PROOF AND THE ART OF MATHEMATICS

EXAMPLES AND EXTENSIONS

JOEL DAVID HAMKINS

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Preface

The best way to learn mathematics is to dive in and do it. Don't just listen passively to a lecture or read a book—you have got to take hold of the mathematical ideas yourself! Mount your own mathematical analysis. Formulate your own mathematical assertions. Consider your own mathematical examples. I recommend play—adopt an attitude of playful curiosity about mathematical ideas; grasp new concepts by exploring them in particular cases; try them out; understand how the mathematical constructions from your proofs manifest in your examples; explore all facets, going beyond whatever had been expected. You will find vast new lands of imagination. Let one example generalize to a whole class of examples; have favorite examples. Ask questions about the examples or about the mathematical idea you are investigating. Formulate conjectures and test them with your examples. Try to prove the conjectures—when you succeed, you will have proved a theorem. The essential mathematical activity is to make clear claims and provide sound reasons for them. Express your mathematical ideas to others, and practice the skill of stating matters well, succinctly, with accuracy and precision. Don't be satisfied with your initial account, even when it is sound, but seek to improve it. Find alternative arguments, even when you already have a solid proof. In this way, you will come to a deeper understanding. Test the statements of others; ask for further explanation. Look into the corner cases of your results to probe the veracity of your claims. Set yourself the challenge either to prove or to refute a given statement. Aim to produce clear and correct mathematical arguments that logically establish their conclusions, with whatever insight and elegance you can muster.

This book is offered as a companion volume to my book *Proof and the Art of Mathematics*, which I have described as a mathematical coming-of-age book for students learning how to write mathematical proofs. Spanning diverse topics from number theory and graph theory to game theory and real analysis, *Proof and the Art* shows how to prove a mathematical theorem, with advice and tips for sound mathematical habits and practice, as well as occasional reflective philosophical discussions about what it means to undertake mathematical proof. In *Proof and the Art*, I offer a few hundred mathematical exercises, challenges to the reader to prove a given mathematical statement, each a small puzzle to figure out; the intention is for students to develop their mathematical skills with these challenges of mathematical reasoning and proof.

Here in this companion volume, I provide fully worked-out solutions to all of the odd-numbered exercises, as well as a few of the even-numbered exercises. In many cases, the solutions here explore beyond the exercise question itself to natural extensions of the ideas. My attitude is that, once you have solved a problem, why not push the ideas harder to see what further you can prove with them? These solutions are examples of how one might write a mathematical proof. I hope that you will learn from them; let us go through them together. The mathematical development of this text follows the main book, with the same chapter topics in the same order, and all theorem and exercise numbers in this text refer to the corresponding statements of the main text. This book was typeset using LaTeX, and all figures were created using TikZ in LaTeX, except in chapter 12 for the Königsberg bridge image, which I drew myself by hand, and the triangulated torus image, released by user AG2gaeh under a Creative Commons license.

Joel David Hamkins
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About the Author

I am an active research mathematician and mathematical philosopher at Oxford University. I work on diverse topics in mathematical logic and the philosophy of mathematics, including especially the mathematics and philosophy of the infinite. For me, mathematics is a lifelong process of learning and exploring. Truly one of life's great joys is to share interesting new mathematical ideas or puzzles with others, and I find myself doing so not only in my research papers and books but also on my blog, on Twitter, and on MathOverflow. I cordially invite you to join the conversations on all these forums (links below). My new book, *Lectures on the Philosophy of Mathematics*, forthcoming with MIT Press, emphasizes a mathematically grounded perspective on the philosophy of mathematics, an approach that I believe will appeal both to mathematicians and to philosophers of mathematics.

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1 A Classical Beginning

The classical theorem that $\sqrt{2}$ is irrational is a gem of antiquity, proved by a beautiful argument that has endured millennia, a pinnacle of human insight and achievement. Here, we begin with exercises that solidify the foundations, establishing facts used in the proof of this classic result, and then move on to exercises that generalize the result beyond $\sqrt{2}$, proceeding in small steps that achieve the same conclusion in more and more cases, until ultimately we provide a complete general criterion for when $\sqrt{n}$ is irrational.

> **1.1** Prove that the square of any odd number is odd.

Following the theorem-proof format, let us make the claim here as a formal theorem statement, for which we then provide proof.

Theorem. *If n is an odd integer, then n^2 also is odd.*

Proof. Assume that n is an odd integer. By definition, this means that $n = 2k + 1$ for some integer k. In this case, we can calculate that

$$n^2 = (2k + 1)^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1.$$

We can write this final sum as $2(2k^2 + 2k) + 1$, which is $2r + 1$, if we let $r = 2k^2 + 2k$. And so n^2 also is odd, as desired. □

More generally, this theorem is a consequence of the following familiar fact:

Theorem. *The product of any two odd integers is odd.*

Proof. Suppose that n and m are odd integers. By definition, this means that $n = 2k + 1$ and $m = 2r + 1$ for some integers k and r. Now observe that

$$nm = (2k + 1)(2r + 1) = 4kr + 2k + 2r + 1.$$

This is equal to $2(2kr + k + r) + 1$, which is 2 times an integer, plus 1; and so nm is odd, as desired. □

1.3 Prove that $\sqrt[4]{2}$ is irrational. Give a direct argument, but kindly also deduce it as a corollary of theorem 1 in the main text.

Theorem. *$\sqrt[4]{2}$ is irrational.*

Proof as direct argument. Suppose toward contradiction that $\sqrt[4]{2}$ is rational. In this case, we may express it as a fraction

$$\sqrt[4]{2} = \frac{p}{q},$$

where p and q are integers, and q is not zero. We may furthermore assume that this fraction is in lowest terms, so that p and q have no common factors. By raising both sides to the 4th power, we conclude that

$$2 = \frac{p^4}{q^4}$$

and therefore that $2q^4 = p^4$. It follows that p^4 is even. Since the product of any number of odd numbers remains odd, it follows that p cannot be odd, and so p is even. So $p = 2k$ for some integer k, and consequently, $2q^4 = p^4 = (2k)^4 = 16k^4$, which implies that $q^4 = 8k^4$. So q^4 also is even, and so q is even. So the fraction p/q was not in lowest terms after all, contradicting our assumption. So $\sqrt[4]{2}$ cannot have been rational in the first place, and so it is irrational. □

Alternative proof as corollary to theorem 1. Theorem 1 in the main text was the assertion that $\sqrt{2}$ is irrational. The key thing to notice is that $\sqrt{2}$ is simply the square of $\sqrt[4]{2}$,

$$\sqrt{2} = \left(\sqrt[4]{2}\right)^2,$$

and so if $\sqrt[4]{2}$ were a rational number p/q, then $\sqrt{2}$ would be the square of this number $\sqrt{2} = (p/q)^2 = p^2/q^2$, which remains rational. This would contradict theorem 1. So we conclude that $\sqrt[4]{2}$ cannot be rational. □

We can generalize this observation to the following theorem:

Theorem. *If r is irrational, then so is $\sqrt{r}$ and indeed $\sqrt[k]{r}$ for any positive integer k.*

The exercise is a special case of this, since $\sqrt[4]{2} = \sqrt{\sqrt{2}}$.

Proof. Notice that $r = (\sqrt[k]{r})^k$, and so if $\sqrt[k]{r}$ were a rational number p/q, then we would have $r = (p/q)^k = p^k/q^k$, which is rational, contrary to our assumption. □

1.5 Prove that $\sqrt{5}$ and $\sqrt{7}$ are irrational. Prove that $\sqrt{p}$ is irrational, whenever p is prime.

Let's warm up with a direct argument for $\sqrt{5}$ and then generalize it to $\sqrt{p}$ for arbitrary primes. We shall make use of the fundamental theorem of arithmetic, stating that every positive integer has a unique prime factorization.

Theorem. *$\sqrt{5}$ is irrational.*

Proof. Suppose toward contradiction that $\sqrt{5} = n/m$ is rational, represented by a fraction with integers n and $m \neq 0$. By squaring both sides and clearing the denominator, we see that

$$5m^2 = n^2.$$

From this, it follows that n^2 is a multiple of 5. This implies that n itself must be a multiple of 5, since the only way to get 5 into the prime factorization of n^2 is to have it already in n, as the prime factorization of n^2 is obtained from the prime factorization of n by squaring every term. And so $n = 5k$ and thus $5m^2 = n^2 = (5k)^2 = 25k^2$. So we can cancel 5 and deduce $m^2 = 5k^2$. So 5 must also appear in the prime factorization of m^2 and hence also in that of m. So n/m was not in lowest terms after all, contrary to our assumption. So $\sqrt{5}$ must be irrational. □

Essentially the same argument works with any prime number, since the only thing we had used about the number 5 in the argument was that it was prime. So let us set out and prove this more general fact.

Theorem. *If p is a prime number, then $\sqrt{p}$ is irrational.*

Proof. Let p be a prime number, and suppose toward contradiction that $\sqrt{p}$ is rational. So we may represent it as a fraction

$$\sqrt{p} = \frac{n}{m},$$

where n and m are integers, and $m \neq 0$. By squaring both sides, we see that $p = n^2/m^2$ and consequently $pm^2 = n^2$. It follows that p appears in the prime factorization of n^2. Therefore, it must also appear in the prime factorization of n. So $n = pk$, and so $pm^2 = n^2 = (pk)^2 = p^2k^2$. By canceling one p, we deduce that $m^2 = pk^2$. And so p must appear in the prime factorization of m^2, and consequently also in that of m. So both n and m are multiples of p, which contradicts our assumption that p/q was in lowest terms. So $\sqrt{p}$ must be irrational. □

We may now deduce the original cases of the question as a corollary.

Corollary. $\sqrt{5}$ *and* $\sqrt{7}$ *are each irrational.*

Proof. This is an instance of the theorem, since 5 and 7 are each prime. □

One might object that we needn't have proved the first theorem above, that $\sqrt{5}$ is irrational, since we've just now deduced it as a corollary to the more general theorem that $\sqrt{p}$ is irrational for every prime p. But I would find that objection mistaken. Just because you can deduce a theorem as a consequence of a more general theorem doesn't mean that you should only do the argument that way. The earlier, more elementary proof retains value simply because it is a more elementary or concrete instance, one that furthermore exhibits the main idea that led to the more general theorem in the first place. It was easier to understand the first argument simply because it has used the particular number 5 instead of the variable p, which meant one fewer abstraction woven into the argument. In addition, let me say categorically that there is absolutely nothing wrong with proving the same fact twice or more times with different arguments. I find it to be good mathematical style, even in a formal mathematical research paper, to warm up with an easier case of what will ultimately be a more general or abstract argument.

> **1.7** Prove that $\sqrt{2m}$ is irrational, whenever m is odd.

For example, $\sqrt{10}$, $\sqrt{14}$, $\sqrt{30}$, and $\sqrt{50}$ are each irrational.

Theorem. $\sqrt{2m}$ *is irrational, whenever m is odd.*

Proof. Suppose toward contradiction that m is odd and that $\sqrt{2m} = p/q$ is rational, where p and $q \neq 0$ are integers, and where this fraction is in lowest terms. Squaring both sides leads to $2mq^2 = p^2$. Therefore, p^2 is even, and so p also must be even. So $p = 2k$ and thus $2mq^2 = p^2 = (2k)^2 = 4k^2$. Canceling the 2 leads to $mq^2 = 2k^2$. So mq^2 is even. But since m is odd, this means that q^2 must be even, and so q also is even. So p/q was not in lowest terms after all, and so $\sqrt{2m}$ cannot have been rational. □

> **1.9** Criticize this "proof." Claim. $\sqrt{n}$ is irrational for every natural number n. Proof. Suppose toward contradiction that $\sqrt{n} = p/q$ in lowest terms. Square both sides to conclude that $nq^2 = p^2$. So p^2 is a multiple of n, and therefore p is a multiple of n. So $p = nk$ for some k. So $nq^2 = (nk)^2 = n^2k^2$, and therefore $q^2 = nk^2$. So q^2 is a multiple of n, and therefore q is a multiple of n, contrary to the assumption that p/q is in lowest terms. □

Let us criticize the proposed proof, which is not correct. To begin, let us consider carefully the claim that is being made.

> ***"Claim."*** *$\sqrt{n}$ is irrational for every natural number n.*

But this claim is not true! We can easily find many counterexamples. Consider the case $n = 25$, for example, for which $\sqrt{n} = \sqrt{25} = 5$, which is certainly a rational number, and similarly $\sqrt{49} = 7$ and $\sqrt{100} = 10$. So sometimes $\sqrt{n}$ is rational, and these are counterexamples that refute the claim. It follows, of course, that the proof cannot be right, and so we should expect to find some kind of mistake. But where exactly does the argument go wrong? Let's go through each sentence of the argument.

> ***"Proof."*** *Suppose toward contradiction that $\sqrt{n} = p/q$ in lowest terms.*

The proof starts out completely fine. It seems that we shall try to prove the claim by contradiction, by supposing that $\sqrt{n}$ is rational and then trying to derive a contradiction from this assumption.

> *Square both sides to conclude that $nq^2 = p^2$.*

This step also is fine; it follows similar reasoning as we used when proving the theorem on $\sqrt{2}$ and the other cases.

> *So p^2 is a multiple of n, and therefore p is a multiple of n.*

Yes, p^2 is a multiple of n, since we observed above that $nq^2 = p^2$, so the first part of this is correct. But the next statement, "and therefore p is a multiple of n" is not right. Just because a number p^2 is a multiple of a number, it doesn't follow that p must be a multiple of the number. For example, 100 is a multiple of 25, but 10 is not. In the main argument about $\sqrt{2}$, we did argue that if p^2 is a multiple of 2, then p also is a multiple of 2. And it is fine in that case, but what makes it correct is that 2 is a prime number, and so if it shows up in the prime factorization of p^2, then it must have been already in the factorization of p itself. In general, if p^2 is a multiple of a number n, this doesn't mean that p has to be a multiple of n. So this is where the argument is incorrect.

> *So $p = nk$ for some k. So $nq^2 = (nk)^2 = n^2k^2$, and therefore $q^2 = nk^2$.*

If we did know that p was a multiple of n, then this part would be correct.

> *So q^2 is a multiple of n, and therefore q is a multiple of n, contrary to the assumption that p/q is in lowest terms.* □

This is the same mistake as earlier, but with q instead of p. We can't conclude in either case that p or q is a multiple of n, and so no contradiction is reached after all; the proof is not valid.

1.11 For which natural numbers n is $\sqrt{n}$ irrational? Prove your answer.

Our answer to this question will generalize and unify all the answers we gave above, which will become special cases of this general result. This process therefore illustrates a frequent pattern in mathematics, where one finds the general argument only after having made arguments in various special cases. Faced with a mathematical question or collection of questions, one proves at first what one can about it; and when the argument is therefore grasped more fully as a result of this progress, one often realizes that a slightly more general argument can prove a more general fact; and then a still more general argument proves a still more general fact, generalizing again and again. Eventually, in the fortunate cases, one arrives at a satisfying summative result, which unifies the earlier arguments while using essentially similar reasoning, reaching the unifying general fact to which the argument was tending, from which all the earlier results follow as immediate special cases. That is the pattern of this case, where at first we proved $\sqrt{2}$ is irrational, but then generalized to $\sqrt{p}$ for prime numbers p, and $\sqrt{2m}$ where m is odd. What is it, really, that is making the arguments work? We can generalize to the following general account.

Theorem. *For any natural number n, the number $\sqrt{n}$ is rational if and only if n is a perfect square. That is, $\sqrt{n}$ is rational if and only if $n = k^2$ for some natural number k, in which case $\sqrt{n} = k$.*

In particular, it follows that $\sqrt{n}$ is a rational number if and only if it is itself a natural number. Much of the argument is contained in the following lemma.

Lemma. *A natural number n is a perfect square if and only if every exponent in the prime factorization of n is even.*

Proof. For the forward direction, notice that if one has a perfect square $n = k^2$, then the prime factorization of n is obtained by simply squaring the prime factorization of k, and this will result in all even exponents. That is, if we have the prime factorization of k as

$$k = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r},$$

then the prime factorization of the square is

$$n = k^2 = \left(p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}\right)^2 = p_1^{2e_1} p_2^{2e_2} \cdots p_r^{2e_r},$$

and this has all exponents even.

Conversely, if the prime factorization of a natural number n has all even exponents

$$n = p_1^{2e_1} p_2^{2e_2} \cdots p_r^{2e_r},$$

A Note to the Instructor

In this book, I have assembled a collection of what I find to be compelling mathematical statements with interesting elementary proofs, illustrating diverse proof methods and intended to develop a beginner's proof-writing skills. All who aspire toward mathematics, who want to engage fully with the mathematical craft by undertaking a mathematical analysis and constructing their own proofs of mathematical statements, will benefit from this text, whether they read it as part of a university proof-writing course or study it on their own.

I should like to emphasize, however, that the book is not an axiomatic development of its topics from first principles. The reason is that, while axiomatic developments certainly involve proof writing, I find that they are also often burdened, especially in their beginnings, with various tedious matters. Think of the need, for example, to establish the associativity of integer addition from its definition. I find it sensible, in contrast, to separate the proof-writing craft in its initial or introductory stages from the idea that an entire mathematical subject can be developed from weak axiomatic principles. I also find it important to teach proof writing with mathematically compelling, enjoyable examples, which can inspire a deeper interest in and curiosity about mathematics; students will then be motivated to work through other examples on their own.

So the proofs in this book are not built upon any explicitly given list of axioms but, rather, appeal to very general mathematical principles with which I expect the reader is likely familiar. My hope is that students, armed with the proof-writing skills they have gained from this text, will go on to undertake axiomatic treatments of mathematical subjects, such as number theory, algebra, set theory, topology, and analysis.

The book is organized around mathematical themes, rather than around methods of proof, such as proofs by contradiction, proofs by cases, proofs of if-then statements, or proofs of biconditionals. To my way of thinking, mathematical ideas are best conceived of and organized mathematically; other organizational plans would ultimately be found artificial. I do not find proofs by contradiction, for example, to be a natural or robust mathematical category. Such a proof, after all, might contain essentially the same mathematical insights

2 Multiple Proofs

Mathematical progress is often achieved when one explores alternative proofs for a theorem. A different argument may reveal different aspects of the problem or different avenues of generalization. Here, we explore several such aspects arising from the various alternative proofs given in the main text of the simple fact that $n^2 - n$ is always even for any natural number.

> **2.1** Prove that the sum, difference, and product of two even numbers is even. Similarly, prove that the sum and difference of two odd numbers is even, but the product of odd numbers is odd.

To provide a solution to this exercise, we need first to know exactly what it means to be an even number or an odd number. What is the definition? The *even* integers are those that are a multiple of 2, the numbers of the form $2k$, where k is an integer. The *odd* integers have the form $2k + 1$, with a remainder of 1 when dividing by 2. By the Euclidean algorithm, every number has a remainder either of 0 or 1 when dividing by 2, and so every number is either even or odd, but never both.

Theorem. *The sum, difference, and product of even numbers remain even.*

Proof. This is quite easy to see. Suppose we have two even numbers, say, $2k$ and $2r$. If we add them, we get $2k + 2r = 2(k + r)$, which is even because it is a multiple of 2. If we subtract, we get $2k - 2r = 2(k - r)$, which is even because it is a multiple of 2. And if we multiply them, we get $2k \cdot 2r = 2(2kr)$, which is even because it is a multiple of 2. So the sum, difference, and product of any two even numbers remain even. □

Theorem. *The sum and difference of two odd numbers are even, but the product of two odd numbers is odd.*

Proof. Suppose that we have two odd numbers, $2k + 1$ and $2r + 1$. If we add them, we get

$$(2k + 1) + (2r + 1) = 2k + 2r + 2 = 2(k + r + 1),$$

which is even, because it is a multiple of 2. If we subtract them, we get

$$(2k + 1) - (2r + 1) = 2k - 2r = 2(k - r),$$

which is even, because it is a multiple of 2. But if we multiply them, we get

$$(2k + 1)(2r + 1) = 2k \cdot 2r + 2k + 2r + 1 = 2(2kr + k + r) + 1,$$

which is odd because it is a multiple of 2 plus 1. □

This latter argument was the same we had considered in exercise 1.1 as a generalization of the claim that the square of any odd number is odd.

> **2.3** True or false: if the product of one pair of positive integers is larger than the product of another pair, then the sum also is larger.

The question is whether $pq > rs$ implies $p + q > r + s$ in the positive integers.

Theorem. *The statement is false. It is not always true that if the product of one pair of positive integers is larger than the product of another pair, then the sum also is larger.*

Proof. To prove that the statement is false, it suffices to give a counterexample. Consider the two pairs $2, 3$ and $1, 5$. The product of the first pair is larger than the product of the second pair, because $2 \cdot 3 = 6 > 5 = 1 \cdot 5$, but the sum of the first pair is only $2 + 3 = 5$, whereas the sum of the second pair is $1 + 5 = 6$. So this pair of pairs is a counterexample to the statement, and therefore the statement is not true in general. □

To refute a universal statement, it suffices to give a particular counterexample, and for clarity it is often best to give a very specific counterexample when possible.

> **2.5** Prove that the product of k consecutive integers is always a multiple of k.

Theorem. *The product of k consecutive integers is always a multiple of k.*

This theorem exhibits a common situation in mathematics, where one proves a theorem by one argument, perhaps by a comparatively straightforward argument, but actually, a considerably stronger result can be proved, by a somewhat trickier argument. In this case, we have the simple result here and a much stronger result, which I shall explain after exercise 2.7. For now, let's prove just the result that is claimed here.

Please use the theorem-proof format of writing even when solving the exercises in this text. For example, suppose the exercise says:

> **Exercise.** Prove that every hibdab is hobnob.

Then you should begin your solution not by starting directly with your proof, or by rewriting those instructions. Rather, you begin by writing a clear statement of what you are proving, like this:

> **Theorem.** *Every hibdab is hobnob.*
>
> *Proof.* And so on with your argument. □

Notice that this turns the instruction statement of the exercise into a new, clear mathematical statement. It would have made no sense to prove the original assertion, "Prove that every hibdab is a hobnob," because that is not a mathematical statement, a statement that might be true or false, but is rather an imperative, an *instruction* about what we should do. We carry out that instruction by formulating a clear mathematical statement as our theorem and then proving this statement. In this way, you shall turn every exercise into your own formal theorem statement and proof.

Let me say lastly that I have also gathered together in this text a collection of what I find to be sound mathematical habits of mind, bits of mathematical wisdom or advice that I believe to be beneficial or even fundamental to sound mathematical practice, highlighted in boxes at the end of each chapter. Adopting these habits, I believe, will help an aspiring mathematician solve a problem, find an elusive proof, or write better proofs. Let me mention one of them right now that we have just discussed.

> **Use the theorem-proof format.** In all your mathematical exercises, write in the theorem-proof style. State a clear claim in your theorem statement. State lemmas, corollaries, and definitions as appropriate. Give a separate, clearly demarcated proof for every formally stated mathematical claim.

About the Author

I am an active research mathematician working on the mathematics and philosophy of the infinite. I have published about one hundred research articles on diverse topics in mathematical logic and set theory, ranging from forcing and large cardinals to infinitary computability and infinite game theory, including infinite chess. My blog, *Mathematics and Philosophy of the Infinite*, features a variety of mathematical posts and commentary, including an accessible series on "Math for Kids," and the reader can discover what pricks my mathematical fancy on Twitter (see links below). In addition, I have posted over one thousand research-level mathematical arguments on MathOverflow, the popular mathematics question-and-answer forum, which is becoming a fundamental tool for mathematical research. Interested readers can therefore find many of my mathematical proofs and arguments online. For me, mathematics has been an enjoyable lifelong learning process; I continually strive to improve.

I am also a mathematical philosopher, working in mathematical and philosophical logic with a focus on infinity, especially in set theory, the philosophy of set theory, and the philosophy of mathematics. In recent work, I have been exploring pluralism in the foundations of mathematics by introducing and investigating the multiverse view in set theory and the mathematics and philosophy of potentialism. My forthcoming book *Lectures on the Philosophy of Mathematics*, upon which I have based my lectures here in Oxford, emphasizes a mathematically grounded perspective on the subject.

I have taught college-level mathematics for over twenty-five years, mostly at the City University of New York, where I have taught essentially every proof-based undergraduate topic in pure mathematics, as well as many graduate-level topics, especially in logic. At CUNY I was Professor of Mathematics, Professor of Philosophy, and Professor of Computer Science.

I have recently and quite happily taken up a new position as Professor of Logic at the University of Oxford, where I am a member of the Faculty of Philosophy, affiliated member of the Mathematics Institute, and the Sir Peter Strawson Fellow in Philosophy at University College. I have also held diverse visiting professor positions over the years, at New York

University, Carnegie Mellon University, the University of Cambridge, the University of Toronto, Kobe University, the University of Amsterdam, the University of Münster, the University of California at Berkeley, and elsewhere. I earned my B.S. in mathematics (1988) from the California Institute of Technology and my Ph.D. in mathematics (1994) at the University of California at Berkeley.

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1 A Classical Beginning

One of the classical gems of mathematics—and to my way of thinking, a pinnacle of human achievement—is the ancient discovery of incommensurable numbers, quantities that cannot be expressed as the

The Pythagoreans discc e side and diagonal of a square have no common unit of measure; there is no smaller unit length of which they are both integral multiples; the quantities are *incommensurable*. If you divide the side of a square into ten units, then the diagonal will be a little more than fourteen of them. If you divide the side into one hundred units, then the diagonal will be a little more than 141; if one thousand, then a little more than 1414. It will never come out exactly. One sees those approximation numbers as the initial digits of the decimal expansion:

$$\sqrt{2} = 1.4142135623730950488016887242096980785 6\ldots$$

The discovery shocked the Pythagoreans. It was downright heretical, in light of their quasi-religious number-mysticism beliefs, which aimed to comprehend all through proportion and ratio, taking numbers as a foundational substance. According to legend, the man who made the discovery was drowned at sea, perhaps punished by the gods for impiously divulging the irrational.

Slightly revised proof of theorem 1. Suppose toward contradiction that $\sqrt{2}$ is rational. So $\sqrt{2} = p/q$ for some integers p and q, and we may assume that the numerator p is chosen as small as possible for such a representation. It follows as before that $2q^2 = p^2$, and so p^2 and hence also p is even. So $p = 2k$ for some k, which implies that $q^2 = 2k^2$ as before, and so q^2 and hence also q is even. So $q = 2r$ for some r, and consequently $\sqrt{2} = p/q = (2k)/(2r) = k/r$. We have therefore found a rational representation of $\sqrt{2}$ using a smaller numerator, contradicting our earlier assumption. So $\sqrt{2}$ is not rational. □

This way of arguing, although very similar to the original argument, does not require putting fractions in lowest terms. Furthermore, an essentially similar idea can be used to prove that indeed every fraction can be put in lowest terms.

1.2 Lowest terms

What does it mean for a fraction p/q to be in lowest terms? It means that p and q are *relatively prime*, that is, that they have no common factor, a number $k > 1$ that divides both of them. I find it interesting that the property of being in lowest terms is not a property of the rational number itself but rather a property of the fractional expression used to represent the number. For example, $\frac{3}{6}$ is not in lowest terms and $\frac{1}{2}$ is, yet we say that they are equal: $\frac{3}{6} = \frac{1}{2}$. But how can two things be identical if they have different properties? These two expressions are equal in that they describe the same rational number; the values of the expressions are the same, even though the expressions themselves are different. Thus, we distinguish between the description of a number and the number itself, between our talk about a number and what the number actually is. It is a form of the *use/mention* distinction, the distinction between syntax and semantics at the core of the subject of mathematical logic. How pleasant to see it arise in the familiar elementary topic of lowest terms.

Lemma 2. *Every fraction can be put in lowest terms.*

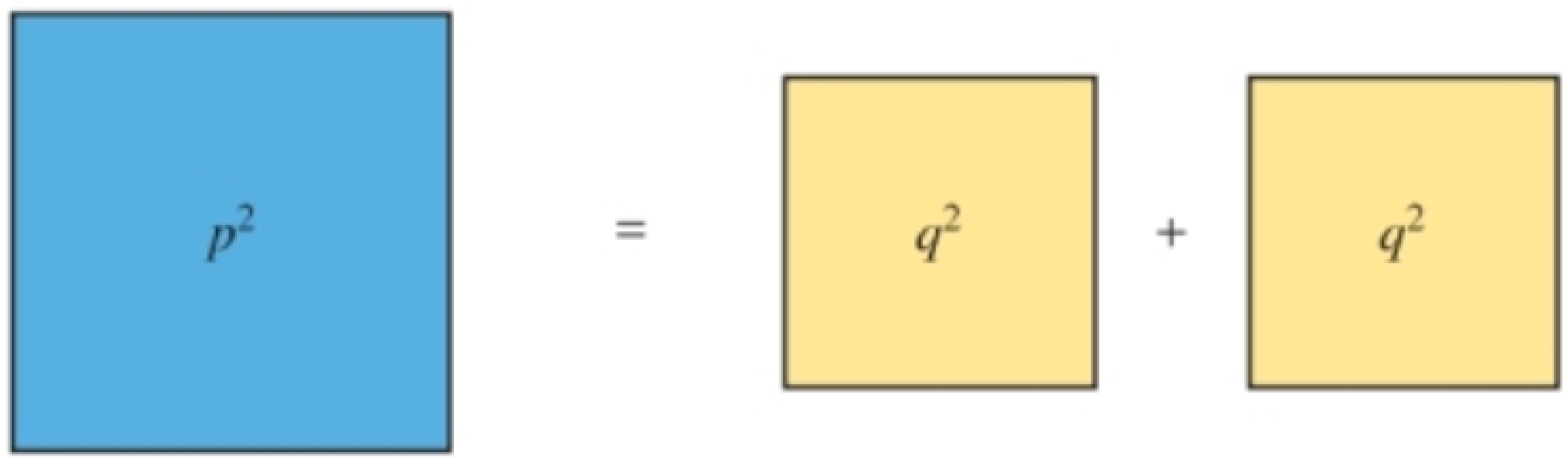
Proof. Consider any fraction p/q, where p and q are integers and $q \neq 0$. Let p' be the smallest nonnegative integer for which there is an integer q' with $\frac{p}{q} = \frac{p'}{q'}$. That is, we consider a representation $\frac{p'}{q'}$ of the original fraction $\frac{p}{q}$ whose numerator p' is as small as possible. I claim that it follows that p' and q' are relatively prime, since if they had a common factor, we could divide it out and thereby make an instance of a fraction equal to p/q with a smaller numerator. But p' was chosen to be smallest, and so there is no such common factor. Therefore, p'/q' is in lowest terms. □

This proof and the previous proof of theorem 1 relied on a more fundamental principle, the least-number principle, which asserts that if there is a natural number with a certain property, then there is a smallest such number with that property. In other words, every nonempty set of natural numbers has a least element. This principle is closely connected with the principle of mathematical induction, discussed in chapter 4. For now, let us simply take it as a basic principle that if there is a natural number with a property, then there is a smallest such number with that property.

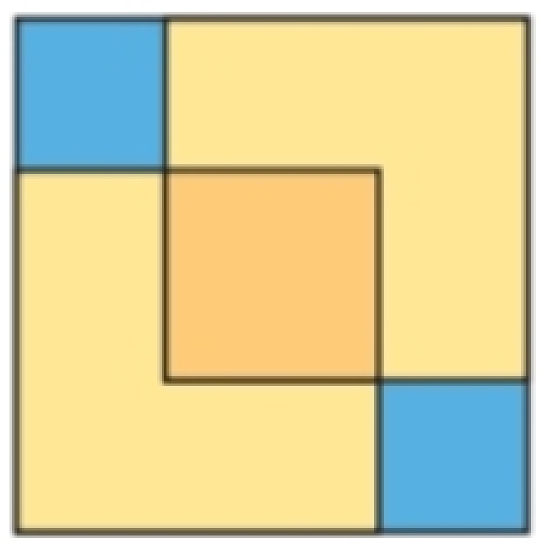
1.3 A geometric proof

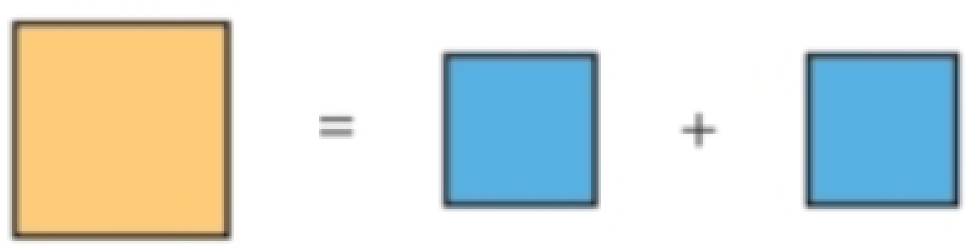
Let us now give a second proof of the irrationality of $\sqrt{2}$, one with geometric character, due to Stanley Tennenbaum. Mathematicians have found dozens of different proofs of this classic result, many of them exhibiting a fundamentally different character from what we saw above.

A geometric proof of theorem 1. If $\sqrt{2}$ is rational p/q, then as before, we see that $p^2 = q^2 + q^2$, which means that some integer square has the same area as two copies of another smaller integer square.



We may choose these squares to have the smallest possible integer sides so as to realize this feature. Let us arrange the two medium squares overlapping inside the larger square, as shown here at the right. Since the large blue square had the same area as the two medium gold squares, it follows that the small orange central square of overlap must be exactly balanced by the two smaller uncovered blue squares in the corners. That is, the area of overlap is exactly the same as the area of the two uncovered blue corner spaces. Let us pull these smaller squares out of the figure to illustrate this relation as follows.





Notice that the squares in this smaller instance also have integer-length sides, since their lengths arise as differences in the side lengths of previous squares. So we have found a strictly smaller integer square that is the sum of another integer square with itself, contradicting our assumption that the original square was the smallest such instance. □