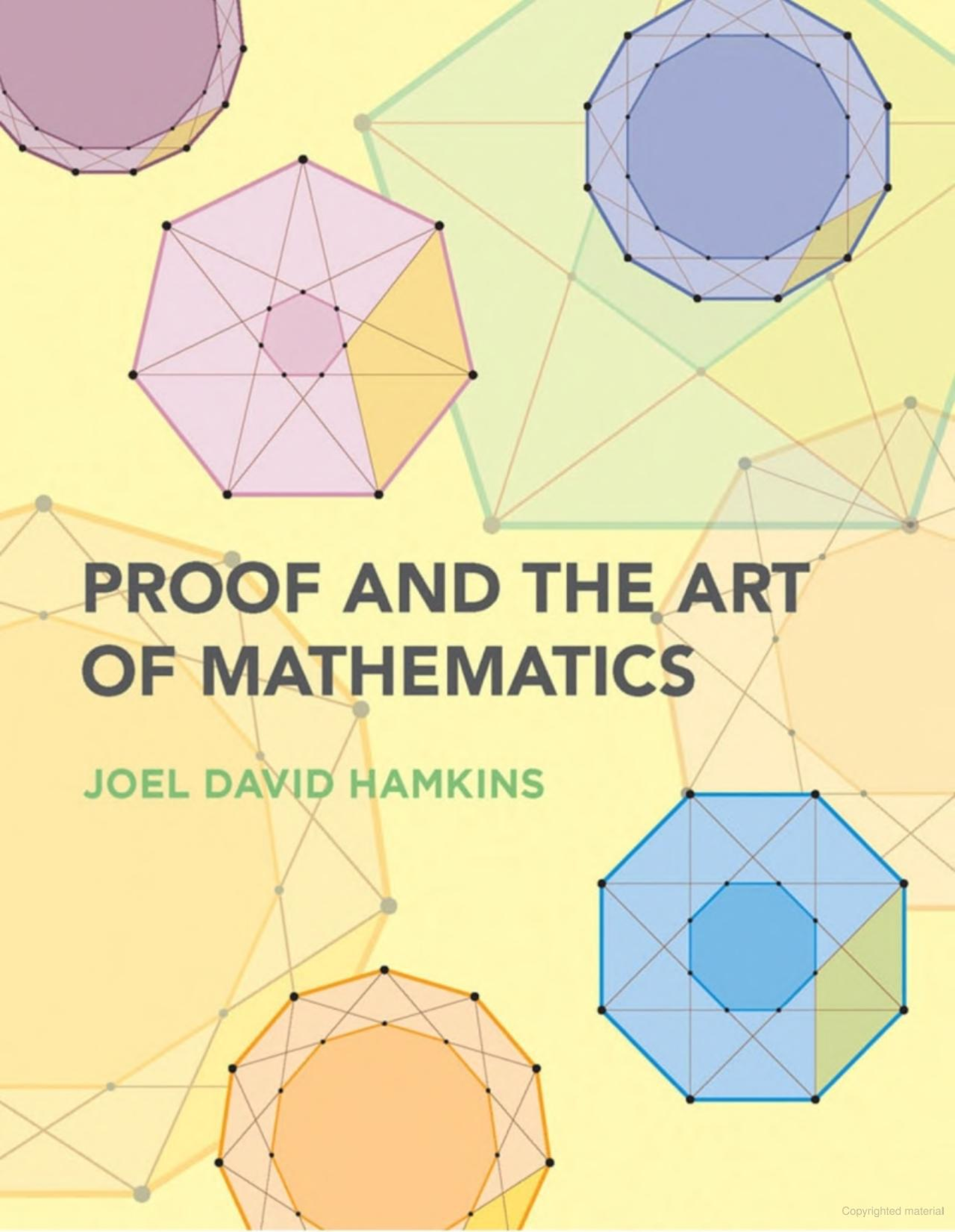

PROOF AND THE ART OF MATHEMATICS

JOEL DAVID HAMKINS

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Preface

This is a mathematical coming-of-age book, for students on the cusp, who are maturing into mathematicians, aspiring to communicate mathematical truths to other mathematicians in the currency of mathematics, which is: proof. This is a book for students who are learning—perhaps for the first time in a serious way—how to write a mathematical proof. I hope to show how a mathematician makes an argument establishing a mathematical truth.

Proofs tell us not only that a mathematical statement is true, but also why it is true, and they communicate this truth. The best proofs give us insight into the nature of mathematical reality. They lead us to those sublime yet elusive *Aha!* moments, a joyous experience for any mathematician, occurring when a previously opaque, confounding issue becomes transparent and our mathematical gaze suddenly penetrates completely through it, grasping it all in one take. So let us learn together how to write proofs well, producing clear and correct mathematical arguments that logically establish their conclusions, with whatever insight and elegance we can muster. We shall do so in the context of the diverse mathematical topics that I have gathered together here in this book for the purpose.

What is a proof, really? Mathematicians are sometimes excused from jury duty, it is said, because according to the prosecutors, they do not know what it means to prove something "beyond a reasonable doubt," which is the standard of evidence that juries follow for conviction in US criminal courts. Indeed, a mathematician's standard of evidence for proof is very high, perhaps too high for the prosecutors to want them on the jury.

Mathematical logicians have a concept of formal proof, which is a detailed form of proof written in a rigid formal language. These proofs, often intended to be processed computationally, understood and verified by a machine, can irrefutably establish the validity of the theorems they prove, yet they are often essentially unreadable by humans, usually providing us with little mathematical insight beyond the truth of the raw statement that is proved. Despite this, the emerging field of automated theorem proving may in future decades profoundly transform mathematical practice.

In high-school geometry, students often learn a standard two-column form of proof, in which certain kinds of statements are allowed in the left column, provided that they are

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justified by admissible reasons in the right column. This form of proof highlights an all-important feature, namely, that proofs must provide a chain of reasoning logically establishing their conclusion from their premises.

Meanwhile, in contemporary mathematical research writing, one finds a more open, flexible proof format, written essay style in complete sentences while retaining the defining feature that a proof must logically establish the truth of its conclusion. In fact, this style of proof writing has endured millennia. If you open Euclid's *Elements*, for example, now translated into nearly every human language, you will find beautiful proofs written in an engaging open-prose style of argument. The same proof-writing style permeates mathematical writing through the ages, up to the present day, filling our mathematical research journals and books.

It is this style of proof writing with which we shall be concerned in this book, the open-prose style of argument that mathematicians use to establish and communicate to one another the truth of a mathematical claim. For the purposes of this book, let me say:

> A *proof* is any sufficiently detailed convincing mathematical argument that logically establishes the conclusion of a theorem from its premises.

Mathematical proofs naturally make use of a variety of proof methods, which may vary depending on the mathematical topic. We shall undertake proofs in number theory, for example, using the principle of mathematical induction; in combinatorics, using various counting arguments; and in game theory, graph theory, real analysis, and so on, using methods common to these diverse areas. Throughout, we shall learn about certain common pitfalls for beginners and emphasize new aspects of proof writing when they arise. There are, of course, a variety of proof styles that one may adopt within the theorem-proof format that I describe and use in this book, and I hope that the reader may grow to develop his or her own mathematical voice.

If I had to describe my own proof-writing style, I would say that I am somewhat less concerned with providing all of the technical details of an argument, especially when these can be easily provided by the reader or when they obscure or distract from the idea of the proof, and somewhat more concerned with providing a deeper explanation, with conveying the essential idea of an argument in such a way as to cultivate mathematical insight and understanding in my reader. I often strive to lead the reader through a series of smaller or plainly stated mathematical truths that in combination make the larger conclusion clear. To my way of thinking, a key value of many mathematical achievements consists in their making certain formerly difficult ideas easy to understand. Ultimately, my aim is mathematical insight; I aspire to induce an *Aha!* moment in my reader. For this reason, I prefer to write in plain language, when I can, or to explain a mathematical idea in ordinary words, when this successfully conveys the intended mathematical idea. But of course, sometimes the only way to convey a mathematical idea accurately is with detailed mathematical notation or formalism, and in this case one must step up with the right tools. So although I strive for

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nontechnical simplicity, I do not use this as an excuse for vagueness, and naturally I strive to provide fuller details or technical notation when the clarity of the argument requires it.

This book was typeset using LaTeX. Except for the two images on pages 41 and 146, which are in the public domain, and the image of the Königsberg bridges on page 133, which I drew myself by hand, I created all the other images in this book using TikZ in LaTeX, specifically for this book.

Joel David Hamkins
July 2019

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A Note to the Instructor

In this book, I have assembled a collection of what I find to be compelling mathematical statements with interesting elementary proofs, illustrating diverse proof methods and intended to develop a beginner's proof-writing skills. All who aspire toward mathematics, who want to engage fully with the mathematical craft by undertaking a mathematical analysis and constructing their own proofs of mathematical statements, will benefit from this text, whether they read it as part of a university proof-writing course or study it on their own.

I should like to emphasize, however, that the book is not an axiomatic development of its topics from first principles. The reason is that, while axiomatic developments certainly involve proof writing, I find that they are also often burdened, especially in their beginnings, with various tedious matters. Think of the need, for example, to establish the associativity of integer addition from its definition. I find it sensible, in contrast, to separate the proof-writing craft in its initial or introductory stages from the idea that an entire mathematical subject can be developed from weak axiomatic principles. I also find it important to teach proof writing with mathematically compelling, enjoyable examples, which can inspire a deeper interest in and curiosity about mathematics; students will then be motivated to work through other examples on their own.

So the proofs in this book are not built upon any explicitly given list of axioms but, rather, appeal to very general mathematical principles with which I expect the reader is likely familiar. My hope is that students, armed with the proof-writing skills they have gained from this text, will go on to undertake axiomatic treatments of mathematical subjects, such as number theory, algebra, set theory, topology, and analysis.

The book is organized around mathematical themes, rather than around methods of proof, such as proofs by contradiction, proofs by cases, proofs of if-then statements, or proofs of biconditionals. To my way of thinking, mathematical ideas are best conceived of and organized mathematically; other organizational plans would ultimately be found artificial. I do not find proofs by contradiction, for example, to be a natural or robust mathematical category. Such a proof, after all, might contain essentially the same mathematical insights

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and ideas as a nearby proof that does not proceed by contradiction. Indeed, mathematical statements will often admit diverse proofs, using diverse proof methods—I explore this point explicitly in chapter 2—and it would seem wrong to me to suggest that a given statement should be proved only with a particular method.

For these reasons, I have organized the book around a selection of mathematically rich topics, having interesting theorems with elementary proofs. I have chosen what I find to be especially enjoyable mathematical topics—at times fanciful, yet always amenable to a serious mathematical analysis—because I feel students will learn best how to write proofs with material that is itself intrinsically interesting. Please enjoy! I have simply treated the various proof methods as they arise naturally within each topic. Some of the topics, such as the theory of games and order theory, seem to be somewhat less often covered in the contemporary mathematics curriculum than I believe is warranted; therefore, kindly let the students learn some of this beautiful mathematics while developing their proof-writing skills.

If the book is to be used as part of an undergraduate proof-writing course, then each chapter could be the basis of a lecture or sometimes several lectures. Some chapters are a little longer, and the earlier chapters are generally a little easier. The chapters on Pick's theorem (chapter 8) and on the polygonal dissection congruence theorem (chapter 10) are each essentially chapter-length developments of a single major theorem. The final chapters, on infinity (chapter 13), order theory (chapter 14), and real analysis (chapter 15), are perhaps a little more abstract or difficult.

Very little of the material depends fundamentally on earlier chapters, except that the method of mathematical induction, covered in chapter 4, is used in many arguments throughout the book, and some of the material in chapters 14 and 15 depends on chapter 13 and on the properties of functions in chapter 11. I would find it sensible to cover relations (chapter 11) before graph theory (chapter 12).

I would recommend that instructors cover any or all of the topics they find appealing, starting with chapter 1. The students can be directed to read the preface and other preliminary material on their own, especially "A Note to the Student," but the theorem-proof format should be discussed explicitly in class, probably in the first lecture. Students can also be instructed to peruse the mathematical habits from later chapters, as they are applicable generally. When I used this book as the textbook for an undergraduate proof-writing course at the City University of New York, we covered the entire book in one fifteen-week semester.

I recommend that essentially all homework and exam work for a proof-writing course using this textbook require the student to write proofs. The students should prove essentially every substantive mathematical statement they make. I have included exercises in each chapter, and I would find it normal for students to attempt every exercise in any chapter that is covered. A few sample answers are provided at the end of the book, and a more

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extensive volume of answers is available from MIT Press. Students should be encouraged to form study groups, to work on the exercises together, since one learns mathematics very effectively by trying out one's ideas on one's peers. In this sense, the practice of mathematics can be considered a social performance activity. Please encourage your students to understand the mathematical idea in the proofs they study, rather than to memorize specific proofs. Indeed, I find little value in having students memorize specific mathematical proofs; in my experience anyone who understands a proof well can reliably reproduce it, even months or years later, without ever having specifically memorized it.

I also recommend that the instructor spend time in class giving careful solutions to some or even all of the exercises, with fully detailed proofs, after the students have attempted them, so that the students can benefit from exposure to further examples of sound mathematical proof. Please do not worry about "giving away the idea," since surely there is an infinite supply of new mathematical questions with which the students might struggle on their own. In my experience and perception of our subject, what students seem to need most is exposure to far more and better examples of high-quality mathematical arguments. Give them an abundant supply of well-written, well-explained proofs in your lectures and in your office hours; I think you will find that they pick it up by example.

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Please use the theorem-proof format of writing even when solving the exercises in this text. For example, suppose the exercise says:

> **Exercise.** Prove that every hibdab is hobnob.

Then you should begin your solution not by starting directly with your proof, or by rewriting those instructions. Rather, you begin by writing a clear statement of what you are proving, like this:

> **Theorem.** *Every hibdab is hobnob.*
>
> *Proof.* And so on with your argument. □

Notice that this turns the instruction statement of the exercise into a new, clear mathematical statement. It would have made no sense to prove the original assertion, "Prove that every hibdab is a hobnob," because that is not a mathematical statement, a statement that might be true or false, but is rather an imperative, an *instruction* about what we should do. We carry out that instruction by formulating a clear mathematical statement as our theorem and then proving this statement. In this way, you shall turn every exercise into your own formal theorem statement and proof.

Let me say lastly that I have also gathered together in this text a collection of what I find to be sound mathematical habits of mind, bits of mathematical wisdom or advice that I believe to be beneficial or even fundamental to sound mathematical practice, highlighted in boxes at the end of each chapter. Adopting these habits, I believe, will help an aspiring mathematician solve a problem, find an elusive proof, or write better proofs. Let me mention one of them right now that we have just discussed.

> **Use the theorem-proof format.** In all your mathematical exercises, write in the theorem-proof style. State a clear claim in your theorem statement. State lemmas, corollaries, and definitions as appropriate. Give a separate, clearly demarcated proof for every formally stated mathematical claim.

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About the Author

I am an active research mathematician working on the mathematics and philosophy of the infinite. I have published about one hundred research articles on diverse topics in mathematical logic and set theory, ranging from forcing and large cardinals to infinitary computability and infinite game theory, including infinite chess. My blog, *Mathematics and Philosophy of the Infinite*, features a variety of mathematical posts and commentary, including an accessible series on "Math for Kids," and the reader can discover what pricks my mathematical fancy on Twitter (see links below). In addition, I have posted over one thousand research-level mathematical arguments on MathOverflow, the popular mathematics question-and-answer forum, which is becoming a fundamental tool for mathematical research. Interested readers can therefore find many of my mathematical proofs and arguments online. For me, mathematics has been an enjoyable lifelong learning process; I continually strive to improve.

I am also a mathematical philosopher, working in mathematical and philosophical logic with a focus on infinity, especially in set theory, the philosophy of set theory, and the philosophy of mathematics. In recent work, I have been exploring pluralism in the foundations of mathematics by introducing and investigating the multiverse view in set theory and the mathematics and philosophy of potentialism. My forthcoming book *Lectures on the Philosophy of Mathematics*, upon which I have based my lectures here in Oxford, emphasizes a mathematically grounded perspective on the subject.

I have taught college-level mathematics for over twenty-five years, mostly at the City University of New York, where I have taught essentially every proof-based undergraduate topic in pure mathematics, as well as many graduate-level topics, especially in logic. At CUNY I was Professor of Mathematics, Professor of Philosophy, and Professor of Computer Science.

I have recently and quite happily taken up a new position as Professor of Logic at the University of Oxford, where I am a member of the Faculty of Philosophy, affiliated member of the Mathematics Institute, and the Sir Peter Strawson Fellow in Philosophy at University College. I have also held diverse visiting professor positions over the years, at New York

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1 A Classical Beginning

One of the classical gems of mathematics—and to my way of thinking, a pinnacle of human achievement—is the ancient discovery of incommensurable numbers, quantities that cannot be expressed as the

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The Pythagoreans disc e side and diagonal of a square have no common unit of measure; there is no smaller unit length of which they are both integral multiples; the quantities are *incommensurable*. If you divide the side of a square into ten units, then the diagonal will be a little more than fourteen of them. If you divide the side into one hundred units, then the diagonal will be a little more than 141; if one thousand, then a little more than 1414. It will never come out exactly. One sees those approximation numbers as the initial digits of the decimal expansion:

$$\sqrt{2} = 1.41421356237309504880168872420969807856\ldots$$

The discovery shocked the Pythagoreans. It was downright heretical, in light of their quasi-religious number-mysticism beliefs, which aimed to comprehend all through proportion and ratio, taking numbers as a foundational substance. According to legend, the man who made the discovery was drowned at sea, perhaps punished by the gods for impiously divulging the irrational.

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Slightly revised proof of theorem 1. Suppose toward contradiction that $\sqrt{2}$ is rational. So $\sqrt{2} = p/q$ for some integers p and q, and we may assume that the numerator p is chosen as small as possible for such a representation. It follows as before that $2q^2 = p^2$, and so p^2 and hence also p is even. So $p = 2k$ for some k, which implies that $q^2 = 2k^2$ as before, and so q^2 and hence also q is even. So $q = 2r$ for some r, and consequently $\sqrt{2} = p/q = (2k)/(2r) = k/r$. We have therefore found a rational representation of $\sqrt{2}$ using a smaller numerator, contradicting our earlier assumption. So $\sqrt{2}$ is not rational. □

This way of arguing, although very similar to the original argument, does not require putting fractions in lowest terms. Furthermore, an essentially similar idea can be used to prove that indeed every fraction can be put in lowest terms.

1.2 Lowest terms

What does it mean for a fraction p/q to be in lowest terms? It means that p and q are *relatively prime*, that is, that they have no common factor, a number $k > 1$ that divides both of them. I find it interesting that the property of being in lowest terms is not a property of the rational number itself but rather a property of the fractional expression used to represent the number. For example, $\frac{3}{6}$ is not in lowest terms and $\frac{1}{2}$ is, yet we say that they are equal: $\frac{3}{6} = \frac{1}{2}$. But how can two things be identical if they have different properties? These two expressions are equal in that they describe the same rational number; the values of the expressions are the same, even though the expressions themselves are different. Thus, we distinguish between the description of a number and the number itself, between our talk about a number and what the number actually is. It is a form of the *use/mention* distinction, the distinction between syntax and semantics at the core of the subject of mathematical logic. How pleasant to see it arise in the familiar elementary topic of lowest terms.

Lemma 2. *Every fraction can be put in lowest terms.*

Proof. Consider any fraction p/q, where p and q are integers and $q \neq 0$. Let p' be the smallest nonnegative integer for which there is an integer q' with $\frac{p}{q} = \frac{p'}{q'}$. That is, we consider a representation $\frac{p'}{q'}$ of the original fraction $\frac{p}{q}$ whose numerator p' is as small as possible. I claim that it follows that p' and q' are relatively prime, since if they had a common factor, we could divide it out and thereby make an instance of a fraction equal to p/q with a smaller numerator. But p' was chosen to be smallest, and so there is no such common factor. Therefore, p'/q' is in lowest terms. □

This proof and the previous proof of theorem 1 relied on a more fundamental principle, the least-number principle, which asserts that if there is a natural number with a certain property, then there is a smallest such number with that property. In other words, every nonempty set of natural numbers has a least element. This principle is closely connected with the principle of mathematical induction, discussed in chapter 4. For now, let us simply

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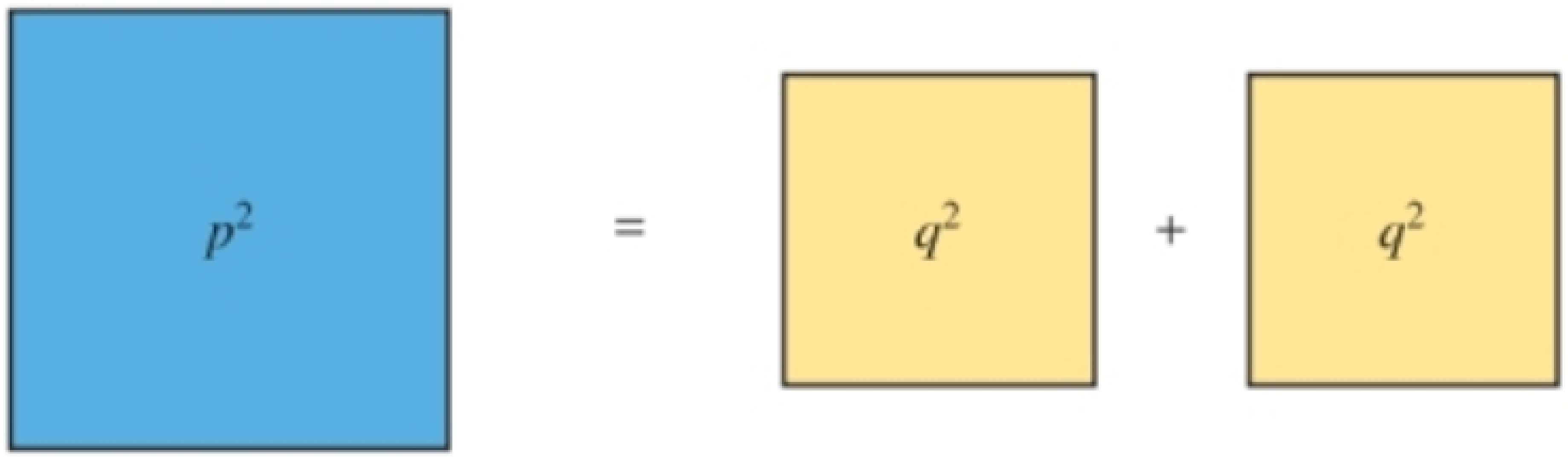
take it as a basic principle that if there is a natural number with a property, then there is a smallest such number with that property.

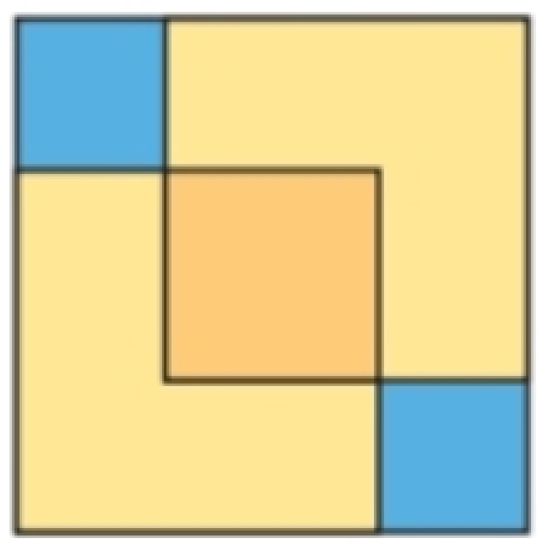
1.3 A geometric proof

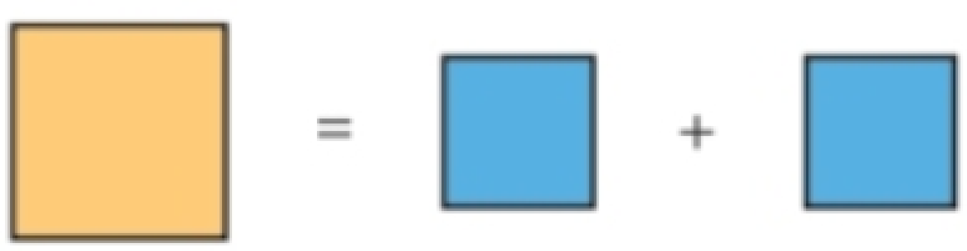
Let us now give a second proof of the irrationality of $\sqrt{2}$, one with geometric character, due to Stanley Tennenbaum. Mathematicians have found dozens of different proofs of this classic result, many of them exhibiting a fundamentally different character from what we saw above.

A geometric proof of theorem 1. If $\sqrt{2}$ is rational p/q, then as before, we see that $p^2 = q^2 + q^2$, which means that some integer square has the same area as two copies of another smaller integer square.



We may choose these squares to have the smallest possible integer sides so as to realize this feature. Let us arrange the two medium squares overlapping inside the larger square, as shown here at the right. Since the large blue square had the same area as the two medium gold squares, it follows that the small orange central square of overlap must be exactly balanced by the two smaller uncovered blue squares in the corners. That is, the area of overlap is exactly the same as the area of the two uncovered blue corner spaces. Let us pull these smaller squares out of the figure to illustrate this relation as follows.





Notice that the squares in this smaller instance also have integer-length sides, since their lengths arise as differences in the side lengths of previous squares. So we have found a strictly smaller integer square that is the sum of another integer square with itself, contradicting our assumption that the original square was the smallest such instance. □

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