REVERSE MATHEMATICS

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PROOFS FROM THE INSIDE OUT

John Stillwell

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Preface

This is a book about the foundations of mathematics—a topic once of interest to outstanding mathematicians, such as Dedekind, Poincaré, and Hilbert, but today sadly neglected. This neglect is unfortunate for several reasons:

- As mathematics splits into more and more specialties, the need for a unifying viewpoint becomes more acute.
- Foundations unify not only mathematics but also the neighboring disciplines of computer science and physics.
- Recent advances in mathematical logic throw new light on the foundations of analysis, and on the elusive concept of mathematical "depth."

This book aims at the last point in particular, by focusing on the topic of *reverse mathematics*.

As its name suggests, reverse mathematics looks at the concept of proof in the opposite to normal direction. Instead of seeking the consequences of given axioms, it seeks the axioms needed to prove given theorems. This is actually an old idea, at least in the foundations of geometry. From the time of Euclid until the nineteenth century it was a burning question whether the parallel axiom was needed to prove theorems such as the Pythagorean theorem. We review the history of the parallel axiom in chapter 1 of this book, as a case study in reverse mathematical ideas, together with the similar story of the axiom of choice in set theory.

Although both these axioms illustrate the idea of reverse mathematics, the subject as it is understood today lies mostly in a narrow but important region *between* geometry and set theory: the theory of real numbers, which is the foundation of calculus, analysis, and most of mathematical physics. (Reverse mathematics has also made interesting contributions to algebra, combinatorics, and topology which we mention more briefly.)

The real numbers, as we understand them today, emerged from nineteenth century efforts to *arithmetize* analysis and geometry. By building real numbers from sets of rational numbers (and hence, ultimately, from sets of natural numbers) it becomes possible to encode sequences of real numbers and arbitrary continuous functions—and hence most of the objects of analysis—by sets of natural numbers. We review the arithmetization of analysis, and also the basic theorems of analysis, in chapters 2 and 3. After this we are ready to ask: which *axioms* do we need to prove these basic theorems? The answer, roughly, is a set of axioms for the natural numbers (the *Peano axioms*) plus a suitable *set existence axiom*.

Now set existence axioms come in various *strengths*, depending on the strength of the theorems we wish to prove. The lowest useful strength turns out to be intimately related to the foundations of *computation*: it asserts the existence of computable sets. This in turn involves a study of the concept of computation, which merges with analysis because both have a common basis in arithmetic. After an informal introduction to computability in chapter 4 we develop a formal concept of computation, and its arithmetization, in chapter 5.

In chapters 6 and 7 we bring together the ideas of analysis, arithmetic, and computation in some axiom systems for analysis, known as RCA_0, WKL_0, and ACA_0. These systems, which are distinguished mainly by set existence axioms of increasing strength, between them prove most of the basic theorems of analysis. More remarkably, they sort the basic theorems into three levels because, once above the "base" level of RCA_0, most theorems are *equivalent* to the set existence axiom of the system that proves them. This makes each of these set existence axioms the "right axiom" in the sense articulated by Friedman (1975):

> When a theorem is proved from the right axioms, the axioms can be proved from the theorem.

We will see, for example, that RCA_0 can prove the intermediate value theorem; the defining axiom of WKL_0 is the right axiom to prove the Heine-Borel theorem and the extreme value theorem; and the defining axiom of ACA_0 is the right axiom to prove the Cauchy convergence criterion and the Bolzano-Weierstrass theorem.

Thus in reverse mathematics we meet the usual cast of characters from an introductory real analysis course, but in an entirely new story.

REVERSE
MATHEMATICS

CHAPTER 1

Historical Introduction

The purpose of this introductory chapter is to prepare the reader's mind for *reverse mathematics*. As its name suggests, reverse mathematics seeks not theorems but the right axioms to prove theorems already known. The criterion for an axiom to be "right" was expressed by Friedman (1975) as follows:

> When the theorem is proved from the right axioms, the axioms can be proved from the theorem.

Reverse mathematics began as a technical field of mathematical logic, but its main ideas have precedents in the ancient field of geometry and the early twentieth-century field of set theory.

In geometry, the parallel axiom is the right axiom to prove many theorems of Euclidean geometry, such as the Pythagorean theorem. To see why, we need to separate the parallel axiom from the *base theory* of Euclid's other axioms, and show that the parallel axiom is not a theorem of the base theory. This was not achieved until 1868. It is easier to see that the base theory can prove the parallel axiom *equivalent* to many other theorems, including the Pythagorean theorem. This is the hallmark of a good base theory: what it cannot prove outright it can prove equivalent to the "right axioms."

Set theory offers a more modern example: a base theory called ZF, a theorem that ZF cannot prove (the well-ordering theorem) and the "right axiom" for proving it—the axiom of choice.

From these and similar examples we can guess at a base theory for analysis, and the "right axioms" for proving some of its well-known theorems.

1.1 EUCLID AND THE PARALLEL AXIOM

The search for the "right axioms" for mathematics began with Euclid, around 300 BCE, when he proposed axioms for what we now call *Euclidean geometry*. Euclid's axioms are now known to be incomplete; nevertheless, they outline a complete system, and they distinguish between really obvious "basic" axioms and a less obvious one that is crucial for obtaining the most important theorems. For historical commentary on the axioms, see Heath (1956).

The basic axioms say, for example, that there is a unique line through two distinct points and that lines are unbounded in length. Also basic, though expressed only vaguely by Euclid, are criteria for *congruence of triangles*, such as what we call the "side angle side" or SAS criterion: if two triangles agree in two sides and the included angle then they agree in all sides and all angles. (Likewise ASA: they agree if they agree in two angles and the side between them.)

Using the basic axioms it is possible to prove many theorems of a rather unsurprising kind. An example is the *isosceles triangle theorem*: if a triangle ABC has side AB = side AC then the angles at B and C are equal. However, the basic axioms fail to prove the signature theorem of Euclidean geometry, the *Pythagorean theorem*, illustrated by figure 1.1.

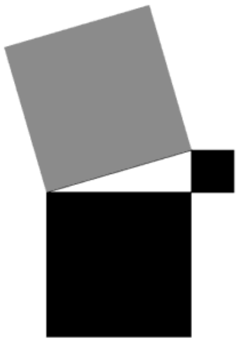

Figure 1.1 : The Pythagorean theorem

As everybody knows, the theorem says that the gray square is equal to the sum of the black squares, but the basic axioms cannot even prove the *existence* of squares. To prove the Pythagorean theorem, as Euclid realized, we need an axiom about infinity: the *parallel axiom*.

The Parallel Axiom

I call the parallel axiom an axiom about infinity because it is about lines that do not meet, *no matter how far they are extended*—and one of Euclid's basic axioms is that lines can be extended indefinitely. Thus parallelism cannot be "seen" unless we have the power to see to infinity, and Euclid preferred not to assume such a superhuman power. Instead, he gave a criterion for lines *not* to be parallel, since a meeting of lines can be "seen" a finite distance away.

Parallel axiom. If a line n falling on lines l and m (figure 1.2) makes angles α and β with $\alpha + \beta$ less than two right angles, then l and m meet on the side on which α and β occur.

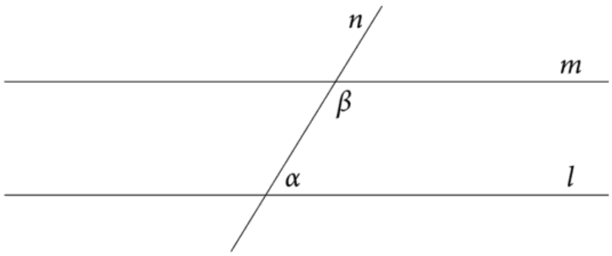

Figure 1.2 : Angles involved in the parallel axiom

It follows that if $\alpha + \beta$ equals two right angles (that is, a straight angle) then l and m do *not* meet. Because if they meet on one side (forming a triangle) they must meet on the other (forming a congruent triangle, by ASA), since there are angles α and β on both sides and one side in common (figure 1.3). This contradicts uniqueness of the line through any two points.

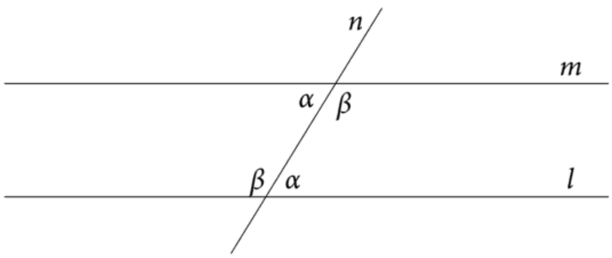

Figure 1.3 : Parallel lines

the same thing, the angle between the tangents to the great circles).

Indeed, it is often easier to describe a spherical triangle by its angles rather than the lengths of its sides. All spherical triangles with the same angles in fact have the same size, because of a famous theorem of Harriot[1] from 1603: *the angle sum of a spherical triangle, minus π, is proportional to its area*. There are several ways to tile the surface of the sphere with congruent triangles. Figure 1.6 shows one in which the sphere is divided into 48 triangles, each of which has angles $\pi/2, \pi/3, \pi/4$. Alternate triangles have been cut out of the sphere, to make it easier to see them all, and the sphere has been illuminated from the inside. This then is the standard

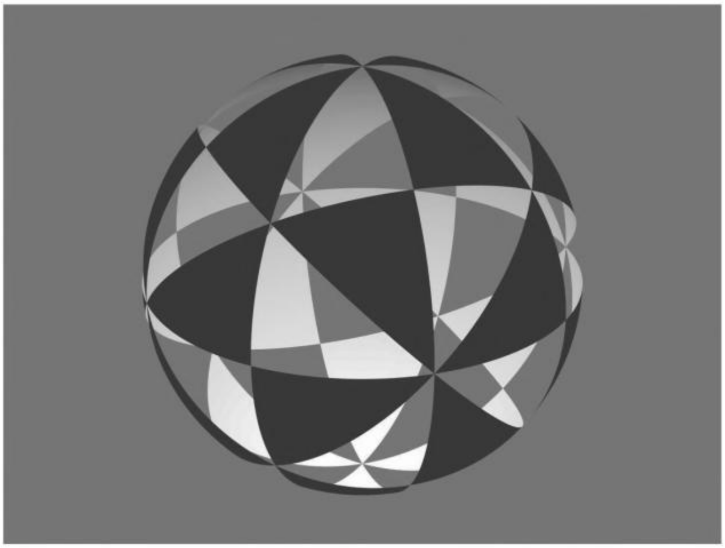

Figure 1.6 : Tiling the sphere with triangles

model of spherical geometry: "points" are ordinary points on the sphere, "lines" are great circles, and "angles" are the angles between the tangents to the great circles at their point of intersection. "Distance," if we wish to use the concept, is the distance between points on the sphere, measured along the (shorter) piece of the great circle connecting them.

Now we move to another model, by *projecting the sphere onto the plane*. Specifically, we use the light inside the sphere (at its north pole) to cast a shadow on the plane. The result is shown in figure 1.7. The pic-

[1]Thomas Harriot was mathematical consultant to Sir Walter Raleigh, and traveled with him on some of his voyages.

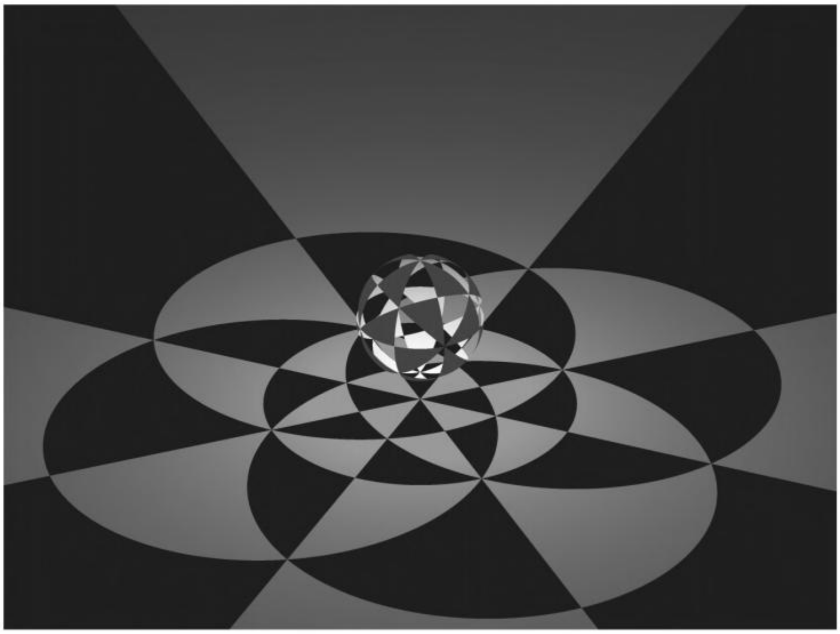

Figure 1.7 : Projecting the sphere onto the plane

ture shows two remarkable features of projection from the north pole, which is known as *stereographic* projection:

- circles map to circles (or, in exceptional cases, to straight lines, which we might call "circles of infinite radius"), and
- angles are preserved.

Thus "points" are still points, "lines" are still circles, and "angle" is still the angle between the tangents to the circles. "Distance," alas, is not a Euclidean distance of any kind, since equal distances on the sphere can be mapped to unequal Euclidean distances in the plane. Likewise, "area" is not Euclidean area, but we can easily measure it by the angle sum minus π.

Strictly speaking, we have not projected the whole sphere onto the plane, but the sphere minus its north pole (the light source). To correct for this we add a *point at infinity* to the plane—a point approached by the shadows of points on the sphere as they approach the north pole. The point at infinity completes each straight line to a closed curve, so that they too become circles. Thus our second interpretation of spherical geometry models all "lines" by circles, and "angles" by angles between circles. In the next subsection we will see a similar model of non-Euclidean geometry.

Models of Non-Euclidean Geometry

Beltrami (1868) discovered several models of non-Euclidean geometry; that is, of Euclid's basic axioms plus a non-Euclidean parallel axiom stating that *for any line l and a point P outside it, there is more than one line m that does not meet l*. The easiest of Beltrami's models to view in its entirety is the one shown in figure 1.8.

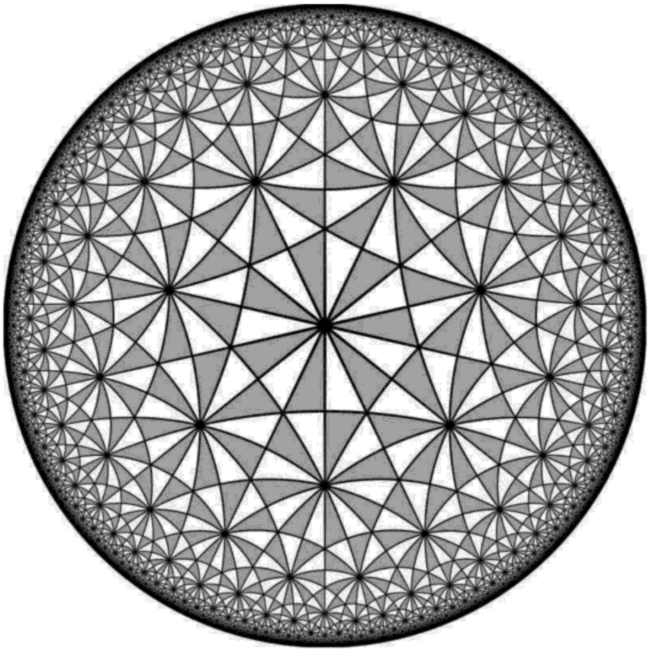

Figure 1.8 : The conformal disk model

In this model, "points" are points in the interior of the disk, "lines" are circular arcs perpendicular to the boundary circle of the disk (counting the straight line segments through the disk center as circles of infinite radius) and "angle" is the angle between circles. As in spherical geometry, triangles are congruent if they have the same angles, so in this picture the disk is filled with infinitely many congruent triangles, each with the angles $\pi/2, \pi/3, \pi/7$. These are the smallest triangles that can tile the non-Euclidean plane and, as in spherical geometry, their area is determined by their angle sum: *π minus the angle sum of a non-Euclidean triangle is* *proportional to its area.*

As with the plane model of spherical geometry, the precise definition of “distance” is complicated. But here one gets a better feel for it because there are so many triangles, each of the same non-Euclidean size. One sees, for example, that infinitely many triangles lie along each “line,” so each “line” is of infinite “length.” It is even possible to accept that each “line” gives the least “distance” between any two points in the disk, since one counts fewer triangles when travelling on a circular arc perpendicular to the boundary than on any other route. Thus one can understand how the model satisfies the basic axioms of Euclid. But it clearly does *not* satisfy the parallel axiom. If one takes the vertical “line” l through the center of the disk and the point P, say, somewhat to its right, then there are different “lines” m and n through P that do not meet l, as is clear from figure 1.9.

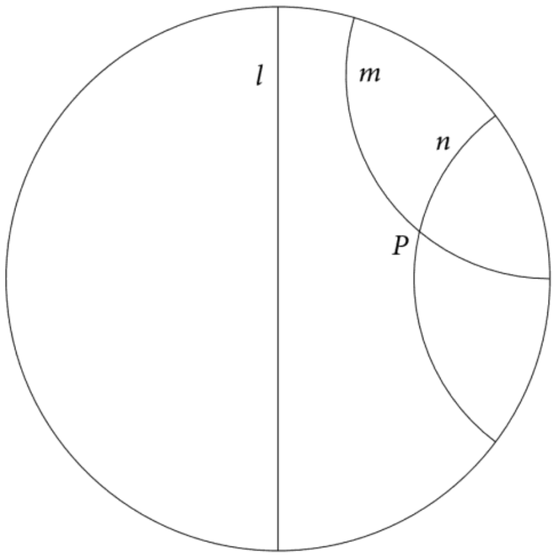

Figure 1.9 : Failure of the parallel axiom

So, when the details of Beltrami’s construction are checked, one has *a model for the basic axioms of Euclid plus a counterexample to the parallel axiom*. Therefore, *the parallel axiom does not follow from the other axioms of Euclid*, and hence the theorems equivalent to the parallel axiom (such as the four mentioned in the previous section) likewise do not follow from Euclid’s other axioms. However, the equivalences between the parallel axiom and these theorems are provable from Euclid's other axioms. This situation is typical of reverse mathematics: we have a *base theory* which is too weak to prove certain desirable theorems, but strong enough to prove *equivalences* between them.

New Foundations of Geometry and Mathematics

The discovery of non-Euclidean geometry shook the foundations of mathematics, which before the nineteenth century had been implicitly based on Euclid's concepts of "line" and "plane." By creating doubts about the meaning of "line" and "plane," non-Euclidean geometry prompted a search for new foundations in *arithmetic*, since the fundamental properties of numbers were not in doubt.

In particular, the "line" was rebuilt as the system $\mathbb{R}$ of *real numbers*, which has both algebraic and geometric properties. The next few sections describe the emergence of geometry based on, or influenced by, the real number concept. In chapter 2 we will see how the real numbers also became the foundation of analysis.

1.3 VECTOR GEOMETRY

The first major advance in geometry after the Greeks was made by Fermat and Descartes in the 1620s, and published in the *Geometry* of Descartes (1637). Their innovation was to use algebra in geometry, describing lines and curves by equations, thereby reducing many problems of geometry to routine calculations. But before they could "algebraicize" geometry they had to *arithmetize* it, a step that already took them far beyond Euclid. In fact, it was the first step towards a sweeping arithmetization of geometry and analysis that occurred in the nineteenth century.

As every mathematics student now knows, the Euclidean plane is arithmetized by assigning real number *coordinates* x and y to each point P in the plane. The numbers x and y are visualized as the horizontal and vertical distances to P from the origin O, in which case the distance $|OP|$ of P from O is $\sqrt{x^2+y^2}$ by the Pythagorean theorem (figure 1.10). But P can be *defined* as the ordered pair[2] $\langle x, y\rangle$, and its distance from O defined as $\sqrt{x^2+y^2}$. More generally, the distance from $P_1 = \langle x_1, y_1\rangle$ to

[2]In this book I use $\langle a, b\rangle$ to denote the ordered pair of a and b, because (a, b) will be on duty to represent the open interval between a and b.

where θ is the angle between the lines from $\mathbf{0}$ to $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively. Thus Grassmann (1847) found another way to describe Euclidean geometry as a "base theory" plus the "right axiom" to derive the Pythagorean theorem. Interestingly, his base theory (the vector space axioms) admits extension by a different axiom that gives *non*-Euclidean geometry.

Making a Vector Space Non-Euclidean

The key property of Grassmann's inner product is that it is *positive definite*; that is, $|\boldsymbol{u}|^2 = \boldsymbol{u} \cdot \boldsymbol{u} > 0$ if $\boldsymbol{u} \neq \mathbf{0}$, so every nonzero vector has positive length. Einstein's theory of special relativity motivated Minkowski (1908) to introduce a *non*-positive definite inner product on the space $\mathbb{R}^4$ of spacetime vectors $\langle t, x, y, z \rangle$, namely

$$\langle t_1, x_1, y_1, z_1 \rangle \cdot \langle t_2, x_2, y_2, z_2 \rangle = -t_1 t_2 + x_1 x_2 + y_1 y_2 + z_1 z_2.$$

With the Minkowski inner product $\boldsymbol{u} = \langle t, x, y, z \rangle$ has "length" $|\boldsymbol{u}|$ given by

$$|\boldsymbol{u}|^2 = -t^2 + x^2 + y^2 + z^2,$$

which clearly is zero or negative for many vectors. To make visualization easier we consider the corresponding concept of length on the space $\mathbb{R}^3$ of vectors $\boldsymbol{u} = \langle t, x, y \rangle$, namely

$$|\boldsymbol{u}|^2 = -t^2 + x^2 + y^2.$$

This means that in $\mathbb{R}^3$ we have a "sphere[3] of radius $\sqrt{-1}$ about O," consisting of the vectors $\boldsymbol{u} = \langle t, x, y \rangle$ such that

$$-t^2 + x^2 + y^2 = -1.$$

This surface in $\mathbb{R}^3$ is the *hyperboloid* $x^2 + y^2 - t^2 = 1$.

It turns out that the Minkowski distance on the surface of the hyperboloid gives a non-Euclidean geometry—the same as that of the Beltrami model in the previous section. Figure 1.11, which is derived from a picture by Konrad Polthier of the Freel University of Berlin, shows the connection between the two. The tiling of the disk projects to a tiling of the hyperboloid by triangles that are congruent in the sense of Minkowski distance.

[3] In a remarkable prophecy, Lambert (1766) conjectured that there might be a geometry on the sphere of imaginary radius for which the angle sum of a triangle is less than π, and where the area of a triangle is proportional to π minus its angle sum. This is indeed what happens in Beltrami's non-Euclidean geometry.

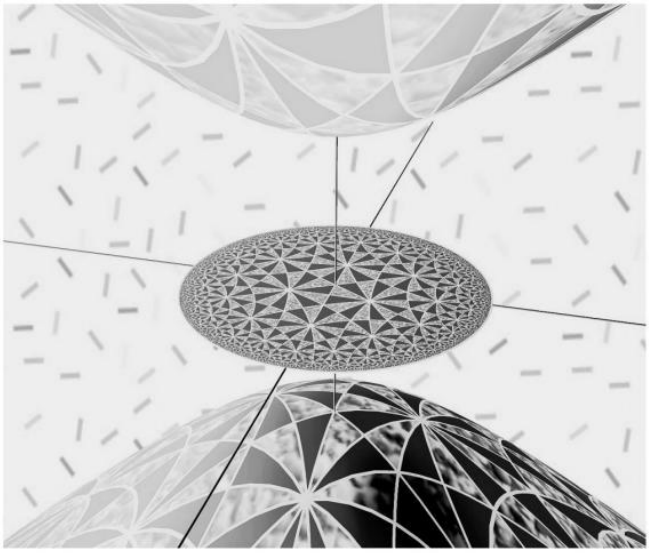

Figure 1.11 : The hyperboloid model of non-Euclidean geometry

1.4 HILBERT'S AXIOMS

Euclid's *Elements* is the first organized presentation of mathematics that survives from ancient times. It is best known for its treatment of geometry, deducing theorems from axioms in a style that became standard for mathematics until the nineteenth century. Then the discovery of non-Euclidean geometry put Euclid's geometry under the microscope, and by the late nineteenth century his axioms were found to have some gaps. But this only strengthened the movement towards axiomatization. The gaps in Euclid were filled by Hilbert (1899) and, in the meantime, axiomatic treatments of number theory and algebra were given by Dedekind, Peano, and others.

Euclid also gave a deductive treatment of numbers in the *Elements*, but it was complicated by the Greek discovery of irrationality, which was thought to disqualify some geometric quantities (such as the diagonal $\sqrt{2}$ of the unit square) from being numbers at all. Irrational quantities were not fully reconciled with whole or rational numbers until the publication of the Dedekind (1872) book on irrational numbers. Dedekind found that Euclid had been on the right track—the only new idea needed to make his theory of irrational quantities part of his theory of numbers was acceptance of *infinite sets* of rational numbers (see section 1.5).

The two main threads of the *Elements*, geometry and the real numbers, were combined in the *Grundlagen der Geometrie* (foundations of geometry) of Hilbert (1899). Here, Hilbert not only filled the gaps in Euclid's geometric axioms, he also introduced two axioms that complete a geometric path to the real number system $\mathbb{R}$. This was a historic achievement, though Hilbert's path is not the best for all mathematical purposes. The *arithmetization* path to real numbers via the rational numbers ultimately proved more useful for analysis, and we will take it up again in chapter 2.

Hilbert (1899) found that Euclid's geometry and the arithmetic of real numbers follow from 17 axioms, described below. All but two of them are purely geometric. The exceptions are the *Archimedean axiom*, which says no line segment is "infinitely large" compared with another, and the *completeness axiom*, which says there are no "gaps" in the points on a line. (These two axioms were not needed by Euclid, who considered only points constructible by ruler and compass.) Their purpose is to prove that any line satisfying the axioms is essentially the line $\mathbb{R}$ of real numbers. It follows that any plane satisfying the axioms is essentially the plane of Descartes, so Euclid's geometry has really only one model—the plane of pairs of real numbers.

This very satisfying convergence of the geometric and arithmetic viewpoints comes about because Hilbert's geometric axioms yield not just Euclid's geometric theorems—they also yield *algebra*, which Euclid did not foresee. In fact, algebraic structure arises in stages corresponding to axiom *groups*, which Hilbert introduces one by one.

Axioms of incidence. These relate lines and points. They include Euclid's axiom that two points determine a line, and a form of the parallel axiom: for any line l and point $P \notin l$ there is exactly one line m through P not meeting l. Also (which went without saying in Euclid) each line has at least two points, and there are three points not in a line.

Axioms of order. The first three of these axioms say the obvious things about the order of three points on a line: if B is between A and C then it is also between C and A; any A and C have a point B between them; for any three points, one is between the other two. The fourth, called *Pasch's axiom*, is about the plane: a line meeting one side of a triangle at an internal point meets exactly one of the other sides.

Axioms of congruence. The first five of these axioms are about equality of line segments or angles, and the addition of line segments. They state the existence and uniqueness of line segments or angles equal to given ones, at a given position. They also say (as Euclid put it) “things equal to the same thing are equal to each other.” The last congruence axiom is the SAS criterion for congruence of triangles.

Circle intersection axiom. Two circles meet if one of them contains points both inside and outside the other. (Euclid overlooked this axiom, even though he assumed it in his very first proposition, constructing an equilateral triangle.) Note that the points “inside” a circle of radius r are those at distance $< r$ from its center.

Archimedean axiom. For any nonzero line segments AB and CD there is a natural number n such that n copies of AB are together greater than CD.

Completeness axiom. Suppose the points of a line l are divided into two nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ such that no point of $\mathcal{A}$ is between two points of $\mathcal{B}$ and no point of $\mathcal{B}$ is between two points of $\mathcal{A}$. Then there is a unique point P, in either $\mathcal{A}$ or $\mathcal{B}$, that lies between any other two points, of which one is in $\mathcal{A}$ and the other is in $\mathcal{B}$. (Thus, there is no “gap” between $\mathcal{A}$ and $\mathcal{B}$.)

These axioms give precise meaning to the idea of a theorem being *equivalent* to the parallel axiom: namely, the equivalence is provable in the *base theory* of Hilbert’s axioms *minus* the parallel axiom. All theorems previously thought to be equivalent to the parallel axiom (such as those mentioned in section 1.1) are equivalent to it in this sense. As suggested at the end of section 1.2, proving equivalences in a weaker system is the hallmark of *reverse mathematics*. We will see further historical examples in the later sections of this chapter. Today, the idea has been most fully developed in systems of analysis, and we will see some its main results in chapters 6 and 7.

Algebraic Content of Hilbert's Axioms

The incidence axioms allow us to define sum and product of points on a line by means of the constructions shown in figures 1.12 and 1.13.

The sum construction chooses a point 0 on the line then, for any points a and b on the line, constructs a point $a + b$ with the help of the parallels shown. In effect, the parallels allow the point b to be "translated" along the line by the distance between 0 and a.

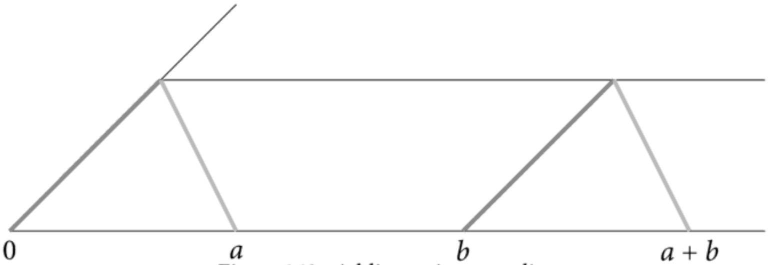

Figure 1.12 : Adding points on a line

The product construction also requires a point 1 on the line (the "unit of length"), and various parallels now allow us to "magnify" the distance from 0 to b by the distance from 0 to a, producing the point ab.

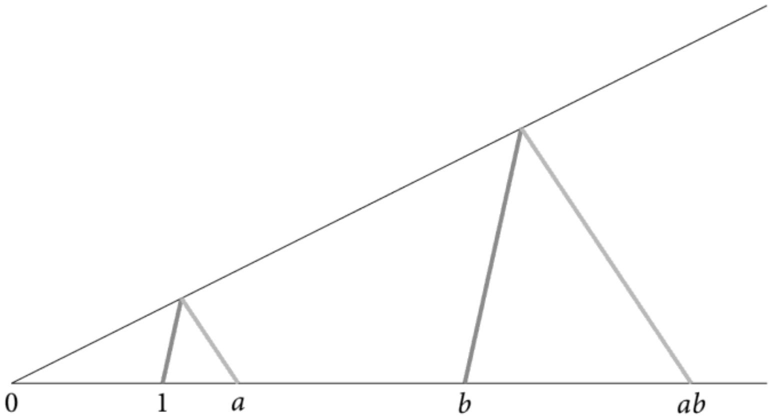

Figure 1.13 : Multiplying points on a line

With the help of the congruence axioms one can prove that the sum and product operations just defined have the following algebraic prop-