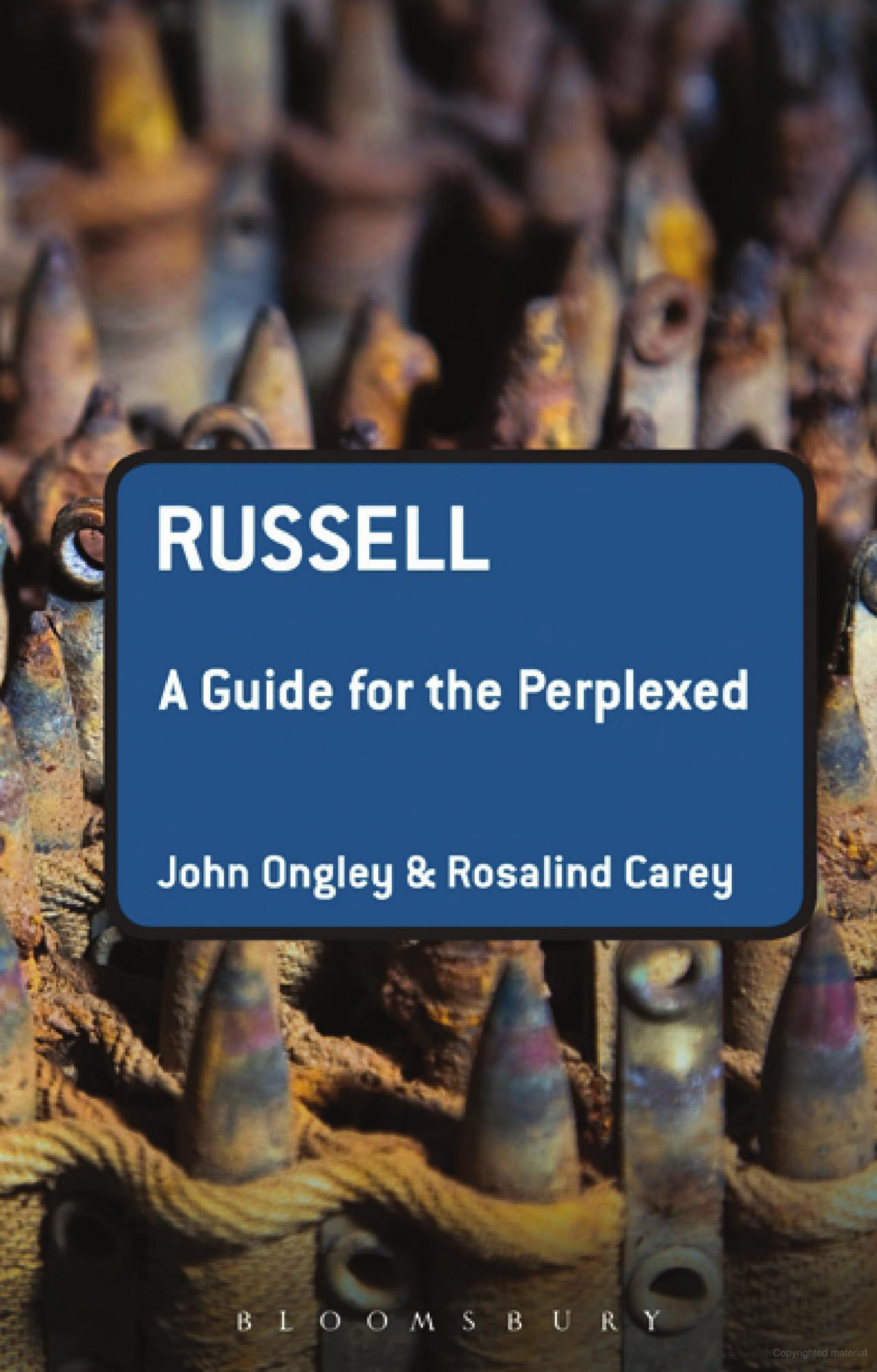

RUSSELL

A Guide for the Perplexed

John Ongley & Rosalind Carey

BLOOMSBURY

A GUIDE FOR THE PERPLEXED

Russell

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ROSALIND CAREY

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PREFACE

This book is a guide to some of Bertrand Russell's more difficult philosophical works and ideas. Russell's most important and at the same time most difficult work is *Principia Mathematica*, the monumental three-volume opus cowritten with Alfred North Whitehead. These volumes present in elaborate detail his ground-breaking logical analysis of the foundations of mathematics. Written almost entirely in logical notation, it is difficult in the extreme to work through and understand.

Russell wrote an informal guide to *Principia Mathematica*—one without logical symbolism, and, he says, one "offering a minimum of difficulty to the reader." This is his *Introduction to Mathematical Philosophy*. Though concise and beautifully written, it is itself not always easy to understand. This guide's first aim is to help the reader master Russell's informal *Introduction*, then, having mastering that, to understand *Principia Mathematica*. This will enable the reader to also understand Russell's earlier masterpiece on the foundations of mathematics, *Principles of Mathematics*.

Russell also had a larger philosophy—one not just about logic and mathematics, but about the world more broadly, one that sought to understand the nature of the universe and the way that we know it. This philosophy especially includes Russell's metaphysics, his theory of knowledge, and his theory of language, which are the subjects of his following works: "Philosophy of Logical Atomism," *Analysis of Mind*, *Analysis of Matter*, *Inquiry into Meaning and Truth*, and *Human Knowledge*. Because his ideas on these subjects are spread out over many works and evolve over time, we take a different approach in covering them and present each subject as it occurs in Russell's early, middle, and late work. Here, we aim to give the reader a broad understanding of Russell's larger philosophy and to see the evolution of his thought as a whole.

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CHAPTER ONE

Introduction

Bertrand Russell (1872–1970) was arguably the greatest philosopher of the twentieth century and the greatest logician since Aristotle. He wrote original philosophy on dozens of subjects, but his most important work was in logic, mathematical philosophy, and analytic philosophy. Russell is responsible more than anyone else for the creation and development of the modern logic of relations—the single greatest advance in logic since Aristotle. He then used the new logic as the basis of his mathematical philosophy called *logicism*.

Logicism is the view that all mathematical concepts can be defined in terms of logical concepts and that all mathematical truths can be deduced from logical truths to show that mathematics is nothing but logic. In his work on logicism, Russell developed forms of analysis in order to analyze quantifiers in logic and numbers and classes in mathematics, but he was soon using them to analyze points in space, instants of time, matter, mind, morality, knowledge, and language itself in what was the beginning of analytic philosophy.

This first chapter introduces Russell's work in logic, logicism, and analysis, and then introduces his broader inquiries of analytic philosophy in metaphysics, knowledge, and meaning. Subsequent chapters treat each subject in detail. However, all of Russell's technical philosophy revolves around his logicism. Because Russell's mathematical philosophy is the key to the rest of his work, and because it is the most difficult part of it, we begin this chapter with a discussion of logicism, then keep circling back to it, relating it to the rest, until it seems to the reader that it is the easiest thing in the world to understand.

1 Logic and logicism: Basic concepts

Let's start with some basic logical concepts. A sentence is a group of words that express a meaning that is a complete thought. A declarative sentence expresses a meaning that is either true or false. A proposition is the meaning expressed by a declarative sentence such as the true proposition "The earth is round" or the false one "The earth is flat." So propositions are either true or false. The declarative sentences that express them are also said to be true or false.

Subjects and predicates follow. The subject of a proposition is who or what the proposition is about. "The earth is flat" is about the earth. So the earth is the subject of that proposition. The predicate is what is said about, or attributed to, the subject. Here, the proposition attributes flatness to the earth, so "____ is flat" is the predicate. Logicians write predicates using blank spaces, or more usually, variables like x, y, or z to indicate where the subject goes in relation to the predicate. Bertrand Russell called predicates *propositional functions*. In this book, we use the terms interchangeably.

The predicate "x is flat" is a *one-place predicate*, because it only has one place where a subject can go—it attributes a property to one thing. Two-place predicates are *relations* like that in "Indiana is flatter than Ohio." Here, the subjects are "Indiana" and "Ohio" and the predicate is "x is flatter than y." (In grammar, the first is the subject and the second is the object; in logic, they are both subjects.) Common two-place relations in mathematics are $x = y$, $x > y$, and $x < y$. There are also three-place relations like that in "Ohio is between Indiana and Pennsylvania," where the predicate is "x is between y and z," which is often used in geometry. There are also four-place relations, and so on.

Before Russell's logic of relations, logic consisted principally of the Aristotelian logic of one-place predicates. This simple logic can analyze sentences that use one-place predicates to attribute properties to objects like "Tom is tall" or "The sky is blue." It can also analyze slightly more complex sentences like "All humans are animals" (if someone is human, that person is an animal) and "Some humans are thoughtful" (at least one person is both human and thoughtful) and from these two sentences infer that "Some animals

Russell's original form of logicism, in his 1903 *Principles of Mathematics*, did not attempt to avoid the paradoxes of the new logic, and so did not contain the complex mechanisms Russell later added to his logic to avoid them. It is a straightforward theory, containing all of logicism's basic elements. We present this basic logicism, which we call *naïve logicism*, in Chapter 2. The complex version meant to avoid paradoxes, which occurs in the 1910–13 *Principia Mathematica*, we call *restricted logicism*. We describe that in Chapter 3.

3 Logicism and analysis

As well as founding the logic of relations, developing the theory of logicism, and discovering fundamental contradictions in logic and set theory, Russell more than anyone else founded the twentieth-century movement of analytic philosophy that still dominates philosophy today. Analytic philosophy as practiced by Russell logically analyzes language to say what there is and how we know it. Analysis is a significant part of analytic philosophy and its role in the movement is largely due to Russell. His logical analysis of mathematics is the primary example of analysis.

Notions of analysis vary from one analytic philosopher to another and from one analysis to another by a single philosopher. This last case is true of Russell himself. Most generally, "analysis" for him means beginning with something that is common knowledge and seeking the fundamental concepts and principles it is based on. This is followed by a synthesis that begins with the basic concepts and principles discovered by analysis and uses them to derive the common knowledge with which one began the analysis.

In Russell's own words (*Introduction to Mathematical Philosophy*): "By analyzing we ask . . . what more general ideas and principles can be found, in terms of which what was our starting-point can be defined or deduced" (p. 1). Similarly, in *Principia Mathematica*, he says "There are two opposite tasks which have to be concurrently performed. On the one hand, we have to analyze existing mathematics, with a view to discovering what premises are employed . . . . On the other hand, when we have decided upon our premisses, we have to build up again [i.e., synthesize] as much as may seem necessary of the data previously analyzed" (vol. 1, p. v).

Immanuel Kant uses the same concepts of analysis and synthesis to describe his *Prolegomena to Any Future Metaphysics* and *Critique of Pure Reason*. "I offer here," he says in the *Prolegomena*, "a plan which is sketched out after an analytical method, while the *Critique* itself had to be executed in the synthetical style" (p. 8). In the *Prolegomena* we start with science (mathematics and physics) and by *analysis*, he says, "proceed to the ground of its possibility," that is, to its fundamental concepts, while in the *Critique*, "they [the sciences] must be derived . . . from [the fundamental] concepts" (p. 24).

Russell's *Introduction to Mathematical Philosophy*, an informal introduction to *Principia*'s logicism, is similarly analytic. About it, he says: "Starting from the natural numbers, we have first defined *cardinal number* and shown how to generalize the conception of number, and have then analyzed the conceptions involved in the definition, until we found ourselves dealing with the fundamentals of logic." About synthesis, he says "In a synthetic, deductive treatment these fundamentals [reached by analysis] come first, and the natural numbers [with which the analysis started] are reached only after a long journey" (p. 195).

And *Principia Mathematica* is a synthesis: it begins with the logical fundamentals found by analysis, and from them deductively builds up the mathematics the analysis started with. As Russell says in *Principia* itself, it is "a deductive system" in which "the preliminary labor of analysis does not appear." Instead, it "merely sets forth the outcome of the analysis . . . making deductions from our premisses . . . up to the point where we have proved as much as is true in whatever would ordinarily be taken for granted" (vol. 1, p. v).

Russell's *Introduction to Mathematical Philosophy* is thus to *Principia Mathematica* what Kant's *Prolegomena* is to the *Critique of Pure Reason*—an analysis that takes common knowledge and finds its basic principles, which synthesis then uses to demonstrate the knowledge analyzed. The *Introduction to Mathematical Philosophy* and *Prolegomena* also both informally introduce the subjects presented more rigorously in the synthetic works. But Kant seeks to justify knowledge with the principles uncovered by analysis. Russell does not. For him, the logical ideas analysis uncovers are less certain than the arithmetic it analyzes.

For Russell, what we analyze—arithmetic—is certain and *inductively* justifies the fundamental principles found by analysis when synthesis deduces arithmetic from them. (If synthesis shows that logic implies arithmetic, and arithmetic is true, then logic is *probably* true. The argument is inductive.) Russell does not think arithmetic is made certain by being derived from logic, but that logic is made more certain by arithmetic being derived from it.

As Russell says in *Principia*: “The chief reason in favor of any theory on the principles of mathematics [the justification of the premisses that imply mathematics] must always be inductive, i.e. it must lie in the fact that the theory in question enables us to deduce ordinary mathematics” (vol. 1, p. v). What is found by analysis is less certain than what is analyzed. Russell does not seek certainty from the analysis of mathematics, but an understanding of the reasons, however uncertain, for accepting what we normally take for granted.

“In mathematics,” Russell further says, “the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point . . . . hence, the early deductions [of *Principia*], until they reach this point, give reasons rather for believing the premisses because true consequences follow from them, than for believing the consequences because they follow from the premisses” (p. v–vi). *Principia* does indeed show that arithmetic follows from logic, which gives us some reason to accept those logical principles as an account of arithmetic’s nature.

4 Logical analysis: The theory of descriptions

These concepts of analysis and synthesis may seem vague, but they will get you a long way in understanding Russell’s *Introduction to Mathematical Philosophy* and *Principia Mathematica*. At some point, however, to understand Russell’s work one must learn his more technical, logical kinds of analysis that are his theory of descriptions and incomplete symbols, his “no-class” theory of classes, his theory of logical types, and his logical constructions.

In the theory of descriptions, Russell analyzes descriptions of objects and classes by translating them into his new logic, where we can see that they do not always mean what they seem to mean in ordinary language. That is, Russell analyzes expressions of ordinary language into more careful logical expressions that are their true meaning. His *Introduction to Mathematical Philosophy* as a whole is the simpler sort of analysis, but within it are several more technical logical analyses using the theory of descriptions.

Russell first published the theory of descriptions in his 1905 article "On Denoting." The theory figures prominently in *Principia Mathematica*, where it is given a fairly clear presentation in the Introduction. Russell's clearest exposition of it is in the 1918 "Philosophy of Logical Atomism," and another is in his 1919 *Introduction to Mathematical Philosophy* (Chapter 16), which is the version most people read in college.

For Russell, the theory of descriptions shows that the grammar of ordinary language is often misleading. Using it, sentences containing singular definite descriptions—descriptions of the form "the so-and-so" such as "the author of *Waverly*" in the sentence "Scott was the author of *Waverly*"—are analyzed so that the description does not occur in the logical analysis of the sentence, but is replaced by a predicate.

For example, "the author of *Waverly*" in "Scott was the author of *Waverly*" is replaced with the predicate "*x* wrote *Waverly*" and the sentence becomes "There is exactly one thing *x* such that *x* is Scott, and *x* wrote *Waverly*," or more briefly, "Scott wrote *Waverly*." The description "the author of *Waverly*" no longer occurs in the logical analysis of the sentence. In particular, the word "the" is gone. That is the whole function of the theory of descriptions.

Why analyze a sentence so that the definite description it contains, and especially the word "the," disappears? Notice that "the author of *Waverly*" seems to function like a name and to denote a particular object. However, the expression that replaces it, "*x* wrote *Waverly*," is a predicate, not a name, and by itself it does not denote any such object. Let us pause here to consider this idea that names denote, but predicates do not. It is an important idea to Russell.

The idea that names refer to, or denote, objects should not be controversial. "Napoleon" refers to the commanding French general at the battle of Waterloo, "Einstein" to the man who created the special and general theories of relativity, and so forth. And as Russell points out, names have these references independently of occurring in propositions. Finally, definite descriptions like "the author of *Waverly*" seem to function like names and refer to particular individuals too, just as "Sir Walter Scott" does.

Predicates, on the other hand, do not name, or refer to, objects. For example, the predicate "x is red" does not name or denote any particular individual by itself independently of occurring in a proposition. It does not specify which object or objects it might be used to apply to. So a predicate is definitely not a name. Because definite descriptions are not names but are predicates, Russell calls them *incomplete symbols*. They appear to name objects, but they really don't.

By showing that definite descriptions, which appear to be names of objects, really aren't, we can see how sentences containing descriptions can be meaningful without the sentence asserting the existence of what is described. For example, we can see how sentences like "The present king of France rolled the round square down the golden mountain" can be meaningful without asserting that any of these things exist.

This solves a general problem of logic for Russell—how to logically analyze sentences containing definite descriptions true of no objects. More significantly, Russell uses a variation of this theory, called his "no-class" theory of classes, to remove all references to classes in his logic by treating names of classes and descriptions of classes as predicates. Then, since logic, so interpreted, does not assume that sets exist, the Russell paradox of the set of all sets that are not members of themselves cannot occur—as we will see next.

5 Logical analysis: The "no-class" theory of classes

In addition to analyzing singular definite descriptions so that what appear to be names are seen to actually be predicates that do not name anything, Russell sometimes treats proper names the same way, for example, in *Principia Mathematica* (in *14.21). He suggests there that words like "Homer" that appear to be proper names are actually concealed definite descriptions like "the author of the Homeric poems." They are then treated like definite descriptions and replaced with predicates. By 1918, in "The Philosophy of Logical Atomism," Russell is using this idea aggressively, insisting that *all* proper names like "Socrates" and "Napoleon" are disguised definite descriptions, but in *Principia*, he only suggests it once.

6 Logical analysis: The theory of logical types

Though the no-class theory does avoid Russell's paradox of classes, there is a paradox similar to it for predicates that the no-class theory does not eliminate. This of course is because the no-class theory eliminates classes, not predicates. Here is the new paradox: Some predicates are true of themselves, for example, "*x* is a predicate" is itself a predicate. Others are not—for example, "*x* is red" is not red. From this, we can form the predicate "*x* is a predicate that is not true of itself." This predicate is true of some predicates and not of others. But is it true of itself or not? If it is, it isn't, and if it isn't, it is. We thus have a contradiction.

So simply eliminating classes from one's logic and logicism using the no-class theory does not eliminate all self-referential paradoxes from logicism, because similar paradoxes arise in it for predicates. We can try to use something analogous to the no-class theory to eliminate predicates. For example, we might replace predicates with propositions. Unfortunately there are also self-referential paradoxes for propositions. And so on.

Fortunately, Russell has another method for avoiding paradoxes called *the theory of logical types*. Notice that both versions of the Russell paradox result from allowing a set to be a member of itself or a predicate to apply to itself. The many other sorts of self-referential paradoxes similarly arise self-referentially, by allowing sets to be members of themselves, predicates to apply to themselves, propositions to be about themselves, and so forth. The theory of types prevents the paradoxes from arising by banning self-reference.

In the mature "restricted" logicism of *Principia*, then, as well as adopting the no-class theory of classes, Russell adopts the rule that a set cannot be a member of itself and a predicate cannot apply to itself, that is, it cannot take itself as an argument. This rule is the *theory of logical types*. And the theory of logical types is justified by the *vicious circle principle*, which says that any sentence formed by a set taking itself as a member or predicate taking itself as an argument is meaningless. By adopting the rule that is based on this principle, namely, the theory of types, the paradoxes for both sets and predicates do not arise.

The theory of types works like this: If sets cannot meaningfully be members of themselves and predicates cannot meaningfully refer to themselves, we end up with a hierarchy of different types, or levels, of sets or of predicates, their level depending on what type of things *they* can meaningfully take as members or arguments, and on what sets or predicates can meaningfully take *them* as members or arguments.

At the first level in the hierarchy are individuals. This is the zero-order. Then, there are predicates that apply to individuals. These are called *first-order* predicates. Anything we call an object is an individual—cars, people, molecules, mountains, what have you. A first-order predicate is something like "x is brave." It applies to individuals to form propositions like "Nelson Mandela is brave."

Since first-order predicates now cannot apply to themselves, predicates that apply to first-order predicates are called *second-order* predicates. If courage is a first-order property, we must use a second-order property, like "x is an important virtue" to say something about it such as "Courage is an important virtue." First-order predicates also cannot take predicates of a higher-order than themselves as arguments. Then there are predicates that apply to second-order predicates—these are third-order predicates. And so on.

Sets are structured similarly with individuals again at the zero-order. Sets that take individuals as members are first-order sets, sets that take sets that take individuals as members are second-order sets, and so on. And propositions about objects are first-order propositions, those about first-order propositions are second-order propositions, and so on.

This is the basic idea. The actual theory of types is a few steps more complicated than this and will be explained in full in Chapter 3. But as you can see, stratifying sets and the things they can take as members, or predicates and the things they can apply to, prevents them from being self-referential, so the paradoxes of logic and set theory cannot arise.

Notice though that *both* the no-class theory of classes *and* the theory of logical types are used to avoid the paradoxes of class theory and logic. Why both methods? First, the no-class theory gets rid of sets by converting them to predicates. But since paradoxes also arise for predicates, the theory of types is needed to stratify predicates and prevent paradoxes for predicates from arising.

There are also predicates that apply to sets, but since the no-class theory transforms these sets into predicates, there is no need to create a separate hierarchy for them. This keeps the theory of types from getting any more complex than it already is. There are also *philosophical* problems with stratifying predicates that apply to sets. By converting the sets to predicates, the philosophical problems such as the one described below are avoided.

Notice that there is still a hierarchy for sets in type theory. Why? Although it is understood that symbols for sets are "really" predicates in *Principia*, the mathematics in it is done using symbols for sets anyway. They still need stratifying in order to be used, even though we know they are really predicates. And because there are self-referential paradoxes that arise for propositions, the hierarchy of propositions is included in the theory of types as well.

One last point: notice that the paradoxes for set theory only arise from some sets. But the no-class theory eliminates all sets. This is clearly overkill. Why do it? Answer: As well as needing to avoid the set-theoretic paradoxes, Russell has separate philosophical reasons for wanting to eliminate classes from his logic altogether, for example, to avoid the ancient problem of the one and the many.

Sometimes symbols for sets are treated as representing many things (its members), other times they are treated as representing one thing (the set itself). But it cannot be both. Because of this and other such philosophical puzzles, as well as in order to simplify the theory of types, Russell eliminates *all* classes from his logic using the no-class theory and the idea of logical fictions to define them away.

These, then, are the broad outlines of Russell's mathematical philosophy called logicism. We have seen that Russell uses several different kinds of analysis in his mathematical philosophies. He also applies these methods outside of mathematics to answer philosophical questions about the world at large. We have already seen four varieties of analysis: the general kind that seeks the most basic concepts and principles, the theory of descriptions, the no-class theory of classes, and the theory of logical types. A fifth kind is Russell's analysis of entities with logical constructions, which he uses to analyze physical points, space and time, mental phenomena, matter, and even moral and political concepts. These topics will be introduced in the remainder of this chapter, and discussed at greater length in Chapters 4 through 6.

7 Analysis and metaphysics

Russell's ideas about the nature of reality are often responses to problems in logic, mathematics, and analysis. His views on reality in early work (1900–17) are expressed in *Principles of Mathematics* (1903), "On the Relations of Universals and Particulars" (1912), and "Analytic Realism" (1911). In them, a defense of analysis is part of his view of reality.

Philosophical monists, who were common in England in Russell's time, argue that analyzing the whole of reality into parts is impossible. They feel that the nature of objects is determined by the role they play in larger wholes, and that analyzing wholes into parts leaves out these larger connections. And if the nature of an object lies in the role it plays in a whole, and the nature of that whole lies in the role it plays in some larger whole, reality is ultimately one undivided whole—the plurality we experience is an illusion.

To defend analysis, Russell rejects the monists' arguments and concludes that reality is plural and "atomistic," that is, composed of parts that can be understood independently of their role in the whole. Details about reality in Russell's atomism are reached by analysis of logical principles—it is a *logical* atomism. He believes that logic and grammar reveal the nature of reality. This avoids beliefs about reality not warranted by logic. For example, if reality consists of things that can be analyzed into parts, the parts themselves are either complex and further analyzable or not complex and simple. If they are complex, they presuppose the existence of still simpler entities.

Russell's logical atomism is also based on understanding grammar. Monists assume that the logic of sentences always has a subject-predicate form, where a predicate applies a property to a subject—as in "Socrates is wise" where the predicate "x is wise" applies the property of wisdom to Socrates. If all sentences are really subject-predicate sentences, relations expressed by verbs in sentences like "Socrates is wiser than Plato" must also be properties.

Instead of understanding "Socrates is wiser than Plato" as expressing the relation "x is wiser than y" between Socrates and Plato, monists understand it as saying that "x is wiser than Plato" is a property of Socrates. Treating relations this way makes being wiser than Plato seem like an essential property of Socrates. Treating all relations this way—as essential properties of objects—-makes everything seem interrelated to every other thing as a part of its essential nature. Thus, they can only be understood as parts of wholes. This view takes relations as "internal" (i.e., essential) properties of objects.

With his logic of relations, Russell can say that verbs are not predicates and relations are not properties of things. Rather, relations are entities in their own right, not part of the things related. Relations between things are thus "external" to the nature of things. They are not facts about the essential nature of the things related. This is the view of "external relations." Complexes of things are thus external relations among simpler things.

Using grammar as a guide, Russell also assumes that entities occur in specific ways in propositions. Some occur only as subjects of propositions. Others occur as relations or properties of propositions but can also occur in other propositions as subjects. Those that can only be subjects he calls "things" or "particulars." Those that can be both subjects and predicates or relations he calls "concepts" or "universals."

Russell also examines logic and grammar to find the basic elements of nature. These include numbers, classes, concepts, properties, propositions, universals, particulars, particles, points, and instants. As analysis develops, the list of elements changes. The theory of descriptions says descriptions are not names, and soon that "Socrates" is a disguised description and not really a name either. Instead, they are properties or relations. Similarly, the no-classes theory replaces classes with properties, so classes need not be assumed to exist. Both theories are metaphysical: they eliminate the need to assume the existence of certain entities, assuming others instead. The theory of types is also metaphysical in distinguishing these elements into different logical types of things.

In Russell's middle period (1918–34) logic and metaphysics continue to be linked in works such as "Philosophy of Logical Atomism" (1918), his introduction to Ludwig Wittgenstein's *Tractatus* (1921), *Analysis of Mind* (1921), and *Analysis of Matter* (1927). He now thinks his earlier ideas are mistaken. In 1911, properties and relations are abstract entities, *universals* that can occur in propositions as predicates or as things and subjects, for example, as "Robert is a man" and "*Man* is a concept." He now thinks relations and properties cannot be subjects and that universals are not among the data of experience. We only experience particulars.

knowledge of them is *a priori*. But this is because they are now viewed as definitions, which are uninformative. With empirical knowledge, he no longer thinks we are conscious of particulars and universals or know them by acquaintance. The proper method of philosophy is still to make as few metaphysical assumptions as possible, and neutral monism lets him avoid assuming a non-physical relation called "awareness." He now defines mental occurrences using logical words, assuming only the particulars of neutral monism.

The construction of minds and objects occurs by gathering particulars together in different ways. At any moment, for example, a star is a class consisting of various sensation-particulars. Your momentary experience of the star, that is, what occurs in you, is a different class of the same particulars. The whole collection of classes over time defines the star, and the whole collection of your experiences of stars and other things defines you.

After constructing mental phenomena in *Analysis of Mind*, Russell returns to the study of matter. This is due to changes in his views he thinks general relativity and quantum theory require. In *Analysis of Matter* (1927), he argues that all experiences—all data—are subjective and determined by a person's standpoint. He now accepts inductive inferences from our experiences to events in the physical world that cause them. He thus gives an account of induction and of scientific reasoning which assumes events continuous with those we perceive and extrapolates from perceived relations to relations among events in physical space-time.

In the 1930s and 1940s, Russell's late period, these themes dominate his discussion of knowledge, especially that of the *a priori* principles that guide scientific reasoning. The principal texts are *Inquiry into Meaning and Truth* (1940) and *Human Knowledge* (1948). The paper "On Verification" (1938) is also important. The postulates are those actually involved when scientists or ordinary people pursue a line of reasoning. Of all possible inferences that might be drawn from the data, what governs the decision to follow one and ignore the others? On his view, it is the presence of *a priori* expectations about the world.

These have a psychological origin. They are caused by experience but not inferred from it and exist as primitive beliefs or habits. For example, if idly watching the path of a cat crossing an empty room, you would be astonished if it winked in and out of sight, or if it should be here and then suddenly somewhere quite different. This is because we bring expectations about continuity and permanence to experience that are created by experiencing certain qualities and general patterns in the world, not just by our psychology. Our expectations, which, made explicit, are postulates of science, are therefore about the world but known *a priori*, since we bring them to experience. His late period also focuses on "linguistic epistemology," that is, with constructing languages to aid us in discovering what the data are and what we must infer.

9 Analysis and the theory of meaning

In his early period, Russell's theories of meaning are confined to what words and sentences denote. These occur in his early metaphysical works such as the *Principles* (1903). Russell thinks the meaning of a name, verb, or predicate, is the entity it denotes, which may be concrete or abstract, in time and space or outside them. Words that occur as subjects of sentences denote either particulars or universals (things or concepts), while predicates and verbs denote only universals.

Though the things corresponding to words and phrases are their *meanings*, this is not to say that we are aware of them as meanings. Russell explains this with his doctrine of acquaintance with universals. We can be acquainted with a patch of color and not know that it is an instance of the word "yellow." For this, the particular patch is not enough: we need to grasp the universal *yellow*. The understanding of meaning is by way of universals.

The above remarks concern words. Until 1910, the meaning of a sentence is also viewed as a single complex entity—the proposition *aRb* of two objects *a* and *b* with relation *R* to one another. On this view, a sentence has a meaning (the complex entity) even if it is not believed or judged. Eventually, Russell finds this doctrine unacceptable and replaces it with the theory that a sentence has no complete meaning until it is judged or supposed or denied by someone. On this view, *judging* is not a relation between a person and a single entity *aRb*, but a relation between a person and *a*, *R*, and *b*. The proposition is broken into parts and enters into a person's belief, which arranges them in a meaningful way.

There is now no single entity *aRb* that is the meaning of a sentence. There are only sentences, which are incomplete symbols, and the context of belief that gives the sentence a complete meaning. This is another analysis using the theory of descriptions: a sentence "*aRb*" is an incomplete symbol that acquires meaning when judged or believed but is otherwise meaningless. That a person has a belief is a fact, and the entities that constitute the meaning of the sentence are gathered together with the believer in that fact. Just as the theory of descriptions replaces descriptions with predicates, so here it replaces propositions with facts of belief.

This theory requires that a person is acquainted with the things that enter into the belief, for example, with *a*, *R*, and *b*. But acquaintance with this data is not enough to make a judgment. To believe or judge, a person must also be acquainted with the *form* in which things are put together. In this case, he or she must grasp what it means to assert a relation.

In his middle period, Russell's analysis of language and meaning develops well beyond his early views, which hardly constitute a theory of meaning at all. Some texts are "Philosophy of Logical Atomism" (1918), "On Propositions" (1919), *Analysis of Mind* (1921), "Vagueness" (1923), "Logical Atomism" (1924), and "The Meaning of Meaning" (1926). The novelty is the attempt to explain language and meaning in terms of causal relations to the world.

For words, Russell adopts a partly behaviorist account where words are classes of sensations (mouth movements, sounds, etc.) and acquire meaning by association with other sensations of the things meant. For example, a child experiences certain sensations that are collectively a toy and learns to make certain sounds that are collectively the word "toy." Departing from behaviorism, Russell says the sensations of the toy give rise to images associated both with the toy and the word "toy." The meaning of "toy" and the images are products of cause and effect where the word or image can come to have the effects the original sensations had.

Russell had, in his early period, resisted reliance on images in his theories of meaning, but in his middle period he embraces them. Belief is no longer a relation among things (*a*, *R*, *b*, and a person). Instead, the content of belief consists of images and feelings (acceptance, doubt, etc.). And verbs occur in sentences under new constraints. They now do not name anything (denote no universal) but merely create a structure of words that is the sentence. Just as an egg carton is not a kind of egg but a means of holding eggs in a pattern, verbs are now merely means of creating a spatial (if written) or temporal (if spoken) relation among words in sentence.

Russell's late period work on language occurs in *Inquiry into Meaning and Truth* (1940). There he tries to solve philosophical problems by constructing proto-languages and artificial languages. As before, we have feelings toward images or words. He now builds on this by developing a psychological or causal theory of a hierarchy of languages having logical constraints. In the logically fundamental language, we use single-word sentences for immediate experiences. But our utterances also convey feelings like doubt or certainty toward beliefs, as when we wonder "Is it true that this is sugar?" With this idea, Russell explains the psychological meaning of logical words like "true."

We also find a new analysis of indexical words like "I," "this," and "here." At the same time, he tries to identify a minimum vocabulary for sciences like physics and to identify the kinds of sentences that can serve as premises. Since he is interested in physics and psychology, he asks whether the words and sentences that report the observations of a physicist will also serve in the same way for psychology.

Philosophers besides Russell have pursued their own conceptions of analysis. Russell's friend G. E. Moore, who influenced Russell as well as later philosophers, is an important example. But there is no doubt that Russell is most responsible for founding the movement of analytic philosophy. In the following pages, Russell's contribution to that philosophy is described in greater detail. The next chapter describes Russell's logicism, and the chapter following describes the elaborations he added to it to avoid paradoxes it faced. Following that, in Chapters 4 through 6, we return to the broader doctrines about things, knowledge, and language sketched above.

CHAPTER TWO

Naïve logicism

Bertrand Russell's greatest achievement after the invention of modern logic was his use of that logic to analyze mathematics and show that its true nature is logic. The view that mathematics is logic is called *logicism*. To demonstrate his logicist thesis, Russell analyzed mathematics to show that all mathematical concepts, and especially the concept *number*, can be defined in terms of logical concepts and that all mathematical truths can be deduced from logical truths. The attempt to demonstrate that mathematics is just logic is called the *logicist program*. Russell first described his logicist program in his 1903 *Principles of Mathematics*, carried it out in elaborate detail with Alfred North Whitehead in their 1910–13 three-volume *Principia Mathematica*, and presented it informally in his 1919 *Introduction to Mathematical Philosophy*.

Russell was not the only person to argue that mathematics can be derived entirely from logic. Gottlob Frege (1848–1925) had argued for the view informally in 1884 and Richard Dedekind in 1888. Frege then argued for it rigorously in 1893 and 1903, though he did so using a strange and difficult notation.[1] And before this, the philosopher Rudolf Hermann Lotze had asserted though had not argued for the view that mathematics is just logic. But Russell's reduction of mathematics to logic is the one that brought the attention of the world to the subject and that developed the idea in its greatest detail and sophistication.

In May 1901, while writing his 1903 book on logicism, Russell discovered a contradiction in his logic. The contradiction is now concepts and five axioms. Dedekind has shown the same thing, with a little less elegance, in 1888. Hence, these axioms are often called the *Dedekind-Peano* axioms.

Since the real numbers and all their properties and operations can be derived from the natural numbers, and the natural numbers can be derived from Peano's five axioms and three primitive concepts, that means that all classical mathematics can be derived from Peano's five axioms and three concepts. Russell then only needed to show that Peano's three undefined concepts can be defined with logical concepts and his five axioms deduced from logical truths to show that all classical mathematics can be derived from logic, thus showing that mathematics is just logic.

The reason why Russell had to define Peano's primitive concepts in terms of logical concepts before deriving Peano's axioms from logical axioms is this: to deduce a statement from other statements, all of the concepts the first statement contains must be possessed by the other statements, or else definable in terms of concepts they possess, otherwise the first statement cannot be deduced from them. Peano had deduced all the principles of arithmetic from his axioms using just three undefined concepts—*zero*, *number*, and *successor*, plus logical concepts such as *any*, *all*, *the same as*, *not*—to express his axioms. Russell thus had to define *zero*, *number*, and *successor* in terms of the logical concepts before he could deduce Peano's axioms from logical propositions.

2 Introducing Peano's axioms

Before looking at how Russell defines Peano's concepts in terms of logical concepts and derives Peano's axioms from logical truths, let us first see how Peano derives the natural numbers and arithmetic from his axioms, and the reasons why Russell thinks Peano's primitive concepts need further defining at all. For now, let *number* mean *natural number*. Peano's axioms are then:

1. 0 is a number
2. the successor of any number is a number
3. no two numbers have the same successor
4. 0 is not the successor of any number

5 any property belonging to 0 and to the successor of any number which has it belongs to all numbers (mathematical induction)

Using these five axioms expressed in terms of *zero*, *number*, and *successor*, Peano defines the natural numbers as follows: 1 is the successor of 0, 2 is the successor of 1, and so on. This gives us an endless series of continually new numbers, because, by axiom 2, we can go on endlessly defining, and by axiom 3 no new successor is the same as an earlier successor, and by axiom 4, no successor is 0. This insures that the series is not circular but is an endless progression beginning with 0 and such that every successor is a new number.

Finally, axiom 5 guarantees that the series contains *all* the natural numbers, because by the first axiom, 0 is a number, and by axiom 2 if something is a number so is its successor. Axiom 5 says that if 0 has a property and the successor of every number with the property also has it, then all numbers have the property. And since 0 is a number and the successor of any number is a number, axiom 5 lets us assert that every natural number is a number in this series. Put another way, saying that all the natural numbers are in the series of numbers defined by Peano's axioms is the same as saying that all natural numbers are in the set N of numbers generated by these axioms and definitions. And because 0 is in N and the successor of any member of N is in N, axiom 5 guarantees that all natural numbers are in the set N of all the numbers generated by Peano's axioms.

Peano began the series of natural numbers with the number 1: his first axiom is "1 is a number." In Russell's version of Peano's system, the natural numbers begin with 0: Russell's first axiom is "0 is a number." We will use Russell's definition of the Peano axioms. Though we have defined 0 and all the natural numbers with them, there are no positive or negative numbers yet, much less fractions or irrational numbers. These are all defined later in terms of natural numbers.

As well as defining the natural numbers, Peano must define addition and multiplication for them. Here are his definitions: Let Sn mean the successor of n. Then $m + n$ for any numbers m and n is the number such that

6 $m + 0 = m$, and

7 $m + Sn = S(m + n)$

From these two rules, we can calculate $m + n$ for any numbers by applying the rules repeatedly until we arrive at the answer. (It is a *recursive* definition.) For a simple example, we calculate $3 + 2$:

$$3 + 2 = 3 + S1$$
$$3 + S1 = S(3 + 1)$$
$$S(3 + 1) = S(3 + S0)$$
$$S(3 + S0) = SS(3 + 0))$$
$$SS(3 + 0)) = SS3$$
$$SS3 = S4$$
$$S4 = 5.$$

This definition presupposes that we have already defined each number in terms of its successor and so know that $S0 = 1, S1 = 2$, and so on. Peano's definition of addition essentially defines $m + n$ as being m applications of the successor function, first to n, then to the successor of n, then to the successor of the successor of n, and so on for m applications.

Multiplication is similarly defined: $m \times n$ is the number such that

8 $m \times 0 = 0$, and

9 $m \times Sn = m + (m \times n)$.

For a simple example, we calculate 2×3:

$$2 \times 3 = 2 \times S2$$
$$2 \times S2 = 2 + (2 \times 2)$$
$$2 + (2 \times 2) = 2 + (2 \times S1)$$
$$2 + (2 \times S1) = 2 + (2 + (2 \times 1))$$
$$2 + (2 + (2 \times 1)) = 2 + (2 + (2 \times S0))$$
$$2 + (2 + (2 \times S0)) = 2 + (2 + (2 + 0))$$
$$2 + (2 + (2 + 0)) = 2 + (2 + 2) = 2 + 4 = 6.$$

In other words, $m \times n$ equals m added to itself n times. And we have already defined addition and so may use it in defining multiplication.

The statement of definitions like "$m + 0 = m$" and "$m + S(n) = S(m + n)$" may appear to be more like axioms than definitions, that is, like extra principles added to the Peano axioms to permit addition and multiplication. And if by "definition" we mean something that adds nothing new to a system, but only introduces new notation for what can already be asserted with other symbols, then because Peano's definitions of addition and multiplication do seem to introduce new principles not stated by Peano's axioms, thus extending the expressive power of Peano's axioms, the definitions themselves seem more like axioms than definitions.

In deriving arithmetic from Peano axioms, we are allowed to use logic (indeed, we must use logic to derive anything from anything else), and for Dedekind, Peano, and Russell, set theory is a part of logic. As Russell shows in *Principles of Mathematics*, addition and multiplication can be defined in terms of set theory using the operation of set union. This makes Peano's definitions of addition and multiplication for arithmetic *theorems* of logic, and since nothing new is introduced by them, they are *definitions* as well. So if set theory is a part of logic, addition and multiplication are both definitions and theorems.

Russell defines addition and multiplication in terms of set theory as follows: The union of two sets is the set whose members belong to either of the original two. Then, $m + n$ is the number of members of a set that is the union of two sets, one having m members and the other n members, where no item belongs to both. Multiplication is similarly defined. Take two classes, one with m members and the other with n members. Let their multiplicative class be the class formed from all possible ordered pairs consisting of one member from each of the two sets. The product $m \times n$ is the number of ordered pairs in the multiplicative class.

Independently of set theory, addition and multiplication can be derived from second-order logic in ways corresponding to these set-theoretical definitions. So second-order logic can also justify Peano's definitions of addition and multiplication as theorems of logic. Addition and multiplication can thus be derived from logic when logic includes either second-order logic or set theory. And from this, Peano's principles of addition and multiplication are both theorems and definitions of the logic.

Following W. V. Quine, many people view set theory as part of mathematics rather than logic. Also following Quine, and to avoid paradoxes of second-order logic, many people limit the logic they use to first-order logic. Then, in a first-order system of logic without set theory, Peano's principles of addition and multiplication cannot be proved from prior logical assumptions but must be explicitly added as axioms to the original five axioms. But when logic includes set theory or second-order logic, and Russell's logic includes both, the Peano definitions of addition and multiplication are both theorems and definitions.

3 The definition of number

We can now define *logicism* in terms of Peano's axioms: logicism is the program of defining Peano's three primitive concepts *zero*, *number*, and *successor* in terms of logical concepts and deriving Peano's five axioms from logical truths. But how, specifically, does Russell do this? Quite simply, he both defines the three concepts and deduces the five axioms from his definition of *number*. Russell's definition of number—commonly called the "Frege-Russell" definition of number because Russell and Frege each proposed it—is this: a number is the set of all sets having the same number of elements in them, 2 being the set of all couples, 3 the set of all triples, and so forth.[3]

At first glance, this definition of a number seems circular. It defines number in terms of itself, and uses specific numbers like 2 to define 2, 3 to define 3, etc. But *number* can be defined without using the concept itself, like this: A number is a class of all classes having the same number of members. Two classes have the same number of members when they are *similar*. A number is thus a set of all sets similar to each another. We then only need to define similarity without the concept of number.

Before doing this, however, we introduce some logical notation: Russell's symbols for relations. We use capital letters R, S, or T to represent relations Then, if R is a two-place relation—if it relates one thing to another—we use two variables, x and y, to represent the two things, and write xRy, which says x has relation R to y. For example, if we let R represent the relation *greater than*, xRy says that x is greater than y. Some people write this as $R(x, y)$ instead of xRy. But again, if R is the relation *greater than*, $R(x, y)$ says x is greater than y. In this guide, we write two-place relations as xRy. Now let's return to the definition of similarity.

Peano's definitions of addition and multiplication are thus also true for the new interpretation. The Peano axioms, including addition and multiplication, are as true for interpretation (b), 0, 2, 4, 6, 8, 10, . . . , as they are for 0, 1, 2, 3, 4, 5, ...

In the same way, other interpretations make Peano's axioms true. For another example, let the numerals, 0, 1, 2, 3, 4, 5 . . . , mean what we usually mean by:

(c) 100, 101, 102, 103, 104, 105 . . .

Here 0 means what we usually mean by 100, but *number* and *successor* stay the same. Addition and multiplication get redefined so that $m + n$ now means what we ordinarily mean by $m + n + 100$, and $m \times n$ means what we ordinarily mean by $m \times n + 100$.

Here is another oddity: how do we say what we ordinarily mean by 0 and 1 with interpretation (c)? If negative numbers are developed, we can then use "–100" to mean what we usually mean by "0." This seems to go beyond the Peano axioms. We have a similar problem with interpretation (b). With it, we can only count couples of things, say, pairs of socks. But we cannot count odd numbers of socks. We must first extend the system beyond the Peano axioms, in this case, by developing fractions. Then 1/2, 3/2, 5/2 can mean what we usually mean by 1, 3, 5.

Differing interpretations that each satisfy a set of axioms are *isomorphic* to one another. Specifically, if each makes the axioms true and there is a 1-1 correspondence between the objects of the interpretations, they are isomorphic. For Peano's axioms, each will be a progression of objects with one object, 0, having no successor. More new words: an interpretation making a set of axioms true is a *model* of those axioms and is said to *satisfy* the axioms. There are an infinite number of interpretations of Peano's axioms that satisfy them. Which interpretation should we use to interpret the axioms and the numerals?

5 The true meaning of "number"

Mathematicians commonly say that for the purposes of pure mathematics, it doesn't matter what the symbols for numbers mean—they can even be left uninterpreted and pure mathematics will get along just fine without them. In fact, alternative geometries have been worked out without clarifying what they mean by *point*, *line*, or *plane*. As long as the axioms are consistent (i.e., don't lead to a contradiction), it is acceptable mathematics. Still, mathematicians also often claim that an intuitive sense of the symbols guides their work. Moreover, logicians and set theoreticians who analyze mathematics into fundamental concepts want to understand what it is all about whether their work matters to mathematics or not. So again, which interpretation should we use?

Russell argued that our numbers cannot be just any of an infinite number of progressions that satisfy the Peano axioms, because we want them to "apply in the right way to common objects. We want to have ten fingers and two eyes and one nose. A system in which 0 means 100, 1 means 101, and so on, might be all right for pure mathematics, but would not suit daily live."[4] Russell's definition of number picks out this standard interpretation. Still, Russell understands that we can use an interpretation in our everyday lives where 0 means what we now mean by 100. The model we choose to interpret our numerals is wholly a matter of convention, and Russell understands that. So what is he saying?

Even though the model we use for arithmetic is a purely conventional choice, there is just one interpretation that almost all members of the human race have chosen, and that is the interpretation that says people have one nose and two eyes. This is the interpretation Russell's definition of number gives us. It is wholly a matter of convention that humans have chosen this model, but it *is* the convention we have chosen. Thus, picking Russell's definition of number to interpret Peano's axioms may entirely be a matter of convention, but being the established human convention, it is still the true meaning of "number" as we use the word.

By saying that his definition of number is the right one, Russell also means that by his definition, numbers apply to classes. By his definition—that a number is a set of similar sets—to say that there are 12 Apostles means that the *class* of Apostles has the property of being 12. And this is just how we use numbers. It cannot be the case, for example, that the number 12 is a property of each Apostle, so that Peter is 12, Paul is 12, and so on. Being the property of a class is how natural numbers actually work. Not all definitions of number make this clear. Only by viewing numbers as properties of sets will numbers apply properly to the world.

Russell's definition of number is thus again the one needed for applied mathematics. This does not mean that his interpretation of Peano's axioms is itself applied mathematics. A system of applied mathematics, Russell says, must contain nonlogical constants, and Russell's definition of number and interpretation of the Peano axioms is purely logical. Logicism is pure mathematics, not applied mathematics. But pure mathematics cannot be used for applied mathematics unless numbers are something like sets of similar sets.

(Note: First we say that for Russell a number is a set of all sets similar to one another, then we say that for Russell a number is the property of a set. But these are just two ways of saying the same thing. Take the set of all the dogs in the world. To say that something belongs to the set of all dogs is to say that it has the property of being a dog. Similarly, to say that the number 12 is the set of all sets having 12 members is to say that each set in it has the property of having 12 members.)

According to one standard theory, the meaning of a word is either its extension, its intension, or both. A word's intension is a definition picking out every object the word applies to. Its extension is the set of things the intension picks out. By Russell's definition, the number 3 is the set of all sets with 3 things. Thus, Russell defines numbers as the extension of a numeral, so for those who believe the meaning of a word is its extension, Russell's definition of *number* gives the true meaning of the word.

For those who think the true meaning of a word is its intension, that is, a definition that picks out all and only those things in its extension, Russell's definition of number—*the set of all sets similar to one another*—is what defines the set of things numerals pick out. And for those who think the meaning of a word is both its intension and its extension, they can have both the definition and the set it defines.

Philosophers sometimes say that Russell's logicism substitutes classes for numbers, and that the use of his definition of number is justified *as a substitute* only because the classes do everything numbers do in mathematics. But this is a misunderstanding. If the meaning of a word is its extension and/or intension, then Russell's definition of number is exactly what we mean by a number, it is not just a substitute for it.

Peano was aware that an infinite number of interpretations satisfied his axioms, but he intended that we use his system with an intuitive understanding of its three fundamental concepts. He also thought that we could define those concepts by "abstraction," where we abstract from the different interpretations what is common to all of them. But what is common to all of them is just that they satisfy Peano's axioms, that is, the axioms themselves are the only things that are common to all the different interpretations. Peano's idea for defining *number* thus does not pick out any particular interpretation. And without a particular interpretation, and especially one that applies numbers to classes in the way that humans actually do, we will not be able to apply mathematics to the world properly, or really, at all. With Russell's definition, we can.

6 The concepts defined and axioms derived

We now show how, with Russell's definition of number, one can define Peano's primitive concepts and deduce his axioms from logic. Russell's most accessible account of this is in his *Introduction to Mathematical Philosophy*. But there are places in it where people sometimes get stuck. We thus take special care in what follows to explain what we feel are the difficult spots of the *Introduction*. In particular, we find that people first encounter difficulty reading it with the definitions of posterity (N-posterity, R-posterity, P-posterity). These are really not hard to understand, but pay attention when we get there. We will make them clear to you. But first we will go through Russell's simpler 1903 version of logicism.

From Peano's five axioms and three undefined concepts, the natural numbers and their properties can be derived along with addition and multiplication. And once we have the natural numbers, we can define negative and positive numbers (when $a = b + x$, x is negative when $a < b$ and positive when $a > b$) and fractions (as ratios of natural numbers in the form m/n). And with the rational numbers, we can define the irrational numbers with what is called a *Dedekind cut*, explained in detail below.

To derive all this from logic, Russell will logically define Peano's three undefined concepts and deduce Peano's five axioms from logic using those definitions and logic. In 1903, in Chapter 14 of his *Principles of Mathematics*, Russell does this in a simple and straightforward way. First he introduces his logical (i.e. set-theoretic) definition of *number* as a set of sets all having the same number of members, that is, that are *similar* to one another. Then he defines the individual natural numbers, beginning with 0 and 1, using his general definition of number. With addition and the number 1, he defines *successor of*. And with these concepts logically defined, he derives the rest of the natural numbers, proving that Peano's axioms are logical truths. Now let's go over this in detail.

First, in the 1903 *Principles*, Russell defines the natural numbers with his general definition of number like this:

(a) 0 is the class of all classes whose only member is the null set

(b) a number is a class of all classes similar to any one of themselves

(c) 1 is the class of all classes not null and such that if x and y belong to them, $x = y$

(d) $n + 1$, the successor of any number n, is the number of the union of a class a of n members and a class of one member x, when n is a number and $x \notin a$

(e) the natural numbers are the members of every class s that 0 belongs to, and that $n + 1$ belongs to if n belongs to it

(Russell sometimes defines 0 as the class of all classes whose only member is the null set, but often says more simply that it is the class whose only member is the null set. These definitions are the same. In set theory, two sets are identical when they have exactly the same members—not the same *number* of members, but exactly the same members. All classes containing only the null set have exactly the same member, the null set, and so are identical, that is, they are all the same set. There can only be one set whose only member is the null set.)

Russell's definitions above define *zero*, *number*, *successor*, and each natural number. And given these definitions one can prove that the five Peano axioms are logical truths, expressed wholly in logical terms and true for all possible cases. (A logical truth, after all, is true for all possible cases of it.) Specifically, from these definitions the following Peano axioms are obviously true:

1 0 is a number

2 if n is a number, $n + 1$ is a number, where $n + 1$ means the successor of n

every inductive property. So to define numbers, we need to define objects that belong to *every* inductive class. And for that, we need the concept of *posterity*.

5 the posterity of *n* =df the class containing all the members of *every* hereditary class that *n* belongs to (where heredity is defined in terms of *successor of*).

And from this we can easily define the posterity of 0.

6 the posterity of 0 =df the class of objects that belong to every hereditary class 0 belongs to (where heredity is defined in terms of *successor of*).

The posterity of 0 is thus the set of numbers that belong to *every* hereditary class 0 belongs to. This is just a precise definition of mathematical induction (axiom 5), that every property that belongs to 0, and to the successor of every number with that property, belongs to all numbers. In other words, natural numbers have *all* the hereditary properties 0 has. We can now define the natural numbers in terms of posterity.

7 the natural numbers =df the posterity of 0

This defines *natural number* in terms of *zero* and *successor* by defining *posterity* in terms of *zero* and *successor* and defining *natural number* in terms of *posterity*.

Two of Peano's axioms can be inferred from the definition of *natural number* in terms of *posterity* alone. The definition of natural number implies Peano axiom 1, that 0 is a number, because the natural numbers are defined as the posterity of 0, and the posterity of a number includes the number itself, so 0 is a natural number. The definition of natural number also implies Peano axiom 5, the principle of mathematical induction, because the posterity of 0 just is the definition of mathematical induction. In other words, since the natural numbers are defined with mathematical induction (posterity), they must imply mathematical induction in return (axiom 5). The definition of natural number also allows one to weaken Peano axiom 2, which says the successor of a natural number is a natural number. We can drop the last part, because we have just defined natural numbers in terms of successors, so the successor of a natural number is already a natural number by definition.

We now define 0 and *successor* with the Frege-Russell definition of number, and derive the other three Peano axioms from them. 0 is the set whose only member is the null set, and the successor of a number n is the number of a class with n items to which an item x is added where x does not belong to the original class. Note: the concept *number of a class* in this definition is already defined by the Frege-Russell definition of number.

We then prove Peano axiom 4, that 0 is not the successor of any number: by the definition of successor, the successor of a number is the number of a class with at least one member, but this is not true of 0, so 0 does not succeed any number. The weakened version of axiom 2—every number has a successor—follows from the definition of successor: since any class can have a member added to it, its number of members is then the successor of its earlier number of members. It only remains to define axiom 3, that no two numbers have the same successor.

If the universe contains an infinite number of things, then for each natural number there will be other natural numbers greater than it, specifically, for any number m, there will always be the number $m+1$. Then we can say that $m+1 = n+1$ only if $m = n$. But if the universe is finite, this is not the case. If there are, say, 10 things in the universe, the successor of 10 is null, because there is no class of 11 things, and the successor of 11 is likewise null. Then $10+1 = 11+1$. Peano axiom 3 holds only for an infinite universe.

In 1903, Russell thought he could logically prove that the universe contains an infinite number of things, if only abstract objects like propositions, numbers, or classes. By 1910, he realizes this cannot be proved logically. He therefore has to accept as an *assumption* that the universe consists of an infinite number of things. This assumption is called the "axiom of infinity." By assuming an infinite universe, Peano's third axiom is valid. In his early logicism, however, Russell did not realize that he needed this assumption.

If the universe is infinite, Russell can define Peano's three basic concepts in terms of logical concepts, and derive Peano's five axioms from them. Granting him this one assumption, his logicist thesis seems vindicated. Also, by defining the basic concepts this way, one can say what it means for something to be finite. Each natural numbers is finite. With the natural numbers, one can define this important property.

8 Ordering the natural numbers

Because *natural number* is defined with *posterity*, and *posterity* is defined with *successor*, natural numbers are finite. Proof: finite natural numbers can be reached from 0 by successive additions of 1. Thus, finite natural numbers obey mathematical induction starting from 0. Thus, finite natural numbers possess every property possessed by 0 and by the successor of every number possessing the property. Thus, finite numbers are the posterity of 0 as defined by the successor relation.[5] And by definition, these are all the natural numbers.

Any natural number can therefore be reached from 0 by a finite number of steps from n to $n + 1$. Or really, any natural number can be reached from any other natural number by a finite number of steps from n to $n + 1$, or from $n + 1$ to n if moving toward 0. So every natural number is finite. This defines *finite number* in terms of mathematical induction.

There are only a finite number of numbers between any two natural numbers, because defining natural numbers with mathematical induction makes them so. The only progressions that make mathematical induction true have a finite number of steps between any two of their members. After all, if there were an infinite number of steps between them, you would never reach one from the other, and could not define the unreachable one with mathematical induction.

For example, moving step-by-step from –1 in the series –1, –1/2, –1/4, –1/8 . . . 1/8, 1/4, 1/2, 1, you will never reach 1, for there are an infinite number of steps between the two—the steps from –1 to 1 go on forever. But a series of numbers defined as those reached by repeatedly adding or subtracting 1 starting from any other member of the series makes every number in it finite, with the number of successor or predecessor steps between any two of them also finite.

Finally (before discussing order), though each natural number is finite and the number of steps with the successor or predecessor function between any two is finite, the class of finite natural numbers itself is infinite. There are an infinite number of finite integers: the successions go on forever, but each element in the progression, being a finite number of steps from any other, is itself finite. The number of natural numbers thus cannot be a natural number, but must be a higher "infinite" number. In fact, it is $\aleph_0$ (aleph null), the smallest infinite number.

In addition to defining the natural numbers and their finitude, mathematical induction (posterity) also defines their order. In fact, it defines the order of magnitude, 0, 1, 2, 3, . . . that we are so familiar with. Simply defining the natural numbers with the Frege-Russell definition of number, as a set of similar sets, does not define their order. What organizes objects into a series is some relation among them. But the natural numbers do not have just one order. As Russell says in *IMP* (p. 29), they have all the orders of which they are capable, and the natural numbers are capable of an infinity of orders. For example, you might start with 0, list all the odd numbers, then all the evens, like this: 0, 1, 3, 5, . . . , 2, 4, 6, and so on. Or you might start with 1, then take all the evens, then all the odd multiples of 3, then the odd multiples of 5 but not 2 or 3, and so on. Each order is defined by a different relation among them. Their most important order is their order of magnitude.

To order any set of objects, an ordering relation *R* must have three properties. It must be:

1. asymmetrical—if *a* has relation *R* to *b*, *b* does not have relation *R* to *a*.
2. transitive—if *a* has relation *R* to *b*, and *b* has *R* to *c*, then *a* has *R* to *c*.
3. closed for the series—for any two objects *x* and *y* being ordered, either *x* has relation *R* to *y* or *y* has *R* to *x*.

The successor relation might at first seem capable of ordering the natural numbers according to magnitude by itself, for 1 is succeeded by 2, 2 by 3, 3 by 4, and so on, in what seems like a series. But the successor relation is not transitive: though 1 is succeeded by 2, and 2 is succeeded by 3, 1 is not succeeded by 3. Nor is the successor function closed: it cannot tell us whether 3 comes before 7 or vice versa, and that is what an ordering relation must do. We can, however, use the successor relation to define a transitive relation, namely, the relation of a number n to its posterity.

Remember, a property is hereditary if, whenever a natural number n has it, its successor n' has it; a class is hereditary if, whenever n is a member of it, so is n'; the posterity of n is the class of numbers with every hereditary property n has; and the posterity of 0 is the class of numbers with every hereditary property 0 has. Now 0 has every hereditary property that 0 has, so 0 belongs to its own posterity, and so does every other natural number, because the natural numbers are *defined* as the posterity of 0. Clearly, then, posterity is the relation "less than or equal to" ($\leq$).

Posterity ($\leq$) is a transitive relation: for any natural number x, y, z, if $x \leq y$ and $y \leq z$, then $x \leq z$. However, it is not asymmetrical: when $x \leq y$, we don't know if $x < y$ or if $x = y$. The relation *less than* ($<$), however, *is* asymmetrical. When $x < y$, y cannot be less than x. To define an ordering relation with the successor relation, we want *less than* ($<$), which Russell calls *proper posterity*, not *less than or equal to* (*posterity*). Here is the definition: a number n is in the *proper posterity* of m if n possesses every hereditary property possessed by m'. This is transitive, asymmetrical, and also closed—for any two numbers m and n, either $m < n$ or $n < m$. With this, we can define the natural numbers in their familiar order of magnitude.

Both posterity and proper posterity are defined with the *successor* relation $n' = n + 1$, because they are defined with heredity and heredity is defined with *successor*. A relation defined with this sense of successor will clearly order the natural numbers: moving from n to $n + 1$ is the very essence of the natural numbers. But not all objects can be ordered using this sense of *successor*, for example, consider the series of kings of England. Here, we move from one member of the series to a successor, but here "successor" means "eldest legitimate son," not $n + 1$.

Or consider fractions. The relation *less than*, being asymmetrical, transitive, and closed, will order any two fractions m/n and p/q, determining whether one precedes the other or the other precedes the one. But fractions do not have successors the way natural numbers do. Fractions are *dense*. For any two fractions there is always a third between them in order. The notion of *less than*, when defined as proper posterity with a notion of *hereditary* which itself

We then generalize this to define all positive and negative integers, not just +1 and –1. When c is the successor of b and b the successor of a, the relation between c and a is 2 applications of the successor function. Denote this as S^2. Then, cS^2a means c is the successor of the successor of a, that is, $c = a+2$ and that means that S^2 defines +2. Now let S^x mean x applications of the successor function where x is any natural number. Then $+x$ will be S^x, that is, the successor function applied x times. Similarly for negative numbers, $-x$ will be S^{*x}, the predecessor function applied x times. Addition and multiplication for positive and negative integers are defined like those for natural numbers.

Fractions (ratios) are as easy to define. The fraction m/n is the relation between two natural numbers x and y when $xn = ym$. For example, $2/4 = x/y$ when $2y = 4x$, which is true when $x = 1$ and $y = 2$, or $x = 2$ and $y = 4$ or $x = 3$ and $y = 6$, and so on. This thus defines fractions in terms of multiplication of natural numbers, and $1/2 = 2/4 = 3/6 = 4/8$, and so on. We then define addition for them in terms of addition and multiplication of natural numbers: $m/p + n/p = (m+n)/p$ and $m/n + p/q = [(m \times q) + (n \times p)]/(n \times q)$, just as in ordinary arithmetic. Multiplication for ratios is similarly straightforward, namely, $m/n \times p/q = (m \times p)/(n \times q)$.

Fractions are relations between natural numbers, and so different from natural numbers. Natural numbers are classes of classes; fractions are relations between natural numbers, and so relations between classes of classes—not the same at all. (When Russell introduces his theory of types—described in the next chapter—natural numbers and fractions become different logical types and even more different from one another.) Positive and negative integers are also relations between natural numbers, but not the same relations, so fractions are also different from positive or negative integers.

The fractions $0/m$ and $m/0$ are zero and infinity for any natural number m. The zero of the fractions is not the zero of the natural numbers, as explained above. And the infinity of the fractions, symbolized as ∞, is not the Cantorian infinite $\aleph_0$. It is a potential infinite; Cantor's is an actual infinite. It is like the series 1, 2, 3, 4, . . . getting progressively larger and larger, rather than like the set of *all* natural numbers {1, 2, 3, 4, . . . } taken at once, as with Cantor's infinity. Cantor's infinity assumes the axiom of infinity—- that there are an infinite number of things. The infinity of the fractions does not.

To order the fractions in a series, Russell defines *greater than* and *less than* for fractions. From here on, let m, n, p, and q be nonzero natural numbers. Then, for two ratios m/n and p/q, m/n is less than p/q if and only if $mq < pn$. Similarly, m/n is greater than p/q if and only if $mq > pn$. In the series of ratios, 0 and ∞ are the smallest and largest numbers. If they are omitted, there is no smallest or largest fraction. For any fraction, $m/2n$ is smaller than it and $2m/n$ is larger, and between any two fractions m/n and p/q when $m/n < p/q$, $(m + p)/(n + q)$ is always greater than m/n and less than p/q.

There are thus always an infinite number of fractions between any two other fractions, unlike the natural numbers and integers. This property is called "compactness" by Russell and density by contemporary mathematicians. Also, there is no fraction p/q that immediately follows another fraction m/n in the series of fractions: no two fractions are consecutive. These are properties of Cantorian infinity, not the infinity of fractions: they cannot be proved without the axiom of infinity.

Our fractions are so far signless. Russell defines positive and negative fractions as he did for the integers: $+p/q$ is the relation $m/n + p/q$ to m/n, where m/n is any fraction at all. This relation means "greater than by p/q," that some number is greater than another by p/q. And $–p/q$ is defined as the converse relation, m/n to $m/n + p/q$, which means "less than by p/q." Positive and negative fractions are clearly different from positive and negative integers: positive and negative integers are relations of classes of classes; positive and negative fractions are relations of relations of classes of classes.

Though the definitions of positive and negative integers and rational numbers are straightforward, it is interesting that both ratios and signed integers are relations of classes of classes but not the same relation, and so not the same, and that signed ratios are relations of relations of classes of classes. Only natural numbers are classes of classes. The difference between the potential infinity of fractions and the actual Cantorian infinite is also of interest.

10 Background to defining real numbers

Other than its answer to the question "What is a natural number?" what is most interesting in logicism is its answer to the question

"What is a real number?" What are new in real numbers are irrational numbers, for example, $\sqrt{2}$ and π. Irrational numbers, or really, irrational lengths or magnitudes, were discovered by the ancient Greeks when they discovered that squares with sides 1 unit by 1 unit long will have diagonals whose lengths cannot be expressed by any fraction or any other numbers known to them, and then discovered many more such magnitudes.

Because irrational lengths cannot be expressed as fractions, and lengths are the subject matter of geometry rather than arithmetic, not all objects of geometry can be expressed in terms of arithmetic. Arithmetic and geometry thus had relatively separate developments from the Greeks on. With the development of algebra, irrational numbers were encountered again, this time as possible solutions to equations. But these made even less sense to mathematicians than irrational lengths in geometry: at least lengths are real. The divide between geometry and arithmetic, and later algebra, thus persisted into the nineteenth century.

Whitehead and Russell define irrational numbers as lengths in *Principia Mathematica* (*PM*), specifically, as the lengths of number lines in analytic geometry, where numbers are correlated with points on a line. They define irrationals as line segments of a number line—as a series of numbers—rather than as single points on it. Russell defines irrational numbers similarly in *Introduction to Mathematical Philosophy* (*IMP*).

Fractions are similarly defined in *Principia* as relations between line segments This differs from *IMP*, where fractions are relations of natural numbers as we defined them in the last section. In *PM*, however, fractions are relations of line segments. For example, 2/3 is the relation between two lines A and B, where A is two-thirds as long as B, and this is defined as being the case when 3 lengths of A equal 2 lengths of B. In *Principles of Mathematics* (*POM*), Russell distinguishes between ratios and fractions (*PM* and *IMP* do not), ratios being relations of natural numbers, as in IMP, and fractions being ratios of lengths as in PM. But Russell's POM definition of fractions as ratios of lengths was unclear, and so is replaced in *PM* with a better one by Whitehead, namely, the one described above.

Three volumes of *PM* were published. A 4th volume—on geometry—was planned but never written. However, one will find places in *POM*, *PM*, and *IMP* where Russell, and Whitehead in *PM*, discuss various geometric subjects as a part of logicism, giving ghost-like hints of what a logicist treatment of geometry might have been like. The definitions of fractions and real numbers in *PM* and of real numbers in *IMP* are examples of this.

The definition of real numbers in *Principia Mathematica* did not originate with Russell and Whitehead, but with Dedekind as a part of the arithmetization of analysis. Dedekind defined certain sets of rational numbers, known as Dedekind cuts, that correspond to points on the number line and represent both rational and irrational real numbers. Russell added the definition of mathematical concepts in terms of logical concepts and the derivation of mathematics from logic.

Mathematicians often conceive of real numbers differently than Russell, as points on a number line. Sometimes that point is described more carefully as the limit of a particular series of fractions that progresses toward it. But for Russell, real numbers are the whole series of ratios approaching that limit, not just the limit itself. In this way, Whitehead and Russell treat them as segments—something geometrical. These make it easier to apply mathematics to the physical world.

11 Real and complex numbers defined

Russell defines both rational and irrational real numbers with what is called a *Dedekind cut*, in honor of Richard Dedekind who first used them to define real numbers. A Dedekind cut divides all the members of a number series into two sets, where every member of the series is in one of the two sets and every number in one set is less than every number in the other. Each set is called a *section.* The section with the lower numbers is the *lower section*, the other is the *upper section.*

There are several different kinds of cuts, depending on how the "point of section"—the point where the two sections are divided —is defined. For a cut in the series of fractions, where α is the lower section, β the upper section, and c a fraction in the series, we might define α as every fraction less than c and β as c and every fraction greater than c. Then, the upper section β has a *minimum* value, namely, c, but the lower section α does not have a maximum value.

The fraction c is thus the minimum value of β; no member of β is less than c. The lower section α has no maximum value, it contains every fraction less than c, but not c itself. As the values of its members increase in size they come closer and closer to c but never reach it; for every one of them, there are always others that are closer to c, and for these too there are still others that are closer. There is no greatest number in α that is greater than all the others.

This lower section α does however have an *upper limit*: it is the number in the series that the members of α approach but never reach as they increase in value, namely, c. Though α has no maximum, it has c for an upper limit. If we reverse the matter and make a cut in the series of fractions where every member of the lower section α is less than or equal to c and every member of the upper section β is greater than c, then α will have a maximum value, namely c, but β will have no minimum. It will, however, have c as a *lower limit*.

Two sets of fractions where one contains every fraction lower than c and the other contains every fraction greater than c would not be a cut. A cut must place *every* member of a series in either the lower or upper section. The two sets just described do not contain every fraction—they leave out c itself. And two sets where one contains every fraction less than or equal to c and the other contains every fraction greater than or equal to c are also not a cut, because the same fraction cannot belong to both sets.

For some cuts, the lower section has a maximum value and the upper section has a minimum value, for example, a cut in the positive integers. Then, if the lower section is every integer less than or equal to 7 and the upper section is every integer greater than 7, the lower section has 7 as its maximum value and the upper section has 8 as its minimum value. Finally, there are cuts with neither a maximum nor upper limit for the lower section and neither minimum nor lower limit for the upper section. These are used to define irrational numbers.

Can we make a cut in the fractions at $\sqrt{2}$? Not exactly. $\sqrt{2}$ is not a fraction at all, so we cannot define a cut in the fractions by letting $c = \sqrt{2}$, $\alpha =$ the fractions less than c, and β the fractions greater than c. But we *can* let $c = 2$ and define α as the set of fractions whose square is less than 2 and β those whose square is greater than 2. Then, α has neither a maximum nor upper limit, and β has neither arithmetic following Cantor. But what we have covered here is the core of logicism, and enough to discuss the modifications Russell made to logicism in *Principia Mathematica* to prevent paradoxes from arising in it. These modifications amount to what we call Russell's *restricted logicism*, described in the next chapter. Russell's logical definition of infinity and its arithmetic, and the problems that arise for it, are discussed in Chapter 7, on "The Infinite."

CHAPTER THREE

Restricted logicism

Russell conceived of logicism—the thesis that mathematics is nothing more than logic—in January 1901 and described it in detail in his 1903 *Principles of Mathematics*. But even before finishing the book, he discovered a contradiction in its logic. Unless he could find a way of expressing his logic so that it did not imply the contradiction, the logicism of the 1903 book, which presupposed this logic, would be unacceptable, for any theory that implies a contradiction contains at least one false premise.

By the time the book was ready to print, Russell still had not found a way to modify his logic so that it would avoid the contradictions. (By this time there were several.) He thus left the book in its original form, contradictions and all, appending a proposed solution to them at the end of the book. The solution, however, was inadequate: only when he presented the mature version of logicism in the 1910–13 *Principia Mathematica*, which contained many complexities in its logic the earlier version did not, was he able to avoid the contradictions and save his logicism.

The original 1903 version of logicism, presented without any of the complexities introduced later, we call *naïve logicism*. This simpler form of logicism was described in the last chapter. In this chapter, we describe Russell's mature logicism of 1910–13. In particular, we describe the complexities added to the theory in order to avoid the contradictions. Essentially, these complexities restrict the use of logic so that the contradictions cannot be stated. We thus call this mature logicism *restricted logicism*.

What are these restrictions? The most important, and the principal difference between naïve and restricted logicism, is Russell's theory of types. But the theory of types makes Russell's logicism *so* complex that the axiom of reducibility must then be introduced to simplify it. This axiom, however, is itself quite complex: Russell has simplified his logicism by adding more complexity to it! Finally, for good measure, Russell defines classes, or really, explains them away, with a "no-class" theory of classes based on his theory of descriptions. He also uses the theory of descriptions to define, or again, explain away, mathematical functions (things like "$f(x) = x^2 + 1$"). These too add a lot of complexity to logicism.

The version of type theory described here is that of *Principia Mathematica*, as Russell's discussion of it in his 1919 *Introduction to Mathematical Philosophy* is inadequate. In this chapter, then, we take the reader straight to the heart of *Principia Mathematica* itself. Similarly, our descriptions of Russell's axiom of reducibility, theory of descriptions, and no-class theory are described as they occur in *Principia*. It is time for the reader to begin understanding that book directly.

1 Background to discovering the Russell paradox

Russell did not begin writing the *Principles of Mathematics* with logicism in mind, much less with any idea of the paradoxes to come. He had written a final draft of it between October and December 1900. Only after that, in January 1901, did he conceive of logicism. In May 1901, he wrote Part One of the book, describing his logic, and began writing Part Two, on logicism, the same month. Then disaster struck: while writing the part on logicism, he discovered a contradiction in the logic of Part One. The contradiction threatened to be fatal to his logic, and so to the theory of logicism based on it.

A contradiction is a statement or set of statements making two claims that cannot both be true. For example, in "It is both raining out and not raining out at the same time and place" at least one claim must be false, and we would have to look out the window or go outside to decide which we should throw out. Similarly, because

Russell's logic implied a contradiction, at least part of it was false and had to be thrown out. But logicism was based on this logic, so if part of the logic was thrown out, the logicism might not work anymore. Russell had to both reject part of the logic and replace that part with something that did not imply any contradictions and yet did all the work of constructing mathematics from logic that the discarded part had done.

At first Russell did not appreciate the difficulty that eliminating the contradiction would present. Though aware of it while writing the section on logicism for his book during June 1901, he did not revise his logic or modify his approach to logicism at that time to avoid the paradox. Similarly, when he rewrote the section on logic in May 1902 again, nothing was modified to avoid the contradiction. In fact, he turned the manuscript into his publisher in its unmodified form, thinking he would find a solution to the contradiction quickly later on, write a brief account of how to avoid it, and get this to his printer and into the book before it went to press.

A solution to the paradoxes, however, was not readily forthcoming, so in November 1902 Russell added a hasty appendix to his book suggesting a tentative solution, while also mentioning the difficulties the proposed solution faced. The solution tentatively proposed was a simpler version of the more complex theory of types he would eventually adopt and publish seven years later in *Principia Mathematica*. Before accepting the theory of types, in *Principia*, Russell would spend five years searching for a less drastic solution. But he could find nothing else that would both eliminate the paradoxes and save logicism.

2 The set of all sets that are not members of themselves

The paradox in the form in which it was first discovered—as the set of all sets that are not members of themselves—arises in Russell's set theory, and also in Frege's, Cantor's, Dedekind's, and Peano's set theories, because, like the others, he assumed what is called the "axiom of comprehension," that for every predicate that can be formulated in the language of logic or set theory, there is a set consisting of all and only those objects the predicate is true of.

In most cases, this assumption is perfectly acceptable; for example, the predicate "x is divisible by 2" defines the class of even numbers. No problem arises here. But some predicates can be defined in set theory that *do* produce contradictions such as Russell's paradox. The axiom of comprehension must thus be restricted somehow so these paradoxes do not arise. The paradox as stated above assumes sets and so is a problem for set theory. Another version of it arises directly from logic itself using predicates, and another still from logic using relations. These are thus problems for logic itself and not just set theory. Similar paradoxes arise within mathematics.

Here is Russell's paradox for set theory in full: Some sets, such as the set of mathematical objects, are members of themselves, because sets are themselves mathematical objects. Most are not, for example, the set of prime numbers is not itself a prime number. There being sets that are not members of themselves, there is a set a of all sets that are not members of themselves, which can easily be defined: a =df the set of all things x such that x is not an element of x. Is a a member of itself or not? If it is, then it isn't, and if it isn't, then it is. So a both is and isn't a member of itself, which is a contradiction.

Here is the same idea expressed a little more formally in the next four paragraphs, with symbolism for those with logic. (Symbols used are "$x \in a$" for "x is a member of a," "$x \notin a$" for "x is not a member of a," "p → q" for "if p is true then q is true," "p ↔ q" for "p is true if and only if q is true," "∀" for "all," "∃" for "some.") Let a be the set of all sets that are not members of themselves: or symbolically, $a = \{x: x \notin x\}$. Then, for all sets x, x is a member of a if and only if x is not a member of x: or symbolically, $(\forall x)[(x \in a) \leftrightarrow (x \notin x)]$. Since, this applies to *any* x, it must apply when x is a. But then a is a member of a if and only if a is not a member of a: or symbolically, $(a \in a) \leftrightarrow (a \notin a)$. This means that if a is a member of a then a is not a member of a, and if a is not a member of a then a is a member of a: or symbolically, $[(a \in a) \rightarrow (a \notin a)$ and $(a \notin a) \rightarrow (a \in a)]$. And this is a contradiction.

Using a few more symbols ("$\sim p$" for "it is false that p" or simply "not-p"; "p v q" for "either p or q is true"), here is why this is a contradiction: Logic translates $p \rightarrow q$ (if p then q) as $\sim p$ v q (not-p or q). Thus, we translate "$p \rightarrow \sim p$" as "$\sim p$ v $\sim p$." And $\sim p$ v $\sim p$ is simply $\sim p$. And we translate $\sim p \rightarrow p$ as p v p, which is simply p. Applying this to $[(a \in a) \rightarrow (a \notin a)$ and $(a \notin a) \rightarrow (a \in a)]$, $(a \in a) \rightarrow (a \notin a)$ becomes $(a \notin a) \text{ v } (a \notin a)$, which is $(a \notin a)$, and $(a \notin a) \rightarrow (a \in a)$