Quantum Social Science

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QUANTUM SOCIAL SCIENCE

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Contents

Such approach presumes of the reader a substantial knowledge of both contexts, that of quantum mechanics, and that of the particular social field of application. That is asking a lot.

I chose to address this issue, that of more needed interdisciplinary competence in education, science, and the general public, in my recent autobiography *The Crossing of Heaven: Memoirs of a Mathematician*, Springer (2012). I have come to the conclusion that we must invoke and enforce a new term, that of Multidisciplinarity. Interdisciplinarity is a weak word. It implies that one is less than one hundred percent committed to each of the two fields. Or that one is slightly weak in one's own field and leaning on an expert from the other field, who is probably a bit weak also in his field. I have worked successfully in several fields of science and I can assure you that you should plan on becoming an expert also in "the other field," and that will take you, say, at least five years before you have a chance of becoming competitive there.

Thus a collateral message of this foreword is that of advancing the concept and indeed the cause of creating more multidisciplinarity in our future mathematicians, physicists, social scientists, and, in a more general sense, throughout the educated public. A tall order! But great opportunities will open up to those who are strong enough.

This book by Haven and Khrennikov is a move in that direction, a pioneering effort.

Karl Gustafson
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Preface

The current level of specialization of knowledge in a variety of fields of inquiry may make it quite challenging for a researcher to be at the same time a "developer" and a "tester" of a theory. Although a theory can exist without a necessary clear and obvious practical end goal, the ultimate test of the validity of a theory (whether it is situated in the exact or social sciences) will always be how measurement can "confirm" or dislodge a theory.

This book is largely dedicated to the *development* of a theory. We will be the very first to accept the accusation that the duo "theory-test" is widely absent in this work, and we believe it necessary to make this statement at the very beginning.

This book is about a very counter-intuitive development. We want to use a physics machinery which is meant to explain sub-atomic behavior, in a setting which is at the near opposite end of the size spectrum, i.e. the world as we know and live it through our senses. We may know about the sub-atomic world, but we do not have human experience of the sub-atomic world. Do we have credible and provable stories which can explain how the sub-atomic engages into the mechanics of the statistical macro-world? Probably not. Why do we bother then about being so exotic? The interested reader will want us to provide for a satisfactory answer to this obvious question, and we want to leave it up to him or her to decide whether we have *begun*, via the medium of this book, to convince that the level of "exoticality" (and "yes" how exotic is that word?) is sensibly less than anticipated. We can possibly give a glimmer of "hope," even at this early stage. Consider the words of one of the towering giants of physics of the twentieth century – Wolfgang Pauli. In an unpublished essay by Pauli, entitled "Modern examples of 'background physics'," which is reproduced in Meier* (pp. 179–196), we can read Pauli's words (Meier* (p. 185)): "Complementarity in physics . . . has a very close analogy with the terms 'conscious' and 'unconscious' in psychology in

* Meier C. A. (2001). *Atom and Archetype: The Pauli/Jung Letters, 1932–1958.* Princeton University Press.

that any 'observation' of unconscious contents entails fundamentally indefinable repercussions of the conscious on these very contents." The words of Pauli are important. They show there is promise for a connection between "concepts" of utmost importance in two very different sciences: complementarity in quantum physics and "complementarity" between consciousness and unconsciousness in psychology.

In this book, we intend to give the reader a flavor of an intellectual development which has taken shape over several years via the usual media many academics use: conference presentations and academic articles. The theory presented here is nowhere complete but we strongly believe that it merits presentation in book form.

The models presented in this book can be called "quantum-like." They do not have a direct relation to quantum physics. We emphasize that in our approach, the quantum-like behavior of human beings is not a consequence of quantum physical processes in the brain. Our basic premise is that information processing by complex social systems can be described by the mathematical apparatus of quantum mechanics. We present quantum-like models for the financial market, behavioral economics, and decision making.

Connecting exact science with social science is not an easy endeavor. What reveals to be most difficult is to dispel an intuition that somehow there *should* exist a natural bridge between physics and the modeling of social systems. This is a very delicate issue. As we have seen above it is possible to think of "complementarity" as a concept which could bridge physics and psychology. However, in some specific areas of social systems, the "physics equivalent" of the obtained results may have very little meaning.

It is our sincere hope that with this book we can convince the brave reader that the intuition of the authors is not merely naive, but instead informative. Hence, may we suggest that "reading on" is the command of the moment? Let the neurons fire!

Acknowledgements

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- p: momentum
- q: position
- $\mathcal{H}(.,.)$: Hamiltonian function
- $\{f, g\}$: Poisson bracket of two functions f and g on an N particle phase space
- $\{f_1, f_2\}$: Poisson bracket for a pair of classical observables f_1, f_2
- $\phi(t, x, y, z,)$: field state at instant t of vector with coordinates x, y and z
- $E(t, x, y, z,)$: electrical field at instant t of vector with coordinates x, y and z
- $B(t, x, y, z,)$: magnetic field at instant t of vector with coordinates x, y and z
- h: Planck's constant
- $\hbar$: rationalized Planck constant
- $\Delta E_{ij} = E_i - E_j$: discrete portion of energy
- L: angular momentum of an electron
- I: intensity of the electromagnetic field
- $A = (a_{ij})$: Hermitian matrix
- $\widehat{\mathcal{H}}$: Hermitian matrix representing the energy observable (quantum Hamiltonian)
- $\hat{q}$: position operator
- $\hat{p}$: momentum operator
- σ_x: standard deviation of position
- σ_p: standard deviations of momentum
- Δ_{q_j}: Laplace operator
- $\psi(t, q)$: probability amplitude on time, t, and position, q
- Γ: phase space of hidden states
- $|\psi\rangle$: element of the Hilbert space H : a ket vector
- $\langle\phi|$: element of the dual space H^*, the space of linear continuous functionals on H: a bra vector
- $\langle\psi_1|\widehat{w}\psi_2\rangle$: Dirac braket, where ψ_1^* denotes the complex conjugate of ψ_1 and $\widehat{w}$ acts on the state function ψ_2.
- k: wave number
- $A(k)$: amplitude function of wave number k
- $\langle p\rangle$: average momentum
- Q: quantum potential
- $\mathbf{P}(.|C)$: conditional probability dependent on the context, C
- D_+: mean forward derivative
- D_-: mean backward derivative

Some economics/finance symbols used in the book

- σ: volatility
- $\alpha(\sigma)$: drift function of volatility
- $\beta(\sigma)$: diffusion function of volatility

- dX, dz, dW: Wiener process
- $\overrightarrow{q} = (q_1, q_2 \ldots q_n)$: n-dimensional price vector
- m_j: number of shares of stock j
- $T_j(t)$: market capitalization of trader j at time t
- $V(q_1, \ldots, q_n)$: interactions between traders as well as interactions from other macro-economic factors
- Π: portfolio value
- F: financial option price
- S: stock price
- $\Delta = \frac{\partial F}{\partial S}$: delta of the option
- f_u; f_d : intrinsic values of the option when the price of the asset is respectively going up and down
- $E(r)$: expected return
- $\delta\Pi$: discrete change in the value of the portfolio, Π
- μ: expected return
- dF: infinitesimal change in F (the option price)
- r_f: risk free rate of interest
- $\phi(S, t)$: part of the premium invested in the stock, S
- S_T: asset price at the expiration of the option contract
- S_0: asset price at the inception of the option contract
- $P(., .|., .)$: conditional probability distribution
- $E[S_T|I_t]$: conditional expectation of a stock price at time $T > t$, given the information you have at time t
- $E(e^{Y_t\lambda})$: moment generating function, λ is some arbitrary parameter, and Y_t follows a probability density function (pdf) with mean μt and $\sigma^2 t$
- $E^{\widetilde{P}}$ [., .]: expectation with respect to a risk neutral probability measure $\widetilde{P}$
- E^P [., .]: expectation with respect to a probability measure P
- C_t: option call value at time t
- P_t: option put value at time t
- $\overrightarrow{\Phi} = (\Phi_1, \Phi_2, \ldots, \Phi_K)$: K-dimensional state price vector
- $\overrightarrow{D_1}, \ldots, \overrightarrow{D_K}$: security price vector at time t_1, if the market is, respectively, in state $1, \ldots, K$
- λ: Lagrangian multiplier
- $E(u(W))$: expected utility of wealth, W
- $\succ$: preference relation
- $\succeq$: weak preference relation
- β_i: CAPM - Beta of asset i

Part I

Physics concepts in social science? A discussion

where:

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{T}. \tag{1.10}$$

Remark that ν indicates the frequency expressed as the number of cycles per time unit. Clearly, as is intuitive, the force constant k and the mass m influence this frequency. The position $x(t)$ depends on this frequency ν but also on the amplitude A and phase ϕ. They can be found from the system of equations:

$$A\cos\phi = x_0, \quad A\sin\phi = -v_0/2\pi\nu. \tag{1.11}$$

In the case of a particle in the three-dimensional case, the force f is a vector $f = (f_x, f_y, f_z)$. It is called conservative if there exists a (real) potential $V(q)$, $q = (x, y, z)$, such that $f_x = -\frac{\partial V}{\partial x}$, $f_y = -\frac{\partial V}{\partial y}$, $f_z = -\frac{\partial V}{\partial z}$. We also recall the notion of the gradient of a function V. This is a vector composed of its partial derivatives and it is denoted as ∇V. Hence, a conservative force can be represented as the "negative gradient" of the potential:

$$f = -\nabla V. \tag{1.12}$$

Although in this book we try to minimize mathematical details as much as possible, we need to point out the theorem of the existence and uniqueness of the solution of the equation (1.4) with the initial conditions (1.5). Such a problem, i.e. an equation with initial conditions, is called the *Cauchy problem*. This is one of the basic mathematical problems of classical mechanics. The simplest version of the aforementioned theorem is that if the force is described by a smooth function f, i.e. differentiable and with continuous derivative, and the derivative is bounded, i.e. there exists a constant $c > 0$ such that, for every $q \in \mathbb{R}^3$, $|f'(q)| \leq c$, then, for any pair (q_0, v_0), a unique solution of the Cauchy problem exists (1.4), (1.5). This mathematical theorem was the main source of the *causal deterministic viewpoint* to classical mechanics: if we know the position and velocity of a particle at $t = t_0$, then we can find them at any instant of time $t > t_0 : q = q(t)$, $v = v(t) = \frac{dq(t)}{dt}$.

Consider the following quote by Laplace [1]:

We ought to regard the present state of the universe as the effect of its antecedent state and as the cause of the state that is to follow. An intelligence knowing all the forces acting in nature at a given instant, as well as the momentary positions of all things in the universe, would be able to comprehend in one single formula the motions of the largest bodies as well as the lightest atoms in the world; provided that its intellect were sufficiently powerful to subject all data to analysis; to it nothing would be uncertain, the future as well as the past would be present to its eyes.

Later interpretations of quantum mechanics also leave the theoretical possibility of such a super intellect contested.

This is a good example of how pure mathematics generates fundamental philosophic principles. As it often happens in science, it is not easy to change philosophic principles which have been established on the basis of some special mathematical results and models. During Laplace's lifetime, the theory of differential equations had not yet been well developed. Nowadays, it is well known that the Cauchy problem (1.4), (1.5) may have a non-unique solution even for continuous forces. If f is smooth, then the solution is unique only locally, i.e. for a small neighborhood of the point (t_0, x_0). However, globally it can be non-unique. Hence, modern mathematics does not imply determinism even in classical mechanics (see [2] for usage of this argument in classical non-deterministic biological dynamics). We also remark that if the dynamics of a particle is even deterministic, but unstable, then a small disturbance of initial conditions, can change crucially the trajectory of such a particle. In such a case, although the principle of determinism is formally valid, it has no usage in real practice, since it is impossible to determine initial conditions with infinite precision. This argument against the uncontrollable usage of the principle of determinism in classical mechanics was presented by Blohinzev [3] in his comparison of classical and quantum mechanics. In conclusion, we can see from the above that Laplace's causal determinism is indeed a mere prejudice.

Besides Laplace's prejudice, we can also mention the Kantian prejudice which says that physical space has to be identified with its Euclidean model [4]. This prejudice was based on two-thousand years of Euclidean geometry. The first blow to the Kantian views of physical space was given by Lobachevsky. However, the genius of Einstein was needed to establish modern views of the geometry of physical space.

The above discussion raises a reasonable recommendation: the reader may want to veer close to mathematics and instead steer away from general physical, metaphysical, and philosophic principles.

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1.3 The Hamiltonian formalism

To proceed from classical to quantum mechanics, one typically uses the Hamiltonian formalism for the description of the motion of classical particles. As usual, let us introduce the momentum $p = mv$ of a particle and consider *phase space* with coordinates (q, p), where q is position. Points of the phase space are interpreted as states of classical particles. We state again that, by Newton's second law, to determine the trajectory of a particle it is necessary to know both initial position q_0 and the velocity v_0. In particular, knowledge of only position is not sufficient. Therefore, it is natural to define the *particle's state* as the pair (q, v). By scaling the velocity by the particle's mass, we introduce its momentum, p, and equivalently we represent the particle's state as a pair (q, p).

We remark that the momentum's definition can be expressed in the form of an ordinary differential equation:

$$\frac{dq}{dt} = \frac{p}{m}. \tag{1.13}$$

Hence, Newton's second law, (1.4), can be written as:

$$\frac{dp}{dt} = -\frac{dV}{dq}. \tag{1.14}$$

Let us introduce the following function on the phase space:

$$\mathcal{H}(q, p) = \frac{p^2}{2m} + V(q). \tag{1.15}$$

$\mathcal{H}(.,.)$ is called the *Hamiltonian function*. This is the total energy of a particle which moves under the action of the force induced by the potential V and the kinetic energy $\frac{p^2}{2m}$. The system of equations (1.13), (1.14) can be written as:

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \ \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q}. \tag{1.16}$$

This is the system of Hamiltonian equations. It is easy to prove that the energy is preserved in the process of motion:

$$\mathcal{H}(q(t), p(t)) = \mathcal{H}(q(t_0), p(t_0)). \tag{1.17}$$

To prove this important fact (the law of energy conservation), it is sufficient to use the basic rule for the differentiation of a composition of functions and then to apply this rule to the system of Hamiltonian equations.

By using the Hamiltonian formalism, we can formulate a feature of classical mechanics, which can be called *locality.* Let us consider a system consisting of N particles with the three-dimensional coordinates $q_j = (x_j, y_j, z_j)$, $j = 1, \ldots, N$ and corresponding momenta p_j. The Hamiltonian function of a system of N particles with masses m_j moving in the potential $V(q_1, \ldots, q_N)$ has the form:

$$\mathcal{H}(q, p) = \sum_{j=1}^{N} \frac{p_j^2}{2m_j} + V(q), \tag{1.18}$$

where $q = (q_1, \ldots, q_N)$, $p = (p_1, \ldots, p_N)$. The above Hamiltonian gives the total energy of this system composed of N particles. The system of Hamiltonian equations describing the dynamics of this composite system can be written as:

$$\frac{dq_j}{dt} = \frac{\partial \mathcal{H}(q, p)}{\partial p_j}, \; \frac{dp_j}{dt} = -\frac{\partial \mathcal{H}(q, p)}{\partial q_j}, \; j = 1, \ldots, N. \tag{1.19}$$

Within the potential V, the interaction between different particles is described by terms containing coordinates of a few particles. We can consider the interaction between particles by writing for instance terms of the form $q_i \ldots q_N$ (various products of different coordinates). But let us consider now a potential which does *not* contain interaction terms, $V(q) = V_1(q_1) + \cdots + V_N(q_N)$. The corresponding system of Hamiltonian equations is:

$$\frac{dq_j}{dt} = \frac{p_j}{m_j}, \; \frac{dp_j}{dt} = -\frac{\partial V_j}{\partial q_j}, \; j = 1, \ldots, N. \tag{1.20}$$

This is a system of N-independent equations.

Hence, an important principle emerges from our discussion so far: *Hamiltonian mechanics is local, i.e. in the absence of interaction between particles, such particles move independently of each other.*

We remark that non-local motion, as is the case with for instance Bohmian mechanics (see Chapter 6), has the following (paradoxical from the viewpoint of our classical intuition) feature. In the absence of interaction, even for $V \equiv 0$, the dynamics of different particles are *dependent* on each other. Changing the state of one particle (q_j, p_j) induces changing the states (q_i, p_i), $i \neq j$, of other particles. In the classical world, we have never seen such a behavior of physical systems.

Let us introduce a mathematical tool which has a key role in the Hamiltonian formalism. The *Poisson bracket* of two functions on the N-particle phase space,

$f(q, p), g(q, p)$, is defined as:

$$\{f, g\} = \sum_{j=1}^{N} \Big(\frac{\partial f(q, p)}{\partial q_j} \frac{\partial g(q, p)}{\partial p_j} - \frac{\partial f(q, p)}{\partial p_j} \frac{\partial g(q, p)}{\partial q_j} \Big). \tag{1.21}$$

As an example, consider functions $f(q, p) = q_j, g(q, p) = p_j$. Then:

$$\{q_j, p_j\} = 1, \ \{q_j, p_k\} = 0, j \neq k. \tag{1.22}$$

$$\{q_j, q_k\} = 0, \ \{p_j, p_k\} = 0. \tag{1.23}$$

By using the Poisson bracket, we rewrite the system of Hamiltonian equations as:

$$\frac{dq_j}{dt} = \{q_j, \mathcal{H}\}, \ \frac{dp_j}{dt} = \{p_j, \mathcal{H}\}. \tag{1.24}$$

This form of the Hamiltonian dynamics will be used to proceed from classical Hamiltonian mechanics to quantum mechanics.

1.4 Statistical mechanics and the Liouville equation

In studying the dynamics of an ensemble of a huge number, say N particles, the presence of the system of Hamiltonian equations plays merely a methodological role. From as early as the nineteenth century until the 1960s, it was simply impossible to solve this system for large N and non-trivial potentials. Nowadays in principle one can solve it numerically and obtain millions of trajectories in the phase space. However, it is not clear how one can use or visualize the results of such computations. Already in the nineteenth century it was proposed that instead of studying the trajectories of individual particles, it would be better to consider the probability to find a particle in some domain, say W, of the phase space. Such an approach meant in effect a move away from the deterministic description of mechanics to a statistical description. Hence, the name *statistical mechanics* was coined to denote this particular area of study.

Let us consider the phase space of the system of N particles, $\mathbb{R}^{2N}$, with points (q, p), where $q = (q_1, \ldots, q_N)$, $p = (p_1, \ldots, p_N)$. What is the probability density function which indicates the probability to find the first particle at point q_1 with momentum p_1, the second particle at point q_2 with momentum $p_2, \ldots,$ the Nth particle at q_N with momentum p_N? Since momenta are mass scalings of velocities, the question can be reformulated as: "What is the probability density function of the first particle at point q_1 with velocity v_1, the second particle at point q_2 with velocity $v_2, \ldots,$ the Nth particle at q_N with velocity v_N?" We state that mathematically a probability density is a function $\rho(q, p)$ which is another, capitalism → socialism → communism. Even Sigmund Freud's psychological determinism was created under the influence of classical mechanics. In [2], we can read about Freud's psychoanalysis: "What is attractive about the theory, even to the layman, is that it seems to offer us long sought-after and much-needed causal explanations for conditions which have been a source of a great deal of human misery. The thesis that neuroses are caused by unconscious conflicts buried deep in the unconscious mind in the form of repressed libidinal energy would appear to offer us, at last, an insight in the causal mechanism underlying these abnormal psychological conditions as they are expressed in human behavior, and further show us how they are related to the psychology of the 'normal' person" (see also [3]). Furthermore, [3] mentions that "In psychology, those, like Freud, who believe in psychic determination in psychiatry, assume that all mental events have causes. Freud believed that the existence of unconscious forces proved psychic determinism to be a fact of mental life . . . he regarded psychoanalysis as a science based on causal-deterministic assumptions." See [4] [5] for an attempt to combine Freudian psychological determinism and free will through the Bohmian quantum model. See also Chapter 6 for the Bohmian mechanics model.

1.6 References

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1.7 Classical fields

The completion of the theory of classical mechanics of particles was performed via the introduction of the notion of the classical mechanics of fields, i.e. *classical field theory* – the local deterministic theory of the electromagnetic field. The field's state (at the instant of time t) is given by the vector:

$$\phi(t, x, y, z)) = (E(t, x, y, z), B(t, x, y, z)), \tag{1.31}$$

where:

$$E(t,x,y,z) = (E_1(t,x,y,z), E_2(t,x,y,z), E_3(t,x,y,z)) \qquad (1.32)$$

and:

$$B(t,x,y,z) = (B_1(t,x,y,z), B_2(t,x,y,z), B_3(t,x,y,z)) \qquad (1.33)$$

are electric and magnetic fields, respectively. The dynamics of the electromagnetic field also can be described by the Cauchy problem (i.e., a dynamical equation combined with initial conditions):

$$\frac{\partial \phi(t,x,y,z)}{\partial t} = L(\phi(t,x,y,z)),\ \phi(t_0,x,y,z) = \phi_0(t,x,y,z), \qquad (1.34)$$

where L is a differential operator of the first order with partial derivatives with respect to coordinates. At the moment, its form is not important. This operator was found by Maxwell. We remark that *the system of Maxwell equation (1.34) can be written as a Hamiltonian system with respect to field components* [1]. The electric component plays a role of position, $q(t,x,y,z) \equiv E(t,x,y,z)$, and the magnetic component plays a role of momentum, $p(t,x,y,z) \equiv B(t,x,y,z)$. What is the phase space of this field system? The energy of the electromagnetic field (at the fixed instant of time) is given by the integral:

$$\mathcal{E}(E,B) = \int_{\mathbb{R}^3} (E^2(x,y,z) + B^2(x,y,z))dxdydz. \qquad (1.35)$$

Since this integral is finite, the field's position $q(x,y,z) = E(x,y,z)$ and the field's momentum $p(x,y,z) = B(x,y,z)$ have to be square integrable (integrals of squared functions are less than infinity). By using mathematical analogy, we can see E^2 relates to a real potential – B^2 relates to kinetic energy – in the non-field setting. Denote the space of square integrable functions by L_2. Thus, the field's phase space is the Cartesian product of two L_2-spaces: $L_2 \times L_2$.

The electric and magnetic components can be combined in a single complex valued field $\phi = E + iB$. This is well known (in classical signal theory) as the *Riemann–Silberstein* representation of the classical electromagnetic field. This representation induces the complex structure on the phase space, $\phi = q + ip$, where all functions depend on spatial coordinates.[3] This is equivalent to the consideration of the space H of square integrable complex valued functions:

$$H = \left\{ \phi : \int |\phi(x,y,z)|^2 dxdydz < \infty \right\}. \qquad (1.36)$$

[3] This complex representation of the classical electromagnetic field was in usage before the creation of quantum mechanics. Nowadays, little attention is paid to this historical fact. The fathers of quantum mechanics, including Schrödinger, were well aware of the Riemann–Silberstein representation. Originally, the complex wave function was invented merely as an analogue of the complex (classical) electromagnetic field.

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[1] Khrennikov, A. (2008). Quantum-like model for classical random electromagnetic field. *Journal of Modern Optics*, 55, 2257–2267.

1.9 The Born–Sommerfeld quantization

Contrary to what could be expected, the first steps towards edifying quantum mechanics were not revolutionary at all! We note that those first steps were not at all accompanied by a fundamental change in the philosophical foundations of science. The story started with a graph representing experimental data, and it contained a spike, which from the viewpoint of classical mechanics was quite difficult to explain. The experimental graph turned out to be about *black body radiation* and it was Max Planck who found that the spike could be explained in a classical statistical mechanics framework, but it required one novel assumption: *radiation is emitted not continuously, but by discrete portions.* Purely formally the energy space was decomposed into cells. The size of a cell depends on the frequency ν of the oscillations of the electromagnetic field. Max Planck postulated that dependence on the frequency is linear and the coefficient does not depend on the frequency:

$$\Delta E_\nu = h\nu. \tag{1.37}$$

This coefficient of proportionality was later called *Planck's constant.* Since the frequency has dimension l/time, the Planck constant is expressed in units of "energy × time." This is the dimension of the classical physical variable called *action.* This constant was measured with very good precision:

$$h \approx 6.6260693(11) \times 10^{-34}\ \text{J} \times \text{sec}. \tag{1.38}$$

The decomposition of the energy space into small cells and the summation over cells is similar to the standard procedure of forming Riemann sums.[4] To calculate the Riemann integral, the size of cells has to go to zero. However, Planck did not make this last step and wrote the answer using cells of a finite size (proportional to h). We note that in quantum folklore, one can find a story that Max Planck obtained the correct answer, because he simply did not know that in order to calculate Riemann's integral one has to consider that the limit of the cell's size goes to zero.

We remark that the tool of discretization of the energy space was not as novel as it is typically presented in textbooks on quantum mechanics. It was actually rather standard in classical statistical mechanics! In particular, Boltzmann used

[4] These sums are used in the construction of the Riemann integral.

discretization $\epsilon, 2\epsilon, \ldots, n\epsilon, \ldots,$ where ϵ was a "minimal quant" of energy. This is maybe the reason why the work of Planck was very welcome in the classical statistical community: nobody considered the introduction of a parameter of energy discretization as an attack against classical statistical mechanics!

The discretization parameter h ceased to be merely a parameter only after Einstein's work [1] (1905). In his work, Einstein claimed that $\Delta E_\nu = h\nu$ is not just a minimal portion of energy which can be transmitted for the frequency ν, but that even in the absence of interaction of the electromagnetic field and matter, the field is "quantized," i.e. it is split into a collection of quanta of lights. Later these quanta were called *photons.* Thus, in opposition to classical field theory, the electromagnetic field has to be decomposed into an ensemble of photons, i.e. it has corpuscular features.[5]

The next step towards quantum theory was performed on the basis of *Bohr's quantization condition.* We want to explicitly state that in classical Hamiltonian mechanics, the energy is preserved on each trajectory (see (1.17)). Suppose now that there exist constraints (of an unknown nature) which forbid some motions and some trajectories, and that the system can move only via a discrete set of trajectories. Denote those possible trajectories (consistent with the constraints) by:

$$\gamma_1, \ldots, \gamma_n, \ldots \tag{1.39}$$

Since the energy is constant on each of them, we obtain a discrete set of possible energies:

$$E_1 = E(\gamma_1), E_2 = E(\gamma_2), \ldots, E_n = E(\gamma_n), \ldots \tag{1.40}$$

This idea was explored by Niels Bohr in his model of the atom. It was known from experiments that atoms can emit and absorb energy only by quantized portions. Bohr proposed a model describing this feature of atoms. In this model, the electron is a classical-like particle which moves around the nucleus. However, in such a motion a purely classical charged particle would continuously emit radiation and lose energy. Finally, it would fall onto the nucleus. This was one of the main unsolved problems of classical electrodynamics. Bohr postulated that an electron can move only on a discrete set of orbits (1.39) and hence its energy can take only a discrete series of values (1.40). Since an electron can only jump from one orbit to another, the atom can emit (absorb) only discrete portions of energy $\Delta E_{ij} = E_i - E_j$. To match experimental data, Bohr postulated that the frequency

[5] There is a piece of irony in this story. Although Albert Einstein introduced quanta of light and hence in this way he made the first step towards modern quantum theory, later (in the 1920s) he gave up and until the end of his life he worked to create a classical field theory which would describe quantum phenomena.

ν of emitted (absorbed) electromagnetic radiation is determined by Planck's law:

$$\Delta E_{ij} = E_i - E_j = h\nu. \tag{1.41}$$

Since these frequencies were known from experiment, he could find energy spacing in the atom. Bohr was also able to "derive" energy spacing even theoretically and he obtained a key result which indicates that the angular momentum L of an electron is to be an integer multiple of $\hbar$

$$L = n\frac{h}{2\pi} = n\hbar, \tag{1.42}$$

where $n = 1, 2, 3, \ldots$ is the quantum number, and $\hbar = h/2\pi$.

Bohr's quantization condition (1.42) determining the electron's orbits in the atom was generalized to the famous Bohr–Sommerfeld quantization rule (which also had been postulated):

$$\int_{\gamma_n} pdq = hn, \tag{1.43}$$

where γ_n is the permitted orbit corresponding to a natural number n. In the original Bohr model, only circular orbits were permitted; in the Bohr–Sommerfeld model orbits can be elliptic.

Thus, the first step towards quantum theory was the recognition that some physical quantities, first of all energy, which were considered as continuous in classical mechanics, are *fundamentally discrete*. Discreteness by itself is less fascinating. However, the concept is fascinating, and even mystical, when it is combined with the wave features of the systems under consideration (see below). Photons are mystical not because they have corpuscular features,[6] but because these features are combined with wave behavior. As was shown in the famous experiment on the interference of quantum light (see Chapter 5, Section 5.3), photons did not lose their wave features. They interfere as usual waves. "Quantumness" was exhibited by the detection procedure. As a consequence of the discreteness of energy in experiments on quantum interference, one registers not intensities of signals and interference of these intensities, but rather clicks of detectors which are "eating" discrete portions of energy. The probability density of the number of clicks presents the interference picture similar to the ordinary wave interference.

[6] Already Newton invented corpuscles of light.

was trivial by its very mathematical nature, but it turned out to play a fundamental role in the further development of quantum theory.

Consider the notion of the *Hermitian matrix* $A = (a_{ij})$: its elements satisfy the condition $a_{ij} = \bar{a}_{ij}$, where, for a complex number $a = a_1 + ia_2$, $\bar{a} = a_1 - ia_2$ denotes its conjugate. Heisenberg identified discrete values of physical observables with eigenvalues of Hermitian matrices. In Heisenberg's approach, all physical observables have to be represented by Hermitian matrices.

As is well known from linear algebra,[10] any Hermitian matrix of finite size, $n \times n$, can be diagonalized in the basis consisting of eigenvectors corresponding to its eigenvalues (eigenvalues are real numbers and eigenvectors are orthogonal). However, matrices in quantum theory are of infinite size. We shall explain later in the book why one cannot proceed with matrices of finite size. In such a case, some Hermitian matrices cannot be diagonalized. Besides eigenvalues, their spectra can contain a non-discrete part. It can even happen that there are no eigenvalues at all, and then such spectra are called continuous. This mathematical formalism matches the physical situation: some physical observables, such as the particle's position and momentum, are still continuous (as it is in classical mechanics).

1.13.1 Canonical commutation relations

Heisenberg performed a formal translation of classical Hamiltonian mechanics (in which observables were given by functions on the phase space) into a new type of mechanics (quantum mechanics), in which observables are represented by Hermitian matrices. He correctly noted the crucial role the Poisson brackets could play in classical formalism. They are defined for any pair of classical observables, f_1, f_2. We remark that Poisson brackets are antisymmetric $\{f_1, f_2\} = -\{f_2, f_1\}$. A natural operation on the set of Hermitian matrices corresponding to Poisson brackets is the commutator of two matrices:

$$[A_1, A_2] = A_1A_2 - A_2A_1, \tag{1.44}$$

defined with the aid of standard matrix multiplication. Note that the commutator is anti-symmetric as well. The transformation of the usual multiplication of functions into matrix multiplication was the great contribution of Heisenberg towards the creation of quantum formalism. Starting with the classical position and momentum observables q_j, p_k (coordinates on the phase space) and the equalities for their Poisson brackets, see (1.22), (1.23), he postulated that corresponding quantum observables denoted by $\hat{q}_j, \hat{p}_k$ (hats are used to distinguish classical and quantum

[10] Please see Chapter 4 where various linear algebra concepts are dealt with in more detail.

observables, functions and matrices) have to satisfy the following commutation relations:

$$[\hat{q}_j, \hat{p}_j] = ihI, \ [\hat{q}_j, \hat{p}_k] = 0, \ j \neq k; \tag{1.45}$$

$$[\hat{q}_j, \hat{q}_k] = 0, \ [\hat{p}_j, \hat{p}_k] = 0, \tag{1.46}$$

where I is the unit matrix. These commutation relations were called the *canonical commutation relations.* The appearance of i in the non-trivial commutator is not surprising. One cannot simply borrow the relation (1.22) and put the unit matrix I, instead of the constant function $f \equiv 1$, on the phase space. Take two Hermitian matrices A_1 and A_2 and form their commutator $B = [A_1, A_2]$. The latter is a skew-Hermitian, i.e. its elements satisfy the condition $b_{ij} = -\bar{b}_{ji}$. However, the unit matrix is simply Hermitian; by multiplying it by i, we obtain a skew-Hermitian matrix iI. The Planck constant h in (1.45) plays a role of scaling factor (the scale of energy for micro-systems under consideration). One can say that Heisenberg introduced a *non-commutative phase space.*

1.13.2 Schrödinger's representation

Later Schrödinger found a concrete representation for the quantum observables of position and momentum, and nowadays authors of textbooks typically simply start with this (Schrödinger) representation. However, this may give the impression that Schrödinger's concrete choice of the operators of position and momentum $\hat{q}_j$, $\hat{p}_j$ played a primary role in the derivation of the canonical commutation relations. This was not the case. As was pointed out, Heisenberg really started with classical Poisson brackets on the phase space variables. As was mentioned in Section 1.11, Schrödinger considered the wave function of a quantum system as a real physical wave. Therefore, we work in the space $L_2(\mathbb{R}^3)$ of square integrable complex valued functions $\psi : \mathbb{R}^3 \to \mathbb{C}$, i.e.:

$$\int_{\mathbb{R}^3} |\psi(q)|^2 dq < \infty. \tag{1.47}$$

For a shorter notation, one can set $H = L_2(\mathbb{R}^3)$. This is a complex Hilbert space with the scalar product:

$$\langle \psi_1, \psi_2 \rangle (\equiv \langle \psi_1 | \psi_2 \rangle) = \int_{\mathbb{R}^3} \overline{\psi_1(q)} \psi_2(q) dq. \tag{1.48}$$

We note that the Hilbert space is defined in Chapter 4, Section 4.3. Contrary to Heisenberg, Schrödinger worked with operators (and not matrices), i.e. he considered a more general framework. For simplicity, we consider here the one-dimensional case, i.e. of a particle moving on the real line. Here $H = L_2(\mathbb{R})$. He introduced the operators of position and momentum:

$$\hat{q}\psi(q) = q\psi(q), \;\; \hat{p}\psi(q) = -i\hbar\frac{\partial\psi(q)}{\partial q}. \tag{1.49}$$

The result of the action of the position operator on the square integrable function $\psi(q)$ (vector in H) leads to another function, $q \to q\psi(q)$. In the same way, the momentum operator $\hat{p}$ transforms $\psi(q)$ into its derivative, up to the constant factor $-i\hbar$. Formally, we can write:

$$\hat{q} = q, \;\; \hat{p} = -i\hbar\frac{\partial}{\partial q}. \tag{1.50}$$

We remark that H was considered by Schrödinger as the space of classical fields. See also Section 1.7 and especially (1.36). Thus, originally the appearance of the complex Hilbert space to represent operators of position and momentum was just the operator reformulation of the theory of classical fields and signals. Of course, this is correct only with respect to views relative to Schrödinger, but not at all relative to Heisenberg or Bohr.

1.13.3 Heisenberg's dynamics

Finally, Heisenberg put operators satisfying the canonical commutation relations in the system of the Hamiltonian equation (1.24), instead of the classical phase space variables. In this way, he derived the basic equations of quantum dynamics:

$$\frac{d\hat{q}_j}{dt} = \frac{i}{h}[\hat{q}_j, \widehat{\mathcal{H}}], \;\; \frac{d\hat{p}_j}{dt} = \frac{i}{h}[\hat{p}_j, \widehat{\mathcal{H}}], \tag{1.51}$$

where $\widehat{\mathcal{H}}$ is a Hermitian matrix representing the energy observable. It is called the *quantum Hamiltonian*. Thus, classical and quantum dynamics have the same form, cf. (1.24) and (1.51). However, the variables have different physical meanings and mathematical representations. In (1.24), the "position" and "momentum" are real valued functions; in particular they commute. In (1.51), the "position" and "momentum" are Hermitian operators (which can be represented by infinite Hermitian matrices); they satisfy the canonical commutation relations. In classical mechanics, the position and momentum are interpreted as objective properties of systems. In quantum mechanics, these are observables which cannot be considered as objective properties of quantum systems.

1.13.4 Quantization procedure

How can one find such a Hamiltonian? Consider the classical Hamilton function of the form (see also Section 1.3):

$$\mathcal{H}(q, p) = \frac{p^2}{2m} + V(q), \tag{1.52}$$

where V is a polynomial function, e.g. $V(q) = kq^2$ and k is a real constant. Then one can construct the corresponding Hermitian matrix, i.e the quantum observable, by formally putting matrices $\hat{q}$ and $\hat{p}$ in the classical Hamiltonian function, instead of classical variables. For the aforementioned potential, we obtain:

$$\widehat{\mathcal{H}} = \frac{1}{2m}\hat{p}^2 + k\hat{q}^2. \tag{1.53}$$

If the potential is not a polynomial, then the mathematics are more complicated. The operator theory and Schrödinger's representation of the canonical commutation relations have to be used, see Section 1.13.1. The main problem arises for classical observables, functions on the phase space, which contain products of position and momentum variables, e.g. $f(q, p) = qp$. In principle, we can form a family of matrix expressions corresponding to this function, $\hat{f} = \alpha\hat{q}\hat{p} + \beta\hat{p}\hat{q}$, where α, β are real numbers, $\alpha + \beta = 1$. However, we obtain the Hermitian matrix only for $\alpha = \beta = 1/2$, $\hat{f} = (\hat{q}\hat{p} + \hat{p}\hat{q})/2$. This rule is known as the *Weyl quantization*.[11]

If one uses the Schrödinger's representation of the canonical commutation relations, i.e. (1.50), the quantum Hamiltonian, (1.52), can be written as:

$$\widehat{\mathcal{H}} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q),$$

where $V(q)$ is the operator of multiplication by the function $V(q)$ in the space of square integrable functions, namely $\psi(q) \to V(q)\psi(q)$. Thus, in Schrödinger's representation all Hamiltonians are simply partial differential operators.

In general, quantization has the meaning of a transition from functions on the phase space, $f(q, p)$, to Hermitian matrices or more generally operators, by using operators of position and momentum, instead of corresponding classical variables. It is important to stress that there is no "reasonable explanation" for such a formal procedure which is required when transiting from classical physical quantities to quantum physical quantities. However, it works well.

We remark that one of the problems in the application of quantum formalism to social sciences (such as economics, finance, psychology, and cognitive science) is that, roughly speaking, we do not have classical (such as Newtonian or Hamiltonian)

[11] Let $f(q, p) = q^2 p$. Find the corresponding matrix representation.

mechanics. In other words, we do not have classical quantities which we can automatically quantize. Therefore, the majority of quantum-like quantities used in the aforementioned domains of science are phenomenological. They are invented by analogy with quantum theory. Hence, and we need to emphasize this, we do not start with a classical model and then quantize it, but we directly mimic the quantum approach. This forms one of the important problems of the quantum-like approach. One possible solution of this problem consists in using phenomenological Hamiltonians.

1.14 Heisenbergian symbolism in physics: a version of symbolism in art

Heisenberg's approach to micro-phenomena was really symbolic (operational). Heisenberg was not able to present physical reasons (in the classical meaning) for the introduction of matrices, instead of functions on the phase space. His calculus was useful to encode observed energy levels in the spectra of Hermitian matrices, but nothing more.[12] Nevertheless, this symbolic approach has been very fruitful and it played a key role. The history of the creation of quantum mechanics is an interesting subject for some social scientists. The main positive impact of Heisenberg's symbolic approach was the novelty in the description of physical phenomena. In fact, this was not a detailed and realistic description as in classical physics, but instead a fuzzy (operational) description of results of measurements. We should also mention Bohr's contribution who emphasized the role of the so-called *experimental context*. For him, it was meaningless to speak about an object outside of the concrete measurement context. The main negative impact was the aggressive anti-classical attitude. From the very beginning, Heisenberg claimed that his symbolic (operational) description of experimental data for micro-systems could not be derived on the basis of a finer classical-like model of micro-reality. Moreover, he and Bohr strongly advertised the viewpoint Mach held that it is meaningless even to try to create such models, since such an activity belongs to the domain of metaphysics and not real physics.[13] One could say that Heisenberg and Bohr were not correct, since a number of "prequantum" (classical-like) models reproducing results of quantum experiments have been created, for example Bohmian mechanics (see Chapter 6) and stochastic electrodynamics. However, it is clear that Heisenberg and Bohr would not agree with such a statement, as for them "prequantum models" are metaphysical.

The behavior and writings of Heisenberg and especially Bohr remind us very much of the manifestos of symbolism and futurism. Please see [1] [2].

[12] We note that to find theoretically these levels, one has to use Schrödinger's approach.

[13] This is a good place to recall that Mach intensively attacked Boltzmann by claiming that, since molecules are not observable (as was the case at that time), they are metaphysical creatures and hence they have to be excluded from real physical theory. Mach's attacks against Boltzmann's realism may have played a role in Boltzmann's tragic death.

additional variables which might improve the quantum description. Nowadays, such variables are known under the name *hidden variables*. When this term was coined, the idea was that position and momentum were known. What was hidden was only the *pair* of those variables, i.e. a point of the phase space. The completeness of quantum mechanics implies that determinism cannot be recovered through a finer description. Thus, instead of the symbolic dynamics of Hermitian matrices, a more detailed dynamics, e.g. in space time, cannot be constructed.

We state again that one has to distinguish determinism as a general philosophic principle from its applications to real phenomena. As was remarked, even in classical mechanics determinism can be violated by dynamics with some (continuous) forces $f(q)$. Moreover, the initial conditions (position and velocity) cannot be determined with infinite precision. If the dynamics are unstable, a small perturbation can change the trajectory crucially. In this case, the principle of determinism has merely a theoretical value.[15] For an ensemble consisting of a huge number of particles, in practice we can operate only with probabilities and the dynamics will be given by the Liouville equation. The fact of the existence of the underlying Hamiltonian mechanics has merely a metaphysical value: positions and momenta can be really assigned to individual particles and these quantities can be imagined as evolving in space independently of our measurements. By the Bohr–Heisenberg approach, it is sufficient to operate only with probabilities for results of measurements.

A reader may be curious and ask the following question: "*Why do we need all these philosophic considerations on the completeness of quantum mechanics?*" The main problem is that we want to use quantum mathematics without having to share the views of Bohr and Heisenberg, i.e. the so-called Copenhagen interpretation of quantum mechanics. Our position is that we consider Heisenberg's discovery as merely a discovery of a *new mathematical formalism* describing results of measurements for systems characterized by a high sensitivity to external influences. The reader can hopefully understand our dilemma: we want to use the total power of the quantum operational (symbolic) approach and at the same time we do not want to give up *realism*. The latter is very important for us. The very subject of this book is tied to this position we adopt. We plan to apply quantum mathematics to social and cognitive phenomena. We cannot forget (even if we wished) that these phenomena are based on *classical* physical processors of information, the brains (individual and collective).

[15] We remark that classical mechanical determinism is rigidly coupled to the mathematical model of space-time, namely to the real continuum. The states of classical systems are given by pairs of real numbers (or by pairs of real vectors). The infinite divisibility of this space plays an important role, cf. with p-adic spaces which have been recently used in theoretical physics [3].