Quantum Field Theory for Economics and Finance

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QUANTUM FIELD THEORY FOR ECONOMICS AND FINANCE

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The International Centre for Education in Islamic Finance

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Foreword

I am aware of the author's work in applying theories of physics to finance since 2003, and the present book is a logical outcome of the author's line of thinking. The presentation of quantum field theory (QFT) given in this book is based on four strategic decisions.

(1) From the very outset it introduces the notion of quantum mathematics. This immediately attracts the attention of readers, with regard two points. First, they realize that in order to feel at home in QFT, they must devote enough time and attention to mastering these techniques. Second, once they have them well in hand, they can also use them outside of physics because they are just mathematical techniques.

(2) The book avoids giving applications of QFT to physics as this does not in the least help to understand QFT as a mathematical discipline.

(3) Throughout the book the formalism of the Feynman path integral is used, which intuitively is indeed the most appealing formalism of QFT.

(4) Last but not least, the book provides applications of QFT to a variety of economic and financial problems. One must realize that this is indeed quite different from calculations tied to high energy physics. Why? Needless to say, the whole machinery of QFT was created for applications to high energy physics; thus, one just follows the track and there is no need to raise any questions. On the contrary, QFT was not created to price options. Thus, instead of just following the track, at each step we have to modify and adapt our understanding of the mathematical tools of QFT.

The book has three distinctive features that are worth highlighting.

(1) There are many books on QFT, but this is a ground-breaking book that connects QFT with concepts in economics and finance.

(2) Almost half the book is devoted to studying models of economics and finance. As the book proceeds with different topics of QFT, chapters on economics and finance are introduced to show the close mathematical connections between these domains of knowledge.

(3) Many of the applications to economics and finance are based on models that can be empirically tested. To me, the most remarkable aspect of the book is that empirical tests show that these models are surprisingly accurate.

Going through the applications of QFT is a highly rewarding exercise as it tests our degree of understanding and expands our view of QFT. When readers grasp the logic of the applications, it will bolster their self-confidence and make them feel at home with QFT, and empower them to apply the mathematics of QFT to new fields of inquiry.

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Preface

Quantum field theory is undoubtedly one of the most accurate and important scientific theories in the history of science. Relativistic quantum fields are the theoretical backbone of the Standard Model of particles and interactions. Relativistic and non-relativistic quantum fields are extensively used in myriad branches of theoretical physics, from superstring theory, high energy physics and solid state physics to condensed matter, quantum optics, nuclear physics, astrophysics and so on.

The mathematics that emerges from the formalism of quantum mechanics and quantum field theory is quite distinct from other branches of mathematics and is termed *quantum mathematics*. Quantum mathematics is a synthesis of linear algebra, calculus of infinitely many independent variables, functional analysis, operator algebras, infinite-dimensional linear vector spaces, the theory of probability, Lie groups, geometry, topology, functional integration and so on.

One of the mathematical bedrocks of quantum mechanics and quantum field theory is the Feynman path integral [Baaquie (2014)]. Unlike functional integration in general, the Feynman path integral is a functional integral with another key feature, which is that the path integral is constructed out of an underlying (infinite-dimensional) linear vector space. Operators are defined on this vector space, including the central operator of theoretical physics, which is the Hamiltonian.

The first application of calculus – made by Newton – was in the study of the dynamics of particles; calculus subsequently has gone on to become the universal language of quantitative modeling. Similarly, although quantum mathematics emerges from the study of quantum phenomena that are intrinsically indeterminate, the mathematical structure is not tied to its origins. Examples discussed below show that the mathematics of quantum field theory extends far beyond only quantum systems and can also be applied to a wide variety of subjects that span natural and social sciences. It is my view that quantum mathematics will, in time, supersede calculus and become the universal framework for quantitative modeling and mathematical thinking.

these ideas are carried over to economics and finance. About 60% of the material of the book is directly an exposition of quantum field theory, with the remaining chapters being focused on its various applications to economics and finance.

The manner of presentation of the two pillars of the book – quantum fields on the one hand and economics and finance on the other– is quite different. Quantum field theory needs no empirical evidence for its utility and validity since the entire domain of particle physics stands as a testament to its empirical success. Hence the focus in the chapters on quantum field theory is on the various mathematical ideas and derivations, and only a fleeting connection is made with other subjects. An integral and pure presentation of quantum field theory is necessary to show that it is free from a bias toward any specific application. In fact, if one skips the Chapters on economics and finance, which are marked by an asterisk, the book then reads as an introductory graduate text on quantum field theory.

Unlike mathematics, which has results of great generality, such as theorems and lemmas, one only needs to flip through the pages of a textbook on quantum mechanics or quantum field theory to see that there are no theorems in quantum physics; instead, what one has are leading models and important examples – with the mathematical analysis flowing naturally in interpreting, explaining and deriving the "physics" of these models. Quantum field theory is illustrated and elaborated on by analyzing a number of exemplary models, such as the scalar, vector and spinor fields. Each of these quantum fields is described by a specific Lagrangian and Hamiltonian – and has distinctive properties on which the book focuses. More advanced chapters such as the structure of the renormalization group are presented later, when the reader has a better grasp of the underlying ideas.

The methodology of the chapters on economics and finance is quite different from the chapters on quantum field theory. In my view, the only justification for the application of quantum mathematics to empirical disciplines outside quantum physics – including economics and finance – is that it must be supported by empirical evidence. In the absence of such evidence – and there are many papers and books that make conceptual connections between quantum mathematics and classical systems with little or no empirical evidence [Bagarello (2013)] – the application in my view is still not complete, and stands only as an interesting mathematical metaphor. For the metaphor to become a concrete mathematical model, empirical evidence is indispensable.

For this reason, topics from economics and finance have been chosen (for inclusion in the book) that have empirical support from market data. Furthermore, a detailed analysis is given on how these quantum mathematical models are adapted to the market – and subsequently calibrated and tested. In chapters on economics and finance, very specific and concrete theoretical models are analyzed – all based on path integrals and Hamiltonians. The introductory chapter on nonlinear interest

rates concentrates on the formalism; the reason is that a quantum finance model of nonlinear interest rates, as realized by the London Interbank Offered Rate (Libor), has been calibrated and exhaustively tested using market data; hence, only certain key features of the formalism are discussed. Two chapters use numerical algorithms and simulations to study nonlinear interest rates; these chapters illustrate a key feature of nonlinear interest rates, which is that in most cases numerical techniques are necessary for obtaining a solution.

The models that have been proposed in economics and finance – all of which are based on work done by myself and collaborators – are quite distinct from those that appear in quantum physics. In particular, all the models in economics and finance have an "acceleration" kinetic term – a term forbidden in quantum mechanics (due to the violation of conservation of probability); it is this term that gives a flavor to all the results in economics and finance that is quite different from what one is familiar with in physics.

The derivations in this book are not tied down to the application of quantum fields to physics – as this would require concepts that are not necessary for understanding the mathematical formalism of quantum fields. Furthermore, topics that apparently have no connection with finance or economics – but have played a pivotal role in quantum field theory – have been included in the hope that these ideas may lay lead to ground-breaking theories and models in economics and finance.

Nonlinearities of quantum fields arise due to self-interactions or because of coupling to other fields – and require the procedure of renormalization for obtaining finite results. The canonical case of a self-interacting nonlinear scalar field is studied in great detail so as to illustrate and analyze the issues that arise in renormalization. The formalism of quantum field theory culminates in the concepts of renormalization, renormalizability and the renormalization group – and which are among the deepest ideas of quantum field theory. It has been shown by Sornette (2003) that ideas from the renormalization group can provide a mathematical framework for understanding, and even predicting, market meltdowns.

Many topics, such as fermions, spinors, ghost fields, bosonic strings and gauge-fixing, are discussed that may seem to have no connection with economics and finance. The reason for including these topics is intentional. The broad range of topics covered gives a flavor to the reader of the great variety and complexity of the models that are a part of quantum field theory. A major omission has been the study of Yang–Mills gauge fields and that of spacetime supersymmetry. These topics need a background far in advance to what has been assumed, and hence could not be covered.

It is impossible and unwise to try to second guess what future directions economics and finance will turn toward; furthermore, gearing the topics discussed

toward what is known closes off many future applications. For this reason, the main thrust of this book is to make the reader aware of, and familiar with, a wide array of quantum mathematical models so that a researcher can make leading edge connections and create new pathways between the domains of quantum fields and economics and finance.

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I would like to acknowledge and express my thanks to many outstanding teachers, scholars and researchers whose work motivated me to study quantum field theory and to grapple with its mathematical formalism.

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I have benefited from discussions with many physicists and field theorists, including but not limited to, in no particular order, Abdus Salam, John Strathdee, M. Ali Namazie, Jamal Nazrul Islam, John Kogut, Leonard Susskind, Michael Fisher, Kurt Gottfried, H. R. Krishnamurthy, Michael Peskin, Steve Shenker, Spenta Wadia, Ashoke Sen, Tan Meng Chwan, Ahmed Zewail, S. G. Rajeev, Cumrun Vafa, Edward Witten, David Gross, Steven Weinberg, Jack Ng, Lay Nam Chang, Frank Wilczek, Nee Pong Chang, Anthony J. Leggett, Sidney Drell, Marvin Weinstein, Phua Kok Khoo, Andrew Strominger, Jayant Narlikar, Juan Maldecena, Bambi Hu, Chen Ning Yang, Bertrand Roehner, Murray Gell-Mann, Gerard 't Hooft, James D. Bjorken, Gyan Bhanot, Nicolas Sourlas, Faheem Hussain, Avinash Dhar, Asghar Qadir, Robert C. Richardson, Francois Martin, Kerson Huang, Syed Wassim, Jean Philippe Bouchaud, Munawar Karim, Brian Miller, Obiyathulla Ismath Bacha, Oh Choo Hiap, Michael Spalinski, Andrew Strominger, Baharom Abdul Hamid, Lai Choy Heng, Mohamed Eskandar Shah Mohd Rasid, Thomas Osipowicz, Emmanuel Haven, Sandro Sozzo, Claudio Coriano, Abbas Mirakhor and many others.

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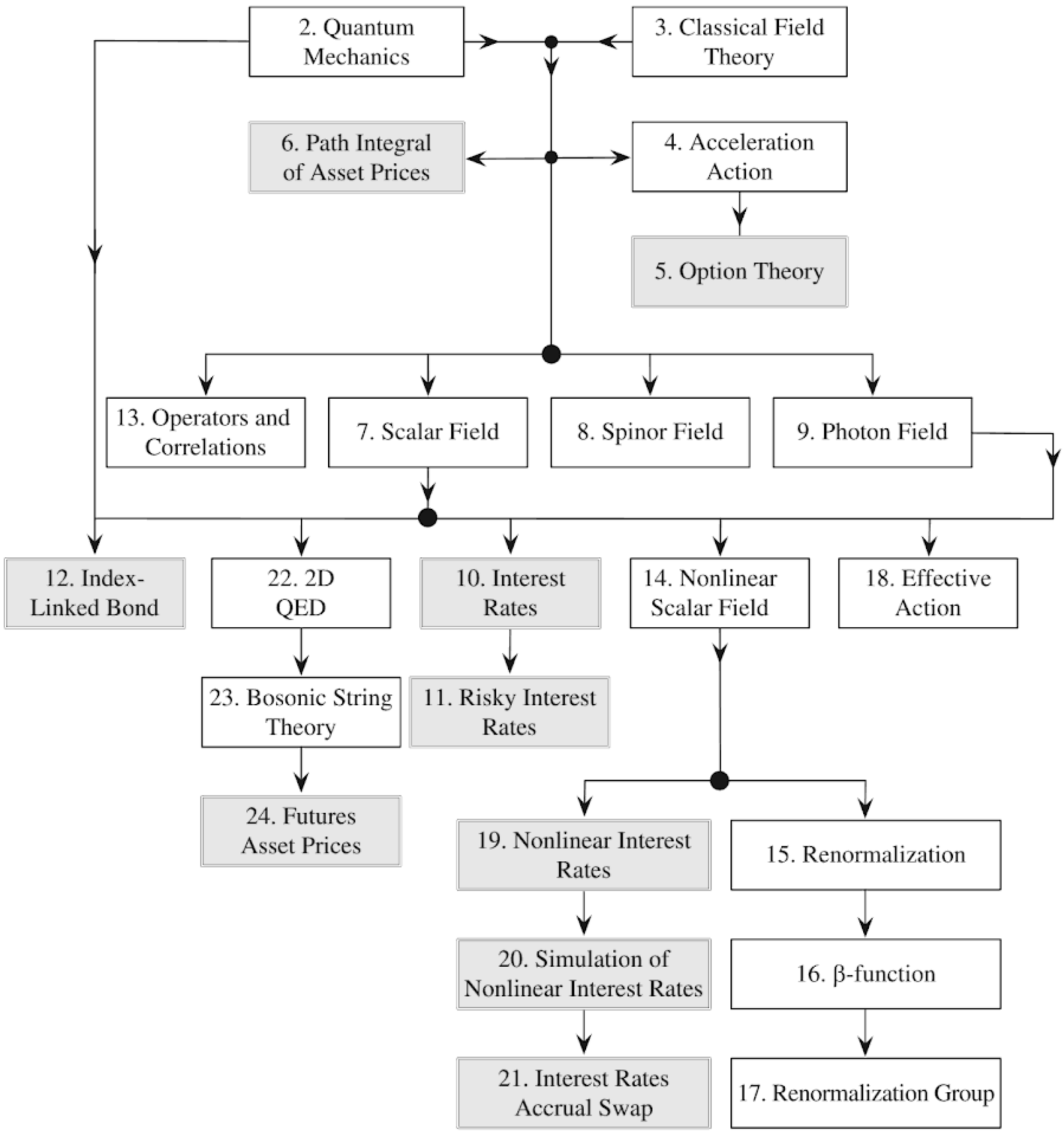


Figure 1.1 The organization of the chapters, with their interconnections. The shaded chapters are on economics and finance.

mathematics of quantum mechanics and foreground the more complex derivations in the next three parts.

Chapter 2 introduces the foundations of quantum mechanics; the quantum principle leads to quantum indeterminacy and to the quantum theory of probability. Quantum probability has emerged in recent years as a major new subfield of decision science and behavioral finance, and the discussion in this chapter is partly to introduce the ideas of quantum probability. A brief discussion of path integrals and Hamiltonians for quantum mechanics prepares the ground for the subsequent analysis.

In Chapter 3 classical field theory is studied to understand the Lagrangian formulation of classical fields. The Lagrangian and action functional are one of the pillars of quantum mathematics, and classical fields are a precursor to quantum fields. Global and local symmetries of the theory are encoded in the symmetries of the Lagrangian. To illustrate the interplay of symmetry breaking and gauge invariance, symmetry breaking for a nonlinear complex scalar field coupled to the Maxwell field is analyzed; it is shown to lead to both the Landau–Ginzburg formulation of superconductivity and the Higgs mechanism of particle physics. The Lorentz group is analyzed to understand the structure of empty spacetime, and it is shown how the Lorentz group classifies the various types of classical relativistic fields.

Chapter 4 studies the evolution kernel for the quantum mechanical acceleration action, which is a higher derivative action. This chapter gives a derivation of the evolution kernel using the state space and Hamiltonian and not a path integral derivation as given by Baaquie (2014). The acceleration action is a key to the studies of asset prices as well as of forward interest rates. The reason is that asset prices are described by the complex branch of the acceleration action, whereas forward interest rates are described by the real branch. The acceleration action yields a pseudo-Hermitian Hamiltonian and, due its higher derivative kinetic term, yields results quite different from quantum mechanics.

Chapter 5 is on option theory. Central ideas such as the martingale condition and option prices free from arbitrage opportunities are discussed in the quantum mechanical framework. The Black–Scholes equation is given a quantum mechanical derivation with no reference to stochastic calculus. The Black–Scholes equation is generalized to the Baaquie–Yang equation using results from the acceleration action. Options for equities and foreign exchange are derived and empirically tested using market data. It is shown that options provide a more accurate gauge of market instabilities than the volatility of the underlying asset.

In Chapter 6 the formulation of statistical microeconomics is reviewed and a Lagrangian is postulated for modeling asset prices. It is shown how the application of Feynman path integrals arises in the study of asset prices. Empirical evidence is discussed to support the applications of quantum mathematics to the study of asset prices. A Monte Carlo simulation is done to study the nonlinear aspect of the Lagrangian, and confirms the validity of perturbatively studying the nonlinear regime using Feynman diagrams. Multiple commodity prices are analyzed and it is shown that a Lagrangian for multiple commodities provides an accurate description of the empirical correlation function of commodity prices.

Part II focuses on *linear quantum fields* and is a necessary preparation for the study of nonlinear quantum fields. Quantum fields come in many varieties and with a great range of underlying degrees of freedom. The simplest, but not unimportant, case of a quantum field is a scalar field. The free quantum field is studied as a

precursor to nonlinear quantum fields. The main difference between a linear and nonlinear quantum field is that the free field does not have any interactions.

Linear fields are important in their own right. The free field comes in many varieties, depending on the nature of the underlying degrees of freedom, with the most important examples being scalar, photon and Dirac fields. One way of decoupling the free field's degrees of freedom is the method of Fourier transform, which resolves the free field into decoupled momentum degrees of freedom. The three most widely used and most useful free quantum fields are the *scalar, spinor* and *vector* quantum fields. These fields have many specific features of great generality and hence need to studied one by one.

Chapter 7 studies the *free scalar* quantum field, which has no self-interaction but nevertheless has many features of a quantum field and is an ideal theoretical laboratory for starting one's study of a system with infinitely many degrees of freedom. Figure 1.1 shows the central position of the free scalar field in developing the more complex and deeper structures of quantum field theory as well as the application of quantum field theory to economics and finance. A scalar quantum field has one degree of freedom for each spacetime point. The scalar field has all the general features of quantum fields and its Lagrangian and Hamiltonian are studied in detail. In particular, the formalism of creation and annihilation operators is carefully analyzed as these are among the most useful mathematical tools for the study of quantum fields. The quantum field in two dimensions is the starting point of this chapter as it is the simplest system quantum field for which the Fock space of states of a quantum field is defined.

Chapter 8 studies the *free spinor* quantum field, of which the Dirac field is a leading example. The Dirac field is based on fermionic degrees of freedom obeying anticommuting fermion statistics. Spinor fields provide a representation of the Lorentz group and are the result of the structure of spacetime. Due to its spinor nature, the quantization of the free Dirac field requires a multicomponent spinor field, having four degrees of freedom at each spacetime point. It is shown how, on the quantization of the Dirac field, two species of particles emerge in its spectrum of states, which are the particle and its *antiparticles*. In fact, the primary motive for studying the Dirac field is to understand the emergence, and the properties, of antiparticles. The relation between the particle and antiparticle states is analyzed and it shown that the Dirac field is invariant under the exchange of particle–antiparticle. Since the Dirac field is a fermionic field, the properties of fermionic variables and the path integral for fermions are briefly reviewed. The Casimir force is evaluated for the Dirac field and leads one to study the boundary conditions for the Dirac field and the associated state space.

Chapter 9 studies the *free photon* field, which is a vector field with the local symmetry of *gauge invariance*. The symmetry of gauge invariance is so important

that the photon field is also referred to as an Abelian gauge field. To quantize the photon field, one has to choose a gauge. Choosing a gauge is necessary for quantizing both Abelian and non-Abelian Yan–Mills gauge fields. The mathematics required for choosing a gauge is studied in great detail, using both the path integral formalism, which leads to Faddeev–Popov quantization, and the Hamiltonian formalism, which leads to the Coulomb gauge. The state space that results for both the path integral and Hamiltonian quantization are discussed. The Becchi–Rouet–Stora–Tyutin (BRST) symmetry exhibited by the gauge-fixed action in the Faddeev–Popov scheme is utilized to define the state space and is shown to be equivalent to the Gupta–Bleuler quantization for a covariant gauge.

Chapter 10 analyzes interest rates in *finance*. Interest rates are modeled using a two-dimensional stochastic field that is mathematically identical to a two-dimensional Euclidean quantum field. The action, Lagrangian and Hamiltonian for the forward interest rates are modeled using a linear (free) two-dimensional Euclidean quantum field. The Lagrangian is a higher order derivative system, and empirical evidence is briefly reviewed to support the modeling of interest rates by a quantum field. The state space and field Hamiltonian operator are both shown to be time dependent. The martingale condition is derived for the forward interest rates using both the path integral and Hamiltonian formulation.

Chapter 11 continues the study of forward interest rates, with the additional coupling of the risk-free to the risky forward interest rates. It is shown how a spread over the risky rates – the spread being a quantum field in its own right – allows one to extend the formalism. The risky forward interest rates are empirically studied, with reasonable support for the model from market data.

Chapter 12 studies a coupon bond with index-linked stochastic coupons. An asset price, represented by a quantum mechanical degree of freedom, determines the amount of payment of the stochastic coupons. The discounting of future cash flows is determined by the zero coupon bonds modeled by the risk-free forward interest rates, which in turn is modeled by a two-dimensional quantum field. The financial instrument is a synthesis of a quantum mechanical degree with a two-dimensional quantum field, and defines a distinct class of financial instruments.

Part III discusses *nonlinear quantum fields*. The nonlinear properties of quantum fields are, in general, mathematically formidable as well as being fairly intractable – and for the same reason also yield novel and unexpected results.

In Chapter 13, a general derivation is given of the connection of operators and state space with the Feynman path integral; in particular, it is shown that all the time-ordered vacuum expectation values of Heisenberg quantum field operators are given by the correlation functions of the quantum field using the Feynman path integral. The Lehmann–Symanzik–Zimmermann (LSZ) formalism is reviewed to show how the scattering of quantum field states can be reduced to the time-ordered vacuum expectation values of the quantum fields, which in turn can be evaluated

using the path integral. These derivations show the centrality of the path integral in the study of quantum fields.

In Chapter 14, the nonlinear scalar quantum field is studied using perturbation theory to understand the divergences of a quantum field. Dimensional regularization is used as an effective cutoff for the quantum field; it is shown that the mass and coupling constant of the quantum field apparently seem to diverge. Feynman diagrams are introduced as a useful bookkeeping device for the terms that appear as one goes to higher and higher order perturbation theory.

Chapter 15 is a key chapter that introduces the idea, as well as a prescription, of renormalization. Four different methods are employed to renormalize the nonlinear scalar field, which are given by bare and renormalized perturbation theory, the background field method and Wilson's thinning of the degrees of freedom. All four methods are shown to yield the same result, but from vastly different perspectives.

The deep and global structures of quantum fields are discussed in Chapters 16 and 17, which address the issues of renormalization and of the renormalization group. The divergences that appear in perturbation theory and the procedure of renormalization are seen to be the natural consequence of the fact that the quantum field describes a system with infinitely many length scales. In Chapter 17 one discovers the rather unexpected connection of quantum field theory to the theory of phase transitions. Recall that quantum field theory was specifically developed to address high energy phenomenon at short distances, whereas phase transitions are determined by the behavior of the system for infinitely separated degrees of freedom.

Another branch of the book leads to the study, in Chapter 18, of effective actions that describe symmetry breaking for nonlinear scalar fields and for scalar quantum electrodynamics. The effective action is evaluated for both cases and it is shown that scalar electrodynamics has spontaneous symmetry breaking that is renormalization group invariant.

Nonlinear scalar fields lead to nonlinear models of interest rates, which is studied in Chapter 19. This chapter concentrates on certain key aspects of the mathematical formalism of nonlinear interest rates. The debt market is driven by Libor simple interest rates. It is shown, due to a nonlinear drift required for fulfilling the martingale condition, that Libor is described by a two- dimensional nonlinear Euclidean quantum field. Due to the higher derivative terms in the Lagrangian, there is no need for renormalizing this nonlinear field. Nonlinear drift is exactly obtained using both the Wilson expansion and the Hamiltonian formulation of the martingale condition. The empirical aspect of nonlinear interest rates has been studied by Baaquie and Yang (2009), Yang (2012) and Baaquie et al. (2014b), and hence is not discussed in this book.

Since perturbation theory is often not effective in studying nonlinear systems, nonlinear interest rates are studied numerically in Chapters 20 and 21– with the

Part I

Introduction

2

Quantum mechanics

2.1 Introduction

Quantum mechanics is based on the quantum principle, which is discussed in the next section. Quantum mechanics is the logical starting point of quantum field theory since ideas such as operators, state space and path integrals can be introduced in a simpler context, with the infinite-dimensional generalizations being made later in the analysis of quantum fields.

In a book addressing applications of quantum field theory to economics and finance, one may question whether there are, at all, any purely quantum effects. This question is partly correct in the sense that all applications to economics and finance considered in this book are based on the view that stochastic and random phenomena in economics and finance are described by *classical probability theory*; quantum mathematics is employed solely as a powerful computational tool for addressing these problems.

However, there has been a major development in social sciences, especially in decision sciences and behavioral finance, where *quantum probability* has been directly invoked in explaining various observed social phenomenon. Quantum probability and quantum superposition, which are some of the unique and enigmatic results that emerge from quantum mechanics [Baaquie (2013b)], have found myriad applications in the social sciences. Two books, by Busemeyer and Bruza (2012) and by Haven and Khrennikov (2013), with references cited therein, provide a comprehensive summary of the varied results, both empirical and theoretical, that have been obtained in applying quantum mechanics to the social sciences.

As reasoned above, for both mathematical and conceptual reasons, the study of quantum fields needs to start from its roots, namely from quantum mechanics.

2.2 Quantum principles

One of the starting points of a quantum field is the principle of quantum mechanics, which states that all physical systems are constituted by degrees of freedom that are fundamentally indeterminate. The indeterminacy is realized in two different manners [Baaquie (2013b)]:

- In the Hamiltonian formulation all physical observables are Hermitian operators acting on a state space built on the underlying degree of freedom. Physically observable quantities are the expectation value of the operators obtained by either using the state space approach of Schrödinger or the operator algebra of Heisenberg.
- In the path integral formulation, the degrees of freedom are integration variables and hence have no fixed value and are intrinsically indeterminate. Physical observables are defined by the matrix elements of operators representing physical quantities, which in turn can be evaluated using the path integral.

Both the path integral and state space/operator formulations of the quantum principle are discussed in this chapter. The mathematics of quantum mechanics is generalized in defining quantum fields. But what remains valid for quantum fields is that they are based on degrees of freedom that are quantum mechanical in nature, being indeterminate. Moreover, the operator algebra and path integral mathematics of quantum mechanics continues to be the mathematical backbone of these objects.

Dirac's notation for linear vector space and operators is used extensively. For those not familiar with the bracket notation, the connection of vector notation with Dirac's bracket notation is briefly discussed in Section 2.10.

The three fundamental principles of quantum mechanics are the following:

- The degree of freedom; this can be discrete or continuous and can be one or infinitely many. The degree of freedom is denoted by ϕ; the collection of all its values forms the space of the degree of freedom $\mathcal{F}$.
- The state space, which in quantum mechanics is a Hilbert space. The state space can be larger than a Hilbert space for systems that do not conserve probability, as is the case for economics and finance. The state space is denoted by $\mathcal{V}$ and an element of $\mathcal{V}$ is $|\psi\rangle \in \mathcal{V}$, where $\mathcal{V} : \mathcal{F} \to \mathcal{V}$. The dual state space $\mathcal{V}_D$ consists of all mappings, denoted by $\langle\chi|$, of $\mathcal{V}$ to the complex numbers $\mathbb{C}$. The expression $\langle\chi|\psi\rangle = \langle\psi|\chi\rangle^* \in \mathbb{C}$ is the *scalar or inner product*.
- Operators $\widehat{O}$ that act on $\mathcal{V}$ and map it to itself $\widehat{O} : \mathcal{V} \to \mathcal{V}$. The space of operators is denoted by $\mathcal{Q} \equiv \mathcal{V} \otimes \mathcal{V}_D$. The *tensor* or *outer product* of two state vectors is given by $|\psi\rangle \otimes \langle\chi| \equiv |\psi\rangle\langle\chi| \in \mathcal{V} \otimes \mathcal{V}_D$.

In summary, quantum mechanics consists of the mathematical triple $\{\mathcal{F}, \mathcal{V}, \mathcal{Q}\}$.

Hence, the coefficients have the important property that

$$0 \leq P_n \leq 1; \quad \sum_n P_n = 1 \tag{2.3.3}$$

Eq. 2.3.3 shows that P_n have the interpretation of the probability of event labeled by n.

Note that quantum theory of measurement requires that only *one* of the detectors, represented by Π_n, detects the quantum state. This is also called the *collapse of the wave function*. Define

$$P_n = |c_n|^2 \in [0, 1] \tag{2.3.4}$$

The interpretation in quantum mechanics is that P_n is the *probability* that the detector – represented by Π_n – detects the quantum state $|\chi\rangle$. Figure 2.1 is a representation of the measurement process in quantum mechanics.

Eqs. 2.3.4 and 2.3.2 show that quantum mechanics is a quantum theory of probability that is a *synthesis* of classical theory of probability with the concept of an underlying linear vector space $\mathcal{V}$ and Hermitian operators acting on it.

Note the fundamental paradox of quantum mechanics, namely that the foundation of the quantum entity, namely the degree of freedom, can never in principle be observed by any experiment.

Furthermore, two orthogonal projection operators Π_n, Π_m can never simultaneously observe the state function ψ. A measurement results in the state function collapsing to *either* the state $|\psi_n\rangle = \Pi_n|\psi\rangle$ or $|\psi_m\rangle = \Pi_m|\psi\rangle$; the state vector $|\psi\rangle$ is *never* simultaneously observed by both the projection operators. If in any experiment, two orthogonal projection operators simultaneously observe the state function ψ, then that would *spell the end* of the current (Copenhagen) interpretation of quantum mechanics.

2.4 Dirac delta function

The Dirac delta function is indispensable in the study of continuous spaces (degrees of freedom), and some of its properties are reviewed. Dirac delta functions are not ordinary Lebesgue measurable functions since they have support on a measure zero set; rather, they are generalized functions also called distributions. The Dirac delta function is the continuum generalization of the discrete Kronecker delta function.

Consider a continuous line labeled by coordinate x such that $-\infty \leq x \leq +\infty$, and let $f(x)$ be an infinitely differentiable function. The Dirac delta function, denoted by $\delta(x - a)$, is defined by

$$\delta(x-a) = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases}$$

$$\delta(x-a) = \delta(a-x): \text{ even function}$$

$$\delta(c(x-a)) = \frac{1}{|c|}\delta(x-a)$$

Furthermore

$$\int_{-\infty}^{+\infty} dx f(x)\delta(x-a) = f(a) \tag{2.4.1}$$

$$\int_{-\infty}^{+\infty} dx f(x)\frac{d^n}{dx^n}\delta(x-a) = (-1)^n \frac{d^n}{dx^n} f(x)|_{x=a} \tag{2.4.2}$$

The Heaviside step function $\Theta(t)$ is defined by

$$\Theta(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \tag{2.4.3}$$

From its definition

$$\Theta(t) + \Theta(-t) = 1 \tag{2.4.4}$$

The following is a representation of the delta-function:

$$\int_{-\infty}^{b} dx\, \delta(x-a) = \Theta(b-a); \quad b > a \tag{2.4.5}$$

$$\Rightarrow \int_{-\infty}^{a} dx\, \delta(x-a) = \Theta(0) = \frac{1}{2} \tag{2.4.6}$$

where last equation is due to the Dirac delta function being an even function. From Eq. 2.4.5,

$$\frac{d}{db}\Theta(b-a) = \delta(b-a)$$

A representation of the delta-function based on the Gaussian distribution is

$$\delta(x-a) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-a)^2\right\} \tag{2.4.7}$$

Moreover

$$\delta(x-a) = \lim_{\mu \to \infty} \frac{1}{2}\mu \exp\{-\mu|x-a|\}$$

From the definition of Fourier transforms

$$\delta(x-a) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x-a)} \tag{2.4.8}$$

A useful identity for option theory is the following. For a stock price given by $S = e^x$, consider a payoff function

$$[e^x - K]_+ = (e^x - K)\Theta(e^x - K)$$

Using definition of the Dirac delta function and Eq. 2.4.8 yields

$$[e^x - K]_+ = \int_{-\infty}^{+\infty} d\xi\, \delta(\xi - x)[e^\xi - K]_+ = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \int_{-\infty}^{+\infty} d\xi\, e^{ip(\xi - x)}[e^\xi - K]_+ \tag{2.4.9}$$

To see the relation of the Dirac delta function to the discrete Kronecker delta, recall for n, m integers

$$\delta_{n-m} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} \tag{2.4.10}$$

Discretize continuous variable x into a lattice of discrete points $x = n\epsilon$, and let $a = m\epsilon$; then $f(x) \to f_n$. Discretizing Eq. 2.4.1 gives

$$\int_{-\infty}^{+\infty} dx f(x)\delta(x - a) \to \epsilon \sum_{-\infty}^{+\infty} f_n \delta(x - a) = f_m$$

$$\Rightarrow \delta(x - a) \to \frac{1}{\epsilon}\delta_{n-m} \tag{2.4.11}$$

Hence, taking the limit of $\epsilon \to 0$ in the equation above,

$$\delta(x - a) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\delta_{n-m} = \begin{cases} 0, & x \neq a \\ \infty, & x = a \end{cases}$$

Considering x as a degree of freedom yields basis state vector $|x\rangle$ and dual basis state $\langle x|$. A function $f(x)$ in Dirac's notation is given by $f(x) = \langle x|f\rangle$ and the scalar product of two functions is

$$\langle f|g\rangle = \int dx f^*(x)g(x) = \int dx \langle f|x\rangle\langle x|g\rangle = \left\langle f \left| \left\{ \int dx |x\rangle\langle x| \right\} \right| g \right\rangle$$

and yields the completeness equation

$$\int dx |x\rangle\langle x| = \mathbb{I} \tag{2.4.12}$$

The completeness equation given in Eq. 2.4.12 plays a central role in analyzing continuous degrees of freedom.

2.5 Schrödinger and Heisenberg formalism

The time evolution of a state vector is given by Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}|\psi_t\rangle = H|\psi_t\rangle \tag{2.5.1}$$

The coordinate eigenstate $|\phi\rangle$ and operator $\widehat{\phi}$ are defined by

$$\widehat{\phi}|\phi\rangle = \phi|\phi\rangle; \;\; \langle\phi'|\phi\rangle = \delta(\phi' - \phi) \tag{2.5.2}$$

and yield the coordinate representation for the Schrödinger equation

$$-\frac{\hbar}{i}\left\langle \phi \left| \frac{\partial}{\partial t} \right| \psi_t \right\rangle = \langle\phi|H|\psi_t\rangle \quad \Rightarrow \quad -\frac{\hbar}{i}\frac{\partial}{\partial t}\psi_t(\phi) = \widehat{H}\left(\phi, \frac{\partial}{\partial\phi}\right)\psi_t(\phi) \tag{2.5.3}$$

Note the Hamiltonian operator acts on the dual basis state $\langle\phi|$, and this is the rule for all differential operators. From Eq. 2.5.1,

$$|\psi_t\rangle = e^{-\frac{it}{\hbar}H}|\psi_0\rangle \tag{2.5.4}$$

The expectation value of a time-independent Schrödinger operator $\widehat{O}$ is given by ($\hbar = 1$)

$$E_\psi[\widehat{O}(t)] = \langle\psi_t|\widehat{O}|\psi_t\rangle = \langle\psi|e^{itH}\widehat{O}e^{-itH}|\psi\rangle = \langle\psi|\widehat{O}(t)|\psi\rangle = tr(\widehat{O}(t)\rho)$$

where

$$\rho = |\psi\rangle\langle\psi|$$

is the *density operator*, also called, for historical reasons, the density matrix.

The time-dependent Heisenberg operator is defined by

$$\widehat{O}(t) = e^{itH}\widehat{O}e^{-itH} \;\; \Rightarrow \;\; i\frac{\partial\widehat{O}(t)}{\partial t} = [\widehat{O}(t), \widehat{H}] : \text{ Heisenberg operator equation}$$

Heisenberg's formulation is more suitable for measurement theory. A device represents physical projection operators Π_n; the quantum state is ρ. The result of repeated measurements yields $P_n = tr(\Pi_n\rho)$.

2.6 Feynman path integral

The Dirac–Feynman formulation of path integrals is derived from the Hamiltonian operator. Although every Hamiltonian yields a path integral, it is not the case that every path integral can in turn be expressed in terms of a Hamiltonian. This aspect of path integrals comes to the fore in studying path integrals for curved manifolds, but will be not addressed in this book.

Consider the continuation of Minkowski time to Euclidean time given by $x_t = x_\tau$, $t = -i\tau$. See Figure 2.2. Hence

$$\mathcal{L}_E = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x) = -\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x)$$

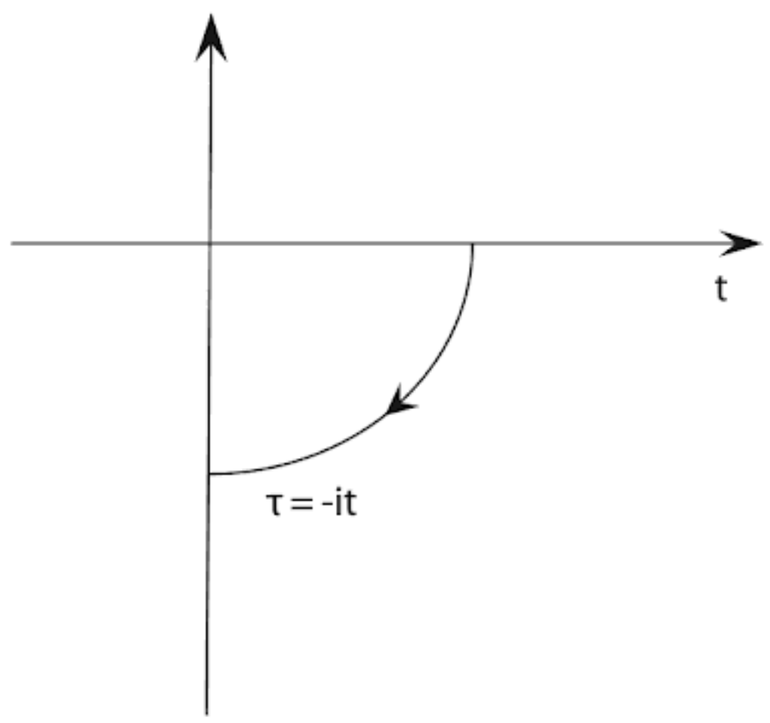


Figure 2.2 Euclidean continuation of real (Minkowski) time.

The canonical momenta changes sign but give the same Hamiltonian

$$H = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x); \;\; H_E = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

The Dirac-Feynman formula is given by

$$\begin{aligned} \langle x'|e^{-i\epsilon H}|x\rangle &= \mathcal{N}(\epsilon)e^{i\epsilon\mathcal{L}(x,x',\epsilon)}\text{: Minkowski} \\ \langle x'|e^{-\epsilon H_E}|x\rangle &= \mathcal{N}(\epsilon)e^{\epsilon\mathcal{L}_E(x,x',\epsilon)}\text{: Euclidean} \end{aligned} \tag{2.6.1}$$

where $\mathcal{N}(\epsilon)$ is a normalization term. The subscript E will be dropped unless necessary.

The Euclidean path integral has the following derivation:

$$\langle x'|e^{-\epsilon H_E}|x\rangle \simeq \langle x'|e^{-\epsilon\frac{p^2}{2m}}|x\rangle e^{-\epsilon V(x)} \tag{2.6.2}$$

Note

$$\int \frac{dp}{2\pi}|p\rangle\langle p| = \mathbb{I} \tag{2.6.3}$$

Hence

$$\begin{aligned} \langle x'|e^{-\epsilon\frac{p^2}{2m}}|x\rangle &= \int \frac{dp}{2\pi}\langle x'|e^{-\epsilon\frac{p^2}{2m}}|p\rangle\langle p|x\rangle = \int \frac{dp}{2\pi}e^{-\epsilon\frac{p^2}{2m}}e^{-ip(x-x')} \\ &= \sqrt{\frac{m}{2\pi\epsilon}}e^{-\frac{m}{2\epsilon}(x-x')^2} \end{aligned}$$

and yields

$$\epsilon\mathcal{L} = -\frac{m}{2\epsilon}(x_{t+\epsilon} - x_t)^2 - \epsilon V(x_t) \;\; \Rightarrow \;\; \mathcal{L} = -\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x)$$

Path integral quantization is more general than starting from the Schrödinger equation and the Hamiltonian operator for following reasons:

- The Schrödinger approach is based on the properties of state space *in addition* to the Hamiltonian driving the Schrödinger equation.
- The spacetime symmetries of the quantum system are explicit in the Lagrangian-based path integral approach, whereas in the Schrödinger approach these are implicit and need to be extracted using the properties of the Hamiltonian and state space. In particular, one has to derive the symmetry operators that commute with the Hamiltonian.
- Path integral quantization yields a transparent formulation of constrained systems, as for example discussed in Baaquie (2014). In the Schrödinger formulation, one needs *both* the Hamiltonian and commutation relations, which for a constrained system are far from obvious and require a fair amount of derivations.

These considerations come to the forefront in the quantization of complicated systems like non-Abelian gauge fields. The starting point is the Lagrangian, and path integral quantization turns out to be more efficient than the Schrödinger approach.

2.8 Hamiltonian from Lagrangian

Recall in Section 2.6, the Lagrangian was derived from the Hamiltonian using the Dirac–Feynman formula. In this section, it is shown how to derive the Hamiltonian H if the Lagrangian is known; one can use the procedure of classical mechanics for the derivation, but instead a quantum mechanical deviation is given in this section.

Option theory, discussed in Chapter 5, is based on classical random processes that is similar to the diffusion equation. Hence, for classical random processes the time parameter t in the path integral appears as "Euclidean time" t, which for option theory is in fact calendar time. A Lagrangian that is more general than the one discussed in Section 2.6 and that arises in the study of spot interest rate model in finance is the Black–Karasinski model [Baaquie (2004)].

Let the degree of freedom be the real variable ϕ; in the Black–Karasinski model, the spot interest rate is given by $r = r_0 e^{\phi}$. Consider the following Lagrangian and action

$$\mathcal{L}(t) = -\frac{1}{2}\left[me^{-2\nu\phi}\left\{\frac{d\phi}{dt} + \alpha(\phi, t)\right\}^2 + V(\phi)\right]$$

$$S = \int_0^\tau dt\mathcal{L}(t) = -\frac{1}{2}\int_0^\tau dt\left[me^{-2\nu\phi}\left\{\frac{d\phi}{dt} + \alpha(\phi, t)\right\}^2 + V(\phi)\right] \quad (2.8.1)$$

For greater generality, a ϕ-dependent mass equal to $me^{-2\nu\phi}$ and a drift term $\alpha(\phi, t)$ have been included in $\mathcal{L}$.

The path integral is given by the following generalization of Eq. 2.6.9:

$$K(\phi_i, \phi_f; \tau) = \int D\phi e^{-\nu\phi} e^S \tag{2.8.2}$$

$$\int D\phi e^{-\nu\phi} \equiv \prod_{t=0}^{\tau} \int_{-\infty}^{+\infty} d\phi(t) e^{-\nu\phi(t)}$$

$$\text{Boundary conditions } \phi(\tau) = \phi_f;\ \phi(t=0) = \phi_i$$

Note the path integral integration measure $\int D\phi$ has a factor of $e^{-\nu\phi}$ needed to obtain a well-defined Hamiltonian.

Recall from the discussion of the evolution kernel in Section 2.6, the path integral is related to the Hamiltonian H by Eq. 2.6.9, namely

$$K(\phi_i, \phi_f; T) = \int D\phi e^{-\nu\phi} e^S = \langle \phi_f | e^{-\tau H} | \phi_i \rangle \tag{2.8.3}$$

One needs to extract the Hamiltonian H from the path integral on the left-hand side of Eq. 2.8.3.

The Hamiltonian propagates the system through infinitesimal time; the time index t is discretized into a lattice with spacing ϵ, where $t = n\epsilon$ with $N = T/\epsilon$ and $\phi(x) \to \phi_n$. The path integral reduces to a finite $(N-1)$-fold multiple integral, analogous to what was obtained in Eq. 2.6.6. Discretizing the time derivative

$$\frac{d\phi}{dt} \to \frac{\phi_{n+1} - \phi_n}{\epsilon}$$

yields the following lattice action and Lagrangian

$$\langle \phi_N | e^{-\epsilon N H} | \phi_0 \rangle = \prod_{n=1}^{N-1} \int d\phi_n e^{-\nu\phi_n} e^{S(\epsilon)} \tag{2.8.4}$$

$$S(\epsilon) = \epsilon \sum_{n=0}^{N-1} L(n)$$

$$L(n) = -\frac{m e^{-2\nu\phi_n}}{2\epsilon^2} \left[\phi_{n+1} - \phi_n + \epsilon\alpha_n\right]^2 - \frac{1}{2}[V(\phi_{n+1}) + V(\phi_n)]$$

As in Section 2.6, the completeness equation given in Eq. 2.4.12 yields

$$\int d\phi_n |\phi_n\rangle\langle\phi_n| = \mathbb{I}$$

and is used $N-1$ times to write out the expression for $e^{-\epsilon N H}$. The Hamiltonian is identified as follows:

$$\langle \phi_{n+1} | e^{-\epsilon H} | \phi_n \rangle = \mathcal{N}(\epsilon) e^{-\nu\phi_n} e^{\epsilon L_n}$$

$$= \mathcal{N}(\epsilon) e^{-\nu\phi_n} \exp\left\{-\frac{m e^{-\nu\phi}}{2\epsilon}\left[\phi_{n+1} - \phi_n + \epsilon\alpha_n\right]^2 - \frac{\epsilon}{2}[V(\phi_{n+1}) + V(\phi_n)]\right\}$$

Since the Hamiltonian depends on the value of ϕ at two different instants, to simplify notation let

$$\phi_{n+1} = \phi; \;\; \phi_n = \phi'; \;\; \alpha_n = \alpha$$

Ignoring terms that are of $O(\epsilon)$ in Eq. 2.8.4, the matrix elements of the Hamiltonian are given by

$$\langle \phi | e^{-\epsilon H} | \phi' \rangle = \mathcal{N}(\epsilon) e^{-\nu\phi} \exp\left\{ -\frac{m e^{-2\nu\phi}}{2\epsilon} \left[\phi - \phi' + \epsilon\alpha \right]^2 - \epsilon V(\phi) \right\} \quad (2.8.5)$$

Note that unlike Eq. 2.6.9, for which the Hamiltonian is known and the Lagrangian was derived from it, in Eq. 2.8.5 one needs to *derive* the Hamiltonian *from* the known Lagrangian. This derivation is the quantum mechanical analog of the derivation of H given by Hamiltonian mechanics in classical mechanics and discussed by Baaquie (2014).

The key feature of the Lagrangian that in general allows one to derive its Hamiltonian is that the Lagrangian contains *only first-order* time derivatives; hence, discretization of the Lagrangian involves only ϕ_n that are nearest neighbors in time, thus allowing it to be represented as the matrix element of $e^{-\epsilon H}$, as in Eq. 2.8.5.

In contrast, for Lagrangians that contain *second-order* or higher order time derivatives, discussed by Baaquie (2014), the derivation of the Hamiltonian from the Lagrangian and path integral is nontrivial since the entire framework of coordinate and canonical momentum is no longer applicable. Instead, one has to employ the Dirac method required for quantizing constrained systems and, in particular, evaluate the Dirac brackets for the system in order to obtain the Hamiltonian and commutation relations.

In Eq. 2.8.5, the time derivative appears in a quadratic form; hence one can use Gaussian integration to rewrite Eq. 2.8.5 in the following manner[1]

$$\begin{aligned} \langle \phi | e^{-\epsilon H} | \phi' \rangle &= e^{-\nu\phi} e^{-\epsilon V(\phi)} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \exp\left\{ -\frac{\epsilon}{2m} p^2 + ip[\phi - \phi' + \epsilon\alpha] e^{-\nu\phi} \right\} \\ &= e^{-\epsilon V(\phi)} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \exp\left\{ -\frac{\epsilon e^{2\nu\phi}}{2m} p^2 + ip\big(\phi - \phi' + \epsilon\alpha\big) \right\} \end{aligned} \quad (2.8.6)$$

where the prefactor of $e^{-\nu\phi}$ has been canceled by rescaling the integration variable $p \to p e^{\nu\phi}$.

The Hamiltonian $H = H(\phi, \partial/\partial\phi)$ is a differential operator and acts on the dual co-ordinate ϕ, as is required for all differential operators, and mentioned earlier after Eq. 2.5.3. Hence, for the state function $|\psi\rangle$, which is an element of the state

[1] Henceforth $\mathcal{N}(\epsilon)$ is ignored since it is an irrelevant constant contributing to only the definition of the zero of energy.

space, the Hamiltonian acts on the *dual basis state* $\langle\phi|$ and yields $\langle\phi|H|\psi\rangle = H(\phi, \partial/\partial\phi)\psi(\phi)$, similar to the result given in Eq. 2.5.3.

The Hamiltonian is hence given by the following representation[2]

$$\langle\phi|e^{-\epsilon H}|\phi'\rangle = e^{-\epsilon H(\phi,\partial/\partial\phi)}\langle\phi|\phi'\rangle = e^{-\epsilon H(\phi,\partial/\partial\phi)}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}e^{ip(\phi-\phi')} \tag{2.8.7}$$

since $\langle\phi|\phi'\rangle = \delta(\phi - \phi')$. Ignoring overall constants and using the property of the exponential function under differentiation, one can rewrite Eq. 2.8.6 as

$$\langle\phi|e^{-\epsilon H}|\phi'\rangle = \exp\left\{\frac{1}{2m}\epsilon e^{2\nu\phi}\frac{\partial^2}{\partial\phi^2} + \epsilon\alpha\frac{\partial}{\partial\phi} - \epsilon V(\phi)\right\}\int_{-\infty}^{+\infty}\frac{dp}{2\pi}e^{ip(\phi-\phi)} \tag{2.8.8}$$

Comparing Eq. 2.8.8 above with Eq. 2.8.7 yields the Black–Karasinski Hamiltonian for spot interest rates r given by [Baaquie (2004)]

$$H = -\frac{1}{2m}e^{2\nu\phi}\frac{\partial^2}{\partial\phi^2} - \alpha(\phi)\frac{\partial}{\partial\phi} + V(\phi);\quad r = r_0 e^{\phi} \tag{2.8.9}$$

The Hamiltonian is quite general since both $V(\phi)$ and $\alpha(\phi)$ can be functions of the degree of freedom ϕ. Note that the Hamiltonian H in general is non-Hermitian – and is Hermitian only for $\nu = 0$ and a pure imaginary α. The path integral has a nontrivial integration measure $\exp\{-\nu\phi\}$ that arises from the underlying state space and needs to be specified in addition to the Hamiltonian.

2.9 Summary

The principles of quantum mechanics are realized by indeterminate degrees of freedom. An entity in quantum mechanics is described by degrees of freedom that, due to quantum indeterminacy, *simultaneously* take all possible values. This is realized by the operator formalism and by the Feynman path integral.

Both the operator formalism and the Dirac–Feynman path integral formalism were briefly reviewed as these form the basis for the generalizations that are required for the description of quantum fields. The path integral was derived starting from the Hamiltonian and in turn the Hamiltonian was obtained starting from path integral quantization.

The interplay of the path integral with the underlying state space and Hamiltonian is one of the foundations and a unique feature of quantum mathematics that distinguishes the Feynman path integral from functional integration in general.

[2] As in Eq. 2.5.3, the convention for scalar product is $\langle p|\phi_n\rangle = \exp(-ip\phi_n)$, and the sign of the exponential in Eq. 2.8.7 reflects this choice. The definition of H requires it to act on the dual state vector $\langle\phi|$; if one chooses to write the Hamiltonian as acting on the state vector $|\phi\rangle$, $H^\dagger$ would then have obtained instead. Since H is not Hermitian, this would lead to an incorrect result.

2.10 Appendix: Dirac bracket and vector notation

Linear algebra is based on the idea of the N-dimensional Euclidean space E_N – which is a finite-dimensional linear vector space. Vectors are elements of E_N and matrices are linear transformations from E_N to E_M. The synthesis of linear algebra with calculus, which is the basis of functional analysis, is most easily carried out by expressing vectors and matrices in Dirac's notation. The discussion is confined to square matrices, which are linear transformations from E_N to E_N, although the notation can also be applied to the general case.

The basic ingredient of the Dirac **bracket** notation is the following

$$\text{ket}: \ |..\rangle = \ \text{vector}; \quad \text{bra}: \ \langle..| = \ \text{dual vector}$$

Taken together they form the complete bracket

$$\text{bracket}: \ \langle..|..\rangle = \text{scalar product} = \text{complex number}$$

Consider, for generality, a complex valued vector $\mathbf{v}$, which is represented by the ket-vector

$$\mathbf{v} = |v\rangle \ ; \ \ \mathbf{v}_i = |v_i\rangle$$

There is no need to make the symbol v boldface inside the ket since the notation makes this clear. The dual bra-vector is given by

$$\mathbf{v}^\dagger = \langle v|$$

The canonical basis and its dual vectors are given by

$$\mathbf{e}_i = |i\rangle; \ \ \mathbf{e}_i^\dagger = \langle i|$$

Linear superposition is written as expected

$$\mathbf{u} = a\mathbf{v} + b\mathbf{w} \ \Rightarrow \ |u\rangle = a|v\rangle + b|w\rangle$$

with the dual expression given by

$$\mathbf{u}^\dagger = a^*\mathbf{v}^\dagger + b^*\mathbf{w}^\dagger \ \Rightarrow \ \langle u| = a^*\langle v| + b^*\langle w|$$

The expansion of a vector into its components is

$$\mathbf{v} = \sum_i v(i)\mathbf{e}_i \quad \Rightarrow \quad |v\rangle = \sum_i v(i)|i\rangle$$

In Dirac's notation, the scalar product is given by

$$\mathbf{v}^\dagger \cdot \mathbf{w} = (\langle v|)(|w\rangle) = \langle v||w\rangle = \langle v|w\rangle$$

functional integration, a few of its key properties, the Gaussian, or the normal random variable, is reviewed.

The basic Gaussian integral is given by

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}\lambda x^2 + jx} = \sqrt{\frac{2\pi}{\lambda}} e^{\frac{1}{2\lambda} j^2} \tag{2.11.1}$$

N-dimensional Gaussian integration

The moment-generating function for the N-dimensional Gaussian random variable is given by

$$Z[j] = \int_{-\infty}^{+\infty} dx_1 \cdots dx_n e^S$$

$$\text{with} \quad S = -\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_i J_i x_i \tag{2.11.2}$$

Let A_{ij} be a symmetric and positive definite matrix that has only positive eigenvalues. A_{ij} can be diagonalized by an orthogonal matrix M

$$A = M^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} M; \quad M^T M = \mathbb{I}$$

Define new variables

$$z_i = M_{ij} x_j; \quad x_i = M^T_{ij} z_j$$

$$\prod_{i=1}^{N} dz_i = \det M \prod_{i=1}^{N} dx_i = \prod_{i=1}^{N} dx_i \equiv Dx$$

Hence

$$Z[j] = \prod_i^N \int dz_i e^{-\frac{1}{2}\lambda_i z_i^2 + (JM^T)_i z_i} = \prod_{i=1}^{N} \left[\sqrt{\frac{2\pi}{\lambda_i}} e^{\frac{1}{2\lambda_i}(J^T M^T)_i (J^T M^T)_i} \right]$$

In matrix notation

$$\sum_i \frac{1}{\lambda_i} (J^T M^T)_i (J^T M^T)_i = J A^{-1} J; \quad \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} = (2\pi)^{N/2} \frac{1}{\sqrt{\det A}}$$

Hence

$$Z[j] = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} e^{\frac{1}{2} J A^{-1} J} \tag{2.11.3}$$

All the moments of the coupled Gaussian random variables can be determined by the generating function given in Eq. 2.11.2, namely

$$E[x_1 x_2 ..x_N] = \frac{\partial^N}{\partial J_1 \partial J_2 ..\partial J_N} Z[J]|_{J=0}$$

Let $t = n\epsilon$, $n = 0, \pm 1, \pm 2, \ldots, \pm N$. The limit yields a continuum number of integration variables $x(t)$, with $-\infty \leq t \leq +\infty$. All summations over n yield integrations and with the "action" given by

$$S = -\frac{1}{2}\int_{-\infty}^{+\infty} dt dt' x(t) A^{-1}(t,t') x(t') + \int_{-\infty}^{+\infty} dt J(t) x(t) \tag{2.11.4}$$

From Eq. 2.11.3, one obtains the generating functional[3]

$$Z[j] = \frac{1}{Z}\int Dx e^{S_0 + \int dt j(t) x(t)} = \exp\left\{\frac{1}{2}\int_{-\infty}^{+\infty} dt dt' j_t A^{-1}(t,t') j(t')\right\} \tag{2.11.5}$$

The normalization $\mathcal{N}$ is now a divergent quantity, which ensures the usual normalization $Z(0) = 1$. In discussions on quantum field theory, Eq. 2.11.5 plays a central role.

The fundamental reason why Gaussian integration generalizes to infinite dimensions is because the measure is invariant under translations, that is, under $x(t) \rightarrow x(t) + \xi(t)$; one can easily verify that this symmetry of the measure yields the result obtained in Eq. 2.11.5.

2.11.1 Quadratic action

Consider the action of the "harmonic oscillator" given by

$$S = -\frac{m}{2}\int_{-\infty}^{+\infty} dt \left[\left(\frac{dx(t)}{dt}\right)^2 + \omega^2 x^2(t)\right] = -\frac{m}{2}\int_{-\infty}^{+\infty} dt x(t)\left(-\frac{d^2}{dt^2} + \omega^2\right) x(t)$$

$$\Rightarrow A^{-1}(t,t') = m\left(-\frac{d^2}{dt^2} + \omega^2\right)\delta(t-t')$$

where an integration by parts was done, discarding boundary terms at $\pm\infty$, to obtain the second expression for S above. The propagator $A(t,t')$ is given by

$$A(t,t') = \frac{1}{2\pi m}\int_{-\infty}^{+\infty} dp \frac{e^{ip(t-t')}}{p^2+\omega^2} = \frac{1}{2m|\omega|} e^{-|\omega||t-t'|}$$

The result above can be verified by using Eq. 2.4.8.

Consider the acceleration action functional given by [Baaquie (2014)]

$$S[x] = -\frac{1}{2}\int_{-\infty}^{+\infty} dt \left[L\left(\frac{d^2x(t)}{dt^2}\right)^2 + \tilde{L}\left(\frac{dx(t)}{dt}\right)^2 + \gamma^2 x^2(t)\right] \tag{2.11.6}$$

[3] The term-generating functional is used instead as generating function as in Eq. 2.11.3 to indicate that one is considering a system with infinitely many variables.

Define the Fourier transform of $x(t), j(t)$ by $\tilde{x}(k), \tilde{j}(k)$ given by

$$x(t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikt} \tilde{x}(k); \;\; j(t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikt} \tilde{j}(k)$$

Eq. 2.11.6 yields

$$S[\tilde{x}] = -\frac{1}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \left[Lk^4 + \tilde{L}k^2 + \gamma^2 \right] x(-k)x(k) \tag{2.11.7}$$

In terms of the Fourier-transformed variables, the generating functional given in Eq. 2.11.5 yields, for action given in Eq. 2.11.7, the following [Baaquie (2014)]:

$$\begin{aligned} E[x(t)x(t')] &= \frac{1}{Z} \int Dx \exp\left\{ S[x] + \int dsj(s)x(s) \right\} x(t)x(t') \\ &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{dk'}{2\pi} e^{ikt+ik't'} \frac{1}{Z} \int D\tilde{x} \exp\left\{ S[\tilde{x}] + \int \frac{dk}{2\pi} \tilde{j}(-k)\tilde{x}(k) \right\} \tilde{x}(k)\tilde{x}(k') \\ \Rightarrow E[x(t)x(t')] &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{e^{ik(t-t')}}{Lk^4 + \tilde{L}k^2 + \gamma^2} \end{aligned} \tag{2.11.8}$$

The correlation function given in Eq. 2.11.8 plays a central role in the study of commodities and interest rates.

2.11.2 Gaussian white noise

The properties of white noise are analyzed as this constitutes the simplest form of Gaussian functional integration; it also shows how the Dirac delta-functions for the correlation functions emerge from functional integration.

The fundamental properties of Gaussian white noise are that

$$E[R(t)] = 0; \quad E[R(t)R(t')] = \delta(t - t') \tag{2.11.9}$$

Figure 2.4 shows that there is an independent (Gaussian) random variable $R(t)$ for each instant of time t.

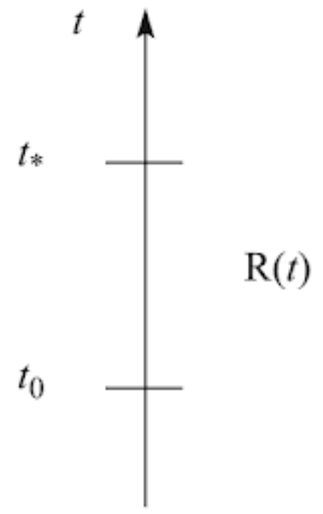


Figure 2.4 One random variable $R(t)$ for each instant of time.

Discretize time, namely $t = n\epsilon$, with $n = 1, 2, \ldots, N$, and with $R(t) \to R_n$. The probability distribution function of white noise is given by

$$P(R_n) = \sqrt{\frac{\epsilon}{2\pi}} \mathrm{e}^{-\frac{\epsilon}{2}R_n^2} \tag{2.11.10}$$

Hence, R_n is a Gaussian random variable with zero mean and $1/\sqrt{\epsilon}$ variance, and is denoted by $N(0, 1/\sqrt{\epsilon})$. The following result is essential in deriving the rules of Ito calculus:

$$R_n^2 = \frac{1}{\epsilon} + \quad \text{random terms of } 0(1) \tag{2.11.11}$$

To write the probability measure for $R(t)$, with $t_1 \leq t \leq t_2$ discretize $t \to n\epsilon$. White noise $R(t)$ has the probability distribution given in Eq. 2.11.10. The probability measure for the white noise random variables in the interval $t_1 \leq t \leq t_2$ is the given by

$$\mathcal{P}[R] = \prod_{n=1}^{N} P(R_n) = \prod_{n=1}^{N} e^{-\frac{\epsilon}{2}R_n^2}; \quad \int dR = \prod_{n=1}^{N} \sqrt{\frac{\epsilon}{2\pi}} \int_{-\infty}^{+\infty} dR_n \tag{2.11.12}$$

Taking the continuum limit of $\epsilon \to 0$ yields, for $t_1 < t < t_2$,

$$\mathcal{P}[R, t_1, t_2] \to e^{S_0}; \;\; S_0 = -\frac{1}{2}\int_{t_1}^{t_2} dt R^2(t) \tag{2.11.13}$$

$$Z = \int DR e^{S_0}; \quad \int dR \to \int DR$$

The action functional S_0 is ultra-local with all the variables being decoupled. Gaussian integration, given in Eq. 2.11.3, yields

$$Z[j, t_1, t_2] = \frac{1}{Z}\int DR e^{\int_{t_1}^{t_2} dt j(t)R(t)} e^{S_0[} = e^{\frac{1}{2}\int_{t_1}^{t_2} dt j^2(t)} \tag{2.11.14}$$

The correlation functions are given by

$$E[R(t)] = 0$$

$$E[R(t)R(t')] = \frac{1}{Z}\int DR\, R(t)R(t')e^{S_0} = \frac{\delta^2}{\delta j(t)\delta j(t')} Z[j]\Big|_{j=0} = \delta(t - t')$$

and yield the result given in Eq. 2.11.9. Functional differentiation is discussed in Noteworthy 7.2.

The results given in Eqs. 2.11.13 and 2.11.14 show that white noise is represented by a path integral with an ultra-local action S_0.

3

Classical field theory

3.1 Introduction

The concept of the classical field is introduced in this chapter. The forms that Nature takes in physics are a variety of fields, including the Standard Model of high energy physics as well as the geometrical theory of gravitation. The key feature of a field – in fact, its most important and *defining property* – is that the field carries both energy and momentum at *every point* of space (and time). This is the reason that a field is considered to be a *physical entity* – as physical as a classical particle – with the difference that the field's energy and momentum can flow from one part to another, unlike a particle for which its energy and momentum are at the point that it occupies (of course, this point can move). Just as is the case for particles, the time evolution of a field exactly conserves the field's total energy and momentum.

As the field evolves in time, the energy and momentum of the field can be redistributed from one point of space to another by the variation in the field's strength. The term "a propagating field" is shorthand for describing the redistribution of the field's energy and momentum at the different points of space; this redistribution can take place in many ways, with the most commonly studied case being the wave-like oscillations of the field's strength at different points of space. A propagating electromagnetic field can, in principle, propagate out to infinite distances. And, conversely, a propagating electromagnetic field impinging on a charged particle can transfer energy to it and cause the particle to accelerate.

The total energy and momentum of the field coupled to charged particles, taken together, is exactly conserved at every point of spacetime.

In general, a classical field is a *determinate entity*, which is completely specified by assigning a numerical value to the field (with appropriate dimensions) at every point in spacetime. The values specifying the field at every point can change as the field evolves in time. A particle of classical physics occupies a single point, whereas a (scalar or a vector) field is spread over space. The numerical value of a field can be a single number, as in the case of a scalar field, or it can consist of several numbers,

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 : \text{ Euler–Lagrange equation} \tag{3.2.8}$$

The boundary term must be zero:

$$\int \frac{d}{dt}\left(\delta q_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) dt = 0 \tag{3.2.9}$$

Note $\mathcal{L}$ is defined only up to a term $d\Lambda/dt$ since

$$\mathcal{L}' = \mathcal{L} + \frac{d\Lambda}{dt}$$

gives the same Euler–Lagrange equation up to a boundary term.

Noteworthy 3.1 Relativistic notation

The metric $\eta_{\mu\nu}$ for Minkowski space is given by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}; \; x^\mu = (x^0, x^1, x^2, x^3) \tag{3.2.10}$$

This yields, using the convention that repeated indices are summed over

$$x_\mu = \eta_{\mu\nu} x^\nu = (ct, -x, -y, -z) = (x_0, x_1, x_2, x_3)$$

A Lorentz-invariant scalar product is defined by using metric $\eta_{\mu\nu}$ and yields

$$A^\mu B_\mu = \eta^{\mu\nu} A_\mu B_\nu = \eta_{\mu\nu} A^\mu B^\nu$$

Furthermore

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right); \;\; \partial^\mu \partial_\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

3.3 Classical field equation

The field φ is defined on spacetime $x,\ t$ and can be density, pressure, temperature fields or electromagnetic and gravitation fields. All fields $\varphi(t,x)$ carry energy and momentum at each spacetime point. The kinetic energy of the field is defined for finite volume $\mathcal{R}^3$ by

$$T = \frac{1}{2}\widetilde{m} \int_{\mathcal{R}^3} d^3x \left(\frac{\partial \varphi(t,x)}{\partial t}\right)^2 \tag{3.3.1}$$

and its potential energy is

$$V = \int_{\mathcal{R}^3} d^3x \left[\frac{1}{2}\widetilde{m} \left(\frac{\partial \varphi}{\partial \vec{x}} \right)^2 + V(\varphi) \right] \tag{3.3.2}$$

The Lagrangian density $\mathcal{L}(t,x)$ is given by

$$\mathcal{L} = \frac{1}{2}\widetilde{m} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \frac{1}{2}\widetilde{m} \left(\frac{\partial \varphi}{\partial \vec{x}} \right)^2 - V(\varphi) \tag{3.3.3}$$

and the action is

$$\mathcal{S} = \int_{t_i}^{t_f} dt \int_{\mathcal{R}^3} d^3x \mathcal{L}(t,x) \equiv \int_V \mathcal{L}(\varphi, \partial_\mu \varphi) \tag{3.3.4}$$

where $V = \mathcal{R}^3 \bigotimes [t_f, t_i]$.

A transformation for the scalar field, keeping the spacetime manifold fixed, has the form

$$\varphi(x) \rightarrow \varphi'(x) = \varphi(x)$$

and leads to the variation

$$\delta\varphi \equiv \varphi'(x) - \varphi(x)$$

with the constraint that the variation at the initial and final surface is zero; that is,

$$\delta\varphi\Big|_{t=t_i} = 0 = \delta\varphi\Big|_{t=t_f}$$

The variation of the action is given by

$$\begin{aligned} \delta\mathcal{S} &= \int_V \left[\frac{\partial \mathcal{L}}{\partial \varphi(t,x)} \delta\varphi(t,x) + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \partial_\mu \varphi \right] \\ &= \int_V \left[\frac{\partial \mathcal{L}}{\partial \varphi(t,x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \right] \delta\varphi(t,x) + \int_V \left[\partial_\mu \left(\delta\varphi \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \right) \right] \\ &= \delta\mathcal{S}_V + \delta\mathcal{S}_{\partial V} \end{aligned} \tag{3.3.5}$$

The *constraint* that the variation on the boundary be zero yields[1]

$$\delta\mathcal{S}_{\partial V} = 0$$

The field equation is then given by the following:

$$\delta\mathcal{S} = \delta\mathcal{S}_V = 0 \;\; \Rightarrow \;\; \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = 0\text{: Classical field equation} \tag{3.3.6}$$

[1] Unlike the case for a particle, the boundary conditions on the field at infinitely distant points of space are required to be separately imposed.

Note the label i in $q_i(t)$ of the canonical coordinate has become a continuous label $\vec{x}$. In this sense, at every point $\vec{x}$, the field has an independent canonical coordinate $\varphi_{\vec{x}}(t)$. Writing Eq. 3.3.6 explicitly yields

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \frac{\partial \varphi}{\partial t}} + \sum_{i=1}^{3}\frac{\partial}{\partial x^i}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial \varphi}{\partial x^i}}\right) = 0 \tag{3.3.7}$$

3.4 Free scalar field

The free scalar field $\varphi(t,x)$ is a real valued function of t,x and is an infinite-dimensional generalization of the simple harmonic oscillator; its Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{m^2}{2}\varphi^2$$

In terms of the space and time coordinates,

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \varphi}{\partial \vec{x}}\right)^2 - \frac{m^2}{2}\varphi^2$$

The Lagrangian yields

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2\varphi;\ \frac{\partial \mathcal{L}}{\partial \frac{\partial \varphi}{\partial t}} = \frac{\partial \varphi}{\partial t};\ \frac{\partial \mathcal{L}}{\partial \frac{\partial \varphi}{\partial \vec{x}}} = -\frac{\partial \varphi}{\partial \vec{x}}$$

In relativistic notation

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi$$

and the Euler–Lagrange equation is

$$-m^2\varphi - \partial_\mu \partial^\mu \varphi = 0 \ \Rightarrow (\partial_\mu \partial^\mu + m^2)\varphi = 0$$

Or equivalently

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \vec{x}^2} + m^2\right]\varphi = 0: \quad \text{Klein–Gordon equation}$$

The Hamiltonian density is

$$H = T + V = \frac{\widetilde{m}}{2}\dot{\varphi}^2 + \frac{\widetilde{m}}{2}(\vec{\nabla}\varphi)^2 + \frac{\omega^2}{2}\varphi^2$$

Hence the Klein–Gordon equation is

$$\left[\partial^\mu \partial_\mu + \frac{m^2c^2}{\hbar^2}\right]\varphi = 0: \quad \text{Klein–Gordon equation}$$

In all the subsequent discussions, the units are chosen so that $c = \hbar = 1$.

3.5 Symmetries

The symmetries of a system are those transformations of the field φ and of the coordinates $x = (t, \vec{x})$ that leave the field equations of the theory unchanged. The symmetry is realized by the transformation leaving *invariant* the Lagrangian (up to a divergence) and hence the action $\mathcal{S}$.

A symmetry transformation by definition leaves the action functional invariant. By considering an infinitesimal symmetry transformation, the invariance of the action yields a volume term that is zero due to the field equations as well as a boundary term that is also zero. The boundary term contains the generators of the symmetry transformations that yield a conserved current. Noether's theorem shows how to extract the conserved currents that result from the symmetry of the action.

Consider a general symmetry transformation that has the form

$$x \to \tilde{x} = \tilde{x}(x); \quad \varphi(x) \to \tilde{\varphi}(\tilde{x}) \tag{3.5.1}$$

The transformation leaves the Lagrangian invariant (up to a divergence), and hence, for any arbitrary volume V, we have

$$S = \int d^4\tilde{x}\mathcal{L}(\tilde{\varphi}(\tilde{x}), \tilde{\partial}\tilde{\varphi}(\tilde{x}), \tilde{x}) = \int d^4x\mathcal{L}(\varphi, \partial\varphi, x)$$

The coordinate transformation yields

$$\tilde{x}^\mu = x^\mu + \delta x^\mu; \quad d^4\tilde{x} = Jd^x$$

where the Jacobian J is given by

$$\frac{\partial\tilde{x}^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu(\delta x^\mu) \quad \Rightarrow \quad J = \det\left(\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\right) = 1 + \partial_\mu(\delta x^\mu)$$

Expanding the action to lowest order in the coordinate yields

$$S = \int d^4\tilde{x}\mathcal{L}(\tilde{\varphi}(\tilde{x}), \tilde{\partial}\tilde{\varphi}(\tilde{x})) = \int d^4x J\{\mathcal{L}(\tilde{\varphi}(x), \partial\tilde{\varphi}(x)) + \partial_\mu\mathcal{L}\delta x^\mu\}$$

Using the expression for the Jacobian J yields

$$\begin{aligned} S &= \int d^4x\{\mathcal{L}(\tilde{\varphi}(x), \partial\tilde{\varphi}(x)) + \partial_\mu\mathcal{L}\delta x^\mu + \mathcal{L}\partial_\mu(\delta x^\mu)\} \\ &= \int d^4x\{\mathcal{L}(\tilde{\varphi}(x), \partial\tilde{\varphi}(x)) + \partial_\mu(\mathcal{L}\delta x^\mu)\} \end{aligned} \tag{3.5.2}$$

Define the variation of the field at the same spacetime point x by

$$\delta\varphi(x) = \tilde{\varphi}(x) - \varphi(x) \tag{3.5.3}$$

Since the transformation is a symmetry, it leaves the action invariant and yields

$$\delta S = 0 = \int d^4x\Big[\mathcal{L}(\tilde{\varphi}(x), \partial\tilde{\varphi}(x)) + \partial_\mu(\mathcal{L}\delta x^\mu) - \mathcal{L}(\varphi(x), \partial_\mu\varphi(x))\Big] \tag{3.5.4}$$

Note the fact that $\delta S = 0$ in Eq. 3.5.4 is a *consequence* of the symmetry of the Lagrangian under the symmetry transformation in question. This is quite different from the derivation in Section 3.3 of the field equations, where one *imposes* the condition of $\delta S = 0$, with this condition constraining the classical field to obey the classical field equation. It is no coincidence that the symmetry transformation and field equation both end up with $\delta S = 0$, since it can be shown that the field equations in fact already contain the symmetries of the Lagrangian [Peskin and Schroeder (1995)]. For an infinitesimal transformation, using Eqs. 3.3.5 and 3.5.4,

$$\begin{aligned}\delta\mathcal{S} &= \int_V \left[\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\partial_\mu\varphi + \partial_\mu(\mathcal{L}\delta x^\mu)\right] \\ &= \int_V \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right]\delta\varphi(t,x) + \int_V \left[\partial_\mu\left(\delta\varphi\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right) + \partial_\mu(\mathcal{L}\delta x^\mu)\right] \\ &= \delta\mathcal{S}_V + \delta\mathcal{S}_{\partial V} = 0\end{aligned}$$

Hence, in general,

$$0 = \delta\mathcal{S} = \delta\mathcal{S}_V + \delta\mathcal{S}_{\partial V}$$

The equation of motion makes $\delta\mathcal{S}_V = 0$, and the symmetry transformation must respect

$$\delta\mathcal{S}_{\partial V} = \int_V \left[\partial_\mu\left(\delta\varphi\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right) + \partial_\mu(\mathcal{L}\delta x^\mu)\right] = 0\text{: Symmetry} \tag{3.5.5}$$

Using Gauss's theorem, the boundary term can be rewritten.[2] For $d\Sigma^\mu$ being the vector of the surface element

$$\begin{aligned}0 = \delta\mathcal{S}_{\partial V} &= \int_V d^4x\partial_\mu\left(\delta\varphi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} + \mathcal{L}\delta x^\mu\right) = \int_{\partial V} d\Sigma_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi + \mathcal{L}\delta x^\mu\right) \\ &= \int_{\partial V} d\Sigma_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}[\delta\varphi + (\partial_\nu\varphi)\delta x^\nu] - \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L}\right]\delta x^\nu\right)\end{aligned} \tag{3.5.6}$$

Note that

$$\delta\varphi + (\partial_\nu\varphi)\delta x^\nu = \tilde{\varphi}(\tilde{x}) - \varphi(x) = \Delta\varphi$$

and yields the final result

$$0 = \delta\mathcal{S}_{\partial V} = \int_{\partial V} d\Sigma_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L}\right]\delta x^\nu\right) \tag{3.5.7}$$

[2] Under a symmetry transformation, the equations of motion are left invariant as long as

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \partial_\mu\mathcal{J}^\mu \quad\Rightarrow\quad \delta\mathcal{L} = \partial_\mu\mathcal{J}^\mu$$

The current $\mathcal{J}^\mu$ given above does not change the equations of motion because, using Gauss's theorem, it integrates to zero in the action. It, however, does contribute to the boundary term $\delta\mathcal{S}_{\partial V}$. We will ignore the extra current $\mathcal{J}^\mu$ as it is not required for subsequent discussions.

Note that the stress tensor in Eq. 3.7.2 is only defined up to a total divergence. An equivalent and modified stress tensor that is conserved can be defined as follows:

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda(K^{\lambda\mu\nu}); \;\; K^{\lambda\mu\nu} = -K^{\mu\lambda\nu} \;\; \Rightarrow \;\; \partial_\mu(\tilde{T}^{\mu\nu}) = 0 \qquad (3.7.4)$$

In some cases, such as electrodynamics, the stress tensor has to be modified to make it symmetric, as required by angular momentum conservation.

3.7.1 Klein–Gordon field

The Lagrangian for the massive scalar field is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2$$

and yields the field equation

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} = \partial^\mu\varphi$$

The stress tensor is

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L} = \partial^\mu\varphi\partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L}.$$

3.7.2 Electromagnetic field

The Maxwell Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

Note that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -F^{\alpha\beta}$$

Hence, from Eq. 3.7.2, the stress tensor is given by

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial_\nu A_\lambda - \delta^\mu_\nu\mathcal{L}$$

and hence

$$T^{\mu\nu} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}\delta^{\nu\mu}F^{\alpha\beta}F_{\alpha\beta} \neq T^{\nu\mu}$$

The stress tensor needs to be symmetric for the conservation of angular momentum. Using the result from Eq. 3.7.4, the new stress tensor is defined by

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

with

$$K^{\lambda\mu\nu} = F^{\lambda\nu}A^{\nu} = -K^{\mu\lambda\nu}$$

The sought-for symmetric stress tensor is given by

$$\tilde{T}^{\mu\nu} = F^{\mu\lambda}F^{\nu}_{\lambda} + \frac{1}{4}\delta^{\nu\mu}F^{\alpha\beta}F_{\alpha\beta} = \tilde{T}^{\nu\mu}$$

3.8 Spontaneous symmetry breaking

The Goldstone model consists of a *complex scalar field* $\varphi(t,x)$ with a nonlinear Lagrangian given by

$$\mathcal{L}_G = \partial_\mu\varphi^*\partial^\mu\varphi - \mu^2\varphi^*\varphi - \lambda[\varphi^*\varphi]^2;\ \ \lambda > 0 \tag{3.8.1}$$

The parameter μ^2 in the Lagrangian yields a well-defined and convergent quantum field theory for both positive and negative values.

Representing the complex scalar field by the following two real scalar fields

$$\varphi(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) + i\varphi_2(x)];\ \ \varphi^*(x) = \frac{1}{\sqrt{2}}[\varphi_1(x) - i\varphi_2(x)]$$

yields the following:

$$\mathcal{L}_G = \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - \frac{1}{2}\mu^2(\varphi_1^2+\varphi_2^2) - \frac{1}{4}\lambda[\varphi_1^2+\varphi_2^2]^2$$

Consider the global U(1) symmetry transformation

$$\varphi(x) \to \varphi'(x) = e^{-i\alpha}\varphi(x);\ \varphi^*(x) \to \varphi'^*(x) = e^{i\alpha}\varphi^*(x)$$

The phase α is constant and hence the transformation is a global one.

The Lagrangian is invariant and hence

$$\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x)$$

The complex field can be represented using polar coordinate as follows:

$$\varphi = \frac{1}{\sqrt{2}}re^{i\theta} \ \ \Rightarrow\ \ \varphi_1 = \frac{1}{\sqrt{2}}r\cos(\theta),\ \ \varphi_2 = \frac{1}{\sqrt{2}}r\sin(\theta)$$

This yields

$$V(r) \equiv \mu^2\varphi^*\varphi + \lambda[\varphi^*\varphi]^2 = \frac{1}{2}\mu^2r^2 + \frac{1}{4}\lambda r^4$$

and hence

$$\frac{\partial V}{\partial r} = r(\mu^2+\lambda r^2)\ ;\ \frac{\partial^2 V}{\partial r^2} = \mu^2 + 3\lambda r^2$$

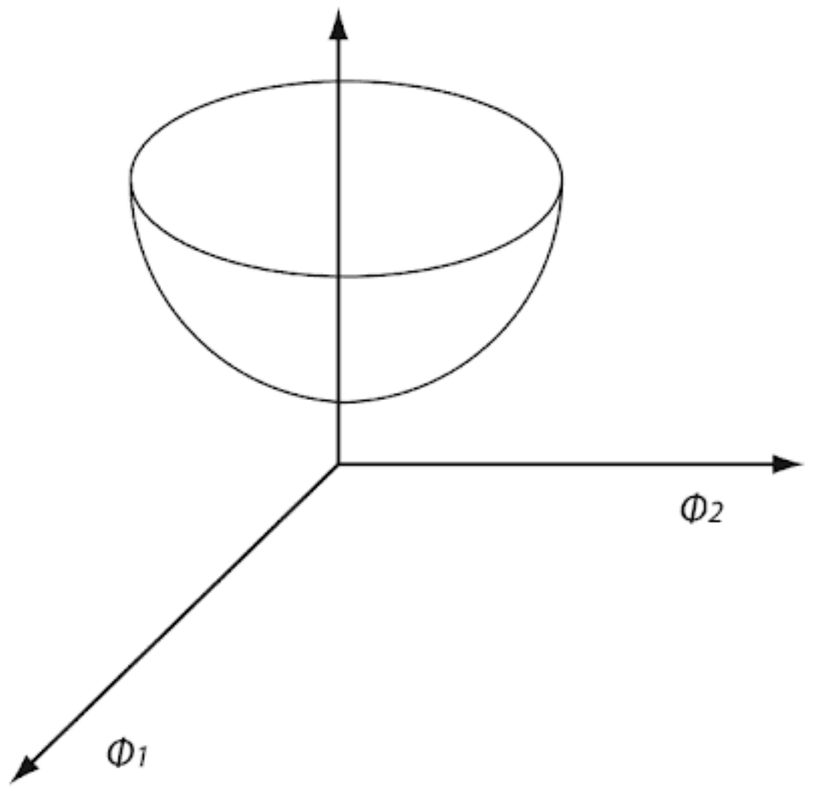


Figure 3.1 Potential for $\mu^2 > 0$ has a unique minimum.

Consider a system undergoing a second-order phase transition at temperature T_c; in the Landau–Ginzburg–Wilson phenomenological approach, μ^2 has the following dependence on temperature T:

$$\mu^2 \propto T - T_c$$

Hence

$$\begin{aligned} &\text{Noncondensed phase: } \mu^2 > 0: \quad T > T_c \\ &\text{Condensed phase: } \mu^2 < 0: \quad T < T_c \end{aligned} \tag{3.8.2}$$

The noncondensed disordered phase has $\mu^2 > 0$ and yields

$$\frac{\partial V}{\partial r} = 0 \;\; \Rightarrow r_0 = 0; \;\; \frac{\partial^2 V}{\partial r^2} = \mu^2 > 0 \;\; \Rightarrow \;\; \text{Minimum}$$

The potential for the noncondensed phase is shown in Figure 3.1.

For the condensed phase $\mu^2 < 0$ and yields

$$\frac{\partial V}{\partial r} = 0 \;\; \Rightarrow r_0^2 = 0, -\frac{\mu^2}{\lambda}$$

Hence

$$\frac{\partial^2 V}{\partial r^2}\Big|_{r_0=0} = \mu^2 < 0 \;\; \Rightarrow \;\; \text{Maximum}$$

and

$$\frac{\partial^2 V}{\partial r^2}\Big|_{r_0^2=-\frac{\mu^2}{\lambda}} = -2\mu^2 > 0 \;\; \Rightarrow \;\; \text{Minimum}$$

In other words, for $\mu^2 > 0$, the minimum of the potential is at $|\varphi| = 0$. The potential V for $\mu^2 < 0$, that is, less than zero, is shown in Figure 3.2; for $\mu^2 < 0$, the minima of the field lie on the circle defined by

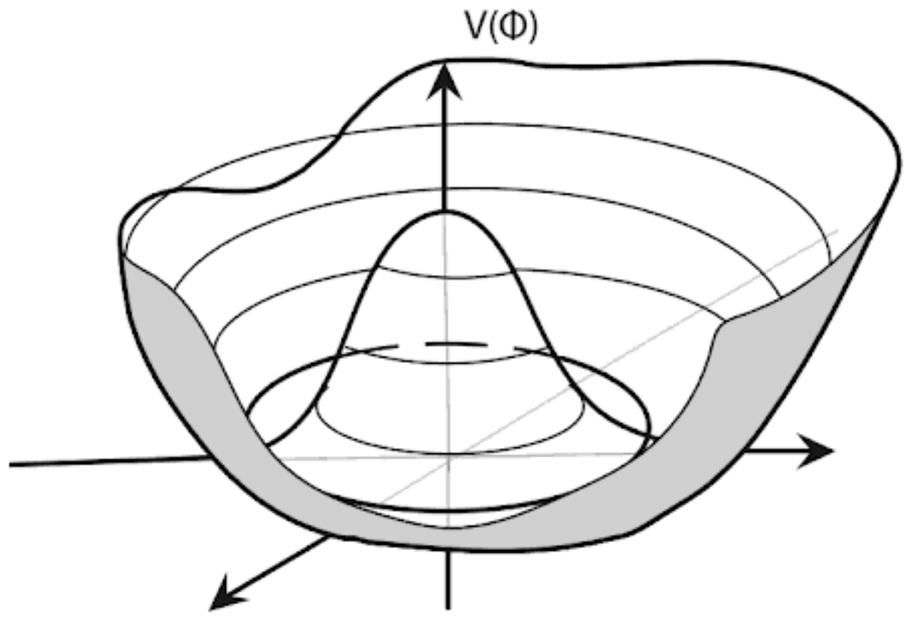


Figure 3.2 There are multiple minima for $\mu^2 < 0$, all of which break global gauge symmetry.

$$r_0^2 = -\frac{\mu^2}{\lambda}$$

Let $\upsilon^2 > 0$ be a constant; in the condensed phase with broken symmetry, to leading order, the value of the field is

$$|\varphi| \simeq |\varphi_0| = \frac{1}{\sqrt{2}}\upsilon \tag{3.8.3}$$

To study the theory around the nonzero minima given by $|\varphi_0|$, consider the change of variables

$$\varphi(x) = \frac{1}{\sqrt{2}}[\upsilon + \sigma(x) + i\pi(x)];\ \varphi^*(x) = \frac{1}{\sqrt{2}}[\upsilon + \sigma(x) - i\pi(x)]$$

In terms of the new field variables the potential is given by

$$\begin{aligned} V &= \frac{1}{2}\mu^2[(\upsilon+\sigma)^2+\pi^2] + \frac{1}{4}\lambda[(\upsilon+\sigma)^2+\pi^2]^2 \\ &= \frac{1}{2}\mu^2[\sigma^2+\pi^2+2\upsilon\sigma+\upsilon^2] + \frac{1}{4}\lambda\Big[\left(\sigma^2+\pi^2\right)^2 \\ &\quad + 2(\upsilon^2\sigma^2+\upsilon^2\pi^2+2\upsilon\sigma^3+2\upsilon\sigma\pi^2) + 4\upsilon^2\sigma^2 + 4\upsilon^3\sigma + \upsilon^4\Big] \\ &= \frac{1}{2}\mu^2[\sigma^2+\pi^2+2\upsilon\sigma+\upsilon^2] + \frac{1}{2}\lambda\Big[\upsilon^2\sigma^2+\upsilon^2\pi^2+2\upsilon^2\sigma^2+2\upsilon^3\sigma\Big] \\ &\quad + \frac{1}{4}\lambda\Big[\left(\sigma^2+\pi^2\right)^2 + 4\upsilon\sigma^3 + 4\upsilon\sigma\pi^2 + \upsilon^4\Big] \end{aligned}$$

and which yields the final result that

$$\begin{aligned} V = (\mu^2+\lambda\upsilon^2)\left[\upsilon\sigma + \frac{1}{2}\pi^2\right] + \frac{1}{2}(\mu^2+3\lambda\upsilon^2)\sigma^2 + \frac{1}{2}\mu^2\upsilon^2 \\ + \frac{1}{4}\lambda\Big[\left(\sigma^2+\pi^2\right)^2 + 4\upsilon\sigma^3 + 4\upsilon\sigma\pi^2 + \upsilon^4\Big] \end{aligned}$$

Choosing υ to be at the minimum of the potential yields

$$\upsilon^2 = r_0^2 = -\frac{\mu^2}{\lambda}$$

Since the field is being expanded about the minimum of the potential, the choice for υ^2 eliminates the first term in the potential – which has *both* the linear term in σ and the quadratic π^2 term. The mass of the π field is zero because the π^2 term is zero and this is due to the excitations of the field along the valley of the potential, which are massless; the massive field σ results from excitations away from the valley.

The field π is massless and is called the "Goldstone boson" field. It is a result of the phase transition having long-range correlation functions.

Replacing μ^2 by $-\lambda\upsilon^2$ in the Lagrangian, we obtain

$$\begin{aligned}\mathcal{L}_G = {} & \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma - \frac{1}{2}(2\lambda\upsilon^2)\sigma^2 : \text{massive scalar} \\ & + \frac{1}{2}\partial_\mu\pi\,\partial^\mu\pi : \text{massless scalar} \\ & - \lambda\upsilon\sigma(\sigma^2+\pi^2) - \frac{\lambda}{4}[\sigma^2+\pi^2]^2 : \text{interaction} \\ & + \frac{1}{4}\lambda\upsilon^4 + \frac{1}{2}\upsilon^2\mu^2 : \text{constant} \end{aligned} \tag{3.8.4}$$

Expanding the field about the nonzero minimum *breaks the global symmetry* of $\varphi \rightarrow e^{i\alpha}\varphi$, which exists in the original Lagrangian, and the vacuum state of the corresponding quantum field theory is said to have *spontaneously broken* the global U(1) symmetry.

The reason the symmetry is broken is because the field variables are now σ, π. The transformation $\sigma \pm i\pi \rightarrow e^{\pm i\alpha}(\sigma \pm i\pi)$ no longer leaves the potential term V invariant. This is because $\upsilon^2 > 0$ is a *physically observable quantity* and it cannot depend on the gauge and hence cannot be changed by a gauge transformation.

Hence, for $\upsilon^2 > 0$, the Lagrangian $\mathcal{L}$ is no longer U(1) invariant. The spontaneous symmetry breaking has given rise to the massless Goldstone boson field π.

3.9 Landau–Ginzburg Lagrangian

Scalar electrodynamics consists of a complex scalar field coupled to an Abelian gauge field. The Landau–Ginzburg Lagrangian provides a phenomenological description of the superconducting phase of ordinary conductors using scalar quantum electrodynamics. The complex scalar field φ is an *order parameter* that describes the phase transition. The development of the microscopic Bardeen–Cooper–Schrieffer (BCS) theory showed that, in the condensed superconducting

phase, the complex field's "mass term" – given by $\varphi^*\varphi$ – represents the density of superconducting Cooper pairs of electrons.

A local (gauge) transformation is one for which the parameter $\alpha = \alpha(x)$ depends on the spacetime point x. Consider the U(1) *local* gauge transformation given by

$$\varphi(x) \to \varphi'(x) = e^{-i\alpha(x)}\varphi(x);\ \varphi^*(x) \to \varphi'^*(x) = e^{i\alpha(x)}\varphi^*(x)$$

which yields

$$\partial_\mu\varphi(x) \to \partial_\mu\varphi'(x) = e^{-i\alpha(x)}\partial_\mu\varphi - i(\partial_\mu\alpha)\varphi$$

Under a U(1) local gauge symmetry, from Eq. 3.8.1 we have

$$\begin{aligned}\mathcal{L}_G \to \mathcal{L}'_G = \mathcal{L}_G &+ \partial_\mu\alpha\partial^\mu\alpha\varphi^*\varphi + i\partial_\mu\alpha\varphi^*\partial^\mu\varphi \\ &-\mu^2\varphi^*\varphi - i\partial_\mu\alpha(\partial^\mu\varphi^*)\varphi - \mu^2\varphi^*\varphi \neq \mathcal{L}_G\end{aligned}$$

Hence the Goldstone Lagrangian $\mathcal{L}_G$ does not have local gauge symmetry.

A gauge field is introduced to obtain exact local gauge symmetry. Consider the Landau–Ginzburg Lagrangian

$$\mathcal{L} = (D_\mu\varphi)^*D^\mu\varphi - \mu^2\varphi^*\varphi - \lambda(\varphi^*\varphi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

where the gauge covariant derivative and gauge field tensor are given by

$$D_\mu\varphi \equiv \partial_\mu\varphi(x) + ieA_\mu(x)\varphi(x);\ \ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

Note that

$$|D_\mu\varphi|^2 \sim |\partial_\mu\varphi|^2 + \varphi^*A\partial_\mu\varphi + (A\varphi)^2$$

Local gauge transformations are defined by

$$\begin{aligned}\varphi(x) \to \varphi'(x) &= e^{-ief(x)}\varphi(x) \\ \varphi^*(x) \to \varphi'^*(x) &= e^{ief(x)}\varphi^*(x) \\ A_\mu(x) \to A'_\mu(x) &= A_\mu(x) + \partial_\mu f(x)\end{aligned}$$

Hence

$$\begin{aligned}D_\mu\varphi \to [\partial_\mu + ie(A_\mu + \partial_\mu f)]&\left(e^{-ief}\varphi\right) \\ = e^{-ief}\partial_\mu\varphi &+ ieA_\mu e^{-ief}\varphi - ie\partial_\mu f e^{-ief}\varphi + ie\partial_\mu f e^{-ief}\varphi \\ = e^{-ief}D_\mu\varphi\end{aligned}$$

which yields

$$(D_\mu\varphi)^*D^\mu\varphi \to (D_\mu\varphi)^*D^\mu\varphi$$

Note that the last equation above states that

$$\partial_\nu F^{\mu\nu} = j^\mu$$

Since $F^{\mu\nu}$ is antisymmetric, one has the identity $\partial_\mu \partial_\nu F^{\mu\nu} = 0$, and hence consistency requires that

$$\partial_\mu j^\mu = 0 \tag{3.9.4}$$

In other words, the gauge field A_μ can only be coupled to a conserved current, reflecting the conservation of electric charge. Eq. 3.9.4 shows that the Noether conserved current, as mentioned earlier, is in fact already contained in the field equations.

3.9.1 Meissner effect

The Meissner effect refers to the phenomenon where a superconductor expels magnetic fields, up to a critical value of the field.

A superconductor is in thermodynamic equilibrium with *no time dependence*; hence all the time derivatives in Landau–Ginzburg Lagrangian are set to zero.

The conserved current, from Eq. 3.9.1, is given by

$$j_\mu = i(\varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi) + 2e\varphi^* \varphi A_\mu$$

For the symmetry-breaking superconductor phase, since the Landau–Ginzburg has the same potential as the Goldstone Lagrangian, the leading order of the field is given from Eq. 3.8.3 by

$$|\varphi_0| = \frac{1}{\sqrt{2}} \upsilon$$

and we obtain

$$\varphi^* \varphi \simeq |\varphi_0|^2 = \frac{1}{2} \upsilon^2$$

Since the fluctuation of the field is small over space, we have that $\varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi$ is negligible. Hence, for a symmetry-breaking superconductor phase we obtain, to leading order, the value for the current, which is given by

$$j_\mu \simeq e\upsilon^2 A_\mu \text{ : London equation}$$

The field equation Eq. 3.9.3 now yields

$$\partial_\nu F_{\mu\nu} = \partial_\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) = e\upsilon^2 A_\mu \tag{3.9.5}$$

Note that

$$\partial_\mu \partial_\nu F_{\mu\nu} = 0 \;\; \Rightarrow \;\; \partial_\mu A_\mu = 0$$

This is a gauge condition on A_μ.

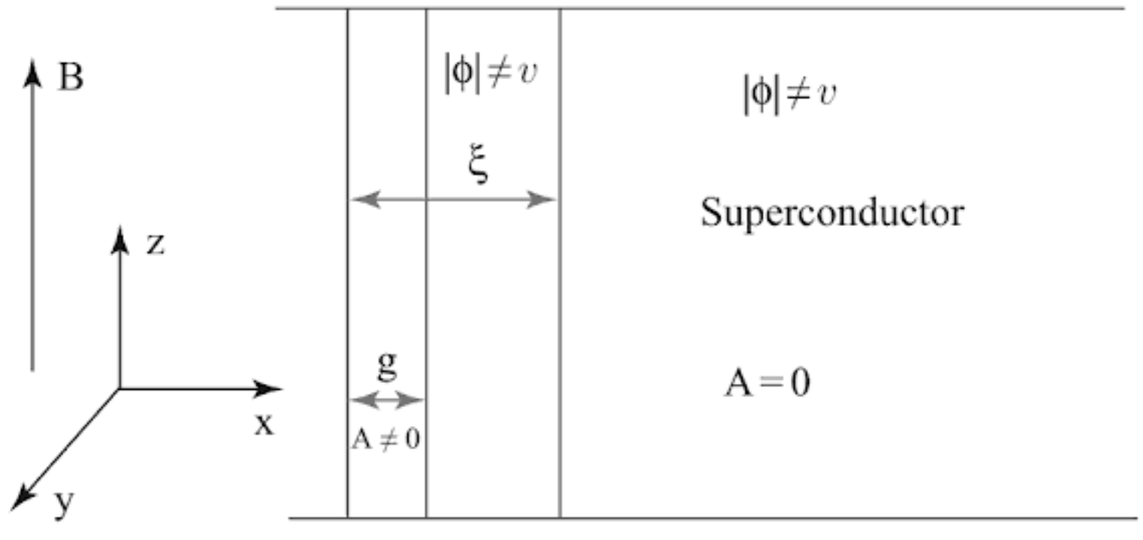


Figure 3.3 The magnetic field penetrates only up to a length of g inside a superconductor. The condensed phase is perturbed up to a skin depth of ξ.

Hence, from Eq. 3.9.5

$$\partial_\nu \partial^\nu A_\mu + e\upsilon^2 A_\mu = 0 : \text{Massive Klein Gordon equation}$$

The gauge field has become massive since the phase transition of the scalar field leads to the breaking of gauge symmetry. We will see in Section 3.10 that this is the same as the Higgs mechanism.

Consider a superconductor occupying the half plane, with its surface at $x = 0$. The magnetic field is given by $\vec{B} = \vec{\nabla} \times \vec{A}$. Imposing a constant magnetic field of strength $\vec{B}_0$ in the z-direction, the London equation yields the magnetic field as given by

$$(-\partial_x^2 + e\upsilon^2)\vec{B} = 0 \;\; \Rightarrow \;\; \vec{B} = e^{-x/g}\vec{B}_0; \;\; g = 1/(e\upsilon^2) : \;\; \text{Meissner effect}$$

As shown in Figure 3.3, the magnetic field penetrates into the superconductor only to a depth of g, called the penetration depth; for a low temperature superconductor, $g \simeq 10^{-9}$ m.

The scalar field φ is the analog of the σ field. For the condensed phase, the mass of the scalar field, similar to the σ field as given in Eq. 3.8.4, has a mass of $2\lambda\upsilon^2$; hence the field equation for the field φ in the condensed phase – the analog of Eq. 3.9.2 – yields

$$(+\partial_x^2 - 2\lambda\upsilon^2)|\varphi| + O(\varphi A_\mu) = 0$$

Using the boundary condition that for φ in the condensed phase, we have

$$\lim_{x\to\infty} |\varphi| = \frac{1}{\sqrt{2}}\upsilon$$

the field equation yields

$$|\varphi(x)| = \frac{1}{\sqrt{2}}\upsilon(1 - e^{-x/\xi}); \;\; \xi = 1/(2\lambda\upsilon^2)$$

Hence, the condensed phase is broken near the boundary of the superconductor, and the length is given by $\xi = 1/(2\lambda\upsilon^2)$. The behavior of the gauge and scalar field for the condensed phase is shown in Figure 3.3.

In particle physics, for spontaneously broken gauge theories, the mass of the gauge field A_μ is equal to the inverse of the penetration depth g and the mass of the Higgs boson is equal to the inverse of the correlation length ξ.

3.10 Higgs mechanism

The interplay between local gauge invariance and spontaneous breaking of symmetry is the basis of the Higgs mechanism. A direct way of seeing this interplay is to choose the so-called *unitary gauge*. Consider polar coordinates for the complex scalar field given by

$$\varphi(x) = \frac{1}{\sqrt{2}} r(x) e^{i\theta(x)}$$

Choose a specific unitary gauge that is determined by the θ field and given by[3]

$$A_\mu \rightarrow A_\mu - \partial_\mu \theta$$

The gauge-covariant derivative term then yields

$$D_\mu \varphi \rightarrow \frac{1}{\sqrt{2}} \Big\{ \partial_\mu + ie(A_\mu(x) - \partial_\mu \theta(x)) \Big\} \Big\{ r(x) e^{i\theta(x)} \Big\}$$

$$= \frac{1}{\sqrt{2}} \Big\{ \partial_\mu r(x) + ieA_\mu(x) r(x) \Big\}$$

In other words, the gauge transformation completely removes one degree of freedom and makes the complex field $\varphi(x)$ into a real field $r(x)$. The choice of gauge produces no change in $F_{\mu\nu}$ and hence we obtain

$$\mathcal{L}(r, A_\mu) = \frac{1}{2} |\partial_\mu r(x) + ieA_\mu(x) r(x)|^2 - \frac{1}{2}\mu^2 r^2(x) - \frac{1}{4}\lambda r^4(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

For the spontaneously broken phase, $\mu^2 = -\lambda\upsilon^2$, with $\upsilon \neq 0$. Let

$$r(x) = \upsilon + \sigma(x)$$

[3] The unitary gauge is a singular limit of the so-called R_ξ gauge; it can be shown [Peskin and Schroeder (1995), Eq. 21.29] that in the unitary gauge, the gauge field A_μ is massless for the case of $\mu^2 > 0$. The R_ξ gauge is defined by the gauge-fixing term

$$\frac{1}{\sqrt{\xi}} \left[\frac{\partial A_\mu}{\partial x_\mu} - \xi e \upsilon r \sin(\theta) \right]$$

The unitary gauge is given by the limit of $\xi \rightarrow \infty$.

The scalar field σ is the Higgs boson and yields

$$\mathcal{L}(r, A_\mu) = \frac{1}{2}[\partial_\mu \sigma(x)]^2 + \frac{1}{2}e^2 A_\mu^2(x)(\upsilon + \sigma(x))^2 - \frac{1}{2}\mu^2(\upsilon + \sigma(x))^2 - \frac{1}{4}\lambda(\upsilon + \sigma(x))^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

This yields the following Lagrangian, using the result obtained earlier for the Goldstone potential:

$$\mathcal{L}_H = \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma - \frac{1}{2}(2\lambda\upsilon^2)\sigma^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2\upsilon^2 A_\mu A^\mu - \lambda\upsilon\sigma^3 - \frac{1}{4}\lambda\sigma^4 + \frac{1}{2}e^2 A_\mu A^\mu(2\upsilon\sigma + \sigma^2)$$

The massless Goldstone field π in Eq. 3.8.4 has been removed using the gauge symmetry of the Lagrangian. Instead of the Goldstone field, one now has the gauge field acquiring a mass equal to $e^2\upsilon^2$; the extra degree of freedom for the massive gauge field mass arises by the gauge field absorbing the Goldstone degree of freedom.

On quantizing the fields, symmetry breaking is the result of the properties of the quantum vacuum $|\Omega\rangle$ of the system. The broken phase is characterized by

$$\langle\Omega|A_\mu|\Omega\rangle = 0; \;\; \langle\Omega|\varphi^*\varphi|\Omega\rangle = \upsilon$$

In the path integral formalism, this condition yields

$$E[A_\mu] = 0; \;\; E[\varphi^*\varphi] = \upsilon$$

where $E[\mathcal{O}]$ is the expectation value of the field variables $\mathcal{O}$.

The symmetry breaking by the Higgs scalar field $\varphi_H(x)$ is given by

$$v = E[\varphi_H(x)] = \langle\Omega|\hat{\varphi}_H(x)|\Omega\rangle = 246 \text{ GeV/c}^2$$

The nonzero value of v is the basis for all the fundamental particles in nature having a nonzero mass [Baaquie and Willeboordse (2015)].

In summary, the Higgs transition and mechanism describe the following phenomenon.

- Before the phase transition, the system has local gauge invariance and consists of a complex scalar plus massless gauge boson. The number of the degrees of freedom before the phase transition is $4 = 2 + 2$.
- After the system undergoes a phase transition, the condensed phase breaks local gauge invariance and consists of a real scalar and a massive gauge boson. After the phase transition, the number of degrees of freedom is also $4 = 1 + 3$.

The process of a massless gauge field acquiring a mass through a phase transition is called the Higgs mechanism. The magic of gauge invariance and symmetry breaking combine to give a new result. In the absence of the gauge field, a phase transition leads to the appearance of the massless Goldstone boson, but when coupled to the gauge field, the Goldstone boson is completely removed from the theory, and, instead, the gauge field becomes massive and the complex massive charged scalar field becomes a massive real scalar field with zero charge.

3.11 Lorentz group

In the previous sections, both scalar and vector fields have been discussed, and in Section 3.7 on the Noether current, the stress tensor was analyzed. Scalar, vector and tensor fields are classified according to their transformation under the Lorentz group, which is a non-compact Lie group that encodes the symmetries of relativistic spacetime. Another field of fundamental importance is the spinor field, of which the Dirac field is the leading exemplar.

The scalar, vector and spinor fields form the backbone of the study of quantum fields. Since all these fields are characterized by their properties under Lorentz transformations, the main features of the Lorentz group are summarized. It is further shown how the various fields are classified using the Lorentz group.

Consider a spacetime point $t, \vec{x}$; the relativistic invariant distance of this point from the origin using the metric given in Noteworthy 3.1 and setting $c = 1$, is given by

$$t^2 - \vec{x}^2 = x_\mu x^\mu = \eta^{\mu\nu} x_\mu x_\mu$$

The Lorentz group in four spacetime dimensions is given by all the transformations $L_{\mu\nu}$ (in matrix notation) that preserve the invariant distance.

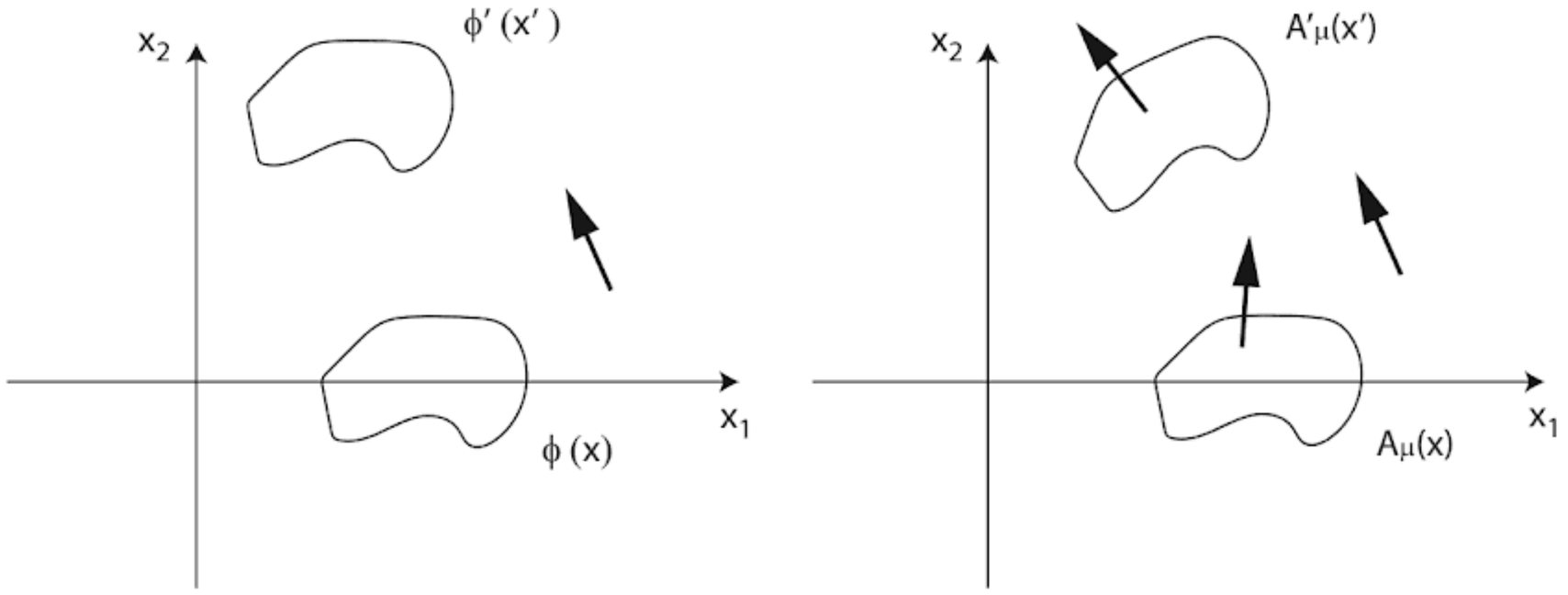
$$\tilde{x}_\nu = L_{\mu\nu} \eta^{\nu\alpha} x_\alpha \quad \Rightarrow \quad x = L\eta x$$

Using $\eta = \eta^T$ and $\eta^2 = \mathbb{I}$ yields the condition that determines $L_{\mu\nu}$:

$$\tilde{x}\eta\tilde{x} = x\eta(L^T\eta L)\eta x = x\eta x \quad \Rightarrow \quad L^T\eta L = \eta \tag{3.11.1}$$

L is a real four-dimensional matrix; Lorentz transformations consist of rotations in three-dimensional space, which require three (compact) parameters, and boosts in three space directions, which require another three (noncompact) parameters. The parameter space of the Lorentz group is a six-dimensional space; the parameter space is not compact, since the boosts take values over an infinite range. Hence, the Lorentz group is a *noncompact group* [Tung (2003)].

The six parameters of the Lorentz group can be organized into a three-dimensional complex vector – consisting of three complex parameters as its components – and is given by



(a) (b)

Figure 3.4 Lorentz transformations for scalar and vector fields.

functions are relativistically invariant. Under a Lorentz transformation, a scalar Lagrangian density yields

$$\mathcal{L}'(x') = \mathcal{L}(x); \;\; d^4x' = \det L \, d^4x = d^4x$$

Hence the action is relativistically invariant since

$$S' = \int d^4x' \mathcal{L}'(x') = \det L \int d^4x \mathcal{L}(x) = S: \text{ invariant}$$

For a vector field,

$$L(\vec{\alpha}) = \mathcal{M}_{\left(\frac{1}{2},0\right)}(\vec{\alpha}) \otimes M_{(\frac{1}{2},0)}(\vec{\alpha})$$

Hence from Eq. 3.12.1, a vector field $A_\mu(x)$ has the following transformation:

$$A'_\mu(x') = L_\mu^{\ \nu} A_\nu(x)$$

The transformation of a vector field is shown in Figure 3.4. The transformed vector field at the transformed point is rotated in relation to the original field at the original field at the original point. Note the additional feature for the vector field. Since it transforms as the tensor product of two representations of the Lorentz group, it is reducible. In particular, using the properties of $su(2)$ Lie algebra, we have

$$\left(\frac{1}{2},0\right) \otimes \left(\frac{1}{2},0\right) = \left(\frac{1}{2} \otimes \frac{1}{2},0\right) = (1,0) \oplus (0,0)$$

and hence

$$\mathcal{M}_{\left(\frac{1}{2},0\right)}(\vec{\alpha}) \otimes M_{\left(\frac{1}{2},0\right)}(\vec{\alpha}) = \mathcal{M}_{(1,0)}(\vec{\alpha}) \oplus 1 \tag{3.12.2}$$

The vector representation of the Lorentz group is equivalent to the direct sum of a spin-one representation $(1,0)$ and spin-zero representation of the rotation group in

three space dimensions. The result given in Eq. 3.12.2 shows that a four-component Lorentz vector field carries a spin-one and spin-zero representation of the Lorentz group. Under rotations in three space dimensions, the time component of the vector field A_0 does not change, and hence we identify it as being the spin-zero component of the Lorentz four vector.

The only vector fields that are renormalizable are gauge fields. For these fields, the scalar particle is removed by choosing the temporal gauge of $A_0 = 0$. The spin-one gauge particle has only two components of the three possible states of spin-one representation due to the gauge invariance of the state space. From the point of view of the Lorentz group, one can now see these features in the derivation of the Hamiltoninan and gauge-invariant state space in Sections 9.2 and 9.3.

3.12.2 Spinor fields[5]

Parity, realized by operator $\mathcal{P}$, is defined by the inversion of only the space dimensions, with the time direction being left unchanged, and is given by

$$\mathcal{P}(t,\vec{x}) \rightarrow (t,-\vec{x})$$

The Dirac field, studied in detail in Chapter 8, is a four-component spinor field. It can be shown [Das (2008)] that the parity operator $\mathcal{P}$ that realizes the effect of space inversion for the Dirac field is given by

$$\psi(t,-\vec{x}) = \mathcal{P}\psi(t,\vec{x}) = \eta\psi(t,\vec{x});\ \ |\eta| = 1 \ \ \Rightarrow \ \ \mathcal{P} = \gamma_0$$

where γ_0 is a Dirac gamma matrix defined in Eq. 3.12.5. Under the parity transformation, the representations j_1, j_2 are interchanged and one has

$$\mathcal{P}\mathcal{M}_{(j_1,j_2)}(\vec{\alpha})\mathcal{P} = \gamma_0\mathcal{M}_{(j_1,j_2)}(\vec{\alpha})\gamma_0 = \mathcal{M}_{(j_2,j_1)}(\vec{\alpha})$$

Hence, for obtaining a system that is invariant under the parity transformation, one has to use the representation given by $\mathcal{M}_{(j,j)}(\vec{\alpha})$ and with the spinors having the dimension $(2j+1)^2$. The Dirac field, which is invariant under parity transformations, transforms under the $\mathcal{M}_{(\frac{1}{2},\frac{1}{2})}(\vec{\alpha})$ representation of the Lorentz group.

The chiral representations of the Lorentz group are given by $\mathcal{M}_{(j,0)}(\vec{\alpha})$, with the chiral field being $(2j+1)$-dimensional. Spacetime spinor fields that violate the symmetry of parity, such as the massless neutrino, are two-dimensional and transform under the $\mathcal{M}_{(\frac{1}{2},0)}(\vec{\alpha})$ representation of the Lorentz group.

The four-dimensional representation of the Lorentz group is given by

$$\mathcal{S} = \mathcal{M}_{\left(\frac{1}{2},\frac{1}{2}\right)}(\vec{\alpha}) = \exp\{i\vec{\alpha}\cdot\vec{\sigma}\}\otimes\exp\{i\vec{\alpha}^*\cdot\vec{\sigma}\}$$

[5] This section should be read after Chapter 8.

$\mathcal{S}$ yields a reducible representation, since each component of the tensor product acts independently on the underlying vector space. Hence, in block diagonal representation using the 2×2 block notation, $\mathcal{S}$ is given by

$$\mathcal{S} = \begin{bmatrix} \exp\{i\vec{\alpha} \cdot \vec{\sigma}\} & 0 \\ 0 & \exp\{i\vec{\alpha}^* \cdot \vec{\sigma}\} \end{bmatrix}$$

The Dirac spinor field, in 2×2 block notation, is given by

$$\psi = \begin{bmatrix} \psi_u \\ \psi_d \end{bmatrix}$$

Under a Lorentz transformation – ignoring the transformation of the spacetime points as these do not enter the discussion – the Dirac spinor transforms, as per Eq. 3.12.1, as follows:

$$\psi' = \mathcal{S}\psi = \begin{bmatrix} \exp\{i\vec{\alpha} \cdot \vec{\sigma}\} & 0 \\ 0 & \exp\{i\vec{\alpha}^* \cdot \vec{\sigma}\} \end{bmatrix} \begin{bmatrix} \psi_u \\ \psi_d \end{bmatrix} \tag{3.12.3}$$

and Hermitian conjugation yields

$$\psi'^{\dagger} = \psi^{\dagger}\mathcal{S}^{\dagger} = \begin{bmatrix} \psi_u^{\dagger} & \psi_d^{\dagger} \end{bmatrix} \begin{bmatrix} \exp\{-i\vec{\alpha}^* \cdot \vec{\sigma}\} & 0 \\ 0 & \exp\{-i\vec{\alpha} \cdot \vec{\sigma}\} \end{bmatrix} \tag{3.12.4}$$

In the representation that has been chosen for the Lorentz transformation, it can be shown that γ_0 is given by [Tung (2003)][6]

$$\gamma_0 = \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} \;\Rightarrow\; \mathcal{S}^{\dagger}\gamma_0\mathcal{S} = \gamma_0 \tag{3.12.5}$$

where the last result follows from Eqs. 3.12.3 and 3.12.4.

The Dirac Lagrangian given in Eq. 8.3.1 is relativistically invariant, and a sketch of the proof is the following. Consider the mass term given by

$$\bar{\psi}\psi = \psi^{\dagger}\gamma_0\psi = [\psi_u^{\dagger}\ \psi_d^{\dagger}] \begin{bmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} \begin{bmatrix} \psi_u \\ \psi_d \end{bmatrix}$$

Hence, from Eq. 3.12.5, Lorentz transformation of the mass term yields

$$\bar{\psi}\psi \rightarrow \bar{\psi}'\psi' = \psi^{\dagger}\mathcal{S}^{\dagger}\gamma_0\mathcal{S}\psi = \bar{\psi}\psi : \text{ invariant}$$

Using an explicit representation of the gamma matrices γ_μ that follows from the representation of the Lorentz group that we are using, it can be shown that [Tung (2003)]

$$\mathcal{S}^{\dagger}\gamma_\mu\mathcal{S} = L_\mu^{\ \nu}\gamma_\nu$$

[6] There are infinitely many equivalent representations of the Dirac matrices and the one chosen here for γ_0 is not the same as that given in Eq. 8.2.3, which is more convenient for analyzing the Hamiltonian of the Dirac field.

from which it follows that

$$\bar{\psi}'\gamma_\mu\psi' = L_\mu^{\ \nu}\bar{\psi}\gamma_\nu\psi$$

Hence, $\bar{\psi}\gamma_\nu\psi$ transforms as the $L(\vec{\alpha})$ vector representation of the Lorentz group – and hence is a Lorentz four vector; this is sufficient to show that the Dirac Lagrangian is a scalar and the Dirac action is Lorentz invariant.

It can be shown that the massless Dirac equation splits into two separate equations, one for ψ_u and the other for ψ_d; the two-component chiral spinors are called Weyl spinors and transform under the $\mathcal{M}_{(\frac{1}{2},0)}(\vec{\alpha})$ representation of the Lorentz group [Tung (2003)]. The massless equation obeyed by the Weyl spinors can be shown to be Lorentz invariant. The massless neutrino in particle physics is taken to be a Weyl spinor and provides the mechanism for parity violation in nature.

Tensor products of the representations of the Lorentz group $\mathcal{M}_{(j1,j2)}(\vec{\alpha})$ give rise to tensor fields and spinor fields with higher spins.

The quantum fields that appear in phase transitions and mathematical finance are not relativistic quantum fields. In particular, the Landau–Ginzburg–Wilson action, which describes phase transitions and is discussed in Section 17.7, is defined in $d = 3$ space dimensions and has no well-defined properties under Lorentz transformations.

3.13 Summary

The notion of classical field sets the stage for the concept of the quantum field. Both quantum and stochastic fields are rooted in the formalism of classical fields, with the idea of Lagrangian, action and Hamiltonian running through all the derivations. Conservation laws and symmetries are features of all quantum fields, and these are initially introduced in the context of classical field theory. Classical concepts of symmetry carry over to quantum fields. Conserved currents and charges are key features of quantum fields, especially in defining the state space of the quantum field.

Important ideas such as symmetry breaking occur in classical physics, and the Meisner effect and Higgs mechanism were discussed to introduce nonlinear Lagrangians that later will be seen to be the starting point of the analysis of various quantum fields.

A brief and condensed discussion of the Lorentz group was given to let readers have a glimpse of the vast application of Lie groups to quantum fields. The Lorentz group provides a classification of spacetime quantum fields determined by how they transform under Lorentz transformations. The scalar, spinor and vector fields are defined by their transformation properties. The more general topic of applying Lie groups to determine the dynamics of quantum fields was not discussed, but is something that the readers should be aware of.

4

Acceleration action

4.1 Action and Hamiltonian

The acceleration Hamiltonian has wide application in many subjects [Baaquie (2014) and references cited therein] and in particular will play an important role in later discussions on option theory, commodities and interest rates. In fact, in all the applications of models based on quantum mathematics to economics and finance, the acceleration Hamiltonian and Lagrangian always seem to be required.

Fundamentally, it is the acceleration term in the action – absent in physics – that seems to be essential in describing the dynamics of phenomena in economics and finance, and this also makes behavior of the models of the social phenomena of economics and finance vastly different from natural phenomena that are studied in physics.

The acceleration Hamiltonian depends on both position x and velocity $v = -dx/dt$, and is given by [Baaquie (2014)]

$$H = -\frac{1}{2a}\frac{\partial^2}{\partial v^2} - v\frac{\partial}{\partial x} + b\dot{x}^2 + \frac{1}{2}cx^2 \tag{4.1.1}$$

The Hamiltonian H has a state space spanned by the completeness equation given by[1]

$$\mathbb{I} = \int dxdv|x,v\rangle\langle x,v| : \quad v = -\frac{dx}{dt} \tag{4.1.2}$$

The Hamiltonian given in Eq. 4.1.1 yields the following "acceleration" Lagrangian, derived in Baaquie (2014) and given by

$$\mathcal{L} = -\frac{1}{2}\left(a\ddot{x}^2 + 2b\dot{x}^2 + cx^2\right) \tag{4.1.3}$$

$$S = \int_0^\tau dt\mathcal{L}; \;\; \ddot{x} = \frac{d^2x}{dt^2}; \;\; \dot{x} = \frac{dx}{dt}$$

[1] The minus sign in defining v is due to the Hamiltonian defined for Euclidean time [Baaquie (2014)].

The pricing kernels for the oscillators are given by

$$\langle\, \xi' | e^{-\tau H_x} | \xi \rangle = \sqrt{\frac{\gamma\omega_1^2\omega_2}{2\pi\,\sinh(\omega_2\tau)}}\exp\left\{-\frac{\gamma\omega_1^2\omega_2}{2\sinh(\omega_2\tau)}[(\xi'^2+\xi^2)\cosh(\omega_2\tau)-2\xi'\xi]\right\}$$
$$\langle\, \eta' | e^{-\tau H_v} | \eta \,\rangle = \sqrt{\frac{\gamma\omega_1}{2\pi\,\sinh(\omega_1\tau)}}\exp\left\{-\frac{\gamma\omega_1}{2\sinh(\omega_1\tau)}[(\eta'^2+\eta^2)\cosh(\omega_1\tau)-2\eta'\eta]\right\} \tag{4.2.8}$$

Summarizing the results yields

$$\begin{aligned}\mathcal{K} &= N\int d\xi' d\eta' d\xi d\eta \exp\{G(x_f v_f+\xi'\eta')-H(x_f\eta'+\xi' v_f)\}\\ &\quad\times\exp\{-G(x_i v_i+\xi\eta)+H(x_i\eta+\xi v_i)\}\\ &\quad\times\exp\left\{-\frac{\gamma\omega_1^2\omega_2}{2\sinh(\omega_2\tau)}[(\xi'^2+\xi^2)\cosh(\omega_2\tau)-2\xi'\xi]\right\}\\ &\quad\times\exp\left\{-\frac{\gamma\omega_1}{2\sinh(\omega_1\tau)}[(\eta'^2+\eta^2)\cosh(\omega_1\tau)-2\eta'\eta]\right\}\\ &= Ne^{-G(x_f v_f-x_i v_i)}\int d\xi' d\eta' d\xi d\eta\exp\left\{-\frac{\gamma}{2}X^T MX+J^T X\right\}\end{aligned} \tag{4.2.9}$$

where the normalization constant is given by

$$N=\left(\frac{i}{2\pi}\gamma\omega_1\sqrt{\omega_1^2-\omega_2^2}\right)^2\sqrt{\frac{\gamma\omega_1^2\omega_2}{2\pi\,\sinh(\omega_2\tau)}}\sqrt{\frac{\gamma\omega_1}{2\pi\,\sinh(\omega_1\tau)}} \tag{4.2.10}$$

The symmetric matrix is given by

$$M=\begin{pmatrix}\frac{\omega_1^2\omega_2\cosh(\omega_2\tau)}{\sinh(\omega_2\tau)} & -\omega_1^2 & -\frac{\omega_1^2\omega_2}{\sinh(\omega_2\tau)} & 0\\ -\omega_1^2 & \frac{\omega_1\cosh(\omega_1\tau)}{\sinh(\omega_1\tau)} & 0 & -\frac{\omega_1}{\sinh(\omega_1\tau)}\\ -\frac{\omega_1^2\omega_2}{\sinh(\omega_2\tau)} & 0 & \frac{\omega_1^2\omega_2\cosh(\omega_2\tau)}{\sinh(\omega_2\tau)} & \omega_1^2\\ 0 & -\frac{\omega_1}{\sinh(\omega_1\tau)} & \omega_1^2 & \frac{\omega_1\cosh(\omega_1\tau)}{\sinh(\omega_1\tau)}\end{pmatrix} \tag{4.2.11}$$

and (T stands for transpose)

$$J^{\mathrm{T}}=\gamma\omega_1\sqrt{\omega_1^2-\omega_2^2}(v_i,x_i,-v_f,-x_f);\;\; X^{\mathrm{T}}=(\xi,\eta,\xi',\eta') \tag{4.2.12}$$

Performing the Gaussian integrations in Eq. 4.2.9 using the result given in Eq. 2.11.3 yields

$$\int d\xi d\xi' d\eta d\eta' \exp\left(-\frac{\gamma}{2}X^{\mathrm{T}}MX+J^{\mathrm{T}}X\right)=\frac{4\pi^2}{\sqrt{\gamma^4\det[M]}}\exp\left(\frac{1}{2\gamma}J^{\mathrm{T}}M^{-1}J\right)$$

The transition amplitude is hence given by

$$\mathcal{K} = \frac{4\pi^2 N}{\sqrt{\gamma^4 \det M}} \exp\left(G(x_f v_f - x_i v_i) + \frac{1}{2\gamma} J^{\mathrm{T}} M^{-1} J \right) \tag{4.2.13}$$

To simplify the notation let

$$s_1 = \sinh(\omega_1 \tau),\ s_2 = \sinh(\omega_2 \tau);\ \ c_1 = \cosh(\omega_1 \tau),\ c_2 = \cosh(\omega_2 \tau).$$

Then, from Eq. 4.2.11

$$M = \begin{pmatrix} \frac{\omega_1^2 \omega_2 c_2}{s_2} & -\omega_1^2 & -\frac{\omega_1^2 \omega_2}{s_2} & 0 \\ -\omega_1^2 & \frac{\omega_1 c_1}{s_1} & 0 & -\frac{\omega_1}{s_1} \\ -\frac{\omega_1^2 \omega_2}{s_2} & 0 & \frac{\omega_1^2 \omega_2 c_2}{s_2} & \omega_1^2 \\ 0 & -\frac{\omega_1}{s_1} & \omega_1^2 & \frac{\omega_1 c_1}{s_1} \end{pmatrix} \tag{4.2.14}$$

and yields the following $\det M$:

$$\det M = \frac{\omega_1^6}{s_1 s_2} \left(\left(\omega_1^2 + \omega_2^2 \right) s_1 s_2 - 2\omega_1 \omega_2 (c_1 c_2 - 1) \right)$$

The normalization, using Eq. 4.2.10, is given by

$$\mathcal{N}(\tau) = \frac{4\pi^2 N}{\sqrt{\gamma^4 \det M}} = \frac{1}{2\pi} \sqrt{\omega_1 \omega_2} \frac{(\omega_1^2 - \omega_2^2)}{\sqrt{\left(\omega_1^2 + \omega_2^2 \right) s_1 s_2 - 2\omega_1 \omega_2 (c_1 c_2 - 1)}} \tag{4.2.15}$$

The inverse matrix M^{-1}, from Eq. 4.2.14, yields the final expression for the transition amplitude:

$$\mathcal{K} = \mathcal{N}(\tau) \exp\{S_c\} \tag{4.2.16}$$

where it is shown in Eq. 4.4.9 that S_c is the classical action. Using the notation

$$x_i = x_1;\ \ x_f = x_3;\ \ v_i = x_2;\ \ v_f = x_4$$

one can write the classical action as

$$S_c = -\frac{1}{2} \sum_{ij=1}^{4} M_{ij} x_i x_j \tag{4.2.17}$$

Note that $S < 0$ since the determinant $\det M$ is **negative** *for all three branches*. The coefficients, using the symmetry of the coefficients, are given by Kleinert (1986):

$$M_{11} = \Gamma \left[\omega_1 \omega_2 \left(\omega_1^2 - \omega_2^2 \right) (\omega_1 s_1 c_2 - \omega_2 s_2 c_1) \right]$$

$$M_{34} = \Gamma \left[2\omega_1 \omega_2 \left(\left(\omega_1^2 + \omega_2^2 \right) (c_1 c_2 - 1) - 2\omega_1 \omega_2 s_1 s_2 \right) \right];\ \ M_{12} = -M_{34}$$

$$M_{14} = -2\Gamma \left[\omega_1 \omega_2 \left(\omega_1^2 - \omega_2^2 \right) (c_1 - c_2) \right];\ \ M_{23} = -M_{14}$$

$$M_{22} = \Gamma\Big[\left(\omega_1^2 - \omega_2^2\right)(\omega_1 c_1 s_2 - \omega_2 c_2 s_1)\Big]$$
$$M_{13} = -2\Gamma\Big[\omega_1\omega_2\left(\omega_1^2 - \omega_2^2\right)(\omega_1 s_1 - \omega_2 s_2)\Big]$$
$$M_{24} = -2\Gamma\Big[\left(\omega_1^2 - \omega_2^2\right)(\omega_1 s_2 - \omega_2 s_1)\Big]$$

where

$$\Gamma = \frac{1}{\left(\omega_1^2 + \omega_2^2\right)s_1 s_2 - 2\omega_1\omega_2(c_1 c_2 - 1)}$$

The final result is given by

$$S_c = -\frac{1}{2}M_{11}\left(x_f^2 + x_i^2\right) - \frac{1}{2}M_{22}\left(v_f^2 + v_i^2\right)$$
$$- M_{14}(x_i v_f - x_f v_i) - M_{12}(x_i v_i - x_f v_f) - M_{13}x_i x_f - M_{24}v_i v_f \qquad (4.2.18)$$

The expression for the classical action given in Eq. 4.2.18 is *real valued* for all three branches, and as mentioned earlier, $S_c < 0$ for all three branches.

Although the Hamiltonian derivation of the transition amplitude is only valid for the real branch, one can analytically continue the results given for M_{ij} to the critical and complex branch as well. The path integral derivation, given in Section 4.4, is valid for all three branches and validates the result found by analytic continuation of the real branch.

4.2.1 Limiting case: $\omega_1 = \omega = \omega_2$: real

The critical branch has $\omega_1 = \omega_2$; it has been shown in Bender and Mannheim (2008) and Baaquie (2014) that the critical Hamiltonian is given by a direct sum of Jordan blocks. The Lagrangian, from Eq. 4.1.6, is given by

$$\mathcal{L} = -\frac{1}{2}\left(\ddot{x}^2 + 2\omega^2\dot{x}^2 + \omega^4 x^2\right) \qquad (4.2.19)$$

Consider

$$\omega_1 = \omega + \epsilon;\ \ \omega_2 = \omega - \epsilon \ \text{ and } \ s = \sinh(\omega\tau);\ \ c = \cosh(\omega\tau)$$

The critical branch is given by the limit of $\epsilon \to 0$ and yields, from Eq. 4.2.18, the following classical action [Kleinert (1986)]:

$$S_c = -\frac{\omega}{s^2 - \omega^2\tau^2}\Big[(sc - \omega\tau)\left(v_i^2 + v_f^2\right) - 2(s - c\omega\tau)v_i v_f$$
$$- \omega\left(s^2 + \omega^2\tau^2\right)(v_f x_f - v_i x_i) + 2\omega s(v_f x_i - v_i x_f)$$
$$+ \omega^2(sc + \omega\tau)\left(x_i^2 + x_f^2\right) - 2\omega^2(s + c\omega\tau)v_f x_i\Big] \qquad (4.2.20)$$

and the transition amplitude is given by

$$\mathcal{K} = \frac{\omega^2}{\sqrt{s^2 - \omega^2\tau^2}} e^{S_c}$$

It is an intractable problem to obtain the result given in Eq. 4.2.20 from the Jordan block-diagonal Hamiltonian. In contrast, the path integral derivation yields Eq. 4.2.17 for all three branches; hence Eq. 4.2.20 provides the transition amplitude for the critical branch by a straightforward limit of the real (or complex) branch – showing the power of the path integral.

4.2.2 Limiting case: $\omega = 0$

One can further take the limit of $\omega \to 0$. From Eq. 4.2.19, the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\ddot{x}^2 \tag{4.2.21}$$

From Eq. 4.2.20 the transition amplitude is given by

$$\mathcal{K} = \frac{\sqrt{3}}{\pi\tau^2} e^{S_c}; \;\; S_c = -\frac{1}{2\tau}(v_f - v_i)^2 - \frac{6}{\tau^3}\left(x_f - x_i - \frac{\tau}{2}(v_f + v_i)\right)^2 \tag{4.2.22}$$

Recall from Eq. 4.1.5 that τ stands for $\beta = a^{-1/3}\tau$; hence, the classical action in terms of remaining time τ is given by

$$S_c = -\frac{a^{1/3}}{2\tau}(v_f - v_i)^2 - \frac{6a}{\tau^3}\left(x_f - x_i - \frac{\tau}{2}(v_f + v_i)\right)^2$$

The result above is the generalization of the Black–Scholes pricing kernel given in Eq. 5.7.6.

4.3 Limiting case: $\tau = 0$

The transition amplitude is given by

$$\mathcal{K} = \langle\, x_f, v_f | e^{-\tau H} | x_i, v_i \rangle$$

In the limit of $\tau \to 0$, it must yield the following Dirac delta-function inner product:

$$\lim_{\tau \to 0} \mathcal{K} = \langle\, x_f, v_f | x_i, v_i \rangle = \delta(x_i - x_f)\delta(v_i - v_f) \tag{4.3.1}$$

The proof of the normalization given in Eq. 4.3.1 is not straightforward because the x_i, v_i (and x_f, v_f) variables have cross-terms and hence the limit of $\tau \to 0$ for the two variables has to be taken simultaneously.

The proof of the normalization given in Eq. 4.3.1 can be shown indirectly in the following manner. Note that in general, for $\tilde{f}(\zeta)$ being the Fourier transform of $f(z)$,

$$\begin{aligned} f(z) &= \int dx \delta(x-z) f(x) = \int d\zeta \left[\int dx \delta(x-z) e^{ix\zeta} \right] \tilde{f}(\zeta) \\ &= \int d\zeta \, e^{iz\zeta} \tilde{f}(\zeta) \end{aligned} \tag{4.3.2}$$

Let a representation of the delta function be given by $\Phi(x,z)$ such that

$$\lim_{\tau \to 0} \Phi(x,z) = \delta(x-z)$$

Then from Eq. 4.3.2 it is sufficient to show that

$$\int dx \lim_{\tau \to 0} \Phi(x,z) e^{ix\zeta} = e^{iz\zeta}$$

and the general result follows from the properties of the Fourier transform. For the transition amplitude, it can be shown by a long and tedious calculation that

$$\lim_{\tau \to 0} \int dx_f dv_f \mathcal{K}(x_f, v_f; x_i, v_i; \tau) e^{izx_f + iwv_f} = e^{izx_i + iwv_i} \tag{4.3.3}$$

which in turn yields the required limit given in Eq. 4.3.1.

The general result given in Eq. 4.3.3 is obtained for the case of $\omega = 0$ given in Eq. 4.2.22:

$$\frac{\sqrt{3}}{\pi \tau^2} \int dx_f dv_f e^{S_c} e^{izx_f + iwv_f} = \frac{\sqrt{3}}{\pi \tau^2} e^{iz(x_i + \frac{\tau}{2} v_i)} \int dx_f dv_f e^{-\frac{1}{2\tau}(v_f - v_i)^2 - \frac{6}{\tau^3} x_f^2} e^{iz(\frac{\tau}{2} v_f + x_f) + iwv_f}$$

Performing the Gaussian integrations yields the expected

$$\lim_{\tau \to 0} \frac{\sqrt{3}}{\pi \tau^2} \int dx_f dv_f e^{S_c} e^{izx_f + iwv_f} = \frac{\sqrt{3}}{\pi \tau^2} \sqrt{\frac{2\pi \tau^3}{12}} \sqrt{2\pi \tau} \; e^{izx_i + iwv_i} = e^{izx_i + iwv_i}$$

Hence, for the special case of $\omega = 0$, Eq. 4.3.1 is given by

$$\lim_{\tau \to 0} \mathcal{K} = \lim_{\tau \to 0} \frac{\sqrt{3}}{\pi \tau^2} e^{S_c} = \delta(x_i - x_f) \delta(v_i - v_f)$$

4.4 Transition amplitude: Path integral derivation

The path integral formulation is reviewed and some equations are repeated for ease of reading. The transition amplitude can be equal to the pricing kernel, as in the case of Black–Scholes given in Eq. 5.4.3. Or the transition amplitude can be used for obtaining the pricing kernel as given in Eq. 5.9.3, which is the case of the acceleration Hamiltonian.

$$\frac{\delta S[x_c(t)]}{\delta x(t)} = 0 \tag{4.4.11}$$

From the Euler–Lagrangian equation given by Eq. 4.4.11, the classical solution $x_c(t)$ satisfies the equation

$$a\dddot{x}_c(t) - 2b\ddot{x}_c(t) + cx_c(t) = 0 \tag{4.4.12}$$

According to the market data, the solution for the index is in the complex branch of Eq. 4.4.12; hence

$$b^2 - ac < 0 \tag{4.4.13}$$

Define y as the four conjugate roots of the equation

$$ay^4 - 2by^2 + c = 0 \tag{4.4.14}$$

The four complex solutions are as

$$y = \pm r \pm i\zeta \tag{4.4.15}$$

where

$$r \equiv Re\left(\sqrt{\frac{b + i\sqrt{ac - b^2}}{a}}\right);\ \ \zeta \equiv Im\left(\sqrt{\frac{b + i\sqrt{ac - b^2}}{a}}\right) \tag{4.4.16}$$

Then the relationship from (a, b, c) to (r, ζ) is as

$$b = +a(r^2 - \zeta^2);\ \ c = a(r^2 + \zeta^2)^2 \tag{4.4.17}$$

Using the notation of r and ζ, the general solution of $x_c(t)$ is given by

$$x_c(t) = e^{rt}(a_1 \sin\zeta t + a_2 \cos\zeta t) + e^{-rt}(a_3 \sin\zeta t + a_4 \cos\zeta t) \tag{4.4.18}$$

where $a_1, \ldots, a_4$ are constants fixed by the boundary conditions and hence depend on x, v, x', v'.

The action S yields

$$\begin{aligned} S &= S[x_c + \epsilon] \\ &= -\frac{1}{2}\int_0^\tau dt\left(a(\ddot{x}_c + \ddot{\epsilon})^2 + 2b(\dot{x}_c + \dot{\epsilon} + j)^2 + c(x_c + \epsilon)^2\right) \\ &= S[x_c] + S[\epsilon] + R \end{aligned} \tag{4.4.19}$$

where S_c is the classical action

$$S[x_c] = -\frac{1}{2}\int_0^\tau dt\left(a\ddot{x}_c^2 + 2b(\dot{x}_c + j)^2 + cx_c^2\right) \tag{4.4.20}$$

$$S[\epsilon] = -\frac{1}{2}\int_0^\tau dt\left(a\ddot{\epsilon}_c^2 + 2b\dot{\epsilon}^2 + c\epsilon^2 + 4bj\epsilon\right)$$

The residual term R is

$$\begin{aligned} R &= -\int_0^\tau dt\ (a\ddot{x}_c\ddot{\epsilon} + 2b\dot{x}_c\dot{\epsilon} + cx_c\epsilon) \\ &= (-a\ddot{x}_c\dot{\epsilon} - 2b\dot{x}_c\epsilon + \dddot{x}_c\epsilon)|_0^\tau - \int_0^\tau dt\ \epsilon(a\ddddot{x} - 2b\ddot{x}_c + cx_c) \end{aligned} \tag{4.4.21}$$

From Eqs. 4.4.12 and 4.4.6,

$$R = 0 \tag{4.4.22}$$

Integrating by parts the classical action S_c in Eq. 4.4.20 and applying the equations of motion, the action can be expressed in terms of only the boundary conditions:

$$\begin{aligned} S_c &= -\frac{1}{2}\int_0^\tau dt\ \Big\{ \mathrm{d}(-a\dddot{x}_c x_c + a\ddot{x}_c\dot{x}_c + 2b\dot{x}_c x_c + 4bjx_c + bj^2) \\ &\quad + x_c(a\ddddot{x}_c - 2b\ddot{x}_c + cx_c)\Big\} \\ &= -\frac{1}{2}\Big(- a\dddot{x}_c x_c + a\ddot{x}_c\dot{x}_c + 2b\dot{x}_c x_c + 4bjx_c + bj^2\Big)\Big|_0^\tau \\ &= -\frac{1}{2}\sum_{I,J=1}^{4} x_I M_{IJ} x_J - 2bjx_1 + 2bjx_3 - bj^2\tau \end{aligned} \tag{4.4.23}$$

where x, v, x', v' are rewritten as

$$x' = x_1;\ v' = x_2;\ \ x = x_3;\ v = x_4$$

To find out coefficient M_{IJ}, assume $j = 0$ and obtain

$$S_c = -\frac{1}{2}\sum_{I,J=1}^{4} x_I M_{IJ} x_J \tag{4.4.24}$$

From Eq. 4.4.23, the derivatives of S_c yield M_{IJ} given by

$$M_{IJ} = -\frac{\partial^2 S}{\partial x_I \partial x_J} \tag{4.4.25}$$

A symmetry of the pricing kernel, for $j = 0$, is the following [Baaquie (2014)]:

$$\mathcal{K}(x, v; x'v') = \mathcal{K}(x', -v'; x, -v)$$

and hence

$$M_{11} = M_{33};\ M_{22} = M_{44};\ M_{12} = -M_{34};\ M_{14} = -M_{23} \tag{4.4.26}$$

The transition amplitude, from Eq. 4.4.9 is, consequently, given by

$$\mathcal{K}(x,v;x',v',\tau) = \mathcal{N}(\tau)e^{S_c(x,v;x',v';\tau)} \tag{4.4.27}$$

$$\begin{aligned} S_c(x,v;x',v',\tau) = &-\frac{1}{2}M_{11}(x'^2+x^2) - \frac{1}{2}M_{22}(v'^2+v^2) - M_{13}x'x \\ &-M_{24}v'v + M_{12}(xv - x'v') + M_{14}(xv' - x'v) - 2bjx' + 2bjx - bj^2\tau \end{aligned}$$

The results for M_{IJ} are given below in terms of the parametrization chosen in Eq. 4.4.17 – as it is more suitable for the classical solution:

$$\begin{aligned} M_{11} &= \Lambda\Big\{2ar\zeta\left(r^2+\zeta^2\right)\left(\left(-1+e^{4r\tau}\right)\zeta + 2e^{2r\tau}r\sin[2\tau\zeta]\right)\Big\} \\ M_{12} &= -\Lambda\Big\{-2a\left(1+e^{4r\tau}\right)r^2\zeta^2 + b\left(\zeta^2+e^{4r\tau}\zeta^2 - 2e^{2r\tau}\left(r^2+\zeta^2\right)\right) \\ &\quad +2e^{2r\tau}r^2\left(b+2a\zeta^2\right)\cos[2\tau\zeta]\Big\} \\ M_{13} &= -\Lambda\Big\{4ae^{r\tau}r\zeta\left(r^2+\zeta^{\,2}\right)\left(\left(-1+e^{2r\tau}\right)\zeta\cos[\tau\zeta] + \left(1+e^{2r\tau}\right)r\sin[\tau\zeta]\right)\Big\} \\ M_{14} &= \Lambda\Big\{4ae^{r\tau}\left(-1+e^{2r\tau}\right)r\zeta\ \left(r^2+\zeta^2\right)\sin[\tau\zeta]\Big\} \\ M_{22} &= -\Lambda\Big\{2ar\zeta\ \left(\zeta - e^{4r\tau}\zeta\ + 2e^{2r\tau}r\sin[2\tau\zeta]\right)\Big\} \\ M_{23} &= -\Lambda\Big\{4ae^{r\tau}\left(-1+e^{2r\tau}\right)r\zeta\left(r^2+\zeta^2\right)\sin[\tau\zeta]\Big\} \\ M_{24} &= \Lambda\Big\{4ae^{r\tau}r\zeta\left(-\left(-1+e^{2r\tau}\right)\zeta\cos[\tau\zeta] + \left(1+e^{2r\tau}\right)r\sin[\tau\zeta]\right)\Big\} \end{aligned}$$

where

$$\Lambda = \frac{1}{\zeta^2 + e^{4r\tau}\zeta^2 - 2e^{2r\tau}\left(r^2+\zeta^2\right) + 2e^{2r\tau}r^2\cos[2\tau\zeta]}$$

Recall from Eq. 4.1.7, the ω_1, ω_2 parameters are given by

$$\omega_1^2 + \omega_2^2 = 2\frac{b}{a^{1/3}};\ \ \omega_1^2\omega_2^2 = ca^{1/3}$$

To write out the normalization $\mathcal{N}(\tau)$, recall from Eq. 4.1 that the Lagrangian has the following three branches. The complex branch is given by $b < \sqrt{ac}$ and yields

$$\omega_1 = e^{i\phi}\omega;\ \ \omega_1 = e^{-i\phi}\omega:\ \ \omega^4 = ca^{1/3};\ \ \cos(\phi) = \frac{b}{\sqrt{ac}}$$

For $\beta = a^{-1/3}\tau$, the normalization is given by Eq. 4.2.15:

$$\mathcal{N} = \frac{\sqrt{\omega_1\omega_2}(\omega_1^2 - \omega_2^2)}{2\pi\sqrt{(\omega_1^2+\omega_2^2)\sinh(\beta\omega_1)\sinh(\beta\omega_2) - 2\omega_1\omega_2\big(\cosh(\beta\omega_1)\cosh(\beta\omega_2) - 1\big)}}$$

4.5 Summary

The acceleration Lagrangian is an important exemplar in its own right and has a vast range of applications. In fact, in Baaquie (2014) three chapters are devoted to studying the intricacies and complexities of this seemingly simple model. The crux of all the new properties of the model comes from the acceleration term; for example, this term leads to a non-Hermitian Hamiltonian and hence is ruled out as an allowed quantum mechanical system.

The importance of the Lagrangian, and of the path integral formulation, is that it is valid for all three branches, whereas the Hamiltonian works well only for the real branch. For the critical and complex branches, the Hamiltonian has to be carefully continued and requires a number of new ideas; in particular, for the critical branch, the Hamiltonian maps into the direct sum of infinitely many Jordan blocks [Baaquie (2014)].

The acceleration action is a key model for all applications of quantum mathematics to economics and finance. There are no issues with the Hamiltonian being non-Hermitian since the interpretation is not based on probability theory. The results given in this chapter are valid for all three branches of the theory and this plays a crucial role in applying the acceleration action to asset pricing, option pricing and interest rate models.

It is worth noting that the kinetic term in physics, given by $m(dx/dt)^2$ for a degree of freedom x, does not change from model to model; what changes is the nature of the degree of freedom as well as the potential term. The kinetic term for all the applications of quantum mathematics that are discussed in this book is given by $L(dx^2/dt^2)^2 + \widetilde{L}(dx/dt)^2$. This kinetic term makes the applications in economics and finance distinct from quantum physics, and yields results that are refreshingly different from what one obtains in quantum physics.

5

Option theory*

5.1 Introduction

It is shown how quantum mechanics provides a natural framework for understanding the theory of option pricing. One of the four famous papers that Einstein (1905) wrote was "On the Movement of Small Particles Suspended in a Stationary Liquid Demanded by the Molecular-Kinetic Theory of Heat". This phenomenon, called Brownian motion, is explained by the theory of random walk, also called a stochastic process. Interestingly enough, the first formalization of random walk was not in Einstein's paper, but instead in the study of finance.

The famous mathematician Henri Poincare assigned one of his graduate students, Louis Bachelier (1900) to study the evolution of a financial security, such as a stock of a company or a bond issued by a government. To price any financial instrument one needs to model the evolution of a stock, and Bachelier assumed that the stock price evolves randomly following a normal distribution. This is very close to the modern approach pioneered by Black and Scholes (1973); except in the modern approach it is the logarithm of the stock price, and not the stock price itself, that is assumed to be normally distributed.

Ideas from theoretical physics have found increasing applications in finance [Bouchaud and Potters (2003); Mantegna and Stanley (1999); Baaquie (2004, 2010)]. The discussion in this chapter is largely based on Baaquie (2008), with the focus on deriving the main results of option theory using the mathematics of quantum mechanics.

5.2 Options on a security

Financial derivatives, or derivatives for short, are important forms of financial instruments that are traded in the financial markets. As its name implies, derivatives are *derived* from other underlying financial instruments: the cash flows of a

5.4 Quantum mechanical pricing of options

The central problem in option pricing is the following: given the payoff function at some future time T, what is the price of the option at an earlier time $t < T$, namely $C(t, S(t))$? The standard approach for addressing option pricing in mathematical finance is based on stochastic calculus [Jarrow and Turnbull (2000)]. An independent derivation for the price of the option is given based on the formalism of quantum mechanics [Baaquie (2008)].

A stock of a company is never negative since the owner of a stock has none of the company's liabilities, and a right to dividends and pro rata ownership of a company's assets. Hence

$$S = e^x \geq 0; \;\; -\infty \leq x \leq +\infty$$

The stock price, at each instant, is considered to have a random value, making it mathematically identical to a quantum particle. The real variable x, similar to a quantum system, can consequently be considered to be a degree of freedom describing the behavior of the stock price.

Financial instruments are functions of x and form a *state space*, which is always taken to be an infinite-dimensional linear vector space. The state space is not a normalizable Hilbert space since fundamental financial instruments such as the stock price $S(x)$ are not normalizable. The state space consists of all possible functions of the degree of freedom x.

Consider a linear vector space $\mathcal{V}$ with elements given by $|\psi\rangle$; the dual space $\mathcal{V}_D$ consists of all mappings of elements $\mathcal{V}$ into the complex numbers. Elements of the dual vector space are denoted by $\langle\chi|$; let $|\psi\rangle$ be an element of $\mathcal{V}$. The mapping to the complex numbers – called the *inner product* – is denoted by $\langle\chi|\psi\rangle$: complex number.

The completeness equation for the degree of freedom, from Eq. 2.4.12, is given by

$$\int_{-\infty}^{\infty} dx|x\rangle\langle x| = \mathbb{I}: \text{ Completeness equation}$$

$\mathbb{I}$ is the identity operator on (function) state space, $|x\rangle$ is a coordinate basis for the state space and $\langle x|$ is the basis of the dual state space. The inner product can be realized by the completeness equation by the following:

$$\langle\chi|\psi\rangle = \langle\chi|\left[\int_{-\infty}^{\infty} dx|x\rangle\langle x|\right]|\psi\rangle = \int_{-\infty}^{\infty} dx\chi^*(x)\psi(x): \; \chi^*(x) = \langle\chi|x\rangle; \psi(x) = \langle x|\psi\rangle$$

Option pricing in the framework of quantum mechanics is based on the following assumptions.