quad \text{or} \quad f(x) \mapsto (x - 1)^2 f[1/(x - 1)]?$$

35. Does the sequence of squares contain an infinite arithmetic subsequence?

36. The integers $1, \ldots, n$ are arranged in any order. In one step you may switch any two neighboring integers. Prove that you can never reach the initial order after an odd number of steps.

37. One step in the preceding problem consists of an interchange of any two integers. Prove that the assertion is still true.

38. The integers $1, \ldots, n$ are arranged in order. In one step you may take any four integers and interchange the first with the fourth and the second with the third. Prove that, if $n(n-1)/2$ is even, then by means of such steps you may reach the arrangement $n, n-1, \ldots, 1$. But if $n(n-1)/2$ is odd, you cannot reach this arrangement.

39. Consider all lattice squares (x, y) with x, y nonnegative integers. Assign to each its lower left corner as a label. We shade the squares $(0, 0)$, $(1, 0)$, $(0, 1)$, $(2, 0)$, $(1, 1)$, $(0, 2)$. (a) There is a chip on each of the six squares (b) There is only one chip on $(0, 0)$.

    Step: If (x, y) is occupied, but $(x+1, y)$ and $(x, y+1)$ are free, you may remove the chip from (x, y) and place a chip on each of $(x+1, y)$ and $(x, y+1)$. The goal is to remove the chips from the shaded squares. Is this possible in the cases (a) or (b)? (Kontsevich, TT 1981.)

40. In any way you please, fill up the lattice points below or on the x-axis by chips. By solitaire jumps try to get one chip to $(0, 5)$ with all other chips cleared off. (J. H. Conway.) The preceding problem of Kontsevich might have been suggested by this problem.

    A solitaire jump is a horizontal or vertical jump of any chip over its neighbor to a free point with the chip jumped over removed. For instance, with (x, y) and $(x, y+1)$ occupied and $(x, y+2)$ free, a jump consists in removing the two chips on (x, y) and $(x, y+1)$ and placing a chip onto $(x, y+2)$.

41. We may extend a set S of space points by reflecting any point X of S at any space point A, $A \neq X$. Initially, S consists of the 7 vertices of a cube. Can you ever get the eight vertex of the cube into S?

42. The following game is played on an infinite chessboard. Initially, each cell of an $n \times n$ square is occupied by a chip. A move consists in a jump of a chip over a chip in a horizontal or vertical direction onto a free cell directly behind it. The chip jumped over is removed. Find all values of n, for which the game ends with one chip left over (IMO 1993 and AUO 1992!).

43. Nine 1×1 cells of a 10×10 square are infected. In one time unit, the cells with at least two infected neighbors (having a common side) become infected. Can the infection spread to the whole square?

44. Can you get the polynomial $h(x) = x$ from the polynomials $f(x)$ and $g(x)$ by the operations **addition, subtraction, multiplication** if
    (a) $f(x) = x^2 + x$, $g(x) = x^2 + 2$; (b) $f(x) = 2x^2 + x$, $g(x) = 2x$;
    (c) $f(x) = x^2 + x$, $g(x) = x^2 - 2$?

45. **Accumulation of your computer rounding errors.** Start with $x_0 = 1$, $y_0 = 0$, and, with your computer, generate the sequences

$$x_{n+1} = \frac{5x_n - 12y_n}{13}, \quad y_{n+1} = \frac{12x_n + 5y_n}{13}.$$

Find $x_n^2 + y_n^2$ for $n = 10^2, 10^3, 10^4, 10^5, 10^6$, and 10^7.

46. Start with two numbers 18 and 19 on the blackboard. In one step you may add another number equal to the sum of two preceding numbers. Can you reach the number 1994 (IIM)?

47. In a regular (a) pentagon (b) hexagon all diagonals are drawn. Initially each vertex and each point of intersection of the diagonals is labeled by the number 1. In one step it is permitted to change the signs of all numbers of a side or diagonal. Is it possible to change the signs of all labels to -1 by a sequence of steps (IIM)?

48. In Fig. 1.7, two squares are neighbors if they have a common boundary. Consider the following operation T: Choose any two neighboring numbers and add the same integer to them. Can you transform Fig. 1.7 into Fig. 1.8 by iteration of T?

1	2	3
4	5	6
7	8	9

Fig. 1.7

7	8	9
6	2	4
3	5	1

Fig. 1.8

49. There are several signs + and − on a blackboard. You may erase two signs and write, instead, + if they are equal and − if they are unequal. Then, the last sign on the board does not depend on the order of erasure.

50. There are several letters e, a and b on a blackboard. We may replace two $e's$ by one e, two $a's$ by one b, two $b's$ by one a, an a and a b by one e, an a and an e by one a, a b, and an e by one b. Prove that the last letter does not depend on the order of erasure.

51. A dragon has 100 heads. A knight can cut off 15, 17, 20, or 5 heads, respectively, with one blow of his sword. In each of these cases, 24, 2, 14, or 17 new heads grow on its shoulders. If all heads are blown off, the dragon dies. Can the dragon ever die?

52. Is it possible to arrange the integers $1, 1, 2, 2, \ldots, 1998, 1998$ such that there are exactly $i - 1$ other numbers between any two $i's$?

53. The following operations are permitted with the quadratic polynomial $ax^2 + bx + c$: (a) switch a and c, (b) replace x by $x + t$ where t is any real. By repeating these operations, can you transform $x^2 - x - 2$ into $x^2 - x - 1$?

54. Initially, we have three piles with a, b, and c chips, respectively. In one step, you may transfer one chip from any pile with x chips onto any other pile with y chips. Let $d = y - x + 1$. If $d > 0$, the bank pays you d dollars. If $d < 0$, you pay the bank $|d|$ dollars. Repeating this step several times you observe that the original distribution of chips has been restored. What maximum amount can you have gained at this stage?

55. Let $d(n)$ be the digital sum of $n \in \mathbb{N}$. Solve $n + d(n) + d(d(n)) = 1997$.

56. Start with four congruent right triangles. In one step you may take any triangle and cut it in two with the altitude from the right angle. Prove that you can never get rid of congruent triangles (MMO 1995).

57. Starting with a point $S(a, b)$ of the plane with $0 < a < b$, we generate a sequence (x_n, y_n) of points according to the rule

$$x_0 = a, \quad y_0 = b, \quad x_{n+1} = \sqrt{x_n y_{n+1}}, \quad y_{n+1} = \sqrt{x_n y_n}.$$

Prove that there is a limiting point with $x = y$. Find this limit.

58. Consider any binary word $W = a_1 a_2 \cdots a_n$. It can be transformed by inserting, deleting or appending any word XXX, X being any binary word. Our goal is to transform W from 01 to 10 by a sequence of such transformations. Can the goal be attained (LMO 1988, oral round)?

59. Seven vertices of a cube are marked by 0 and one by 1. You may repeatedly select an edge and increase by 1 the numbers at the ends of that edge. Your goal is to reach (a) 8 equal numbers, (b) 8 numbers divisible by 3.

60. Start with a point $S(a, b)$ of the plane with $0 < b < a$, and generate a sequence of points $S_n(x_n, y_n)$ according to the rule

$$x_0 = a, \quad y_0 = b, \quad x_{n+1} = \frac{2x_n y_n}{x_n + y_n}, \quad y_{n+1} = \frac{2x_{n+1} y_n}{x_{n+1} + y_n}.$$

Prove that there is a limiting point with $x = y$. Find this limit.

Solutions

1. In one move the number of integers always decreases by one. After $(4n - 2)$ steps, just one integer will be left. Initially, there are $2n$ even integers, which is an even number. If two odd integers are replaced, the number of odd integers decreases by 2. If one of them is odd or both are even, then the number of odd numbers remains the same. Thus, the number of odd integers remains even after each move. Since it is initially even, it will remain even to the end. Hence, one even number will remain.

2. (a) $(0.6a - 0.8b)^2 + (0.8a + 0.6b)^2 = a^2 + b^2$. Since $a^2 + b^2 + c^2 = 3^2 + 4^2 + 12^2 = 13^2$, the point (a, b, c) lies on the sphere around O with radius 13. Because $4^2 + 6^2 + 12^2 = 14^2$, the goal lies on the sphere around O with radius 14. The goal cannot be reached.
(b) $(x - 4)^2 + (y - 6)^2 + (z - 12)^2 < 1$. The goal cannot be reached.
The important invariant, here, is the distance of the point (a, b, c) from O.

3. (a) Repainting a row or column with b black and $8 - b$ white squares, you get $(8 - b)$ black and b white squares. The number of black squares changes by $|(8 - b) - b| = |8 - 2b|$, that is an even number. The parity of the number of black squares does not change. Initially, it was even. So, it always remains even. One black square is unattainable. The reasoning for (b) is similar.

4. Here is a solution valid for natural, rational and irrational numbers. With the invariant $a + b = n$ the algorithm can be reformulated as follows:

If $a < n/2$, replace a by $2a$.

If $a \ge n/2$, replace a by $a - b = a - (n - a) = 2a - n \equiv 2a \pmod{n}$.

Thus, we double a repeatedly modulo n and get the sequence

$$a, 2a, 2^2 a, 2^3 a, \ldots \pmod{n}. \tag{1}$$

Divide a by n in base 2. There are three cases.
(a) The result is terminating: $a/n = 0.d_1 d_2 d_3 \ldots d_k, \quad d_i \in \{0, 1\}$. Then $2^k \equiv 0$

(mod n), but $2^i \not\equiv 0 \pmod{n}$ for $i < k$. Thus, the algorithm stops after exactly k steps.

(b) The result is nonterminating and periodic.

$$a/n = 0.a_1a_2\dots a_pd_1d_2\dots d_kd_1d_2\dots d_k\dots.$$

The algorithm will not stop, but the sequence (1) has period k with tail p.

(c) The result is nonterminating and nonperiodic: $a/n = 0.d_1d_2d_3\dots$. In this case, the algorithm will not stop, and the sequence (1) is not periodic.

5. This is a special case of problem **E10** on *shrinking squares*. Addition is done mod 2: $0+0=1+1=0$, $1+0=0+1=1$. Let $(x_1, x_2, \dots, x_n)$ be the original distribution of zeros and ones around the circle. One step consists of the replacement $(x_1, \dots, x_n) \leftarrow (x_1+x_2, x_2+x_3, \dots, x_n+x_1)$. There are two special distributions $E = (1, 1, \dots, 1)$ and $I = (0, 0, \dots, 0)$. Here, we must **work backwards**. Suppose we finally reach I. Then the preceding state must be E, and before that an alternating n-tuple $(1, 0, 1, 0, \dots)$. Since n is odd such an n-tuple does not exist.

   Now suppose that $n = 2^kq$, q odd. The following iteration

$$(x_1,\dots,x_n) \leftarrow (x_1+x_2, x_2+x_3, \dots + x_n+x_1) \leftarrow (x_1+x_3, x_2+x_4, \dots x_n+x_2)$$
$$\leftarrow (x_1+x_2+x_3+x_4, x_2+x_3+x_4+x_5, \dots) \leftarrow (x_1+x_5, x_2+x_6, \dots) \leftarrow \cdots$$

   shows that, for $q = 1$, the iteration ends up with I. For $q > 1$, we eventually arrive at I iff we ever get q identical blocks of length 2^k, i.e., we have period 2^k. Try to prove this.

   The **problem-solving strategy** of **working backwards** will be treated in Chapter 14.

6. All three numbers a, b, c change their parity in one step. If one of the numbers has different parity from the other two, it will retain this property to the end. This will be the one which remains.

7. (a, b, c) will be transformed into one of the three triples $(a+2, b-1, c-1)$, $(a-1, b+2, c-1)$, $(a-1, b-1, c+2)$. In each case, $I = a - b \bmod 3$ is an invariant. But $b - c = 0 \bmod 3$ and $a - c = 0 \bmod 3$ are also invariant. So $I = 0 \bmod 3$ combined with $a + b + c = 0 \bmod 3$ is the condition for reaching a monochromatic state.

8. If there are numbers equal to 1 in the first column, then we double the corresponding rows and subtract 1 from all elements of the first column. This operation decreases the sum of the numbers in the first column until we get a column of ones, which is changed to a column of zeros by subtracting 1. Then we go to the next column, etc.

9. Consider the remainder mod 9. It is an invariant. Since $10^6 = 1 \bmod 9$ the number of ones is by one more than the number of twos.

10. From $(a-d)(b-c) < 0$, we get $ab + cd < ac + bd$. The switching operation increases the sum S of the products of neighboring terms. In our case $ab + bc + cd$ is replaced by $ac + cb + bd$. Because of $ab + cd < ac + bd$ the sum S increases. But S can take only finitely many values.

11. The product I of the eight boundary squares (except the four corners) is -1 and remains invariant.

12. The numbers starting with the second in each column are an increasing and bounded sequence of integers.

13. (a) Let S be the sum of all numbers except the third and sixth row. $S \bmod 2$ is invariant. If $S \not\equiv 0 \pmod 2$ initially, then odd numbers will remain on the chessboard.

    (b) Let S be the sum of all numbers, except the fourth and eight row. Then $I = S \bmod 3$ is an invariant. If, initially, $I \not\equiv 0 \pmod 3$ there will always be numbers on the chessboard which are not divisible by 3.

14. We have $7^3 = 1 \bmod 9 \Rightarrow 7^{1996} \equiv 7^1 \bmod 9$. This digital sum remains invariant. At the end all digits cannot be distinct, else the digital sum would be $0+1+\cdots+9 = 45$, which is $0 \bmod 9$.

15. The point (x, y) can be reached from $(1, 1)$ iff $\gcd(x, y) = 2^n$, $n \in \mathbb{N}$. The permitted moves either leave $\gcd(x, y)$ invariant or double it.

16. Here, $I(x_1, x_2, \ldots, x_6) = 2x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 + 12x_6 \bmod 10$ is the invariant. Starting with $I(1, 0, 1, 0, 1, 0) = 8$, the goal $I(0, 1, 0, 1, 0, 1) = 4$ cannot be reached.

17. Suppose $\gcd(m, n) = 1$. Then, in Chapter 4, **E5**, we prove that $nx = my + 1$ has a solution with x and y from $\{1, 2, \ldots, m - 1\}$. We rewrite this equation in the form $nx = m(y - 1) + m + 1$. Now we place any m positive integers $x_1, \ldots, x_m$ around a circle assuming that x_1 is the smallest number. We proceed as follows. Go around the circle in blocks of n and increase each number of a block by 1. If you do this n times you get around the circle m times, and, in addition, the first number becomes one more then the others. In this way, $|x_{max} - x_{min}|$ decreases by one. This is repeated each time placing a minimal element in front until the difference between the maximal and minimal element is reduced to zero.

    But if $\gcd(x, y) = d > 1$, then such a reduction is not always possible. Let one of the m numbers be 2 and all the others be 1. Suppose that, applying the same operation k times we get equidistribution of the $(m + 1 + kn)$ units to the m numbers. This means $m + 1 + kn \equiv 0 \bmod m$. But d does not divide $m + kn + 1$ since $d > 1$. Hence m does not divide $m + 1 + kn$. Contradiction!

18. We proceed by contradiction. Suppose all the remainders $0, 1, \ldots, 2n - 1$ occur. The sum of all integers and their place numbers is

$$S_1 = 2\,(1 + 2 + \ldots + 2n) = 2n\,(2n + 1) \equiv 0 \pmod{2n}.$$

    The sum of all remainders is

$$S_2 = 0 + 1 + \ldots + 2n - 1 = n\,(2n - 1) \equiv n \pmod{2n}.$$

    Contradiction!

19. Let the numbering of the prongs be $i_1, i_2, \ldots, i_n$. Clearly $i_1 + \cdot + i_n = n(n+1)/2$. If n is odd, then the numbering $i_j = n + 1 - j$ works. Suppose the numbering is good. The prong and hole with number i_j coincide if the plug is rotated by $i_j - j$ (or $i_j - j + n$) units ahead. This means that $(i_1 - 1) + \cdots + (i_n - n) = 1 + 2 + \cdots n \pmod n$. The LHS is 0. The RHS is $n(n + 1)/2$. This is divisible by n if n is odd.

20. Invariants of this transformation are

$$P : \gcd(x, y) = \gcd(x - y, x) = \gcd(x, y - x),$$

$$Q : xv + yu = 2ab,\ R : x > 0,\ y > 0.$$

P and R are obviously invariant. We show the invariance of Q. Initially, we have $ab + ab = 2ab$, and this is obviously correct. After one step, the left side of Q becomes either $x(v+u)+(y-x)u = xv+yu$ or $(x-y)v+y(u+v) = xv+yu$, that is, the left side of Q does not change. At the end of the game, we have $x = y = \gcd(a, b)$ and

$$x(u + v) = 2ab \to (u + v)/2 = ab/x = ab/\gcd(a, b) = \operatorname{lcm}(a, b).$$

21. Initially, if all components are greater than 1, then they will remain greater than 1. Starting with the second triple the largest component is always the sum of the other two components diminished by 1. If, after some step, we get (a, b, c) with $a \le b \le c$, then $c = a + b - 1$, and a backward step yields the triple $(a, b, b - a + 1)$. Thus, we can retrace the last state (17, 1967, 1983) uniquely until the next to last step: $(17, 1967, 1983) \leftarrow (17, 1967, 1951) \leftarrow (17, 1935, 1951) \leftarrow \cdots \leftarrow (17, 15, 31) \leftarrow (17, 15, 3) \leftarrow (13, 15, 3) \leftarrow \cdots \leftarrow (5, 7, 3) \leftarrow (5, 3, 3)$. The preceding triple should be (1, 3, 3) containing 1, which is impossible. Thus the triple (5, 3, 3) is generated at the first step. We can get from (3, 3, 3) to (5, 3, 3) in one step, but not from (2, 2, 2).

22. Let a_i be the number of chips on the circle #i. We consider the sum $S = \sum ia_i$. Initially, we have $S = \sum i * 1 = n(n + 1)/2$ and, at the end, we must have kn for $k \in \{1, 2, \dots, n\}$. Each move changes S by 0, or n, or $-n$, that is, S is invariant mod n. At the end, $S \equiv 0 \bmod n$. Hence, at the beginning, we must have $S \equiv 0 \bmod n$. This is the case for odd n. Reaching the goal is trivial in the case of an odd n.

23. **Solution 1.** Suppose we get only integer n-tuples from $(x_1, \dots, x_n)$. Then the difference between the maximal and minimal term decreases. Since the difference is integer, from some time on it will be zero. Indeed, if the maximum x occurs k times in a row, then it will become smaller than x after k steps. If the minimum y occurs m times in a row, then it will become larger after m steps. In a finite number of steps, we arrive at an integral n-tuple $(a, a, \dots, a)$. We will show that we cannot get equal numbers from pairwise different numbers. Supppose $z_1, \dots, z_n$ are not all equal, but $(z_1 + z_2)/2 = (z_2 + z_3)/2 = \cdots = (z_n + z_1)/2$. Then $z_1 = z_3 = z_5 = \cdots$ and $z_2 = z_4 = z_6 = \cdots$. If n is odd then all z_i are equal, contradicting our assumption. For even $n = 2k$, we must eliminate the case $(a, b, \dots, a, b)$ with $a \ne b$. Suppose

$$\frac{y_1 + y_2}{2} = \frac{y_3 + y_4}{2} = \cdots = \frac{y_{n-1} + y_n}{2} = a, \quad \frac{y_2 + y_3}{2} = \cdots = \frac{y_n + y_1}{2} = b.$$

But the sums of the left sides of the two equation chains are equal, i.e., $a = b$, that is, we cannot get the n-tuple $(a, b, \dots, a, b)$ with $a \ne b$.

Solution 2. Let $\vec{x} = (x_1, \dots, x_n)$, $T\vec{x} = \vec{y} = (y_1, \dots, y_n)$. With $n + 1 = 1$,

$$\sum_{i=1}^{n} y_i^2 = \frac{1}{4}\sum_{i=1}^{n}(x_i^2 + x_{i+1}^2 + 2x_i x_{i+1}) \le \frac{1}{4}\sum_{i=1}^{n}(x_i^2 + x_{i+1}^2 + x_i^2 + x_{i+1}^2) = \sum_{i=1}^{n} x_i^2.$$

We have equality if and only if $x_i = x_{i+1}$ for all i. Suppose the components remain integers. Then the sum of squares is a strictly decreasing sequence of positive integers until all integers become equal after a finite number of steps. Then we show as in solution 1 that, from unequal numbers, you cannot get only equal numbers in a finite number of steps.

Another Solution Sketch. Try a geometric solution from the fact that the sum of the components is invariant, which means that the centroid of the n points is the same at each step.

24. If you find a negative sum in any row or column, change the signs of all numbers in that row or column. Then the sum of all numbers in the table strictly increases. The sum cannot increase indefinitely. Thus, at the end, all rows and columns will have nonnegative signs.

25. The diagonals partition the interior of the polygon into convex polygons. Consider two neighboring polygons P_1, P_2 having a common side on a diagonal or side XY. Then P_1, P_2 both belong or do not belong to the triangles without the common side XY. Thus, if P goes from P_1 to P_2, the number of triangles changes by $t_1 - t_2$, where t_1 and t_2 are the numbers of vertices of the polygon on the two sides of XY. Since $t_1 + t_2 = 2m + 2$, the number $t_1 - t_2$ is also even.

26. You cannot get rid of an odd divisor of the difference $b - a$, that is, you can reach $(1, 50)$ from $(5, 19)$, but not $(1, 100)$.

27. The three automata leave $\gcd(x, y)$ unchanged. We can reach $(19, 79)$ from $(1, 2)$, but not $(819, 357)$. We can reach (p, q) from (a, b) iff $\gcd(p, q) = \gcd(a, b) = d$. Go from (a, b) down to $(1, d + 1)$, then, up to (p, q).

28. From the inequality $1/a + 1/b \geq 4/(a + b)$ which is equivalent to $(a + b)/2 \geq 2ab/(a + b)$, we conclude that the sum S of the inverses of the numbers does not increase. Initially, we have $S = n$. Hence, at the end, we have $S \leq n$. For the last number $1/S$, we have $1/S \geq 1/n$.

29. The permissible transformations leave the sides of the polygon and their directions *invariant*. Hence, there are only a finite number of polygons. In addition, the area strictly increases after each reflection. So the process is finite.

    Remark. The corresponding problem for line reflections in AB is considerably harder. The theorem is still valid, but the proof is no more elementary. The sides still remain the same, but their direction changes. So the finiteness of the process cannot be easily deduced. (In the case of line reflections, there is a conjecture that $2n$ reflections suffice to reach a convex polygon.)

30. Let $f(x) = x^2 - 3x + 3$. We are asked to solve the equation $f(f(x)) = x$, that is to find the fixed or invariant points of the function $f \circ f$. First, let us look at $f(x) = x$, i.e. the fixed points of f. Every fixed point of f is also a fixed point of $f \circ f$. Indeed,

$$f(x) = x \Rightarrow f(f(x)) = f(x) \Rightarrow f(f(x)) = x.$$

    First, we solve the quadratic $f(x) = x$, or $x^2 - 4x + 3 = 0$ with solutions $x_1 = 3$, $x_2 = 1$. $f[f(x)] = x$ leads to the fourth degree equation $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$, of which we already know two solutions 3 and 1. So the left side is divisible by $x - 3$ and $x - 1$ and, hence, by the product $(x - 3)(x - 1) = x^2 - 4x + 3$. This will be proved in the chapter on polynomials, but the reader may know this from high school. Dividing the left side of the 4th-degree equation by $x^2 - 4x + 3$ we get $x^2 - 2x + 1$. Now $x^2 - 2x + 1 = 0$ is equivalent to $(x - 1)^2 = 0$. So the two other solutions are $x_3 = x_4 = 1$. We get no additional solutions in this case, but usually, the number of solutions is doubled by going from $f[x] = x$ to $f[f(x)] = x$.

31. Suppose the product P is odd. Then, each of its factors must be odd. Consider the sum S of these numbers. Obviously S is odd as an odd number of odd summands. On the other hand, $S = \sum(a_i - i) = \sum a_i - \sum i = 0$, since the a_i are a permutation of the numbers 1 to n. Contradiction!

32. We partition the participants into the set E of even persons and the set O of odd persons. We observe that, during the hand shaking ceremony, the set O cannot change its parity. Indeed, if two odd persons shake hands, O increases by 2. If two even persons shake hands, O decreases by 2, and, if an even and an odd person shake hands, $|O|$ does not change. Since, initially, $|O| = 0$, the parity of the set is preserved.

33. Consider the number U of inversions, computed as follows: Below each 1, write the number of zeros to the right of it, and add up these numbers. Initially $U = 0$. U does not change at all after each move, or it increases or decreases by 2. Thus U always remains even. But we have $U = 1$ for the goal. Thus, the goal cannot be reached.

34. Consider the trinomial $f(x) = ax^2 + bx + c$. It has discriminant $b^2 - 4ac$. The first transformation changes $f(x)$ into $(a + b + c)x^2 + (b + 2a)x + a$ with discriminant $(b + 2a)^2 - 4(a + b + c) \cdot a = b^2 - 4ac$, and, applying the second transformation, we get the trinomial $cx^2 + (b - 2c)x + (a - b + c)$ with discriminant $b^2 - 4ac$. Thus the discriminant remains invariant. But $x^2 + 4x + 3$ has discriminant 4, and $x^2 + 10x + 9$ has discriminant 64. Hence, one cannot get the second trinomial from the first.

35. For three squares in arithmetic progression, we have $a_3^2 - a_2^2 = a_2^2 - a_1^2$ or $(a_3 - a_2)(a_3 + a_2) = (a_2 - a_1)(a_2 + a_1)$. Since $a_2 + a_1 < a_3 + a_2$, we must have $a_2 - a_1 > a_3 - a_2$.

    Suppose that $a_1^2,\ a_2^2,\ a_3^2, \ldots$ is an infinite arithmetic progression. Then

$$a_2 - a_1 > a_3 - a_2 > a_4 - a_3 > \cdots.$$

    This is a contradiction since there is no infinite decreasing sequence of positive integers.

36. Suppose the integers $1, \ldots, n$ are arranged in any order. We will say that the numbers i and k are out of order if the larger of the two is to the left of the smaller. In that case, they form an *inversion*. Prove that interchange of two neighbors changes the parity of the number of inversions.

37. Interchange of any two integers can be replaced by an odd number of interchanges of neighboring integers.

38. The number of inversions in $n, \ldots, 1$ is $n(n-1)/2$. Prove that one step does not change the parity of the inversions. If $n(n-1)/2$ is even, then split the n integers into pairs of neighbors (leaving the middle integer unmatched for odd n). Then form quadruplets from the first, last, second, second from behind, etc.

39. We assign the weight $1/2^{x+y}$ to the square with label (x, y). We observe that the total weight of the squares covered by chips does not change if a chip is replaced by two neighbors. The total weight of the first column is

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2.$$

47. (a) No! The parity of the number of $-1's$ on the perimeter of the pentagon does not change.

(b) No! The product of the nine numbers colored black in Fig. 1.11 does not change.

48. Color the squares alternately black and white as in Fig. 1.12. Let W

10^n	$x_n^2 + y_n^2$
10	1.0000000000
10^2	1.0000000001
10^3	1.0000000007
10^4	1.0000000066
10^5	1.0000000665
10^6	1.0000006660
10^7	1.0000066666

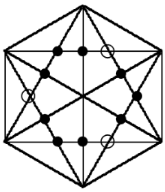


Fig. 1.11

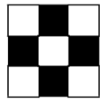


Fig. 1.12

and B be the sums of the numbers on the white and black squares, respectively. Application of T does not change the difference $W - B$. For Fig. 1.7 and Fig. 1.8 the differences are 5 and -1, respectively. The goal -1 cannot be reached from 5.

49. Replace each $+$ by $+1$ and each $-$ by -1, and form the product P of all the numbers. Obviously, P is an invariant.

50. We denote a replacement operation by $\circ$. Then, we have

$$e \circ e = e, \; e \circ a = a, \; e \circ b = b, \; a \circ a = b, \; b \circ b = a, \; a \circ b = e.$$

The $\circ$ operation is commutative since we did not mention the order. It is easy to check that it is also associative, i.e., $(p \circ q) \circ r = p \circ (q \circ r)$ for all letters occurring. Thus, the product of all letters is independent of the the order in which they are multiplied.

51. The number of heads is invariant mod 3. Initially, it is 1 and it remains so.

52. Replace 1998 by n, and derive a necessary condition for the existence of such an arrangement. Let p_k be the position of the first integer k. Then the other k has position $p_k + k$. By counting the position numbers twice, we get $1 + \cdots + 2n = (p_1 + p_1 + 1) + \cdots + (p_n + p_n + n)$. For $P = \sum_{i=1}^{n} p_i$, we get $P = n(3n+1)/4$, and P is an integer for $n \equiv 0$, 1 mod 4. Since $1998 \equiv 2$ mod 4, this necessary condition is not satisfied. Find examples for $n = 4$, 5, and 8.

53. This is an invariance problem. As a prime candidate, we think of the discriminant D. The first operation obviously does not change D. The second operation does not change the difference of the roots of the polynomial. Now, $D = b^2 - 4ac = a^2((b/a)^2 - 4c/a)$, but $-b/a = x_1 + x_2$, and $c/a = x_1 x_2$. Hence, $D = a^2(x_1 - x_2)^2$, i.e., the second operation does not change D. Since the two trinomials have discriminants 9 and 5, the goal cannot be reached.

54. Consider $I = a^2 + b^2 + c^2 - 2g$, where g is the current gain (originally $g = 0$). If we transfer one chip from the first to the second pile, then we get $I' = (a-1)^2 + (b+1)^2 + c^2 - 2g'$ where $g' = g + b - a + 1$, that is, $I' = a^2 - 2a + 1 + b^2 + c^2 + 2b + 1 - 2g - 2b + 2a - 2 = a^2 + b^2 + c^2 - 2g = I$. We see that I does not

change in one step. If we ever get back to the original distribution (a, b, c), then g must be zero again.

The invariant $I = ab + bc + ca + g$ yields another solution. Prove this.

55. The transformation d leaves the remainder on division by 3 invariant. Hence, modulo 3 the equation has the form $0 \equiv 2$. There is no solution.

56. We assume that, at the start, the side lengths are $1, p, q, 1 > p, 1 > q$. Then all succeeding triangles are similar with coefficient $p^m q^n$. By cutting such a triangle of type (m, n), we get two triangles of types $(m + 1, n)$ and $(m, n + 1)$. We make the following translation. Consider the lattice square with nonnegative coordinates. We assign the coordinates of its lower left vertex to each square. Initially, we place four chips on the square $(0, 0)$. Cutting a triangle of type (m, n) is equvalent to replacing a chip on square (m, n) by one chip on square $(m + 1, n)$ and one chip on square $(m, n + 1)$. We assign weight 2^{-m-n} to a chip on square (m, n). Initially, the chips have total weight 4. A move does not change total weight. Now we get problem 39 of Kontsevich. Initially, we have total weight 4. Suppose we can get each chip on a different square. Then the total weight is less than 4. In fact, to get weight 4 we would have to fill the whole plane by single chips. This is impossible in a finite number of steps.

57. Comparing x_{n+1}/x_n with y_{n+1}/y_n, we observe that $x_n^2 y_n = a^2 b$ is an invariant. If we can show that $\lim x_n = \lim y_n = x$, then $x^3 = a^2 b$, or $x = \sqrt[3]{a^2 b}$.

Because of $x_n < y_n$ and the arithmetic mean-geometric mean inequality, y_{n+1} lies to the left of $(x_n + y_n)/2$ and x_{n+1} lies to the left of $(x_n + y_{n+1})/2$. Thus, $x_n < x_{n+1} < y_{n+1} < y_n$ and $y_{n+1} - x_{n+1} < (y_n - x_n)/2$. We have, indeed, a common limit x. Actually for large n, say $n \geq 5$, we have $\sqrt{x_n y_n} \approx (y_n + x_n)/2$ and $y_{n+1} - x_{n+1} \approx (y_n - x_n)/4$.

58. Assign the number $I(W) = a_1 + 2a_2 + 3a_3 + \cdots + na_n$ to W. Deletion or insertion of any word XXX in any place produces $Z = b_1 b_2 \cdots b_m$ with $I(W) \equiv I(Z)$ modulo 3. Since $I(01) = 2$ and $I(10) = 1$, the goal cannot be attained.

59. Select four vertices such that no two are joined by an edge. Let X be the sum of the numbers at these vertices, and let y be the sum of the numbers at the remaining four vertices. Initially, $I = x - y = \pm 1$. A step does not change I. So neither (a) nor (b) can be attained.

60. *Hint*: Consider the sequences $s_n = 1/x_n$, and $t_n = 1/y_n$. An invariant is $s_{n+1} + 2t_{n+1} = s_n + 2t_n = 1/a + 2/b$.

2

Coloring Proofs

The problems of this chapter are concerned with the partitioning of a set into a finite number of subsets. The partitioning is done by *coloring* each element of a subset by the same color. The prototypical example runs as follows.

In 1961, the British theoretical physicist M.E. Fisher solved a famous and very tough problem. He showed that an 8×8 chessboard can be covered by 2×1 dominoes in $2^4 \times 901^2$ or 12,988,816 ways. Now let us cut out two diagonally opposite corners of the board. In how many ways can you cover the 62 squares of the mutilated chessboard with 31 dominoes?

The problem looks even more complicated than the problem solved by Fisher, but this is not so. The problem is trivial. There is no way to cover the mutilated chessboard. Indeed, each domino covers one black and one white square. If a covering of the board existed, it would cover 31 black and 31 white squares. But the mutilated chessboard has 30 squares of one color and 32 squares of the other color.

The following problems are mostly ingenious impossibility proofs based on coloring or parity. Some really belong to Chapter 3 or Chapter 4, but they use coloring, so I put them in this chapter. A few also belong to the closely related Chapter 1. The mutilated chessboard required two colors. The problems of this chapter often require more than two colors.

Problems

1. A rectangular floor is covered by 2×2 and 1×4 tiles. One tile got smashed. There is a tile of the other kind available. Show that the floor cannot be covered by rearranging the tiles.
2. Is it possible to form a rectangle with the five tetrominoes in Fig. 2.1?
3. A 10×10 chessboard cannot be covered by 25 T-tetrominoes in Fig. 2.1. These tiles are called from left to right: straight tetromino, T-tetromino, square tetromino, L-tetromino, and skew tetromino.

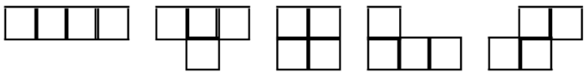


Fig. 2.1

4. An 8×8 chessboard cannot be covered by 15 T-tetrominoes and one square tetromino.
5. A 10×10 board cannot be covered by 25 straight tetrominoes (Fig. 2.1).
6. Consider an $n \times n$ chessboard with the four corners removed. For which values of n can you cover the board with L-tetrominoes as in Fig. 2.2?
7. Is there a way to pack 250 $1 \times 1 \times 4$ bricks into a $10 \times 10 \times 10$ box?
8. An $a \times b$ rectangle can be covered by $1 \times n$ rectangles iff $n|a$ or $n|b$.
9. One corner of a $(2n + 1) \times (2n + 1)$ chessboard is cut off. For which n can you cover the remaining squares by 2×1 dominoes, so that half of the dominoes are horizontal?
10. Fig. 2.3 shows five heavy boxes which can be displaced only by rolling them about one of their edges. Their tops are labeled by the letter ⊤. Fig. 2.4 shows the same five boxes rolled into a new position. Which box in this row was originally at the center of the cross?
11. Fig. 2.5 shows a road map connecting 14 cities. Is there a path passing through each city exactly once?

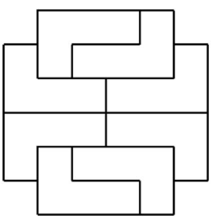


Fig. 2.2

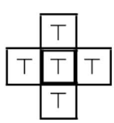


Fig. 2.3

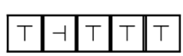


Fig. 2.4

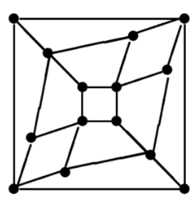


Fig. 2.5

12. A beetle sits on each square of a 9×9 chessboard. At a signal each beetle crawls diagonally onto a neighboring square. Then it may happen that several beetles will sit on some squares and none on others. Find the minimal possible number of free squares.

13. Every point of the plane is colored *red* or *blue*. Show that there exists a rectangle with vertices of the same color. Generalize.

14. Every space point is colored either *red* or *blue*. Show that among the squares with side 1 in this space there is at least one with three red vertices or at least one with four blue vertices.

15. Show that there is no curve which intersects every segment in Fig. 2.6 exactly once.

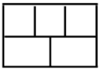


Fig. 2.6

16. On one square of a 5×5 chessboard, we write -1 and on the other 24 squares $+1$. In one move, you may reverse the signs of one $a \times a$ subsquare with $a > 1$. My goal is to reach $+1$ on each square. On which squares should -1 be to reach the goal?

17. The points of a plane are colored *red* or *blue*. Then one of the two colors contains points with any distance.

18. The points of a plane are colored with three colors. Show that there exist two points with distance 1 both having the same color.

19. All vertices of a convex pentagon are lattice points, and its sides have integral length. Show that its perimeter is even.

20. n points ($n \geq 5$) of the plane can be colored by two colors so that no line can separate the points of one color from those of the other color.

21. You have many 1×1 squares. You may color their edges with one of four colors and glue them together along edges of the same color. Your aim is to get an $m \times n$ rectangle. For which m and n is this possible?

22. You have many unit cubes and six colors. You may color each cube with 6 colors and glue together faces of the same color. Your aim is to get a $r \times s \times t$ box, each face having different color. For which r, s, t is this possible?

23. Consider three vertices $A = (0, 0)$, $B = (0, 1)$, $C = (1, 0)$ in a plane lattice. Can you reach the fourth vertex $D = (1, 1)$ of the square by reflections at A, B, C or at points previously reflected?

24. Every space point is colored with exactly one of the colors *red*, *green*, or *blue*. The sets R, G, B consist of the lengths of those segments in space with both endpoints *red*, *green*, and *blue*, respectively. Show that at least one of these sets contains all nonnegative real numbers.

25. *The Art Gallery Problem*. An art gallery has the shape of a simple n-gon. Find the minimum number of watchmen needed to survey the building, no matter how complicated its shape.

26. A 7×7 square is covered by sixteen 3×1 and one 1×1 tiles. What are the permissible positions of the 1×1 tile?

27. The vertices of a regular $2n$-gon $A_1, \ldots, A_{2n}$ are partitioned into n pairs. Prove that, if $n = 4m + 2$ or $n = 4m + 3$, then two pairs of vertices are endpoints of congruent segments.

28. A 6×6 rectangle is tiled by 2×1 dominoes. Then it has always at least one *fault-line*, i.e., a line cutting the rectangle without cutting any domino.

1	2	3	0	1	2	3	0	1	2
0	1	2	3	0	1	2	3	0	1
3	0	1	2	3	0	1	2	3	0
2	3	0	1	2	3	0	1	2	3
1	2	3	0	1	2	3	0	1	2
0	1	2	3	0	1	2	3	0	1
3	0	1	2	3	0	1	2	3	0
2	3	0	1	2	3	0	1	2	3
1	2	3	0	1	2	3	0	1	2
0	1	2	3	0	1	2	3	0	1

Fig. 2.10

7. Assign coordinates (x, y, z) to the cells of the box, $1 \le x, y, z \le 10$. Color the cells in four colors denoted by 0, 1, 2, 3. The cell (x, y, z) is assigned color i if $x + y + z \equiv i \mod 4$. This coloring has the property that a $1 \times 1 \times 4$ brick always occupies one cell of each color no matter how it is placed in the box. Thus, if the box could be filled with two hundred fifty $1 \times 1 \times 4$ bricks, there would have to be 250 cells of each of the colors 0, 1, 2, 3, respectively. Let us see if this necessary packing condition is satisfied. Fig. 2.10 shows the lowest level of cells with the corresponding coloring. There are 26, 25, 24, 25 cells with color 0, 1, 2, 3 respectively. The coloring of the next layer is obtained from that of the preceding layer by adding 1 mod 4. Thus the second layer has 26, 25, 24, 25 cells with colors 1, 2, 3, 0, respectively. The third layer has 26, 25, 24, 25 cells with colors 2, 3, 0, 1, respectively, the fourth layer has 26, 25, 24, 25 cells with colors 3, 0, 1, 2, respectively, and so on. Thus there are $(26 + 25 + 24 + 25) \cdot 2 + 26 + 25 = 251$ cells of color 0. Hence there is no packing of the $10 \times 10 \times 10$ box by $1 \times 1 \times 4$ bricks.
8. If $n|a$ or $n|b$, the board can be covered by $1 \times n$ tiles in an obvious way. Suppose $n \nmid a$, i.e., $a = q \cdot n + r,\ 0 < r < n$. Color the board as indicated in Fig. 2.9. There are $bq + b$ squares of each of the colors $1, 2, \ldots, r$, and there are bq squares of each of the colors $1, \ldots, n$. The h horizontal $1 \times n$ tiles of a covering each cover one square of each color. Each vertical $1 \times n$ tile covers n squares of the same color. After the h horizontal tiles are placed, there will remain $(bq + b - h)$ squares of each of the colors $1, \ldots, r$ and $bq - h$ of each of the colors $r + 1, \ldots, n$. Thus $n|bq + b - h$ and $n|bq - h$. But if n divides two numbers, it also divides their difference: $(bq + b - h) - (bq - h) = b$. Thus, $n|b$. Space analogue: *If an $a \times b \times c$ box can be tiled with $n \times 1 \times 1$ bricks, then $n|a$ or $n|b$ or $n|c$.*

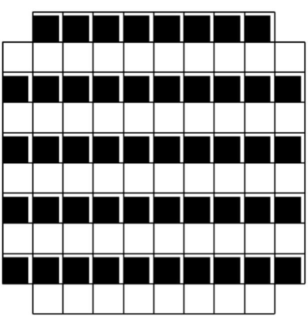


Fig. 2.11

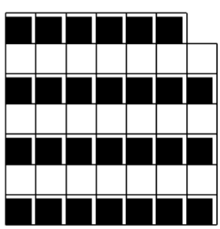


Fig. 2.12

Fig. 2.13

9. Color the board as in Fig. 2.12. There are $2n^2+n$ white squares and $2n^2+3n$ black squares, a total of $4n^2+4n$ squares. $2n^2+2n$ dominoes will be required to cover all of these squares. Since one half of these dominoes are to be horizontal, there will be n^2+n vertical and n^2+n horizontal dominoes. Each vertical domino covers one black and one white square. When all the vertical dominoes are placed, they cover n^2+n white squares and n^2+n black squares. The remaining n^2 white squares and n^2+2n black squares must be covered by horizontal dominoes. A horizontal domino covers only squares of the same color. To cover the n^2 white squares n^2, i.e., n must be even. One easily shows by actual construction that this necessary condition is also sufficient. Thus, the required covering is possible for a $(4n+1)\times(4n+1)$ board and is impossible for a $(4n-1)\times(4n-1)$ board.

10. Suppose the floor is ruled into squares colored black and white like a chessboard. Further suppose that the central box of the cross covers a black square. Then the four other boxes stand on white squares. It is easy to see that the transition $\top \to \top$ requires an even number of flips whereas a transition $\top \to \vdash$ requires an odd number of flips. Hence the boxes #1, 3, 4, 5 in Fig. 2.13 originally stand on squares of the same color. Now the squares occupied by boxes #1, 3, 5 are the same color, and so boxes #1, 3, 5 must have originated on squares of the same color. Since there are not three boxes which originated on black squares, these boxes must stand on white squares. Box #2 must have been flipped an odd number of times. It is now on a black square. Hence it was originally on a white square. Box #4 is now on a black square. Since it was flipped an even number of times, it was originally on a black square. Thus #4 is the central box.

11. Color the cities black and white so that neighboring cities have different colors as shown in Fig. 2.14. Every path through the 14 cities has the color pattern bwbwb-wbwbwbwbw or wbwbwbwbwbwbwb. So it passes through seven black and seven white cities. But the map has six black and eight white cities. Hence, there is no path passing through each city exactly once.

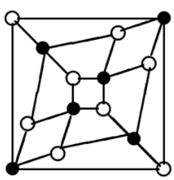


Fig. 2.14

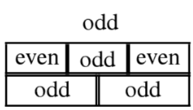


Fig. 2.15

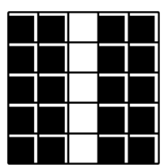


Fig. 2.16

12. Color the columns alternately *black* and *white*. We get 45 black and 36 white squares. Every beetle changes its color by crawling. Hence at least nine black squares remain empty. It is easy to see that exactly nine squares can stay free.

13. Consider the lattice points (x, y) with $1 \le x \le n+1,\ \ 1 \le y \le n^{n+1}+1$. One row can be colored in n^{n+1} ways. By the box principle, at least two of the $(n^{n+1}+1)$ rows have the same coloring. Let two such rows colored the same way have ordinates k and m. For each $i \in \{1, \ldots, n+1\}$, the points (i, k) and (i, m) have the same color. Since there are only n colors available, one of the colors will repeat. Suppose (a, k) and (b, k) have the same color. Then the rectangle with the vertices $(a, k),\ (b, k),\ (b, m),\ (a, m)$ has four vertices of the same color.

The problem can be generalized to parallelograms and to k-dimensional boxes. Instead of the lattice rectangle with sides n and n^{n+1}, we have a lattice box with lengths $d_1 - 1,\ d_2 - 1, \dots d_k - 1$, and

$$d_1 = n + 1, \quad d_{i+1} = n^{d_1 \cdots d_i} + 1.$$

14. Denote by B the property that there is a unit square with four blue vertices.

    Case 1: All points of space are blue $\Rightarrow B$.

    Case 2: There exists a red point P_1. Make of P_1 the vertex of a pyramid with equal edges and the square $P_2P_3P_4P_5$ as base.

    Case 2.1: The four points $P_i,\ i = 2, 3, 4, 5$ are blue $\Rightarrow B$.

    Case 2.2: One of the points $P_i,\ i = 2, 3, 4, 5$ is red, say P_2. Make of P_1P_2 a lateral edge of an equilateral prism, with the remaining vertices $P_6,\ P_7,\ P_8,\ P_9$.

    Case 2.2.1: The four points $P_j,\ j = 6, 7, 8, 9$ are blue$\Rightarrow B$.

    Case 2.2.2: One of the points $P_j,\ j = 6, 7, 8, 9$ is red, say P_6. Then $P_1,\ P_2$, and P_6 are three red vertices of a unit square.

15. The map in Fig. 2.15 consists of three faces each bounded by five segments (labeled *odd*). Suppose there exists a curve intersecting every segment exactly once. Then it would have three points inside the odd faces, where it starts or ends. But a curve has zero or two endpoints.

16. Color the board as in Fig. 2.16. Every permitted subsquare contains an even number of black squares. Initially if -1 is on a black square, then there are always an odd number of -1's on the black squares. Rotation by 90° shows that the -1 can be only on the central square.

    If -1 is on the central square, then we can achieve all $+1$'s in 5 moves

    1. Reverse signs on the lower left 3×3 square.
    2. Reverse signs on the upper right 3×3 square.
    3. Reverse signs on the upper left 2×2 square.
    4. Reverse signs on the lower right 2×2 square.
    5. Reverse signs on the whole 5×5 square.

17. Suppose the theorem is not true. Then the red points miss a distance a and the blue points miss a distance b. We may assume $a \le b$. Consider a blue point C. Construct an isosceles triangle ABC with legs $AC = BC = b$ and $AB = a$. Since C is blue, A cannot be blue. Thus, it must be red. The point B cannot be red since its distance to the red point A is a. But it cannot be blue either, since its distance to the blue point C is b. Contradiction!

18. Call the colors black, white, and red. Suppose any two points with distance 1 have different colors. Choose any red point r and assign to it Fig. 2.17. One of the two points b and w must be white and the other black. Hence, the point r' must be red.

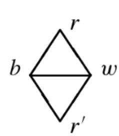


Fig. 2.17

Rotating Fig. 2.17 about r we get a circle of red points r'. This circle contains a chord of length 1. Contradiction!

Alternate solution. For Fig. 2.18 consisting of 11 unit rods, you need at least four colors, if vertices of distance 1 are to have distinct colors.

19. Color the lattices as in a chess board. Erect right triangles on the sides of the pentagon as longest sides. With the two other sides along the sides of the squares, trace the ten shorter sides. Since, at the end, we return to the vertex we left, we must have traced an even number of lattice points (on transition from one lattice point to the next the color of the lattice point changes). Hence the sum of shorter sides is even. The parity of the longer sides (i.e., the sides of the pentagon) is equivalent to the parity of the sums of the shorter sides. Hence the perimeter of the pentagon has the same parity as the sum of the shorter sides.

20. Of $n \geq 5$ points, it is always possible to choose four vertices of a convex polygon. If we color two opposite vertices the same color, then no line will separate the two sets of points.

21. *Result*: We can glue together an $m \times n$ rectangle iff m and n have the same parity.

(a) m and n are both odd. Then we can glue together an $1 \times n$ rectangle as in Fig. 2.19. From these strips, we can glue together the rectangle in Fig. 2.20.

(b) m and n are even. Consider the rectangles with odd side lengths of dimensions $(m-1) \times (n-1)$, $1 \times (n-1)$, $(m-1) \times 1$, and 1×1, respectively. They can be assembled into the rectangle $m \times n$.

(c) m is even, and n is odd. Suppose we succeeded in gluing together a rectangle $m \times n$ satisfying the conditions of the problem. Consider one of the sides of the rectangle with odd length. Suppose it is colored red. Let us count the total number of red sides of the squares. On the perimeter of the rectangle, there are n and in the interior there is an even number, since another red neighbor belongs to one red side of a square. Thus the total number of red sides is odd. The total number of squares is the same as the number of red sides, i.e., odd. On the other hand this number is $m\,n$, that is, an even number. Contradiction!

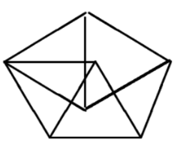


Fig. 2.18

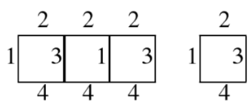


Fig. 2.19

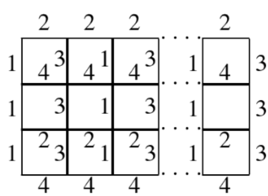


Fig. 2.20

22. The solution is similar to that of the preceding problem.

23. Color the lattice points black and white such that points with odd coordinates are black and the other lattice points are white. By reflections you always stay on lattices

of the same color. Thus it is not possible to reach the opposite vertex of the square $ABCD$.

24. Let $P_1,\ P_2,\ P_3$ be the three sets. We assume on the contrary that a_1 is not assumed by P_1, a_2 is not assumed by P_2, and a_3 is not assumed by P_3. We may assume that $a_1 \geq a_2 \geq a_3 > 0$.

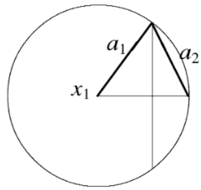


Fig. 2.21

Let $x_1 \in P_1$. The sphere S with midpoint x_1 and radius a_1 is contained completely in $P_2 \cup P_3$. Since $a_1 \geq a_3$, $S \not\subset P_3$. Let $x_2 \in P_2 \cap S$. The circle $\{y \in S | d(x_2, y) = a_2\} \subset P_3$, since P_2 does not realize a_2. But in Fig. 2.21, $a_2 \leq a_1 \Rightarrow r = a_2\sqrt{1 - a_2^2/4a_1^2} \geq a_2\sqrt{3}/2$, and $a_3 \leq a_2 \leq a_2\sqrt{3} \leq 2r$. Thus a_3 is assumed in P_3.

Another ingenious solution will be found in Chapter 4 (problem 67). It will be good training for the more difficult plane problem 68 of that chapter. Both solutions make nontrivial use of the box principle.

25. The gallery is triangulated by drawing nonintersecting diagonals. By simple induction one can prove that such a triangulation is always possible. Then we color the vertices of the triangles properly with three colors, so that any vertex of a triangle gets a different color. By trivial induction, one proves that the triangles of the triangulation can always be properly colored. Now we consider the color, which occurs least often. Suppose it is *red*. The watchmen at the red vertices can survey all walls. Thus the minimum number of watchmen is $\lfloor n/3 \rfloor$.

26. Color the squares diagonally by colors $0,\ 1,\ 2$. Then each 3×1 tile covers each of the colors once. In Fig. 2.22 we have 17 zeros, 16 ones and 16 twos. The monomino must cover one of the squares labeled "0". In addition, it must remain a "0" if we make a quarter-turn of the board. As possible positions there will remain only the central square, the four corners, and the centers of the outer edges in Fig. 2.22. A different coloring yields a different solution. We use the three colors 0, 1, 2 as in Fig. 2.23. That is, the squares colored 0 will be the center, the four corners, and the centers of the outer edges. The tiles 1×3 are of two types, those covering one square of color 0 and two squares of color 1 and those covering one square of color 1 and two squares of color 2. Suppose all squares of color 0 are covered by 1×3 tiles. There will be 9 tiles of type 1 and 7 tiles of type 2. They will cover $9 \cdot 2 + 7 = 25$ squares of color 1 and $7 \cdot 2 = 14$ squares of color 2. This contradiction proves that one of the squares of color 0 is covered by the 1×1 tile.

0	1	2	0	1	2	0
2	0	1	2	0	1	2
1	2	0	1	2	0	1
0	1	2	0	1	2	0
2	0	1	2	0	1	2
1	2	0	1	2	0	1
0	1	2	0	1	2	0

0	1	1	0	1	1	0
1	2	2	1	2	2	1
1	2	2	1	2	2	1
0	1	1	0	1	1	0
1	2	2	1	2	2	1
1	2	2	1	2	2	1
0	1	1	0	1	1	0

Suppose k is the **smallest** white number. From the preceding result, we conclude that all multiples of k are also white. We prove that there are no other white numbers. Suppose n is white. Represent n in the form $qk + r$, where $0 \leq r < k$. If $r \neq 0$, then r is black since k is the smallest white number. But we have proved that qk is white. Hence, $qk + r$ is black. This contradiction proves that the white numbers are all multiples of some $k > 1$.

3

The Extremal Principle

A successful research mathematician has mastered a dozen general heuristic principles of large scope and simplicity, which he/she applies over and over again. These principles are not tied to any subject but are applicable in all branches of mathematics. He usually does not reflect about them but knows them subconsciously. One of these principles, *the invariance principle* was discussed in Chapter I. It is applicable whenever a transformation is given or can be introduced. **If you have a transformation, look for an invariant!** In this chapter we discuss the **extremal principle**, which has truly universal applicability, but is not so easy to recognize, and therefore must be trained. It is also called *the variational method*, and soon we will see why. It often leads to extremely short proofs.

We are trying to prove the existence of an object with certain properties. The *extremal principle* tells us to pick an object which *maximizes* or *minimizes* some function. The *resulting object* is then shown to have the desired property by showing that a slight perturbation (variation) would further increase or decrease the given function. If there are several optimizing objects, then it is usually immaterial which one we use. In addition, the *extremal principle* is mostly constructive, giving an algorithm for constructing the object.

We will learn the use of the *extremal principle* by solving 17 examples from geometry, graph theory, combinatorics, and number theory, but first we will remind the reader of three well known facts:

(a) Every *finite* nonempty set A of nonnegative integers or real numbers has a minimal element $\min A$ and a maximal element $\max A$, which need not be unique.

(b) Every nonempty subset of positive integers has a smallest element. This is called the *well ordering principle*, and it is equivalent to the *principle of mathematical induction.*

(c) An infinite set A of real numbers need not have a minimal or maximal element. If A is bounded above, then it has a smallest upper bound $\sup A$. Read: supremum of A. If A is bounded below, then it has a largest lower bound $\inf A$. Read: infimum of A. If $\sup A \in A$, then $\sup A = \max A$, and if $\inf A \in A$, then $\inf A = \min A$.

E1. *(a) Into how many parts at most is a plane cut by n lines? (b) Into how many parts is space divided by n planes in general position?*

Solution. We denote the numbers in (a) and (b) by p_n and s_n, respectively. A beginner will solve these problems recursively, by finding $p_{n+1} = f(p_n)$ and $s_{n+1} = g(s_n)$. Indeed, by adding to n lines (planes) another line (plane) we easily get

$$p_{n+1} = p_n + n + 1, \qquad s_{n+1} = s_n + p_n.$$

There is nothing wrong with this approach since recursion is a fundamental idea of large scope and applicability, as we will see later. An experienced problem solver might try to solve the problems in his head.

In (a) we have a counting problem. A fundamental counting principle is one-to-one correspondence. The first question is: Can I map the p_n parts of the plane bijectively onto a set which is easy to count? The $\binom{n}{2}$ intersection points of the n lines are easy to count. But each intersection point is the deepest point of exactly one part. (Extremal principle!) Hence there are $\binom{n}{2}$ parts with a deepest point. The parts without deepest points are not bounded below, and they cut a horizontal line h (which we introduce) into $n + 1$ pieces (Fig. 3.1). The parts can be uniquely assigned to these pieces. Thus there are $n + 1$, or $\binom{n}{0} + \binom{n}{1}$ parts without a deepest point. So there are altogether

$$p_n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} \quad \text{parts of the plane.}$$

(b) Three planes form a vertex in space. There are $\binom{n}{3}$ vertices, and each is a deepest point of exactly one part of space. Thus there are $\binom{n}{3}$ parts with a deepest point. Each part without a deepest point intersects a horizonal plane h in one of p_n plane parts. So the number of space parts is

$$s_n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}.$$

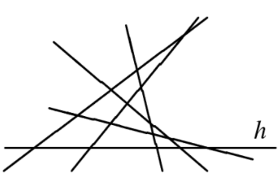


Fig. 3.1

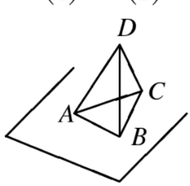


Fig. 3.2

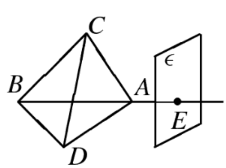


Fig. 3.3

E2. Continuation of 1b. *Let $n \geq 5$. Show that, among the s_n space parts, there are at least* $(2n-3)/4$ *tetrahedra* (HMO 1973).

Telling the result simplifies the problem considerably. An experienced problem-solver can often infer the road to the solution from the result.

Let t_n be the number of tetrahedra among the s_n space parts. We want to show that $t_n \geq (2n-3)/4$.

Interpretation of the numerator: On each of the n planes rest at least two tetrahedra. Only one tetrahedron need rest on each of three exceptional planes.

Interpretation of the denominator: Each tetrahedron is counted four times, once for each face. Hence, we must divide by four.

Using these guiding principles we can easily find a proof. Let ϵ be any of the n planes. It decomposes space into two half-spaces H_1 and H_2. At least one half-space, e.g., H_1 , contains vertices. In H_1, we choose a vertex D with smallest distance from ϵ (extremal principle). D is the intersection point of the planes $\epsilon_1, \epsilon_2, \epsilon_3$. Then $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3$ define a tetrahedron $T = ABCD$ (Fig. 3.2). None of the remaining $n-4$ planes cuts T, so that T is one of the parts, defined by the n planes. If the plane ϵ' would cut the tetrahedron T, then ϵ' would have to cut at least one of the edges AD, BD, CD in a point Q having an even smaller distance from ϵ than D. Contradiction.

This is valid for any of the n planes. If there are vertices on both sides of a plane, at least two tetrahedra then must rest on this plane.

It remains to be shown that among the n planes there are at most three, so that all vertices lie on the same side of these planes.

We show this by *contradiction*. Suppose there are four such planes ϵ_1, ϵ_2, ϵ_3, ϵ_4. They delimit a tetrahedron $ABCD$ (Fig. 3.3). Since $n \geq 5$, there is another plane ϵ. It cannot intersect all six edges of the tetrahedron $ABCD$ simultaneously. Suppose it cuts the continuation of AB in E. Then B and E lie on different sides of the plane $\epsilon_3 = ACD$. *Contradiction!*

E3. *There are n points given in the plane. Any three of the points form a triangle of area* ≤ 1. *Show that all n points lie in a triangle of area* ≤ 4.

Solution. Among all $\binom{n}{3}$ triples of points, we choose a triple A, B, C so that $\triangle ABC$ has maximal area F. Obviously $F \leq 1$. Draw parallels to the opposite sides through A, B, C. You get $\triangle A_1B_1C_1$ with area $F_1 = 4F \leq 4$. We will show that $\triangle A_1B_1C_1$ contains all n points.

Suppose there is a point P outside $\triangle A_1B_1C_1$. Then $\triangle ABC$ and P lie on different sides of at least one of the lines A_1B_1, B_1C_1, C_1A_1. Suppose they lie on different sides of B_1C_1. Then $\triangle BCP$ has a larger area than $\triangle ABC$. This contradicts the maximality assumption about ABC (Fig. 3.4).

E4. $2n$ *points are given in the plane, no three collinear. Exactly n of these points are farms* $F = \{F_1, F_2, \dots, F_n\}$. *The remaining n points are wells:* $W = \{W_1, W_2, \dots, W_n\}$. *It is intended to build a straight line road from each* *farm to one well. Show that the wells can be assigned bijectively to the farms, so that none of the roads intersect.*

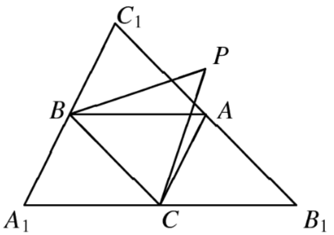


Fig. 3.4

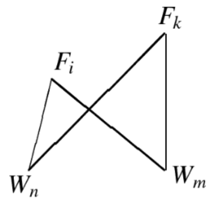


Fig. 3.5

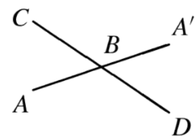


Fig. 3.6

Solution. We consider any bijection: $f : F \mapsto W$. If we draw from each F_i a straight line to $f(F_i)$, we get a road system. Among all $n!$ road systems, we choose one of minimal total length.Suppose this system has intersecting segments $F_i W_m$ and $F_k W_n$ (Fig. 3.5). Replacing these segments by $F_k W_m$ and $F_i W_n$, the total road length becomes shorter because of the triangle inequality. Thus it has no intersecting roads.

E5. *Let Ω be a set of points in the plane. Each point in Ω is a midpoint of two points in Ω. Show that Ω is an infinite set.*

First proof. Suppose Ω is a finite set. Then Ω contains two points A, B with maximal distance $|AB| = m$. B is a midpoint of some segment CD with $C, D \in \Omega$. Fig. 3.6 shows that $|AC| > |AB|$ or $|AD| > |AB|$.

Second proof. We consider all points in Ω farthest to the left, and among those the point M farthest down. M cannot be a midpoint of two points $A, B \in \Omega$ since one element of $\{A, B\}$ would be either left of M or on the vertical below M.

E6. *In each convex pentagon, we can choose three diagonals from which a triangle can be constructed.*

Solution. Fig. 3.7 shows a convex pentagon $ABCDE$. Let BE be the longest of the diagonals. The triangle inequality implies $|BD|+|CE| > |BE|+|CD| > |BE|$, that is, we can construct a triangle from BE, BD, CE.

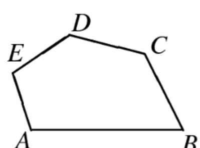


Fig. 3.7

E7. *In every tetrahedron, there are three edges meeting at the same vertex from which a triangle can be constructed.*

Solution. Let AB be the *longest* edge of the tetrahedron $ABCD$. Since $(|AC| + |AD| - |AB|) + (|BC| + |BD| - |BA|) = (|AD| + |BD| - |AB|) + (|AC| + |BC| -$