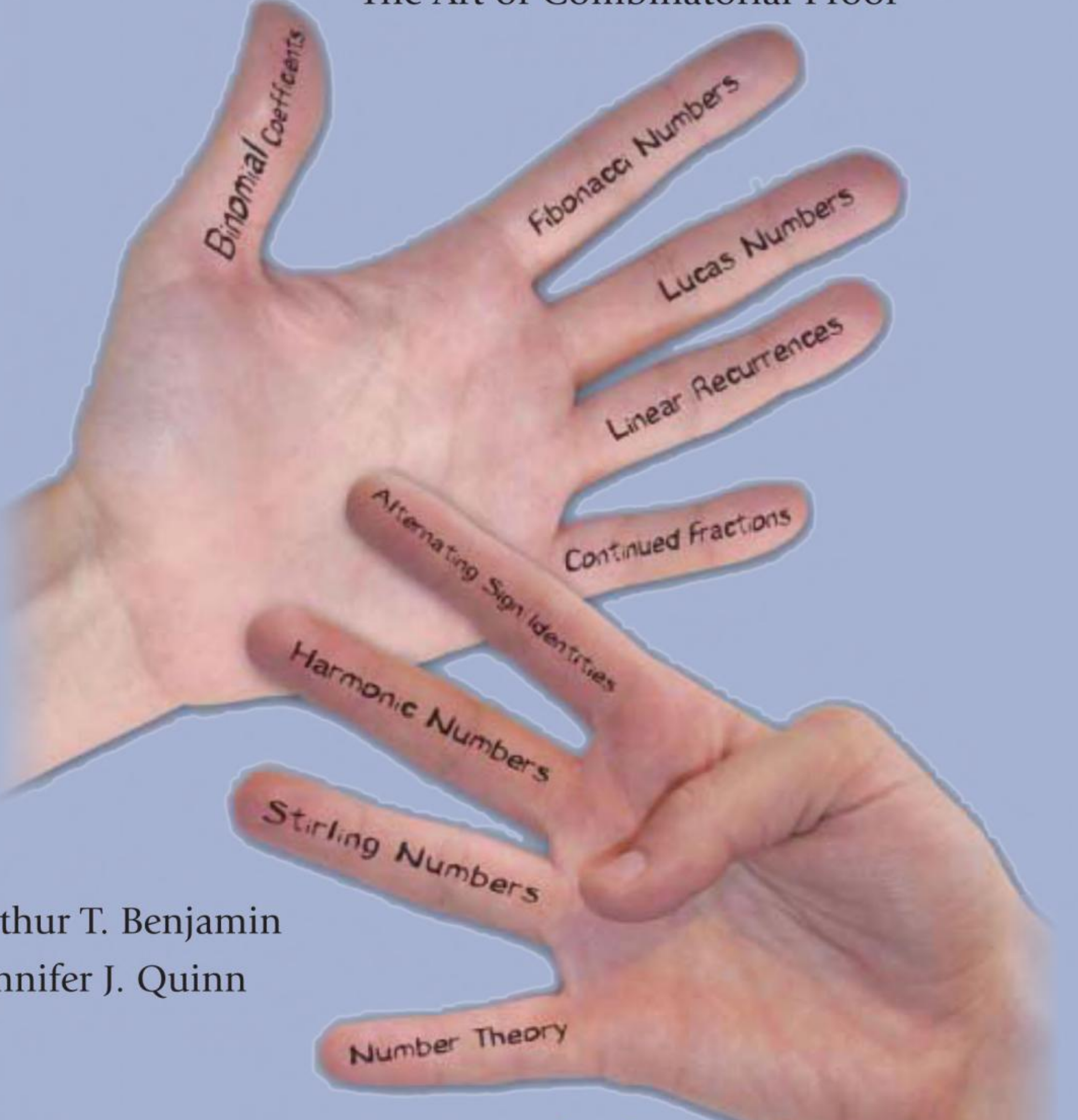
AMS / MAA | DOLCIANI MATHEMATICAL EXPOSITIONS VOL 27

Proofs the Really Count

The Art of Combinatorial Proof



Arthur T. Benjamin
Jennifer J. Quinn

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CHAPTER 1

Fibonacci Identities

Definition The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

The first few numbers in the sequence of Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, $89, 144, \ldots$.

1.1 Combinatorial Interpretation of Fibonacci Numbers

How many sequences of 1s and 2s sum to n? Let's call the answer to this counting question f_n. For example, $f_4 = 5$ since 4 can be created in the following 5 ways:

$$1+1+1+1, \quad 1+1+2, \quad 1+2+1, \quad 2+1+1, \quad 2+2.$$

Table 1.1 illustrates the values of f_n for small n. The pattern is unmistakable; f_n begins like the Fibonacci numbers. In fact, f_n will continue to grow like Fibonacci numbers, that is for $n > 2$, f_n satisfies $f_n = f_{n-1} + f_{n-2}$. To see this combinatorially, we consider the first number in our sequence. If the first number is 1, the rest of the sequence sums to $n - 1$, so there are f_{n-1} ways to complete the sequence. If the first number is 2, there are f_{n-2} ways to complete the sequence. Hence, $f_n = f_{n-1} + f_{n-2}$.

For our purposes, we prefer a more visual representation of f_n. By thinking of the 1s as representing *squares* and the 2s as representing *dominoes*, f_n counts the number of ways to *tile* a board of length n with squares and dominoes. For simplicity, we call a length n board an *n-board.* Thus $f_4 = 5$ enumerates the tilings:

Figure 1.1. All five square-domino tilings of the 4-board

We let $f_0 = 1$ count the empty tiling of the 0-board and define $f_{-1} = 0$. This leads to a combinatorial interpretation of the Fibonacci numbers.

Combinatorial Theorem 1 *Let f_n count the ways to tile a length n board with squares and dominoes. Then f_n is a Fibonacci number. Specifically, for $n \geq -1$,*

$$f_n = F_{n+1}.$$

1	2	3	4	5	6
1	11	111	1111	11111	111111
	2	12	112	1112	11112
		21	121	1121	11121
			211	1211	11211
			22	122	1122
				2111	12111
				212	1212
				221	1221
					21111
					2112
					2121
					2211
					222
$f_1 = 1$	$f_2 = 2$	$f_3 = 3$	$f_4 = 5$	$f_5 = 8$	$f_6 = 13$

Table 1.1. f_n and the sequence of 1s and 2s summing to n for $n = 1, 2, \ldots, 6$.

1.2 Identities

Elementary Identities

Mathematics is the science of patterns. As we shall see, the Fibonacci numbers exhibit many beautiful and surprising relationships. Although Fibonacci identities can be proved by a myriad of methods, we find the combinatorial approach ultimately satisfying.

For combinatorial convenience, we shall express most of our identities in terms of f_n instead of F_n. Although other combinatorial interpretations of Fibonacci numbers exist (see exercises 1–9), we shall primarily use the tiling definition given here.

In the proof of our first identity, as with most proofs in this book, one of the answers to the counting question breaks the problem into disjoint cases depending on some property. We refer to this as *conditioning* on that property.

Identity 1 *For* $n \geq 0$, $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

Question: How many tilings of an $(n+2)$-board use at least one domino?

Answer 1: There are f_{n+2} tilings of an $(n+2)$-board. Excluding the "all square" tiling gives $f_{n+2} - 1$ tilings with at least one domino.

Answer 2: Condition on the location of the last domino. There are f_k tilings where the last domino covers cells $k+1$ and $k+2$. This is because cells 1 through k can be tiled in f_k ways, cells $k+1$ and $k+2$ must be covered by a domino, and cells $k+3$ through $n+2$ must be covered by squares. Hence the total number of tilings with at least one domino is $f_0 + f_1 + f_2 + \cdots + f_n$ (or equivalently $\sum_{k=0}^{n} f_k$). See Figure 1.2.

Identity 2 *For* $n \geq 0$, $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}$.

Question: How many tilings of a $(2n+1)$-board exist?

Answer 1: By definition, there are f_{2n+1} such tilings.

Answer 2: Condition on the location of the last square. Since the board has odd length, there must be at least one square and the last square occupies an odd-numbered cell. There are f_{2k} tilings where the last square occupies cell $2k+1$, as illustrated in Figure 1.3. Hence the total number of tilings is $\sum_{k=0}^{n} f_{2k}$.

Many Fibonacci identities depend on the notion of breakability at a given cell. We say that a tiling of an n-board is *breakable* at cell k, if the tiling can be decomposed into two tilings, one covering cells 1 through k and the other covering cells $k+1$ through n. On the other hand, we call a tiling *unbreakable* at cell k if a domino occupies cells k and $k+1$.

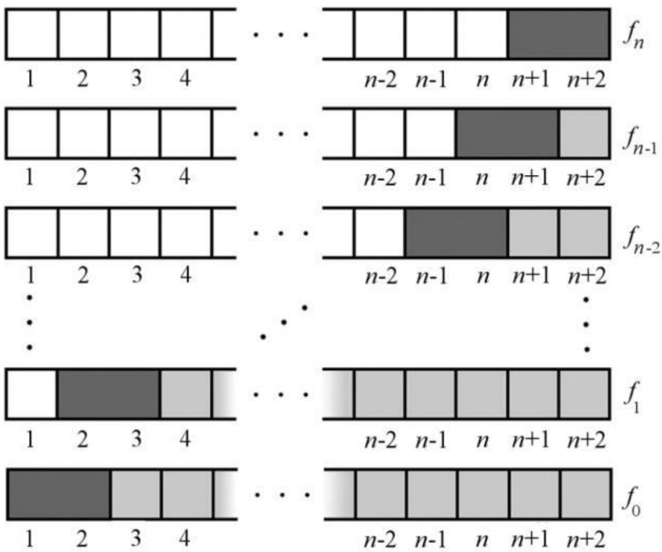


Figure 1.2. To see that $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$, tile an $(n+2)$-board with squares and dominoes and condition on the location of the last domino.

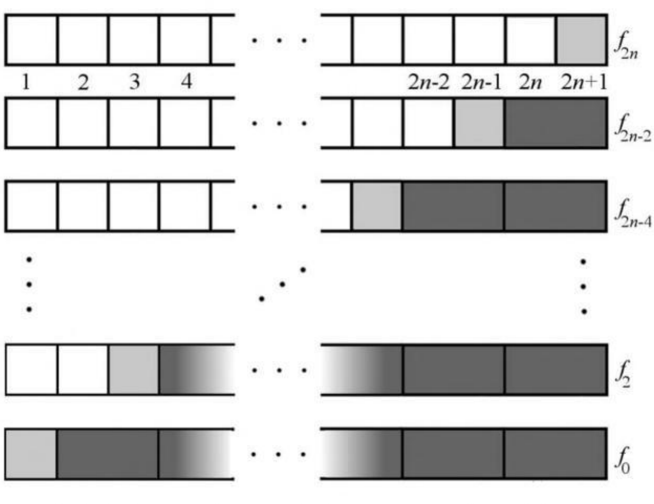


Figure 1.3. To see that $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}$, tile a $(2n+1)$-board with squares and dominoes and condition on the location of the last square.

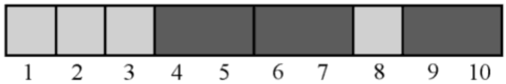


Figure 1.4. A 10-tiling that is breakable at cells $1, 2, 3, 5, 7, 8, 10$ and unbreakable at cells $4, 6, 9$.

For example, the tiling of the 10-board in Figure 1.4 is breakable at cells $1, 2, 3, 5, 7, 8, 10$, and unbreakable at cells $4, 6, 9$. Notice that a tiling of an n-board (henceforth abbreviated an *n-tiling*) is always breakable at cell n. We apply these ideas to the next identity.

Identity 3 *For* $m, n \geq 0$, $f_{m+n} = f_m f_n + f_{m-1} f_{n-1}$.

Question: How many tilings of an $(m+n)$-board exist?

Answer 1: There are f_{m+n} $(m+n)$-tilings.

Answer 2: Condition on breakability at cell m.

An $(m+n)$-tiling that is breakable at cell m, is created from an m-tiling followed by an n-tiling. There are $f_m f_n$ of these.

An $(m+n)$-tiling that is unbreakable at cell m must contain a domino covering cells m and $m+1$. So the tiling is created from an $(m-1)$-tiling followed by a domino followed by an $(n-1)$-tiling. There are $f_{m-1} f_{n-1}$ of these.

Since a tiling is either breakable or unbreakable at cell m, there are $f_m f_n + f_{m-1} f_{n-1}$ tilings altogether. See Figure 1.5.

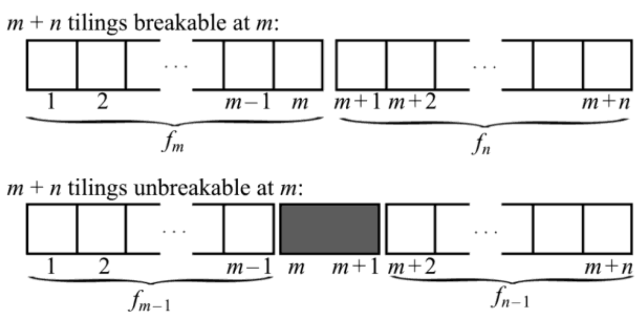


Figure 1.5. To prove $f_{m+n} = f_m f_n + f_{m-1} f_{n-1}$ count $(m+n)$-tilings based on whether or not they are breakable or unbreakable at m.

The next two identities relate Fibonacci numbers to binomial coefficients. We shall say more about combinatorial proofs with binomial coefficients in Chapter 5. For now, recall the following combinatorial definition for binomial coefficients.

Definition The *binomial coefficient* $\binom{n}{k}$ is the number of ways to select k elements from an n-element set.

Notice that $\binom{n}{k} = 0$ whenever $k > n$, so the sum in the identity below is finite.

Identity 4 *For* $n \geq 0$, $\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = f_n$.

Question: How many tilings of an n-board exist?

Answer 1: There are f_n n-tilings.

Answer 2: Condition on the number of dominoes. How many n-tilings use exactly i dominoes? For the answer to be nonzero, we must have $0 \le i \le n/2$. Such tilings necessarily use $n-2i$ squares and therefore use a total of $n-i$ tiles. For example, Figure 1.6 is a 10-tiling that uses exactly three dominoes and four squares. The dominoes occur as the fourth, fifth, and seventh tiles. The number of ways to select i of these $n-i$ tiles to be dominoes is $\binom{n-i}{i}$. Hence there are $\sum_{i\ge 0}\binom{n-i}{i}$ n-tilings.

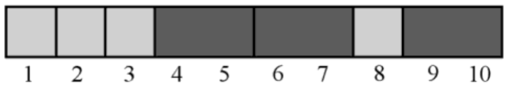


Figure 1.6. There are $\binom{7}{3}$ 10-tilings that use exactly three dominoes. Such a 10-tiling uses exactly seven tiles and is defined by which three of the seven tiles are dominoes. Here the fourth, fifth, and seventh tiles are dominoes.

Identity 5 *For* $n \ge 0$, $\displaystyle\sum_{i\ge 0}\sum_{j\ge 0}\binom{n-i}{j}\binom{n-j}{i} = f_{2n+1}.$

Question: How many tilings of a $(2n+1)$-board exist?

Answer 1: There are f_{2n+1} $(2n+1)$-tilings.

Answer 2: Condition on the number of dominoes on each side of the *median* square.

Any tiling of a $(2n+1)$-board must contain an odd number of squares. Thus one square, which we call the median square, contains an equal number of squares to the left and right of it. For example, the 13-tiling in Figure 1.7 has five squares. The median square, the third square, is located in cell 9.

How many tilings contain exactly i dominoes to the left of the median square and exactly j dominoes to the right of the median square? Such a tiling has $(i+j)$ dominoes and therefore $(2n+1)-2(i+j)$ squares. Hence the median square has $n-i-j$ squares on each side of it. Since the left side has $(n-i-j)+i = n-j$ tiles, of which i are dominoes, there are $\binom{n-j}{i}$ ways to tile to the left of the median square. Similarly, there are $\binom{n-i}{j}$ ways to tile to the right of the median square. Hence there are $\binom{n-i}{j}\binom{n-j}{i}$ tilings altogether.

As i and j vary, we obtain the total number of $(2n+1)$-tilings as

$$\sum_{i\ge 0}\sum_{j\ge 0}\binom{n-i}{j}\binom{n-j}{i}.$$

Figure 1.7. The 13-tiling above has three dominoes left of the median square and one domino to the right of the median square. The number of such tilings is $\binom{5}{3}\binom{3}{1}$.

The next identity is 'prettier' when stated as $F_{2n} = \sum_{k=0}^{n}\binom{n}{k}F_k$.

Identity 6 *For* $n \geq 0$, $f_{2n-1} = \sum_{k=1}^{n} \binom{n}{k} f_{k-1}$.

Question: How many $(2n-1)$-tilings exist?

Answer 1: f_{2n-1}.

Answer 2: Condition on the number of squares that appear among the first n tiles. Observe that a $(2n-1)$-tiling must include at least n tiles, of which at least one is a square. If the first n tiles consist of k squares and $n-k$ dominoes, then these tiles can be arranged $\binom{n}{k}$ ways and cover cells 1 through $2n-k$. The remaining board has length $k-1$ and can be tiled f_{k-1} ways. See Figure 1.8.

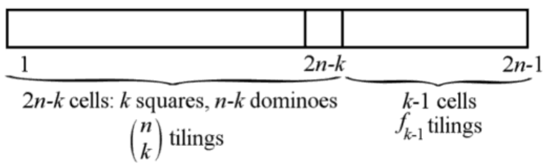


Figure 1.8. There are $\binom{n}{k} f_{k-1}$ tilings of a $(2n-1)$-board where the first n tiles contain k squares and $n-k$ dominoes.

For the next identity, we use the combinatorial technique of finding a correspondence between two sets of objects. In particular, we use a 1-to-3 correspondence between the set of n-tilings and the set of $(n-2)$-tilings and $(n+2)$-tilings.

Identity 7 *For* $n \geq 1$, $3f_n = f_{n+2} + f_{n-2}$.

Set 1: Tilings of an n-board. By definition, this set has size f_n.

Set 2: Tilings of an $(n+2)$-board or an $(n-2)$-board. This set has size $f_{n+2} + f_{n-2}$.

Correspondence: To prove the identity, we establish a *1-to-3 correspondence* between Set 1 and Set 2. That is, for every object in Set 1, we can create three objects in Set 2 in such a way that every object in Set 2 is created exactly once. Hence Set 2 is three times as large as Set 1.

Specifically, for each n-tiling in Set 1, we create the following three tilings that have length $n+2$ or length $n-2$. The first tiling is an $(n+2)$-tiling created by appending a domino to the n-tiling. The second tiling is an $(n+2)$-tiling created by appending two squares to the n-tiling. So far, so good. But what about the third tiling? This will depend on the last tile of the n-tiling. If the n-tiling ends with a square, we insert a domino before that last square to create an $(n+2)$-tiling. If the n-tiling ends with a domino, then we remove that domino to create an $(n-2)$-tiling. See Figure 1.9.

To verify that this is a 1-to-3 correspondence, one should check that every tiling of length $n+2$ or length $n-2$ is created exactly once from some n-tiling. For a given $(n+2)$-tiling, we can find the n-tiling that creates it by examining its ending and removing

i) the last domino (if it ends with a domino) or

ii) the last two squares (if it ends with two squares) or

iii) the last domino (if it ends with a square preceded by a domino).

For a given $(n-2)$-tiling, we simply append a domino for the n-tiling that creates it.

Since Set 2 is three times the size of Set 1, it follows that $f_{n+2} + f_{n-2} = 3f_n$.

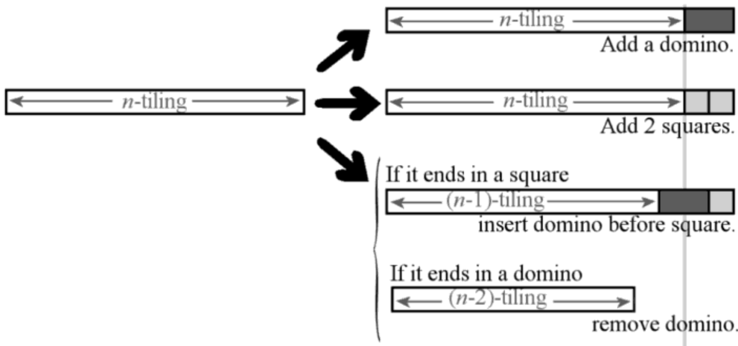


Figure 1.9. A one-to-three correspondence.

Pairs of Tilings

In this subsection, we introduce the technique of *tail swapping*, which will prove to be very useful in several settings.

Consider the two 10-tilings offset as in Figure 1.10. The first one tiles cells 1 through 10; the second one tiles cells 2 through 11. We say that there is a *fault* at cell i, for $2 \leq i \leq 10$, if both tilings are breakable at cell i. We say there is a fault at cell 1 if the first tiling is breakable at cell 1. Put another way, the pair of tilings has a fault at cell i, for $1 \leq i \leq 10$, if neither tiling has a domino covering cells i and $i+1$. The pair of tilings in Figure 1.10 has faults at cells 1, 2, 5, and 7. We define the *tails* of a tiling pair to be the tiles that occur after the last fault. Observe that if we swap the tails of Figure 1.10 we obtain the 11-tiling and the 9-tiling in Figure 1.11, and it has the same faults.

Tail swapping is the basis for the identity below, sometimes referred to as Simson's Formula or Cassini's Identity. At first glance, it may appear unsuitable for combinatorial proof due to the presence of the $(-1)^n$ term. Nonetheless, we will see that this term is merely the "error term" of an "almost" one-to-one correspondence.

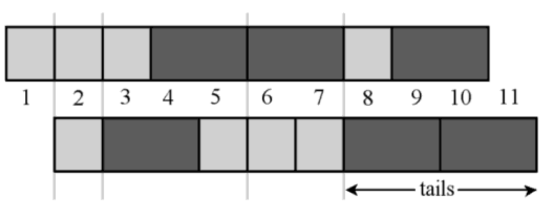


Figure 1.10. Two 10-tilings with their faults (indicated with gray lines) and tails.

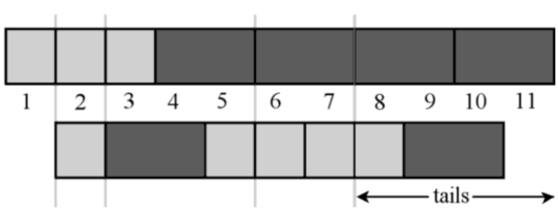


Figure 1.11. After tail swapping, we have an 11-tiling and a 9-tiling with exactly the same faults.

Identity 8 *For* $n \geq 0$, $f_n^2 = f_{n+1}f_{n-1} + (-1)^n$

Set 1: Tilings of two n-boards (a *top* board and a *bottom* board.) By definition, this set has size f_n^2.

Set 2: Tilings of an $(n+1)$-board and an $(n-1)$-board. This set has size $f_{n+1}f_{n-1}$.

Correspondence: First, suppose n is odd. Then the top and bottom board must each have at least one square. Notice that a square in cell i of either board ensures that a fault must occur at cell i or cell $i-1$. Swapping the tails of the two n-tilings produces an $(n+1)$-tiling and an $(n-1)$-tiling with the same faults. This produces a 1-to-1 correspondence between all pairs of n tilings and all tiling pairs of sizes $n+1$ and $n-1$ that have faults. Is it possible for a tiling pair of sizes $n+1$ and $n-1$ to be "fault-free"? Yes, precisely when all dominoes are in "staggered formation" as in Figure 1.12. Thus, when n is odd, $f_n^2 = f_{n+1}f_{n-1} - 1$.

Similarly, when n is even, tail swapping creates a 1-to-1 correspondence between faulty tiling pairs. The only fault-free tiling pair is the all domino tiling of Figure 1.13. Hence when n is even, $f_n^2 = f_{n+1}f_{n-1} + 1$. Considering the odd and even case together produces our identity.

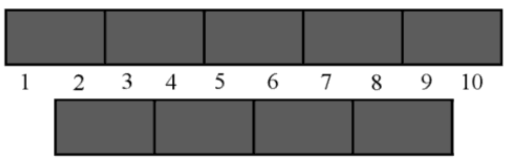


Figure 1.12. When n is odd, there is only one fault-free tiling pair.

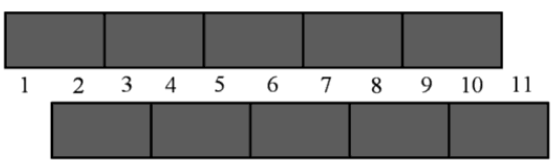


Figure 1.13. When n is even, there is only one fault-free tiling pair.

Identity 9 *For* $n \geq 0$, $\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}$.

Question: How many tilings of an n-board and $(n+1)$-board exist?

Answer 1: There are $f_n f_{n+1}$ such tilings.

Answer 2: Place the $(n+1)$-board directly above the n board as in Figure 1.14, and condition on the location of the last fault. Since both boards begin at cell 1, we

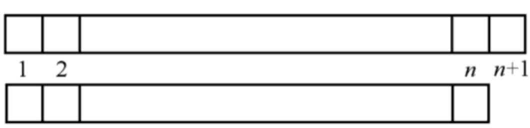


Figure 1.14. There are $f_n f_{n+1}$ ways to tile these two boards.

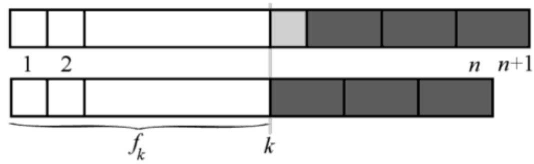


Figure 1.15. There are f_k^2 tilings with last fault at cell k.

shall consider any tiling pair to have a fault at "cell 0". How many tiling pairs have their last fault at cell k, where $0 \leq k \leq n$? There are f_k^2 ways to tile both boards through cell k. To avoid future faults, there is exactly one way to finish the tiling, as in Figure 1.15. (Specifically, all tiles after cell k will be dominoes except for a single square placed on cell $k+1$ in the row whose tail length is odd.) Summing over all possible values of k, gives us $\sum_{k=0}^{n} f_k^2$ tilings.

Advanced Fibonacci Identities

In this subsection we present identities that in our opinion require extra ingenuity. For the first identity, we utilize a method of encoding tilings as binary sequences.

Specifically, for any m-tiling, create the length m binary sequence by converting each square into a "1" and converting each domino into a "01". Equivalently, the ith term of the binary sequence is 1 if and only if the tiling is breakable at cell i. The resulting binary sequence will have no consecutive 0s and will always end with 1. For example, the 9-tiling in Figure 1.16 has binary representation 011101011.

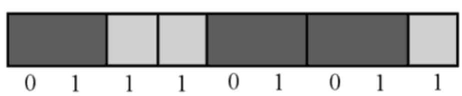


Figure 1.16. The 9-tiling above has binary representation 011101011.

Conversely, a length n binary sequence with no consecutive 0s that ends with 1 represents a unique n-tiling. If such a sequence ends with 0, then it represents an $(n-1)$-tiling (since the last 0 is ignored).

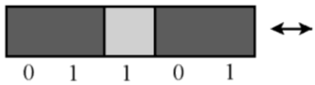
We may now interpret the following identity.

Identity 10 *For* $n \geq 0$, $f_n + f_{n-1} + \sum_{k=0}^{n-2} f_k 2^{n-2-k} = 2^n$.

Question: How many binary sequences of length n exist?

Answer 1: There are 2^n length n binary sequences.

Answer 2: For each binary sequence, we identify a tiling. If a sequence has no consecutive zeros, we identify it with a unique tiling of length n or $n-1$ depending on whether it ended with 1 or 0, respectively. Otherwise, the sequence contains a 00 whose first occurrence appears in cells $k+1$ and $k+2$ for some k, $0 \leq k \leq n-2$. For such a sequence we associate the k-tiling defined by the first k terms of the binary sequence (note that if $k > 0$, then the kth digit must be 1.) For example, the length 11 binary sequence 01101001001 is identified with the 5-tiling "domino-square-domino", as would any binary sequence of the form 0110100$abcd$ where a, b, c, d



01101000000
01101000001
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01101000111
01101001000
01101001001
01101001010
01101001011
01101001100
01101001101
01101001110
01101001111

Figure 1.17. The 5-tiling shown is generated by 16 different binary sequences of length 11, all beginning with 0110100.

can each be 0 or 1. See Figure 1.17. In general, for $0 \le k \le n-2$, each k-tiling will be listed 2^{n-2-k} times. In particular, the empty tiling will be listed 2^{n-2} times.

The next identity is based on the fact that for any $t \ge 0$ a tiling can be broken into segments so that all but the last segment have length t or $t+1$.

Identity 11 *For* $m, p, t \ge 0$, $f_{m+(t+1)p} = \sum_{i=0}^{p} \binom{p}{i} f_t^i f_{t-1}^{p-i} f_{m+i}$.

Question: How many $(m + (t+1)p)$-tilings exist?

Answer 1: $f_{m+(t+1)p}$.

Answer 2: For any tiling of length $m + (t+1)p$, we break it into $p+1$ segments of length $j_1, j_2, \ldots, j_{p+1}$. For $1 \le i \le p$, $j_i = t$ unless that would result in breaking a domino in half—in which case we let $j_i = t+1$. Segment $p+1$ consists of the remaining tiles. Count the number of tilings for which i of the first p segments have length t and the other $p-i$ segments have length $t+1$. These p segments have total length $it + (p-i)(t+1) = (t+1)p - i$. Hence $j_{p+1} = m + i$. Since segments of length t can be covered f_t ways and segments of length $t+1$ must end with a domino and can be covered f_{t-1} ways, there are exactly $\binom{p}{i} f_t^i f_{t-1}^{p-i} f_{m+i}$ such tilings. See Figure 1.18.

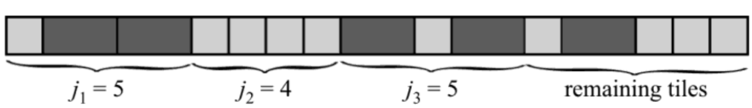

Figure 1.18. When $t = 4$ and $p = 3$, the tiling above is broken into segments of length $j_1 = 5$, $j_2 = 4$, $j_3 = 5$, and $j_4 = 6$.

The next identity reads better when stated in terms of the traditional definition of Fibonacci numbers (where $F_0 = 0$ and $F_1 = 1$ and thus $f_{n-1} = F_n$ for all $n \ge 0$).

Theorem 1 *For* $m \ge 1, n \ge 0$, *if* $m|n$, *then* $F_m | F_n$.

Our combinatorial approach allows us to prove more.

Theorem 2 *For $m \geq 1, n \geq 0$, if m divides n, then f_{m-1} divides f_{n-1}. In fact, if $n = qm$, then $f_{n-1} = f_{m-1} \sum_{j=1}^{q} f_{m-2}^{j-1} f_{n-jm}$.*

Question: When $n = qm$, how many $(n-1)$-tilings exist?

Answer 1: f_{n-1}.

Answer 2: Condition on the smallest j for which the tiling is breakable at cell $jm - 1$. Such a j exists and has value at most q since the tiling is breakable at cell $n-1 = qm-1$. Given j, there are $j-1$ dominoes ending at cells $m, 2m, \ldots, (j-1)m$. The cells preceding these dominoes can be tiled in f_{m-2}^{j-1} ways. Cells $(j-1)m + 1, (j-1)m + 2, \ldots, (jm-1)$ can be tiled f_{m-1} ways. The rest of the board can then be tiled f_{n-jm} ways. See Figure 1.19.

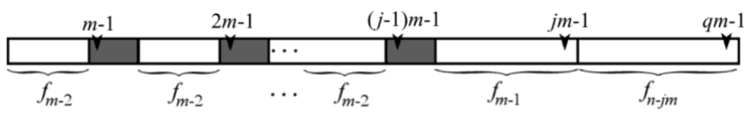

Figure 1.19. There are $f_{m-2}^{j-1} f_{m-1} f_{n-jm}$ ways to tile an $(n-1)$-board when j is the smallest integer for which the tiling is breakable at $jm - 1$.

1.3 A Fun Application

Although the application in this section is not proved entirely by combinatorial means, it utilizes some of the identities from this chapter. Since we have done most of the work to prove it already, it would be a shame to omit it.

For integers a and b, the greatest common divisor, denoted by $\gcd(a, b)$, is the largest positive number dividing both a and b. It is easy to see that for any integer x,

$$\gcd(a, b) = \gcd(b, a - bx), \tag{1.1}$$

since any number that divides both a and b must also divide b and $a - bx$, and vice versa. Two special cases are frequently invoked:

$$\gcd(a, b) = \gcd(b, a - b) \tag{1.2}$$

and

Theorem 3 (Euclidean Algorithm) *If $n = qm + r$, then $\gcd(n, m) = \gcd(m, r)$.*

In the Euclidean algorithm we typically choose $q = \lfloor \frac{n}{m} \rfloor$, so that $0 \leq r < m$. For example, when we apply the Euclidean algorithm to find $\gcd(255, 68)$, we get

$$\gcd(255, 68) = \gcd(68, 51) = \gcd(51, 17) = \gcd(17, 0) = 17.$$

It immediately follows that consecutive Fibonacci numbers are relatively prime, that is

Lemma 4 *For $n \geq 1$, $\gcd(F_n, F_{n-1}) = 1$.*

Proof. This is the world's fastest proof by induction. When $n = 1$, $\gcd(F_1, F_0) = \gcd(1, 0) = 1$. Assuming the lemma holds for the number n, then using (1.2), we get

$$\gcd(F_{n+1}, F_n) = \gcd(F_n, F_{n+1} - F_n) = \gcd(F_n, F_{n-1}) = 1. \qquad \diamond$$

Next we exploit Identity 3 to obtain

Lemma 5 *For* $m, n \geq 0$, $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$.

Proof. $F_{m+n} = f_{m+(n-1)} = f_mf_{n-1} + f_{m-1}f_{n-2} = F_{m+1}F_n + F_mF_{n-1}$. ◇

Finally, we recall that Theorem 1 states if m divides n, then F_m divides F_n. We are now ready to state and prove one of the most beautiful properties of Fibonacci numbers.

Theorem 6 *For* $m \geq 1$, $n \geq 0$, $\gcd(F_n, F_m) = F_{\gcd(n,m)}$.

Proof. Suppose $n = qm + r$, where $0 \leq r < m$. By Lemma 5, $F_n = F_{qm+r} = F_{qm+1}F_r + F_{qm}F_{r-1}$. Thus

$$\gcd(F_n, F_m) = \gcd(F_m, F_{qm+1}F_r + F_{qm}F_{r-1})$$

but by (1.1), we can subtract multiples of F_m from the second term and not change the greatest common divisor. Since by Theorem 1, F_{qm} is a multiple of F_m, it follows that

$$\gcd(F_n, F_m) = \gcd(F_m, F_{qm+1}F_r) = \gcd(F_m, F_r), \tag{1.3}$$

where the last equality follows since F_m (a divisor of F_{qm}) is relatively prime to F_{qm+1} by Lemma 4.

But what do we have here? Equation (1.3) is the same as the Euclidean Algorithm, but with Fs on top. Thus, for example,

$$\gcd(F_{255}, F_{68}) = \gcd(F_{68}, F_{51}) = \gcd(F_{51}, F_{17}) = \gcd(F_{17}, F_0) = F_{17},$$

since $F_0 = 0$. The theorem immediately follows. ◇

For the reader interested in seeing even more advanced Fibonacci identities, we recommend reading Chapters 2 and 9. One of the treats in store is a proof of Binet's Formula, an exact formula for the nth Fibonacci number. Specifically

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

An eager reader actually has all the tools necessary to tackle the combinatorial proof and can jump straight to Identity 240.

1.4 Notes

Fibonacci numbers have a long and rich history. They have served as mathematical inspiration and amusement since Leonardo Pisano (filius de Bonacci) first posed his original rabbit reproduction question at the beginning of the 13th century. Fibonacci numbers have touched the lives of mathematicians, artists, naturalists, musicians and more. For a peek at their history, we recommend Ron Knott's impressive web site, Fibonacci Numbers and the Golden Section [32]. Extensive collections of Fibonacci identities are available in Vajda's *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications* [58] and Koshy's *Fibonacci and Lucas Numbers with Applications* [33].

The Fibonacci Society is a professional organization focusing on Fibonacci numbers and related mathematics, emphasizing new results, research proposals, challenging problems, and new proofs of old ideas. They publish a professional journal, *The Fibonacci Quarterly,* and organize a biennial international conference.

Combinatorial interpretations of Fibonacci numbers have existed for a long time and can be surveyed in Basin and Hoggatt's article [1] in the inaugural issue of the *Fibonacci Quarterly* or Stanley's *Enumerative Combinatorics Vol. 1* Chapter 1 exercise 14 [51]. We've chosen the tiling interpretation and notation presented in Brigham et. al. [15] and further developed in [8].

Finally, a bijective proof of Cassini's formula similar to the one given for Identity 8 without tilings was given by Werman and Zeilberger [60].

1.5 Exercises

Prove each of the identities below by a direct combinatorial argument.

Identity 12 *For* $n \geq 1$, $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n} - 1$.

Identity 13 *For* $n \geq 0$, $f_n^2 + f_{n+1}^2 = f_{2n+2}$.

Identity 14 *For* $n \geq 1$, $f_n^2 - f_{n-2}^2 = f_{2n-1}$.

Identity 15 *For* $n \geq 0$, $f_{2n+2} = f_{n+1}f_{n+2} - f_{n-1}f_n$.

Identity 16 *For* $n \geq 2$, $2f_n = f_{n+1} + f_{n-2}$.

Identity 17 *For* $n \geq 2$, $3f_n = f_{n+2} + f_{n-2}$.

Identity 18 *For* $n \geq 2$, $4f_n = f_{n+2} + f_n + f_{n-2}$.

Identity 19 demonstrates how any four consecutive Fibonacci numbers generate a Pythagorean Triple.

Identity 19 *For* $n \geq 1$,

$$(f_{n-1}f_{n+2})^2 + (2f_nf_{n+1})^2 = (f_{n+1}f_{n+2} - f_{n-1}f_n)^2 = (f_{2n+2})^2.$$

Identity 20 *For* $n \geq p$, $f_{n+p} = \sum_{i=0}^{p} \binom{p}{i} f_{n-i}$.

Identity 21 *For* $n \geq 0$, $\sum_{k=0}^{n} (-1)^k f_k = 1 + (-1)^n f_{n-1}$.

Identity 22 *For* $n \geq 0$, $\displaystyle\prod_{k=1}^{n} \left(1 + \frac{(-1)^{k+1}}{f_k^2}\right) = \frac{f_{n+1}}{f_n}$.

Identity 23 *For* $n \geq 0$, $f_0 + f_3 + f_6 + \cdots + f_{3n} = \frac{1}{2} f_{3n+2}$.

Identity 24 *For* $n \geq 1$, $f_1 + f_4 + f_7 + \cdots + f_{3n-2} = \frac{1}{2}(f_{3n} - 1)$.

Identity 25 *For* $n \geq 1$, $f_2 + f_5 + f_8 + \cdots + f_{3n-1} = \frac{1}{2}(f_{3n+1} - 1)$.

Identity 26 *For* $n \geq 0$, $f_0 + f_4 + f_8 + \cdots + f_{4n} = f_{2n}f_{2n+1}$.

Identity 27 *For* $n \geq 1$, $f_1 + f_5 + f_9 + \cdots + f_{4n-3} = f_{2n-1}^2$.

Identity 28 *For* $n \geq 1$, $f_2 + f_6 + f_{10} + \cdots + f_{4n-2} = f_{2n-1}f_{2n}$.

Identity 29 *For* $n \geq 1$, $f_3 + f_7 + f_{11} + \cdots + f_{4n-1} = f_{2n-1}f_{2n+1}$.

Identity 30 *For* $n \geq 0$, $f_{n+3}^2 + f_n^2 = 2f_{n+1}^2 + 2f_{n+2}^2$.

Identity 31 *For* $n \geq 1$, $f_n^4 = f_{n+2}f_{n+1}f_{n-1}f_{n-2} + 1$.

There are many combinatorial interpretations for Fibonacci numbers. Show that the interpretations below are equivalent to tiling a board with squares and dominoes by creating a one-to-one correspondence.

1. For $n \geq 0$, f_{n+1} counts binary n-tuples with no consecutive 0s.
2. For $n \geq 0$, f_{n+1} counts subsets S of $\{1, 2, \ldots, n\}$ such that S contains no two consecutive integers.
3. For $n \geq 2$, f_{n-2} counts tilings of an n-board where all tiles have length 2 or greater.
4. For $n \geq 1$, f_{n-1} counts tilings of an n-board where all tiles have odd length.
5. For $n \geq 1$, f_n counts the ways to arrange the numbers 1 through n so that for each $1 \leq i \leq n$, the ith number is $i - 1$ or i or $i + 1$.
6. For $n \geq 0$, f_{2n+1} counts length n sequences of 0s, 1s, and 2s where 0 is never followed immediately by 2.
7. For $n \geq 1$, $f_{2n-1} = \sum a_1 a_2 \cdots a_r$, where $r \geq 1$ and $a_1, \ldots, a_r$ are positive integers that sum to n. For example, $f_5 = 3 + 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 1 \cdot 1 = 8$. (Hint: $a_1 a_2 \cdots a_r$ counts n-tilings with tiles of any length, where a_j is the length of the jth tile, and one cell covered by each tile is highlighted.)
8. For $n \geq 1$, f_{2n} counts $\sum 2^{\text{number of } a_i \text{ that equal } 1}$, summed over the same set as before. For example, when $n = 3 = 2 + 1 = 1 + 2 = 1 + 1 + 1$, $f_6 = 2^0 + 2^1 + 2^1 + 2^3 = 13$.
9. For $n \geq 1$, f_{n+1} counts binary sequences $(b_1, b_2, \ldots, b_n)$, where $b_1 \leq b_2 \geq b_3 \leq b_4 \geq b_5 \cdots$.

Uncounted Identities

The identities listed below are in need of combinatorial proof.

1. For $n \geq 1$, $f_0^3 + f_1^3 + \cdots + f_n^3 = \dfrac{f_{3n+4} + (-1)^n 6 f_{n-1} + 5}{10}$.
2. For $n \geq 0$, $f_1 + 2f_2 + \cdots + nf_n = (n+1)f_{n+2} - f_{n+4} + 3$.

3. There are identities for mf_n analogous to Identities 16–18 for every integer m.

   (a) For $n \geq 4$, $5f_n = f_{n+3} + f_{n-1} + f_{n-4}$.

   (b) For $n \geq 4$, $6f_n = f_{n+3} + f_{n+1} + f_{n-4}$.

   (c) For $n \geq 4$, $7f_n = f_{n+4} + f_{n-4}$.

   (d) For $n \geq 4$, $8f_n = f_{n+4} + f_n + f_{n-4}$.

   (e) For $n \geq 4$, $9f_n = f_{n+4} + f_{n+1} + f_{n-2} + f_{n-4}$.

   (f) For $n \geq 4$, $10f_n = f_{n+4} + f_{n+2} + f_{n-2} + f_{n-4}$.

   (g) For $n \geq 4$, $11f_n = f_{n+4} + f_{n+2} + f_n + f_{n-2} + f_{n-4}$.

   (h) For $n \geq 6$, $12f_n = f_{n+5} + f_{n-1} + f_{n-3} + f_{n-6}$.

   These identities are examples of Zeckendorf's Theorem which states that every integer can be uniquely written as the sum of nonconsecutive Fibonacci numbers. The coefficients in the above formulas are the same as in the unique expansion of positive integers in nonconsecutive integer powers of $\phi = (1+\sqrt{5})/2$. For example $5 = \phi^3 + \phi^{-1} + \phi^{-4}$ and $6 = \phi^3 + \phi^1 + \phi^{-4}$. Is there a unifying combinatorial approach for all of these identities?

4. For $n \geq 4$, $f_n^3 + 3f_{n-3}^3 + f_{n-4}^3 = 3f_{n-1}^2 + 6f_{n-2}^3$. Jay Cordes has shown us a combinatorial proof that requires breaking the tiling triples into over a dozen different cases. Does something simpler exist?

5. Find a combinatorial interpretation for the *Fibonomial coefficient*

$$\binom{n}{m}_F = \frac{(n!)_F}{(m!)_F((n-m)!)_F},$$

   where $(0!)_F = 1$, and for $k \geq 1$, $(k!)_F = F_k F_{k-1} \cdots F_1$.

CHAPTER 2

Gibonacci and Lucas Identities

Definition The *Gibonacci numbers* G_n are defined by nonnegative integers G_0, G_1 and for $n \geq 2$, $G_n = G_{n-1} + G_{n-2}$.

Definition The *Lucas numbers* L_n are defined by $L_0 = 2$, $L_1 = 1$ and for $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$.

The first few numbers in the sequence of Lucas numbers are 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...

In this chapter, we pursue identities involving *Gibonacci numbers*, which is shorthand for generalized Fibonacci numbers. There are many ways to generalize the Fibonacci numbers, and we shall pursue many of these generalizations in the next chapter, but for our purposes, we say a sequence of nonnegative integers $G_0, G_1, G_2, \ldots$ is a Gibonacci sequence if for all $n \geq 2$,

$$G_n = G_{n-1} + G_{n-2}.$$

Of all the Gibonacci sequences, the initial conditions that lead to the most beautiful identities correspond to the Fibonacci and Lucas numbers.

2.1 Combinatorial Interpretation of Lucas Numbers

As we shall see Lucas numbers operate like Fibonacci numbers running in circles. Define ℓ_n to be the number of ways to tile a circular board composed of n labeled cells with curved squares and dominoes. For example $\ell_4 = 7$ as illustrated in Figure 2.1. Clearly there are more ways to tile a *circular n-board* than a straight n-board since it is now possible for a single domino to cover cells n and 1. We define an *n-bracelet* to be a tiling of a circular n-board. A bracelet is *out-of-phase* when a single domino covers cells n and 1 and *in-phase* otherwise. In Figure 2.1, we see that there are five in-phase 4-bracelets and two out-of-phase 4-bracelets. Figure 2.2 illustrates that $\ell_1 = 1$, $\ell_2 = 3$, and $\ell_3 = 4$. Notice that there are two ways to create a 2-bracelet with a single domino—either in-phase or out-of-phase.

From our initial data, the number of n-bracelets looks like the Lucas sequence. To prove that they continue to grow like the Lucas sequence, we must argue that for $n \geq 3$,

$$\ell_n = \ell_{n-1} + \ell_{n-2}.$$

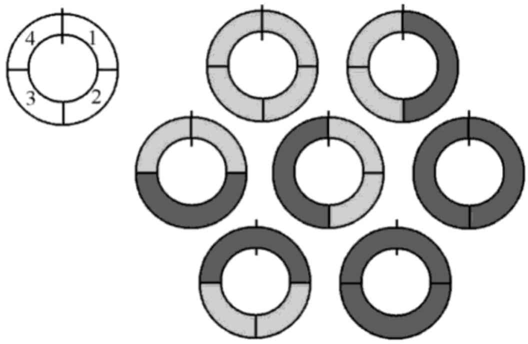

Figure 2.1. A circular 4-board and its seven bracelets. The first five bracelets are in-phase and the last two are out-of-phase.

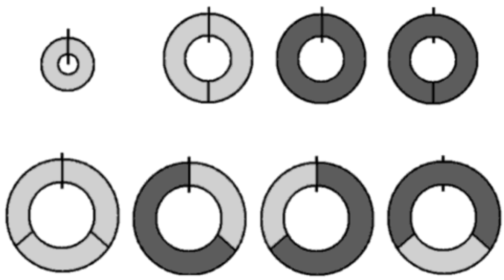

Figure 2.2. There is one 1-bracelet and there are three 2-bracelets and four 3-bracelets.

To see this we simply condition on the *last tile* of the bracelet. We define the *first* tile to be the tile that covers cell 1, which could either be a square, a domino covering cells 1 and 2, or a domino covering cells n and 1. The second tile is the next tile in the clockwise direction, and so on. The last tile is the one that precedes the first tile. Since it is the first tile, not the last, that determines the phase of the tiling, there are ℓ_{n-1} n-bracelets that end with a square and ℓ_{n-2} n-bracelets that end with a domino. By removing the last tile and closing up the resulting gap, we produce smaller bracelets.

To make the recurrence valid for $n = 2$, we define $\ell_0 = 2$, and interpret this to mean that there are two empty tilings of the circular 0-board, an in-phase 0-bracelet and an out-of-phase 0-bracelet. This leads to a combinatorial interpretation of Lucas numbers.

Combinatorial Theorem 2 *For $n \geq 0$, let ℓ_n count the ways to tile a circular n-board with squares and dominoes. Then ℓ_n is the nth Lucas number; that is*

$$\ell_n = L_n.$$

As one might expect, there are many identities with Lucas numbers that resemble Fibonacci identities. In addition, there are many beautiful identities where Lucas and Fibonacci numbers interact.

2.2 Lucas Identities

Identity 32 *For $n \geq 1$, $L_n = f_n + f_{n-2}$.*

Question: How many tilings of a circular n-board exist?

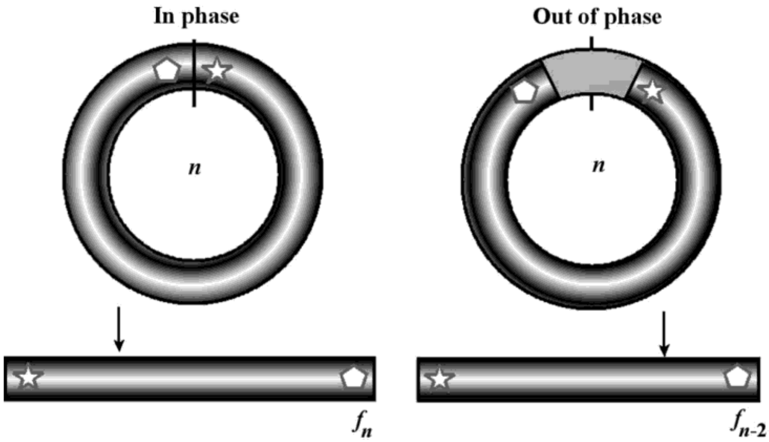

Figure 2.3. Every circular n-bracelet can be reduced to an n-tiling or an $(n-2)$-tiling, depending on its phase.

Answer 1: By Combinatorial Theorem 2, there are L_n n-bracelets.

Answer 2: Condition on whether the tiling is in-phase or out-of-phase. Since an in-phase tiling can be straightened into an n-tiling, there are f_n in-phase bracelets. Likewise, an out-of-phase n-bracelet must have a single domino covering cells n and 1. Cells 2 through $n-1$ can then be covered as a straight $(n-2)$-tiling in f_{n-2} ways. Hence the total number of n-bracelets is $f_n + f_{n-2}$. See Figure 2.3.

The next identity associates an odd-length board tiling with a board and bracelet pair.

Identity 33 *For* $n \geq 0$, $f_{2n-1} = L_n f_{n-1}$.

Set 1: Tilings of a $(2n-1)$-board. This set has size f_{2n-1}.

Set 2: Bracelet-tiling pairs (B, T) where the bracelet has length n and the tiling has length $n-1$. This set has size $L_n f_{n-1}$.

Correspondence: Given a $(2n-1)$-board T^*, there are two cases to consider, as illustrated in Figure 2.4.

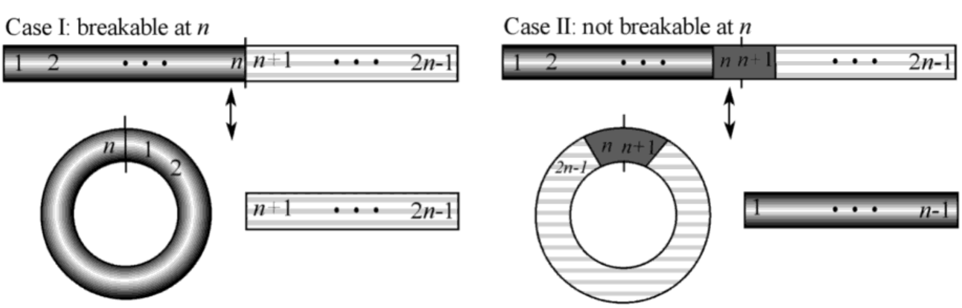

Figure 2.4. A $(2n-1)$-tiling can be converted to an n-bracelet and $(n-1)$-tiling. In our correspondence, the n-bracelet is in-phase if and only if the $(2n-1)$-tiling is breakable at cell n.

Case 1. If T^* is breakable at cell n, then glue the right side of cell n to the left side of cell 1 to create an in-phase n-bracelet B, and cells $n+1$ through $2n-1$ form an $(n-1)$-tiling T.

Case 2. If T^* is unbreakable at cell n, then cells n and $n+1$ are covered by a domino which we denote by d. Cells 1 through $n-1$ become an $(n-1)$-tiling T and cells n through $2n-1$ are used to create an out-of-phase n-bracelet with d as its first tile.

This correspondence is easily reversed since the phase of the n-bracelet indicates whether Case 1 or Case 2 is invoked.

Identity 34 *For* $n \geq 0$, $5f_n = L_n + L_{n+2}$.

Set 1: Tilings of an n-board. This set has size f_n.

Set 2: Tilings of a circular n-board or a circular $(n+2)$-board. This set has size $L_n + L_{n+2}$.

Correspondence: To prove the identity, we establish a 1-*to*-5 *correspondence* between Set 1 and Set 2. That is, for every tiling in Set 1, we can create five bracelets in Set 2 in such a way that every bracelet in Set 2 is created exactly once. Hence Set 2 is five times as large as Set 1.

Given an n-tiling, four of the five bracelets arise naturally. See Figure 2.5. We can create

1. an in-phase n-bracelet by gluing cell n to cell 1, or
2. an in-phase $(n+2)$-bracelet ending with two inserted squares, or
3. an in-phase $(n+2)$-bracelet ending with an inserted domino, or
4. an out-of-phase $(n+2)$-bracelet ending with an inserted domino.

At this point we pause to investigate which bracelets have not yet been created. We are missing out-of-phase n-bracelets and $(n+2)$-bracelets that end with a square

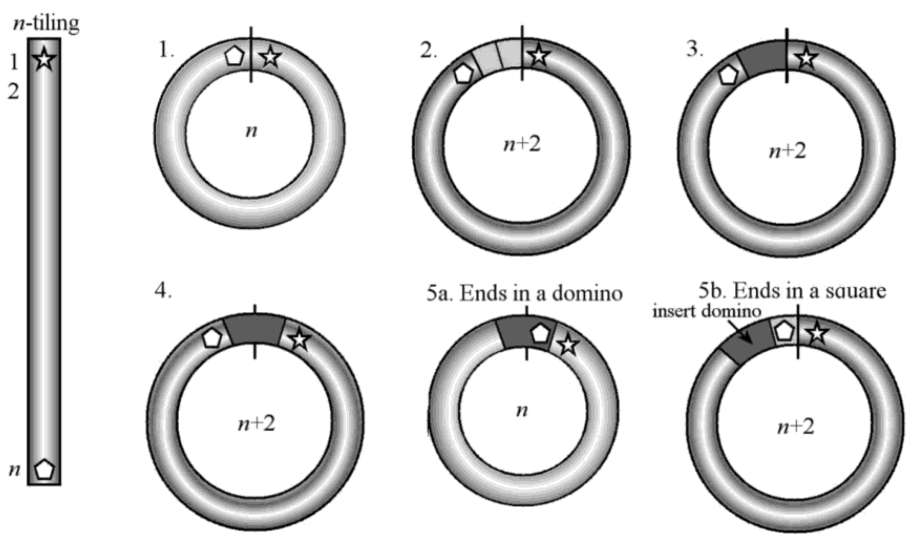

Figure 2.5. Every n-tiling generates five bracelets of size n or $n+2$.