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Ramanujan's Place in the World of Mathematics

Essays Providing a Comparative Study

Second Edition

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Cover illustration: The eight great mathematicians encircling Ramanujan were chosen because of the special significance of their link with Ramanujan. The two at the bottom are Hardy (on the left) and Littlewood (on the right) of Cambridge University; they were the two who analyzed Ramanujan's letters, and came to the conclusion that he was a genius of first magnitude. Immediately above them are Rogers (on the left) and Schur (on the right) who had independently proved what are now called the Rogers-Ramanujan identities and went beyond these identities in their own ways. Above Rogers and Schur are Abel (on the left) and Galois (on the right), geniuses like Ramanujan who died very young, but had made path-breaking contributions in their youth. At the top are Euler and Jacobi, two of the greatest mathematicians in history, with whom Ramanujan was compared for sheer manipulative ability.

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Chapter 1
Ramanujan: An Estimation

Ancient India has a rich mathematical tradition. The Hindus understood the role of zero within the algebraic framework of numbers. This led them to the decimal system which was transmitted to Europe by the Arabs. As in other civilisations, astronomy provided the Indians a motivation for mathematical exploration. In studying the duration of the eclipses, Aryabhatta, Bhaskara and Brahmagupta systematically investigated a class of equations in number theory. In the post-Newtonian era, although great strides were made in Europe, for various reasons, nothing of scientific significance emanated from India. With the emergence of Ramanujan during the beginning of this century, this long period of hibernation came to an end, and Indian mathematical research was rejuvenated mainly in the realms of analysis and number theory.

Ramanujan is admired, and rightly so, for having achieved so much in such a short lifetime that was filled with the impediments of poverty. But examples abound of persons who did outstanding work under the most formidable circumstances. What makes Ramanujan unique is that in spite of the trammels of superstition and orthodox traditions that surrounded him, his untutored genius produced mathematics of the very highest quality.

Contempt expressed by peers strikes a more cruel blow than poverty. When Euclidean geometry was considered to be the only sensible geometry, a young Hungarian Bolyai ventured dangerously against established beliefs and sent his work on non-Euclidean geometry to Gauss, the unquestioned leader of mathematics during the early eighteenth century. After a prolonged silence, Gauss replied that he too had made similar attempts but had given them up because they were of no consequence. Bolyai succumed to the severity of the verdict and did not live to see the day when Einstein utilised non-Euclidean geometry in the theory of

This article appeared in the center page of *The Hindu*, India's National Newspaper, on 19 December 1987, for the Ramanujan Centenary. The article was accompanied by a short note by Hindu Reporter M. Prakash which appeared as a boxed inset.

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relativity. On the contrary, Ramanujan was lucky to be recognised by the British mathematician Hardy.

Is sound knowledge of related fields necessary for research? A moot question indeed! A sophisticated mathematician looks at a problem not in isolation, but in terms of its relationship with other questions and then chooses the appropriate tools to solve it. Sometimes when such approaches have not made headway, a radically new idea from a relatively inexperienced researcher produces a breakthrough. Ramanujan was an extreme example of a mathematician whose contributions were of such high calibre that they belied his lack of formal training. But the penalty he paid was that much of his work turned out to be rediscovery. Whether Ramanujan would have reached greater heights had he been provided rigorous training is debatable. Hardy was of the opinion that such training would have made Ramanujan less of a genius. Instead of taking sides on this issue, we make note that the brilliance of Ramanujan combined with the sophistication of Hardy was the key to their successful collaboration.

Mathematics has grown so vast and intricate that it is unlikely that a Ramanujan-like phenomenon will ever surface again as it will be difficult to make fundamental contributions without a clear understanding of connections between different fields. In the contemporary scene, Paul Erdös is one leading mathematician who defies sophistication. Now past seventy, Erdos continues to be the most itinerant of scientists. His unconventional approach is marked with such distinction that an "Erdosian proof" is instantly recognisable. He is one of the principal architects of probabilistic number theory whose origins can be traced back to a joint paper that Ramanujan wrote with Hardy in 1917.

Ramanujan has to be measured only by the impact of his contributions. One may feel that the most important papers are those that solve longstanding problems. These are significant, but so also are those that trigger new questions and open up fresh avenues of thought. The great mathematician Bernhard Riemann, who worked mainly in analysis, wrote just one paper in number theory with the intention of proving the prime number theorem that was conjectured nearly a century earlier by Gauss. Although Riemann did not prove this theorem, he created a totally new approach and raised several questions that have kept mathematicians busy ever since. Even Ramanujan pursued some ideas akin to those in Riemann's paper. Certain questions raised in that paper have been solved, but one known as "The Riemann Hypothesis" remains unsolved to this day, and the very effort to find a solution has been rewarding.

Ramanujan has also raised several fundamental questions that have engaged mathematicians for decades. Some of his papers have created fruitful areas of research such as the one which led to probabilistic number theory. In another paper he raised a problem which Hardy later called "The Ramanujan Hypothesis", and this was settled only recently by Pierre Deligne in Paris. That the solution came after more than half a century is a measure of the depth of the problem. Deligne was honoured with the Fields Medal which is awarded once every four years at the International Congress of Mathematicians. There is no Nobel Prize for mathematics, but the Fields Medal is considered to be equivalent in prestige although not as lucrative.

Ramanujan lives today through the many questions he has raised. Deligne's achievement is a monumental inscription to his illustrious memory.

100 Percent Pure Talent

Paul Erdös, a distinguished member of the Hungarian Academy of Sciences, told The Hindu in Gainesville, Florida, USA, that he was greatly inspired by Ramanujan's work. He said that Ramanujan's contribution to number theory in collaboration with Hardy formed the basis for his work which led to the creation of probabilistic number theory.

Paul Erdös said that when Hardy was asked what was his greatest contribution to mathematics, he unhesitatingly said "The discovery of Ramanujan". Hardy told him that Ramanujan went far beyond his theorems.

Hardy once gave an estimation of mathematicians on the basis of pure talent on a scale of 1–100. Professor Erdös said that Hardy gave Ramanujan 100, 80 to the famous mathematician David Hilbert, 30 to colleague Littlewood, and only 25 to himself. "Although Hardy was modest in giving himself only 25, the fact that he gave 100 to Ramanujan revealed the regard he had for Ramanujan's work."

A man who lives by numbers, Prof. Erdös noted that Ramanujan was so multifaceted that no mathematician could fully decipher or comprehend all of his creations. "I wish I had a chance to meet Ramanujan. Unfortunately he died when I was seven." Incidentally Ramanujan's Centenary also coincided with the 75th birthday of Professor Erdös. His admirers are planning to celebrate it in a big way.

Hardy was of the opinion that education would not have made Ramanujan a greater mathematician; it might have stifled his genius. However Professor Erdös expressed the view that education would have proved a great deal more for Ramanujan. "He would not have wasted so much time rediscovering the work of other mathematicians."

M. Prakash
Reporter, The Hindu

Chapter 2
Ramanujan: The Second Century

When European mathematicians first came to know of Ramanujan's spectacular results during the early part of this century, they perceived him as a singular genius who produced numerous beautiful but mysterious identities. To a mathematician a result is mysterious if he is not able to understand it in terms of well-known theorems or see it as part of a general theory. Lacking formal education, Ramanujan was in no position to motivate his results or supply rigorous proofs. Even Professor G.H. Hardy could not fully understand many of these identities on infinite series and products. Although Hardy compared Ramanujan to Euler and Jacobi for sheer manipulative ability, he expressed the opinion that Ramanujan's results lacked the simplicity of the very greatest works. But during the last half a century, many of Ramanujan's identities have been studied in detail and put in proper perspective with respect to contemporary theories. Hence his results do not appear now to be quite that mysterious, and in fact by the time his centenary was celebrated, it became clear that his work compared well with those of the very greatest mathematicians. But the study of Ramanujan's formulae is by no means over. As Professor Atle Selberg of The Institute for Advanced Study, Princeton, remarked during the Ramanujan Centenary, it will take many more decades, possibly even more than a century, to completely understand Ramanujan's contributions. Mathematicians know well that Selberg is not given to hyperbole, and so this is very high praise! I will now describe some features of Ramanujan's work which continue to excite researchers today and will engage them in the near future.

Mock Theta Functions Ramanujan's work on mock theta functions is considered to be one of his deepest contributions. These results were discovered by him just before he died, and he communicated them to Hardy in his last letter dated January 1920. A major portion of The Lost Notebook is devoted to mock theta functions.

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In his letter Ramanujan listed several mock theta functions of orders three, five, and seven. Hardy passed on to Professor G.N. Watson the task of analysing Ramanujan's mock theta identities. Watson wrote two papers on this topic, the first of which was his presidential address to The London Mathematical Society entitled "The Final Problem: An Account of the Mock Theta Functions." Watson explained the choice of the title as follows: "I doubt whether a more suitable title could be found for it than used by John H. Watson, M.D., for what he imagined to be his final memoir on Sherlock Holmes." Watson's first paper (1936) dealt with mock theta functions of third order, and the second (1937) with those of fifth order. Watson did not consider the seventh-order functions, but these were investigated by Selberg in 1938. In the last two decades Professor George Andrews has analysed and explained combinatorially many of Ramanujan's mock theta identities. In collaboration with his former student Frank Garvan, Andrews was led to conjecture that some of Ramanujan's mock theta identities were equivalent to certain results on partitions (a partition of a positive integer n is a representation of n as a sum of positive integers not exceeding n). These were called the "Mock Theta Conjectures." These conjectures were settled by Dean Hickerson in 1989, after the Ramanujan Centenary. Just this year Andrews and Hickerson have completed the study of eleven identities of Ramanujan on sixth-order mock theta functions in The Lost Notebook. Another recent advance is the work of Henri Cohen who explained certain mock theta identities in the context of Algebraic Number Theory.

In spite of these breakthroughs, several fundamental questions remain. For instance, no one knows what Ramanujan meant by the "order" of a mock theta function. Ramanujan divided his list of functions into those of third, fifth, and seventh orders. Known identities indicate that these are related to the numbers 3, 5, and 7, but a precise definition of order is yet to be given. So for now, the order of a mock theta function is a convenient label which may or may not have deeper significance. Ramanujan had defined mock theta functions to be those satisfying two conditions. But no one has rigorously shown yet that any of these mock theta functions actually satisfy the second of Ramanujan's conditions. Also, in dealing with mock theta functions, special techniques have been used based on the specific function being discussed. There are attempts to find a unified approach to deal with mock theta functions like the theory of modular forms that is used in the study of theta functions.

Ramanujan's Congruences Some of the most surprising observations by Ramanujan concern congruences or divisibility properties for the partition function. Hardy had asked MacMahon to prepare a table of first two hundred values of the partition function using a certain formula of Euler. As soon as Ramanujan saw this table, he pointed out three congruences involving the primes 5, 7, and 11. The first congruence states that the number of partitions of an integer of the form $5n + 4$ is divisible by 5. For example, there are 30 partitions of 9, and 30 is divisible by 5. Hardy was simply stunned, because partitions represent an additive process, and so he did not expect such divisibility properties. MacMahon had prepared the table, and Hardy had checked it, but neither observed such a relation! Ramanujan had the eye for such connections, and this is an example of the element of surprise that is present throughout Ramanujan's work. Ramanujan generalised his congruences to the powers of 5, 7, and 11. Watson (1938) proved the congruences for the powers of 5 and (in a slightly modified form) for the powers of 7; the congruences involving the powers of 11 were established later by Atkin.

In 1944, Freeman Dyson, then a young student at Cambridge University, conjectured a combinatorial explanation for the congruences involving the primes 5 and 7 using the concept of "rank" for partitions. Dyson published his conjecture in "Eureka", a Cambridge student journal. The Dyson rank conjectures were proved in 1954 by A.O.L. Atkin and H.P.F. Swinnerton-Dyer using the theory of modular forms. Dyson had pointed out that the rank does not explain the third (and deeper) congruence involving the prime 11, but he conjectured the existence of a statistic, which he called the "crank", that would explain the third congruence combinatorially. But he had no idea of what the crank would be. Freeman Dyson has humorously remarked that this was the only instance in mathematics when an object had been named before it had been found! The crank sought by Dyson was found in 1987, one day after the Ramanujan centenary conference in Urbana, Illinois, by Andrews and Garvan. The solution was based on Garvan's Ph.D. thesis at Pennsylvania State University.

Whenever Ramanujan pointed out a relation, it was usually one of many that existed, and often the most striking among those. It has been shown that the coefficients in the expansions of various modular forms satisfy such congruences. During the last two years, Garvan, who is now at the University of Florida, has developed the idea of the crank to combinatorially prove and explain many such congruences. Also, Garvan, Kim and Stanton have applied ideas from Group Theory to explain deeper congruences. Thus the study of Ramanujan-type congruences will continue to be an active line of research in the future.

Rogers–Ramanujan Identities This pair of identities (discovered independently by Rogers in 1894 and Ramanujan around 1910) are considered among the most beautiful in mathematics. The combinatorial description of the first identity is that the number of partitions of an integer n into parts differing by at least two equals the number of partitions of n into parts which when divided by 5 leave remainder 1 or 4. The second identity has a similar description. The simplicity of the identities belies their depth. Several proofs have been given, but none can be considered simple or straightforward. In an attempt to understand these identities, a rich theory has developed, concerning, on the one hand, partitions whose parts satisfy gap conditions and, on the other, partitions whose parts satisfy congruence conditions. For a quarter century beginning around 1960, considerable work has been done in this direction, especially by Professor Basil Gordon of the University of California, Los Angeles, and by Professors George Andrews and David Bressoud of Pennsylvania State University.

Rogers actually found several elegant companions to the Rogers–Ramanujan identities. Fundamental discoveries always find applications eventually. In 1979, the Australian mathematical physicist Rodney Baxter showed that the Rogers–Ramanujan identities and these companions are the solutions to the Hard Hexagon

Model in Statistical Mechanics. For this work, Professor Baxter was awarded the Boltzman medal of the American Physical Society. In recent years, identities of the Rogers–Ramanujan type have found more applications to problems in mathematical physics.

Ramanujan considered the Rogers–Ramanujan identities as arising out of a continued fraction possessing a product representation. It was Ramanujan's insight to have realised the importance of this continued fraction in the theory of modular forms. This is only one of many continued fractions studied by Ramanujan, but perhaps the most appealing. Professor Bruce Berndt of The University of Illinois has analysed several continued fractions of Ramanujan. These continued fractions can be approached in various ways and offer a wide range of problems for exploration.

Products of the Ramanujan type established for this continued fraction are of interest in themselves. In 1980, Andrews and Bressoud showed that there was a pattern among the coefficients of certain Rogers–Ramanujan-type products that had value zero. Professor Gordon and I have recently extended these results to general Rogers–Ramanujan-type products, and there is scope for more work in this area.

Special Functions Ramanujan wrote down several beautiful formulae involving various special functions (like the Beta and Gamma functions). For the past two decades, Professor Richard Askey of the University of Wisconsin, with his students and co-workers, systematically studied q-analogues of various special functions. In the course of this study, many of Ramanujan's identities found in his original notebooks and in the Lost Notebook were extremely useful. Many of the q-analogues found by Askey and others are now finding important applications in Physics, through the idea of Quantum Groups.

The Notebooks When Bruce Berndt began editing the notebooks of Ramanujan, he envisaged publishing three volumes. Springer-Verlag has brought out three volumes, but the work is not over yet. Professor Berndt has almost completed work on the fourth volume, and there will be a fifth! This clearly demonstrates the depth and scope of Ramanujan's contributions. It is now possible to offer courses on Ramanujan's work since much of his work has been edited and books available on the subject. Thus a greater number of bright students will take to a study of Ramanujan's formulae in the decades to follow. In this connection, Robert Kanigel's recent book "The man who knew infinity" will open the eyes of the general public to the wonder that Ramanujan was.

The Lost Notebook It was George Andrews who discovered the Lost Notebook in 1976 at the Wren Library in Cambridge University. Since then, he has analysed hundreds of incredible identities contained in it and published several papers on them, most notably in the journal Advances in Mathematics. On 22 December 1987, Ramanujan's hundredth birthday, the printed version of the Lost Notebook was released. Professors Andrews and Berndt are planning an edited version of the Lost Notebook, much like Berndt's edited version of the original notebooks. This project will have great impact in the coming decades.

The Undying Magic In closing we emphasise certain features about Ramanujan's mathematics.

Ramanujan had the knack of spotting seemingly unexpected relations. Thus there is always an element of surprise for someone who studies his work. Quite often, a closer analysis reveals that there are many more such relations and that Ramanujan was pointing out only the most striking cases. So, one begins to suspect whether Ramanujan had a method to generate such relations. In an effort to find such methods, interesting theories emerge, sometimes leading to connections between different areas of mathematics. Some of the most intriguing connections recently found are between root systems of Lie algebras and the theory of q-series and modular forms. This is the work of Professors V.G. Kac, I.G. MacDonald, and D.H. Peterson. Such connections not only enrich the two areas but also offer several fruitful research projects.

Ramanujan's mathematics remains youthful even in the modern world of the computer. His modular equations were used by Canadians Jonathan and Peter Borwein to calculate π (the ratio of the circumference of a circle to its diameter) to several million decimal places. The Borweins showed that these modular equations produce efficient algorithms to obtain approximations to π and other numbers. More recently, Ramanujan's transformations for elliptic functions were used by David and Gregory Chudnovsky to produce very rapidly convergent algorithms to compute π; in fact the Chudnovskys have now calculated π to the order of about a billion digits!

Finally there is a lasting quality about Ramanujan's mathematics and about fundamental research in general. In the mid-eighteenth century, the British mathematician Stirling did calculations to produce the table of logarithms. With the advent of modern computers, such tables are among the least useful possessions of a library. But the mathematics that went into the construction of such tables never loses its lustre. Indeed, Stirling had developed methods for the acceleration of convergence of series, and this has been the basis for William Gosper's recent program to generate identities using computer algebra packages like MACSYMA. Motivated by Gosper's ideas, Ira Gessel and Dennis Stanton have used q-Lagrangian inversion to generate many identities of the Rogers–Ramanujan type.

Thus Ramanujan has left behind enough ideas to keep mathematicians busy well into the twenty-first century. Professor Dyson has remarked that we should be grateful to Ramanujan not only for discovering so much but also for providing others plenty to discover! What sets Ramanujan apart from the rest of the mathematical giants is that feeling of astonishment he creates with his stunningly beautiful identities. Ramanujan is like a gem with many faces. His identities can be studied from different viewpoints. Each face of this gem dazzles the beholder with its array of colours!

Chapter 3
L.J. Rogers: A Contemporary of Ramanujan

Rogers

Most of us associate the name of L.J. Rogers with the celebrated Rogers–Ramanujan identities and rightly so, because these identities which are unmatched in simplicity of form, elegance and depth, are representative of the best discoveries

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by both these mathematicians. Although Rogers had proved these identities in 1894 nearly twenty years before Ramanujan discovered them, his work was neglected even by his British peers. Indeed, it was only after Ramanujan's rediscovery of Rogers' paper in 1917 that Rogers received belated recognition leading eventually to his election as Fellow of The Royal Society (F.R.S.) in 1924. In spite of this, the mathematical world remained largely unaware of the true significance of Rogers' work. We owe primarily to George Andrews and Richard Askey our present understanding of the range of Rogers' contributions to the theory of q-series and special functions.

L.J. Rogers was a first rate mathematician and a man of many talents ranging from music to linguistics. In Hardy's own admission, Rogers was a mathematician whose talents in the manipulation of series were not unlike Ramanujan's. For sheer manipulative ability, Ramanujan had no rival, except for Euler and Jacobi of an earlier era. But if there was one mathematician in Ramanujan's time who came closest to the Indian genius in his mastery over infinite series and products, it was Rogers. In this article I will describe some of the mathematical contributions of Rogers, their significance and impact on current problems and how they relate to Ramanujan's work. I first describe the fascinating personality of this multi-talented man. This article would not have been possible without the help of Professor George Andrews of Pennsylvania State University, who provided me with several documents relating to Rogers including the Obituary Notices of the Royal Society of 1934.

Man of Many Talents Leonard James Rogers was born in Oxford, England, on 30 March, 1862. His father J.E. Thorold Rogers was a well-known Professor of Economics. In his childhood Rogers had a serious illness and although he recovered completely, he was not sent to school. J. Griffith, an Oxford mathematician, noticed that Rogers had superior mathematical ability and taught him in his boyhood. Rogers had a brilliant undergraduate career at Oxford University. In 1888 an independent Chair of Mathematics was created at Yorkshire College (now the University of Leeds), and Rogers was appointed Professor. He held that position with distinction until 1919 when ill-health forced him to retire prematurely. In 1921 he returned to Oxford, where he lived in retirement until his death on 12 September 1933 at the age of 71.

Rogers was tall, loose limbed and a rather gaunt figure. He was bespectacled and had a beard. He was careless of his appearance and said that his drab clothes were in keeping with his complexion.

Rogers was extra-ordinarily gifted and indeed a genius. His interests extended well beyond mathematics into music, languages, phonetics, skating and even rock-gardens! In this sense he was very different from Ramanujan who had few interests outside of his obsessive love of mathematics. Whatever Rogers studied, he not only acquired a full knowledge of it, but had enough mastery to have it at his fingertips. Combined with this was his whimsical wit and ironic humour, heightened by his ability to keep a serious countenance. With such combination of talent and humour, he was in constant demand at various social gatherings.

Of his varied talents, special mention must be made of his ability to play the piano, helped by his long and nimble fingers. He was a multi-linguist and could speak French, German, Italian and Spanish fluently. Some have attributed this to his interest in phonetics, and the study of various dialects gave him the opportunity of exercising his wonderful ear to the differences of sound. His students enjoyed listening to his lectures, and to some the boredom of mathematical calculations was relieved by his sparkling sense of humour.

Although he was admired and respected by those around him for his many-sided brilliance, he complained that people were ignorant of his real interest, namely, mathematics. Even British mathematicians of his day paid little attention to his papers, and it was not until Ramanujan drew attention to Rogers' work in 1917 that Hardy realised the fundamental nature of Rogers' contributions. Rogers was then conferred the Fellowship of The Royal Society in 1924. In spite of this recognition, in an obituary published in Nature in 1933, it was said that apart from the Rogers–Ramanujan identities, he found little of mathematical value. The writer even expressed the opinion that had not Rogers wasted his time with his other interests but approached mathematics with a single minded purpose, he would have achieved a good deal more and therefore could have been considered a success in life. The research of Andrews in the realm of q-series and that of Askey on special functions have demonstrated that these opinions on Rogers were far from the truth. In fact, in his fundamental papers Rogers had anticipated the discoveries of many noted mathematicians.

The Rogers–Ramanujan Identities During 1893–1895, Rogers published three memoirs in The Proceedings of The London Mathematical Society on the expansion of certain infinite products. In these papers the Rogers–Ramanujan identities and several related results are proved. Ramanujan discovered these two identities in India between 1910 and 1913 and communicated them in letters to Hardy. Ramanujan did not have a proof of these identities and could not supply one when asked by Hardy. Neither Hardy nor his British colleagues had any idea how to approach these identities. The combinatorial version of the first identity is as follows: *The number of partitions of a positive integer into parts which differ by at least 2 is equal to the number of partitions of that integer into parts which when divided by 5 leave remainder 1 or 4.* The second identity has a similar description. This combinatorial description is due to MacMahon and Schur. Neither Rogers nor Ramanujan viewed these identities in terms of partitions.

In 1917, while going through some old issues of the Proceedings of The London Mathematical Society, Ramanujan came across Rogers' papers accidentally. Hardy said that Ramanujan expressed great appreciation for the work of Rogers. A correspondence between Rogers and Ramanujan followed, resulting in a considerable simplification of the proof. At about the same time, the German mathematician I. Schur, who was cut off from England by World War I, discovered these identities independently.

Rogers alluded to his re-emergence ironically, as was characteristic of him, in a letter to F.H. Jackson dated 13 February 1917: " It was with a certain amusement that a theorem which I proved nearly 24 years ago should have remained in obscurity so long and recently brought into prominence as a conjecture. MacMahon wrote to me on the publication of his book regretting that he overlooked my work before it was too late. Since then, I have other ways of proving both identities in a more direct way … .”

In the last few decades several proofs of the Rogers–Ramanujan identities have been given. A major advance was made by Basil Gordon (University of California, Los Angeles) in 1961, who produced an elegant generalisation. Spurred by this, Andrews made great progress and discussed a whole class of related identities. Yet, none of these proofs of the Rogers–Ramanujan identities are simple. In some sense, the simplest proof so far is the one due to David Bressoud (Pennsylvania State University) in 1983. Although the identities have a combinatorial interpretation, no simple combinatorial proof is available converting partitions of one type to another. A combinatorial proof was given by Garsia and Milne in 1981, but it runs to 50 printed pages! Soon after, a shorter combinatorial bijective proof was given by Zeilberger and Bressoud in 1982.

The Hard Hexagon Model In 1979, the Australian mathematical physicist Rodney Baxter was working on a problem in Statistical Mechanics concerning the behaviour of liquid helium over a graphite plate. The Rogers–Ramanujan identities arose as one set of solutions to the model he considered. While struggling to understand these identities he found a proof. Baxter then worked out the full set of solutions and six other companion identities came up. He then contacted George Andrews, the leading authority on this subject, who pointed out that all these identities were contained in Rogers' papers of 1893–1895. A fruitful collaboration between Andrews and Baxter followed, and they obtained significant extensions of Baxter's original model. Thus Rogers' work found application to a problem in Physics nearly a century later. For this work, Baxter was awarded The Boltzman Medal of The American Physical Society. In a series of lectures given in Tempe, Arizona, in 1985, jointly sponsored by The American Mathematical Society and The National Science Foundation of U.S.A., Andrews discussed the eight identities of Rogers and their role in Baxter's Hard Hexagon Model.

Of these six companion identities of Rogers, two of them bear remarkable resemblance to the original Rogers–Ramanujan identities. Andrews and Askey asked for a combinatorial explanation of these two identities of Rogers. An elegant explanation was found by Bressoud in 1978, and these ideas led him to provide combinatorial insights into many partition problems.

The Mechanism Rogers had beaten Ramanujan in his own game by proving two identities that Ramanujan could not! What was the mechanism behind his proof that worked so well? There were several ideas. Rogers viewed these identities as coming out of a process of comparing coefficients in certain general expansions. He considered various combinations of sine and cosine, the trigonometric functions, and compared expansions obtained in two different ways. Such tricks were right up Ramanujan's alley but had escaped his attention. These ideas of Rogers have proved extremely fruitful and have led to the discovery of many partition identities.

W.N. Bailey pursued certain ideas of Rogers and invented a powerful technique now known as The Bailey Chain. But Bailey did not realise the true potential of this technique which has now become the standard process to generate identities of Rogers–Ramanujan type. In two papers which appeared in The Proceedings of The London Mathematical Society (1951–1952), Lucy Slater carried out Bailey's programme and listed over one-hundred such identities.

Anticipated Others The Rogers–Ramanujan identities were not the only instance when Rogers anticipated the work of other great mathematicians. In their famous book "Inequalities" published by Cambridge University Press in 1934, Hardy, Littlewood and Polya observed that Rogers had discovered Hölder's inequality in 1888 one year prior to Hölder. In 1936, Atle Selberg (now at The Institute for Advanced Study, Princeton) discovered certain identities related to the number 7 just as the Rogers–Ramanujan identities are related to the number 5. Freeman Dyson (also at Princeton now) noticed in 1943 that these identities were buried in Rogers' papers of 1895 and 1917.

In his third memoir of 1895, Rogers had considered certain extensions of the famous Hermite polynomials. In 1978, Richard Askey and Mourad Ismail made a systematic study of various results on Special Functions in this memoir and pointed out that Rogers had anticipated the work of many noted mathematicians in this important area.

Invariant Theory During 1886–1887, Rogers wrote four papers in The Theory of Invariants, a subject dealing with algebraic expressions which remain invariant under certain transformations. Such invariance is of great interest to physicists, especially to those working in Relativity. In MacMahon's words, "The theory of invariants sprang into existence under the strong hand of Cayley, but that it emerged finally into a complete work of art for the admiration of future generations of mathematicians, was largely owing to the flashes of inspiration with which Sylvester's intellect illuminated it." Rogers' papers on Invariant Theory published in 1886 formed the main topic of lectures given that year by Sylvester, who was Savillian Professor of Geometry at Oxford University. Invariant Theory was one area where Rogers' work drew immediate attention.

False Theta Functions As is now well known, the mock theta functions are among Ramanujan's deepest contributions. In his last letter to Hardy dated January 1920, Ramanujan said that he had discovered a new class of functions called the mock theta functions which arise more naturally than the false theta functions of Rogers. While it is true that Ramanujan's discoveries on mock theta functions are deeper, the false theta identities of Rogers have beautiful partition theoretic interpretation as was shown by Andrews in his 1979 New Zealand lectures.

Continuing Rediscovery Thus the rediscovery of the mathematical contributions of Rogers is continuing, and he is finally getting the recognition he deserved. Rogers published 35 papers spanning q-series, invariant theory and elliptic function theory. These papers contain important ideas which have sown the seeds of many developments in the succeeding decades.

It would be fair to say that had not Ramanujan unearthed the Rogers papers of 1893–1895, Rogers would not have been elected Fellow of The Royal Society. India owed a great debt to Cambridge University and to Professor Hardy in particular, for encouraging Ramanujan. And this debt has been repaid with Ramanujan's rediscovery of the work of Rogers, thereby showing the British mathematical community that they had a mathematician among them whom they needed to recognise!

Chapter 4
P.A. MacMahon: Ramanujan's Distinguished Contemporary

MacMahon

Percy Alexander MacMahon was an unusual and noteworthy British mathematician, unusual because he had a distinguished career both in mathematics and in the military. Well known as Major MacMahon, he made fundamental contributions to combinatorics and the theory of algebraic forms and received several honours from

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The Royal Society and The London Mathematical Society. He is best remembered for his work on symmetric functions, permutations and partitions. MacMahon's investigations in the theory of partitions not only brought him into contact with Srinivasa Ramanujan but, had a direct effect on some of the brilliant contributions of the Indian genius. In this article I will first describe briefly MacMahon's life in the military owing to connections with India. Next I will discuss some of his major contributions to mathematics and finally emphasise those which have connections with Ramanujan. For biographical facts, I have benefitted from two obituaries of MacMahon, one by H.F. Baker which appeared in 1930 in the Proceedings of The London Mathematical Society, and another by H.H. Turner presented to the Royal Astronomical Society of which MacMahon was president. I have also referred to the Collected Works of MacMahon which have been edited by George Andrews. Finally, I have profited from several stimulating conversations with Professor George Andrews of The Pennsylvania State University and comments from Professor Richard Askey of the University of Wisconsin.

Military Career and India P.A. MacMahon was born in Malta on 26 September 1854 as the second son of Brigadier General P.W. MacMahon. He went to school in Cheltenham. Following in his father's footsteps, he entered the Royal Military Academy in Woolwich in 1871 as a gentleman cadet. He was swiftly made a Lieutenant and posted in the 5th Brigade at St. Thomas Mount in Madras. His stay in Madras was brief because he was transferred to the 8th Brigade in Lucknow soon after. He was again transferred to Meerut in the Punjab but there he stayed for three years. In 1877 he was posted in the Northwestern Frontier Force in Kohat in the Punjab. For his service in the military, he was promoted to the rank of Captain in 1881 and Major in 1889, the title by which he was known since then even in the mathematical world. In 1882 MacMahon returned to his alma mater, The Royal Military Academy, as an Instructor in Mathematics, a post that he held until 1888. He was then appointed as Professor of Physics at The Royal Artillery College in Woolwich, a position that he held with distinction until 1898. From 1906 to 1920 he was Deputy Warden of Standards under the Board of Trade.

Contributions to Mathematics Ever since his youth, MacMahon was absorbed by mathematics. But it was his contact in 1882 with Professor George Greenhill of The Royal Artillery College that changed his whole life for he was exposed to the theory of algebraic forms which at that time was undergoing a full flight of development under masters like Cayley and Sylvester. (Incidentally, Greenhill may have also had an effect on Ramanujan, because according to Hardy, Greenhill's book on elliptic functions was perhaps the source in India from which Ramanujan acquired a knowledge of that wonderful subject). To quote Sir Joseph Larmor, "MacMahon threw himself with indomitable zeal and insight into the great problems of this rising edifice of science. In a short time he was being counted as conspicuous among the leaders largely by the invention of new methods." MacMahon's great strength was his mastery over combinatorial techniques. In particular MacMahon made fundamental contributions to the theory of permutations and the theory of symmetric functions with which it was intimately connected.

In simple terms, a *permutation* of a set is a rearrangement of its elements, and *symmetric functions* are those whose values do not change under permutations of the variables. Permutations and symmetric functions occur in a wide variety of settings in different areas of mathematics. In the theory of probability, which often deals with games of chance, one is always interested in the number of possible outcomes. Permutations often dominate such calculations. In linear algebra, while solving systems of equations, the *determinant* is of paramount importance. As is well known, a system of n linear equations in n unknowns has a unique solution if and only if the determinant of the matrix of coefficients is non-zero. And permutations play a crucial role in the definition of the determinant of a square matrix. As is known to most students of mathematics, symmetric functions arise while solving polynomial equations. More precisely, the coefficients of the polynomial can be expressed in terms of various symmetric functions involving the roots of the equation. In a paper that appeared in the Proceedings of The London Mathematical Society in 1884, MacMahon obtained an elegant extension of a famous formula of Newton involving symmetric functions and the powers of the roots of a polynomial equation.

MacMahon's influence can be seen throughout the theory of permutations. He studied special types of permutations such as those called *derangements*. These are permutations in which every object is in the wrong position. For example, consider n persons going to a restaurant and each turning in his hat at the entrance. When the guests are ready to leave, the clerk returns the hats at random. The question now is in how many ways can the hats be returned such that no guest receives his hat. What is asked here is the number of derangements of n objects. In a paper of 1912 that appeared in the Transactions of The Cambridge Philosophical Society, MacMahon obtained several pretty formulas involving derangements.

MacMahon realised that it was important to study indices of permutations. Roughly speaking an index of a permutation is a measure of how much things have gone wrong in the rearrangement. In a massive paper that appeared in 1913 in The American Journal of Mathematics, MacMahon studied a variety of indices: the greater index, the lesser index, etc. For example, in the permutation 2, 1, 3, 6, 5, 4 of the integers 1, 2, 3, 4, 5, 6, consider those pairs of consecutive terms (k, l) in the permutation for which k is greater than l and k comes before l. In the above example, the required pairs are $(2, 1)$, $(6, 5)$ and $(5, 4)$. For each such pair, note the position where the first entry k occurs and add up the values of these positions. That gives the *greater index* of the permutation. In the above example, the greater index is $1 + 4 + 5 = 10$. MacMahon established a variety of important results on the indices of permutations. In particular, he showed that the generating function of the greater index is the same as the generating function of the number of inversions, both being equal to the *q-multinomial coefficient*. MacMahon was the first to study systematically the combinatorial properties of the q-multinomial coefficients and in particular the *q-binomial coefficient*. Today we understand the q-binomial coefficient as arising out of the expansion of $(x+y)^n$, where the variables x and y satisfy the q-commutation relation $yx = qxy$. Such variations in the familiar commutative law of multiplication are of great interest to physicists, and conditions such as the q-commutation relation are the type one encounters in the study of *Quantum Groups* which play an important role in modern physics.

The results on derangements as well as other results on permutations were all special cases of a rather general result that MacMahon called as the *Master Theorem*, which he proved in his paper of 1912 in The Philosophical Transactions of The Royal Society of London. The master theorem asserts the equality of the coefficients of monomials in the expansion of certain products to the coefficients that occur in the expansion of the reciprocal of a certain determinant. MacMahon himself demonstrated the usefulness of this in two memoirs. Subsequently several proofs of the master theorem have been given the most important in recent years being that of Dominique Foata of the University of Strasbourg in France, who explained its real combinatorial significance.

MacMahon's results on symmetric functions were extensive, and the great majority of his papers relate to them in some way. The basic tool that he used in studying symmetric functions were the Hammond operators. One of the active areas of research today is algebraic combinatorics, and two of the most prominent mathematicians in this field currently are Professors Gian-Carlo Rota and Richard Stanley of the Massachusetts Institute of Technology. The widespread influence of MacMahon's research can be seen in algebraic combinatorics while studying group representations or the Young tableaux.

In 1915 and 1916, Cambridge University Press published his book on *Combinatory Analysis* in two volumes. These volumes contain an extensive treatment of symmetric functions, partitions, compositions, and his work on the Master Theorem and on the enumeration of multipartite numbers.

As professor of physics at the Artillery College in the 1890s, it is not surprising that MacMahon was interested in problems in physics as well. In particular MacMahon published a paper in 1909 in the London Astronomical Society Monthly Notices dealing with the determination of the apparent diameter of a fixed star. This was his only paper in Astronomy. The great British cosmologist Sir Arthur Eddington criticised MacMahon's technique, and this held back its application for quite a long time. We know now that MacMahon's technique was a good one and that he had been correct even in the details. At this juncture it may be worth pointing out that Eddington was also very critical of Chandrasekhar's idea that massive stars would eventually contract under the influence of their own gravity. As it turned out, not only was Chandrasekhar correct, but his ideas became the starting point for the study of neutron stars and black holes.

For his significant contributions to mathematics, MacMahon received several honours. He was conferred Fellowship of The Royal Society (FRS) in 1890. He was elected as President of The London Mathematical Society for a two-year term in 1894. He served as President of The Royal Astronomical Society in 1917–1918. In 1900, The Royal Society first gave him the Royal Medal and later in 1919, the Sylvester Medal. In addition he received the de Morgan Medal of The London Mathematical Society in 1923. In his retiring Presidential address to The London Mathematical Society he gave a superb review of the progress in Combinatory

Analysis during the second half of the nineteenth century. He died in Bognor on Christmas day in 1929.

Connection with Ramanujan It was MacMahon's work in the theory of partitions that brought him into contact with Ramanujan. A *partition* of an integer is a representation of that integer as a sum of positive integers, two partitions being considered the same if they differ only in the order of the parts. Thus we do not distinguish between $2 + 1 + 1$ and $1 + 1 + 2$ as partitions of 4. So the five partitions of 4 are 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$. If we make a distinction between two representations having the same parts but occurring in different order, then we are dealing with *compositions*. MacMahon studied both partitions and compositions combinatorially. Whereas MacMahon used combinatorial techniques, Ramanujan employed analytic methods. MacMahon was very quick in computation and was equal to Ramanujan in this regard. The number of partitions of an integer n, denoted by $p(n)$, grows quite rapidly, and MacMahon used a recurrence formula of Euler involving the pentagonal numbers to compute the first two hundred values of the partition function. This computation of MacMahon had a direct effect on Ramanujan's work as we shall see presently.

The Hardy–Ramanujan Formula The asymptotic formula obtained by Hardy and Ramanujan for the number of partitions of an integer is the finest example of what was achieved when the brilliance of Ramanujan combined with the sophistication of Hardy. Ramanujan realised even in India prior to his departure to England that there are analytic series representations for $p(n)$ and other related functions. It really stunned Hardy when Ramanujan asserted that such a series would give the exact value of $p(n)$. Hardy initially disbelieved this claim because $p(n)$ being a function on the integers has jumps, whereas the series that Ramanujan wrote down involved continuous functions. Ramanujan's spectacular discoveries are full of unexpected relations such as this. But Ramanujan had no proof of this claim. What was required was an ingenious use of the theory of functions of a complex variable, and here the mastery of Hardy over such sophisticated techniques was crucial. Hardy and Ramanujan obtained a series representation for the number of partitions of an integer n by an entirely new technique called the *circle method*. This method was later improved substantially by Hardy and his distinguished colleague Littlewood and is the standard technique used now in a wide variety of additive problems in number theory. Hardy and Ramanujan wanted to check their formula numerically, and for this, MacMahon's table of values for $p(n)$ proved useful. The value $p(200) = 3972999029388$ given by the series coincided with the value that MacMahon had calculated using Euler's recurrence. Actually a significant improvement over the Hardy–Ramanujan formula for $p(n)$ was obtained subsequently by Hans Rademacher, but I will discuss this in a separate article devoted to the life and contributions of Rademacher and highlight connections with Ramanujan's work.

The Ramanujan Congruences MacMahon's table of values of the partition function had another effect on Ramanujan's work, this one being much more

Chapter 5
Fermat and Ramanujan: A Comparison

Fermat

Although Pierre Fermat (1601–1665), one of the founding fathers of Number Theory, and Srinivasa Ramanujan (1887–1920), the legendary Indian genius, are separated by centuries, there are many similarities between the two in style and substance. An important part of the legacy of both Fermat and Ramanujan are

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their many observations recorded informally which have inspired several succeeding generations of mathematicians. Fermat's Last Theorem, which until recently was the most famous unsolved problem in mathematics, was just a marginal entry made by Fermat in a book written by Bachet on the work of the Greek mathematician Diophantus. The statement of the Last Theorem is that for any integer n greater than 2, there are no positive nth powers which are the sum of two other positive nth powers. Fermat claimed that he had a truly marvelous proof of this assertion but unfortunately the margin was too small to contain it! This naturally added a spirit of intrigue to the problem. The simplicity of Fermat's Last Theorem belies its depth and difficulty. For the last three centuries, mathematicians both amateur and professional, have attempted to find a proof of Fermat's assertion but did not succeed. These attempts however did yield plenty of new techniques which have proved immensely useful elsewhere. For example, the subject Algebraic Number Theory was born owing to efforts by Kummer and others to understand the unique factorisation property in very general settings, being motivated to study this question while attempting to solve Fermat's Last Theorem. The proof of Fermat's Last Theorem announced by Andrew Wiles in June 1993 and now completed by Wiles and Taylor is the culmination of years of effort by many illustrious mathematicians and is the result of the fusion of the Theory of Elliptic Curves and Number Theory. In a similar vein, Ramanujan's incomplete entries in his two notebooks and in the Lost Notebook have engaged mathematicians since the beginning of this century. Several branches of mathematics such as Number Theory, The Theory of Partitions, The Theory of Modular Forms, The Theory of Elliptic and Theta Functions, Hyper-Geometric Series and others have been enriched out of attempts to understand Ramanujan's jottings.

Both Fermat and Ramanujan communicated their wonderful findings in letters. Ramanujan wrote letters to mathematicians in England, desperately seeking recognition for his work. Fortunately, Hardy responded favourably. Fermat communicated regularly with his French peers Pascal and Mersenne among others, as well as with British mathematicians. Fermat's challenge to the British mathematicians was what ultimately led to the complete solution of what is now known as Pell's equation. The name Pell's equation is due to Euler who was under the mistaken impression that the British mathematician Pell had done most of the work on this; but we know now that several centuries earlier, the Indian mathematicians Bhaskara and Brahmagupta had made significant progress on such questions.

There are many number theoretic problems which interested both Fermat and Ramanujan. Fermat stated that every positive integer is a sum of no more than three triangular numbers, four squares, five pentagonal numbers, and so on. Special types of numbers like these had been of interest since the days of the Pythagoreans, but no one before Fermat had made such a fundamental observation about them. This assertion of Fermat attracted the attention of many outstanding mathematicians. Gauss gave a proof of the statement that every positive integer is a sum of no more than three triangular numbers, while Lagrange proved the assertion about sums of four squares. The general assertion that every positive integer is a sum of n or fewer n-gonal numbers was established by Cauchy.

Ramanujan was also interested in the representation of integers as sums of squares. But he viewed it from an entirely different angle, in terms of infinite series identities for theta functions. Although Ramanujan seldom stated number-theoretic forms of his identities, such interpretations were natural consequences of his results.

Just as Carr's Synopsis, the first book to make an impression on Ramanujan, so strongly influenced his style of writing, Bachet's Diophantus was the book that dominated Fermat's mathematical life. And it was in this book that Fermat copiously made marginal notes, commenting on possible extensions and improvements of results contained therein. In particular, Pythagorean triangles fascinated Fermat.

It is a fact well known to all high school students that if x, y and z denote the lengths of the three sides of a right-angled triangle with z being the length of the hypotenuse, then $x^2 + y^2 = z^2$. Pythagorean triangles are those where the x, y and z are integers without a common factor. An example is the triangle with sides 3, 4, and 5. Another example is provided by the triple 5, 12 and 13. There are infinitely many Pythagorean triangles, and a formula to generate all of them has been known since antiquity and can be found in Bachet's book. Fermat was interested in finding whether there were any Pythagorean triangles whose area was also a square, and this problem was not discussed by Diophantus or Bachet. Fermat proved by the *method of infinite descent* that such triangles could not exist. The key idea in the method of infinite descent is to show that any positive solution will generate a smaller such solution. By iteration, an infinite sequence of decreasing positive integer solutions would be generated, and this is clearly impossible.

Fermat was not the first person to investigate areas of Pythagorean triangles. As early as the tenth century A.D., the Arabs were interested in determining those numbers which arise as areas of Pythagorean triangles. Fermat proved that squares cannot be areas of Pythagorean triangles. This apparently idle question as to which rational numbers can be realised as areas of right-angled triangles with rational sides has been shown to have significant implications in the modern theory of elliptic curves.

With the exception of The Last Theorem, Fermat is most famous for his method of infinite descent. Fermat used the method of infinite descent to show that various equations do not have solutions among the positive integers, and perhaps believed that this method could be used to give a "truly marvelous proof" of the Last Theorem. In the course of proving that it is impossible to have Pythagorean triangles whose area is a square, Fermat showed that $x^4 + y^4 = z^4$ has no positive integer solutions. He then observed that he could also apply the method of descent on the equation $x^3 + y^3 = z^3$, which is the first case of The Last Theorem. But it was left to Euler to supply the arguments for this case.

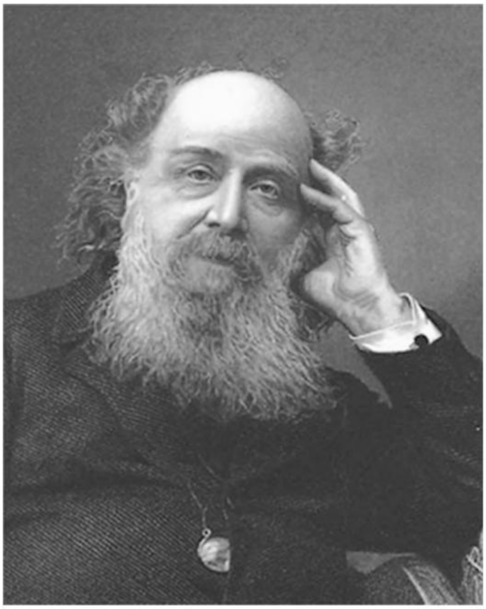
In contrast, Ramanujan was interested mainly in equations which had solutions, and especially providing algorithms or formulas for the solutions. The famous Ramanujan taxi-cab number 1729 is a solution to what appears like a mild variation of the Fermat equation. Indeed 1729 is interesting because $1729 = 12^3 + 1^3 = 10^3 + 9^3$, and this is a solution to $x^3 + y^3 = z^3 + w^3$. In other words, addition of an extra variable w to the Fermat equation provides a solution. Euler had provided a formula for the general solution to this equation, but in Ramanujan's third notebook there is also a formula for the general solution to the taxi-cab equation which is in some ways more elegant than Euler's.

During the Ramanujan Centennial, Atle Selberg of the Institute for Advanced Study at Princeton pointed out that raising fundamental questions is just as important as solving long-standing problems. Both Fermat and Ramanujan have raised several important questions which have engaged mathematicians of the highest calibre in the decades that followed. Fermat's assertions interested Euler, who systematically supplied proofs for many of them. And in doing so, Euler noticed improvements and generalisations. Gauss then took over where Euler had left off. Number Theory as it is taught today, is based on Gauss' *Disquisitiones Arithmeticae*, which is the finished product of the foundation laid by Fermat and the structure erected by Euler. Ramanujan's observations have been food for thought since the beginning of this century. At first, Hardy, Watson and Mordell supplied proofs and explanations for many of Ramanujan's observations. In the past few decades, the work of Erdös, Selberg, Deligne, Askey, Andrews, Berndt and others have revealed the grandeur of Ramanujan's discoveries. However, a complete understanding of Ramanujan's writings will take many decades, possibly more than a century.

The importance of the work that has been generated by a study of the writings of Fermat and Ramanujan can be judged by the kind of recognition that the international mathematical community has given to such efforts. In 1986, Gerd Faltings of Germany was awarded the Fields Medal (the equivalent of the Nobel Prize in mathematics for prestige) for proving the Mordell–Weil conjecture. From this work of Faltings it followed that every Fermat equation had at most a finite number of solutions. Faltings' methods have now led to the creation of a new field known as Arithmetic Geometry. Similarly, Andrew Wiles' proof of Fermat's Last Theorem contains many new ideas which will be developed to yield more results in the future. The most famous of Ramanujan's problems was what Hardy called *The Ramanujan Hypothesis* concerning the size of Ramanujan's tau function. Pierre Deligne was honoured with the Fields Medal in 1978 for proving the Ramanujan hypothesis.

Freeman Dyson of The Institute for Advanced Study in Princeton has said that we should be thankful to Ramanujan for not only discovering so much, but also leaving plenty for others to discover! Similarly, we should be grateful to Fermat for raising questions which have kept mathematicians busy for the past three centuries.

Chapter 6
J.J. Sylvester: Ramanujan's Illustrious Predecessor



Sylvester

The Theory of Partitions, an active area of research today, has gone through four major periods of development. The first era is that of Euler who founded the subject in the 18th century. The second period in the latter half of the 19th century is that of

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paper dealt with complex roots of certain algebraic equations. This problem was first considered by Newton and had defeated even giants like Euler, but Sylvester was successful with it. Sylvester also worked on partitions during this period, prior to his stay at Johns Hopkins. MacMahon observes that in June/July 1859, Sylvester delivered a series of seven lectures on partitions at Kings College, London. Printed outlines of these lectures were distributed to the members of the audience and some others. Much later, in 1897, The London Mathematical Society published these lecture outlines, and they attracted considerable attention.

Unfortunately, at Woolwich Sylvester was in constant dispute with the authorities who considered his manners eccentric and irritable. Therefore, in 1870, even though his research was still going strong, he was forcibly "super-annuated" at the age of 56. But in a few years a new direction in his career was about to begin.

Sylvester at Johns Hopkins In 1875, the Johns Hopkins University was founded in Baltimore, Maryland, under the brilliant leadership of President Gilman, who felt that the ideal way to start would be to appoint two outstanding professors, one in Classics and the other in Mathematics. This way they would be able to ensure excellence without a huge financial commitment. The exact words of Daniel Gilman were: "Enlist a great Mathematician and a distinguished Grecian; your problem will be solved. Such men can teach in a dwelling house as well as in a palace. Part of the apparatus they will bring, part we will furnish. Others will follow." So the famous Joseph Henry sent a letter to Sylvester in August 1875 inviting him to build the mathematics department at Johns Hopkins and offered him a handsome salary, $5000 per annum paid in gold. Sylvester was free to teach whatever he wanted in any manner he saw fit. It was a bold experiment in the educational method. Sylvester, who was still full of fire and enthusiasm, once again crossed the Atlantic.

At Johns Hopkins University, Sylvester founded The American Journal of Mathematics, one of the leading mathematical research journals today. He found the freedom given to him to be so conducive to research. Perhaps the happiest years of his life were the ones he spent at Johns Hopkins. He lectured on a wide variety of topics, the choices being made at the spur of the moment on what he was thinking at that time. Undoubtedly this was difficult for some, but the spontaneity had a profound positive effect on many of his students and associates. The experiment in education taken by the regents of the university was a success. The contents of many of Sylvester's lectures became papers in the newly formed journal. In fact Sylvester published as many as 30 long papers in this journal during the first five years.

Many students and associates of Sylvester in Baltimore have commented on his lecturing style. W.P. Durfee has written: "His manner of lecturing was highly rhetorical and elocutionary. When about to enunciate an important or remarkable statement, he would draw himself up till he stood on the very tip of his toes and in deep tones thunder out his sentences. He preached at us at such times, and not infrequently he wound up quoting a few lines of poetry to impress upon us the importance of what he had been declaring."

A.S. Hathaway says: "I can see him now with his white beard and a few locks of grey hair, his forehead wrinkled o'er with thoughts, writing rapidly his formulae on the board, sometimes explaining as he wrote, while we, his listeners, caught the reflected sounds from the board. But stop, something is not right; he pauses, his hand goes over his forehead to help his thought; he goes over the work again, emphasises the leading points and finally discovers his difficulty ... But at the next lecture, we would hear of some new discovery that was the outcome of that difficulty and some article for the journal that he had begun."

It was during this period that Sylvester wrote a long and important paper entitled "A constructive theory of Partitions arranged in three Acts, an Interact and an Exodion" (American Journal of Mathematics, 1882). This paper contains several seminal ideas of Sylvester and members of his group of whom the most prominent were F. Franklin and W.P. Durfee. In this massive paper, Sylvester laid the foundations of the combinatorial theory of partitions by studying properties of partition graphs. He provided combinatorial proofs of Euler's assertions and extended them. In particular he obtained a significant refinement of Euler's theorem on odd parts and non-repeating parts. Contained in this paper is Franklin's famous proof of Euler's celebrated Pentagonal Numbers Theorem. This proof by Franklin is considered to be the first significant achievement in American mathematics. Yet another idea in this paper which is useful even today, is the concept of a Durfee square (named after W.P. Durfee).

By studying Durfee squares for partitions into non-repeating parts, Sylvester proved a general identity from which Euler's Pentagonal Numbers Theorem followed as a special case. It is normally the experience that one discovers a q-series identity first and later obtains a combinatorial explanation or proof. In this situation, Sylvester had obtained his general identity combinatorially but was unable to provide a generating function proof. So he challenged the mathematical community to find such a proof. His long time friend Cayley responded to the challenge and produced a beautiful generating function proof. This has a connection with the Rogers–Ramanujan identities as we shall soon see. George Andrews has remarked that Sylvester's identity holds the distinction of being the first q-series identity whose first proof was purely combinatorial!

Sylvester stayed in Baltimore for only 8 years. In 1883 he was elected to the Savillian Professorship of Geometry at Oxford University. He occupied this position until 1893 when his health began to decline. He died in 1897 in England at the age of 83. Ramanujan was then a ten year old boy in India.

Connection with Ramanujan's Work As is well known, the Rogers–Ramanujan identities are among the deepest and most elegant in the theory of partitions and q-series. Ramanujan discovered these identities in India prior to his departure to England but was unable to prove them. The British mathematician L.J. Rogers had proved them in 1894. Rogers later gave a second proof which was considerably simpler in detail than his first. After Ramanujan arrived in England and studied Rogers' paper in 1917, he was able to give his own proof. When Rogers and Ramanujan corresponded in 1917, it turned out that Rogers had a third proof which was remarkably similar to Ramanujan's. In view of the importance of these identities, Professor G.H. Hardy arranged for Rogers and Ramanujan to write a joint paper in which both their proofs were presented. This paper appeared in The Proceedings of The Cambridge Philosophical Society in 1919. The paper opens with Hardy's introductory remarks. Rogers' proof is given in Section 1, and Ramanujan's in Section 2. It is interesting that both Rogers' and Ramanujan's proofs make use of an identity which can be considered as the next case beyond the general identity that Sylvester found. But the method employed by Rogers and Ramanujan is not the combinatorial method of Sylvester, but the method underlying Cayley's proof of Sylvester's identity.

In the 1960s Basil Gordon obtained an elegant and important generalisation of the Rogers–Ramanujan identities. One way to prove Gordon's generalisation is to establish a suitable extension of Sylvester's identity.

It should be noted that I. Schur, the German mathematician, independently proved the Rogers–Ramanujan identities in 1917. Owing to World War I, he was cut off from England and so was unaware of the work of Rogers and Ramanujan. Schur's proof is very much combinatorial and in fact goes along the lines of Franklin's proof of Euler's Pentagonal Numbers Theorem. Although several proofs of the Rogers–Ramanujan identities are known today, none are simple. For a long time, it was an open question whether a combinatorial bijective proof of the Rogers–Ramanujan identities can be found. In 1980, Garsia and Milne found a bijective proof, but even this is very complicated. The key idea in the Garsia–Milne proof is based on Schur's combinatorial constructions which can be traced back to the techniques put forth by Sylvester and his students. Thus we see that Sylvester's paper has an influence in current research.

There are other instances where Sylvester's work intersects with that of Ramanujan, as Andrews has pointed out in his New Zealand Lectures. In Ramanujan's Lost Notebook there are several identities which have partition interpretations, but Ramanujan, as was his custom, never stated their combinatorial significance. Ramanujan was more an analyst than a number theorist. So he preferred the analytic form of his identities and did not make their number-theoretic or combinatorial forms explicit. Andrews has shown that many of Ramanujan's identities have elegant partition interpretations and that by using Durfee squares one can not only prove some of these identities but extend them as well.

George Andrews, who has explained the significance of many of Sylvester's ideas on partitions in the context of current research, has said often that Sylvester's paper of 1882 is still worthy of deeper study. On the subject of partitions Sylvester said: "Partitions constitute the sphere in which analysis lives, moves and has its being; and no power of language can exaggerate or paint too forcibly the importance of this till recently almost neglected, but vast, subtle and universally permeating, element of algebraical thought and expression." The theory of partitions continues to be active area of research because of the wealth of ideas handed down by mathematicians of eminence like Euler, Sylvester and Ramanujan.