ALSO BY JORDAN ELLENBERG

How Not to Be Wrong
The Grasshopper King

PENGUIN PRESS
An imprint of Penguin Random House LLC
penguinrandomhouse.com

Maps on this page by Caitlin Bourbeau

Portions of this book appeared in different form as "How Computers Turned Gerrymandering Into a Science" and "Five People. One Test. This is How You Get There" in the *New York Times*; as "The Supreme Court's Math Problem" and "Building a Better World Series" on Slate.com; and as "A Fellow of Infinite Jest" in the *Wall Street Journal.*

LIBRARY OF CONGRESS CATALOGING-IN-PUBLICATION DATA

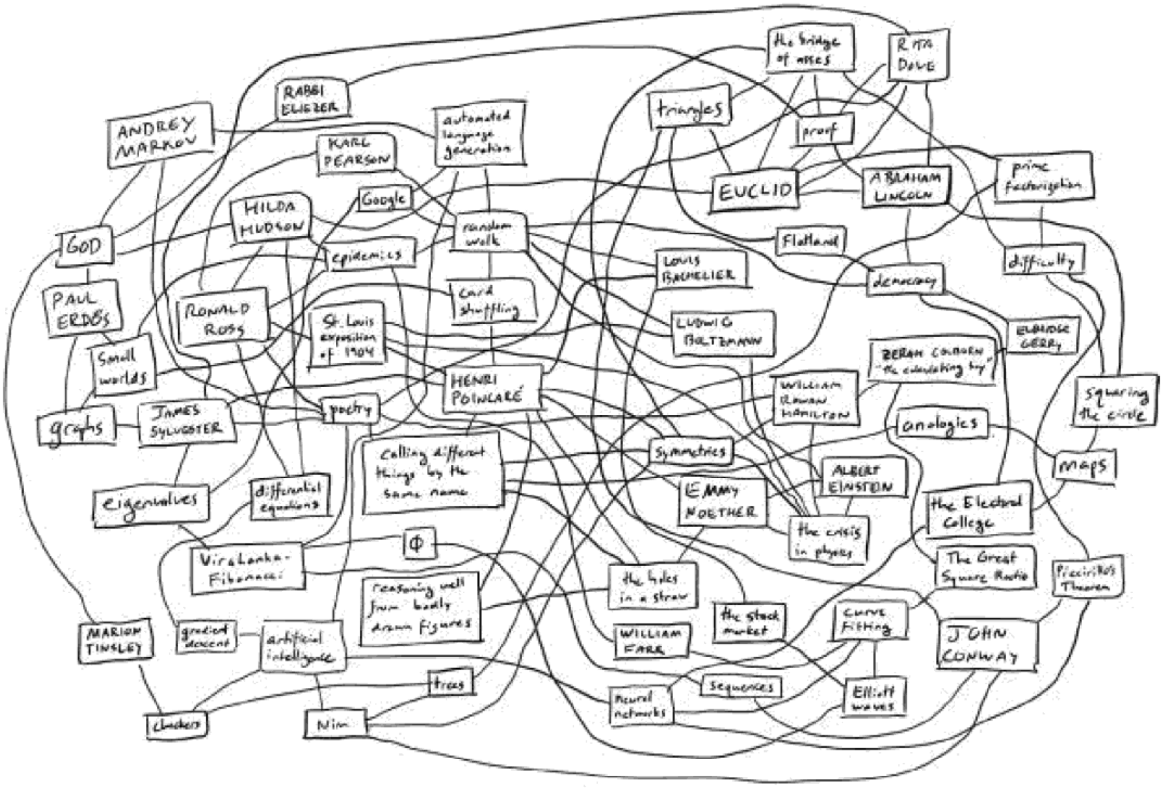
Names: Ellenberg, Jordan, 1971– author.
Title: Shape : the hidden geometry of information, biology, strategy,
democracy, and everything else / Jordan Ellenberg.
Description: First. | New York : Penguin Press, [2021] | Includes bibliographical references and index.
Identifiers: LCCN 2020054440 (print) | LCCN 2020054441 (ebook) | ISBN 9781984879059 (hardcover) | ISBN 9781984879066 (ebook) | ISBN 9780593299739 (export edition)
Subjects: LCSH: Geometry. | Shapes.
Classification: LCC QA446 .E45 2021 (print) | LCC QA446 (ebook) | DDC 516—dc23
LC record available at https://lccn.loc.gov/2020054440
LC ebook record available at https://lccn.loc.gov/2020054441

Designed by Amanda Dewey, adapted for ebook by Cora Wigen

Cover design: Stephanie Ross

pid_prh_5.7.0_c0_r0

CONTENTS



the bridge of asses
RITA DOVE
RABBI ELIEZER
ANDREY MARKOV
automated language generation
KARL PEARSON
triangles
proof
EUCLID
ABRAHAM LINCOLN
prime factorization
Google
HILDA HUDSON
GOD
random walk
epidemics
Flatland
LOUIS BACHELIER
democracy
difficulty
PAUL ERDŐS
RONALD ROSS
card shuffling
St. Louis exposition of 1904
LUDWIG BOLTZMANN
ELBRIDGE GERRY
small worlds
ZERAH COLBURN "the calculating boy"
HENRI POINCARÉ
WILLIAM ROWAN HAMILTON
squaring the circle
graphs
JAMES SYLVESTER
poetry
analogies
calling different things by the same name
symmetries
maps
ALBERT EINSTEIN
EMMY NOETHER
eigenvalues
differential equations
the Electoral College
the crisis in physics
Φ
Virahanka-Fibonacci
The Great Square Rootie
Piccirillo's Theorem
reasoning well from badly drawn figures
the holes in a straw
the stock market
curve fitting
MARION TINSLEY
gradient descent
artificial intelligence
WILLIAM FARR
JOHN CONWAY
trees
sequences
neural networks
Elliott waves
checkers
Nim

INTRODUCTION

Where Things Are and What They Look Like

I am a mathematician who talks about math in public, and this seems to unlock something in people. They tell me things. They tell me stories I sense they haven't told anyone in a long time, maybe ever. Stories about math. Sometimes sad stories: a math teacher rubbing a kid's ego in the mud for no reason but meanness. Sometimes the story is happier: an experience of abrupt illumination that burst open a child's mind, an experience the grown-up wanted to find a path back to, but never quite could. (Actually, this one is kind of sad, too.)

Often these stories are about geometry. It seems to stand out in people's high school memories like a weird loud out-of-scale note in a chorus. There are people who hate it, who tell me geometry was the moment math stopped making sense to them. Others tell me it was the *only* part of math that made sense to them. Geometry is the cilantro of math. Few are neutral.

What makes geometry different? Somehow it's primal, built into our bodies. From the second we exit hollering from the womb we're reckoning where things are and what they look like. I'm not one of those people who will tell you everything important about our inner lives can be traced back to the needs of a shaggy band of savannah-dwelling hunter-gatherers, but it's hard to doubt that those folks had to develop knowledge of shapes, distances, and locations, probably before they had the words to talk about them. When South American mystics (and their non–South American imitators) drink ayahuasca, the sacred hallucinogenic tea, the first thing that happens —okay, the first thing that happens after the uncontrollable vomiting—is the perception of pure geometric form: repeating two-dimensional patterns like the latticework in a classical mosque, or full three-dimensional visions of hexahedral cells clustered into pulsating honeycombs. Geometry is still there when the rest of our reasoning mind is stripped away.

Reader, let me be straight with you about geometry: at first I didn't care for it. Which is weird, because I'm a mathematician now. Doing geometry is literally my job!

It was different when I was a kid on the math team circuit. Yes, there was a circuit. My high school's team was called the Hell's Angles and we filed into every meet in matching black T-shirts with a boom box playing "Hip to Be Square" by Huey Lewis and the News. And on that circuit I was well-known among my peers for balking whenever presented with "show angle APQ is congruent to angle CDF," or the like. Not that I didn't do those questions! But I did them in the most cumbersome possible way, which meant assigning numerical coordinates to each of the many points in the diagram, then grinding out pages of algebra and numerical computation in order to compute the areas of triangles and lengths of line segments. Anything to avoid actually doing geometry in the approved manner. Sometimes I got the problem right, sometimes I got the problem wrong. But it was ugly every time.

If there's such a thing as being geometric by nature, I'm the opposite. You can give a geometry test to a baby. You present a series of pairs of pictures; most of the time the two pictures are of the same shape, but every third time or so, the shape on the right-hand side is reversed. The babies spend more time looking at the reversed shapes. They know *something's* going on and their novelty-seeking minds strain toward it. And the babies that spend more time gaping at the mirrored shapes tend to score higher on math and spatial reasoning tests when they're preschoolers. They're quicker and more accurate at visualizing shapes and what they would look like if rotated or glued together. Me? I lack this ability almost completely. You know the little picture on the credit card machine at the gas station that shows you how to orient the card when you swipe it? That picture is useless to me. It's beyond my mental capabilities to translate that flat drawing into a three-dimensional action. Every single time, I have to run through each of the four possibilities—magnetic stripe up and to the right, magnetic stripe up and to the left, magnetic stripe down and to the right, magnetic stripe down and to the left—until the machine consents to read my card and sell me some gas.

And yet geometry is felt, generally, to be at the heart of what's required for real figuring in the world. Katherine Johnson, the NASA mathematician now well-known as the hero of the book and movie *Hidden Figures*, described her early success at the Flight Research Division: "The guys all had graduate degrees in mathematics; they had forgotten all the geometry they ever knew. . . . I still remembered mine."

MIGHTY IS THE CHARM

William Wordsworth, in the long, mostly autobiographical poem *The Prelude*, tells a somewhat implausible story about a shipwreck victim hurled ashore on an uninhabited island with nothing in his possession but a copy of Euclid's *Elements*, the book of geometric axioms and propositions that launched geometry as a formal subject about two and a half millennia ago. Good luck for shipwreck guy: depressed and hungry though he is, he consoles himself by working his way through Euclid's demonstrations one by one, tracing out the diagrams in the sand with a stick. That's just what it was like to be young, sensitive, poetic Wordsworth, writes middle-aged Wordsworth! Or to let the poet speak:

> Mighty is the charm
> of those abstractions to a mind beset
> With images, and haunted by itself.

(Ayahuasca drinkers have a similar take—the drug reboots the brain, and lifts the mind above the tortured labyrinth it thinks it's stuck in.)

The strangest thing about Wordsworth's shipwreck-geometry story is that it's basically true. Wordsworth borrowed it, with several phrases lifted intact, from the memoirs of John Newton, a young apprentice slave merchant who, in 1745, found himself, not shipwrecked per se, but left on Plantain Island off Sierra Leone by his boss, with little to do and less to eat. The island wasn't uninhabited; the enslaved Africans lived there with him, and his chief tormentor was an African woman who controlled the flow of food: "a person of some consequence in her own country," Newton describes her, then complains, in a truly astonishing failure to grasp the situation, "This woman (I know not for what reason) was strangely prejudiced against me from the first."

A few years later, Newton almost dies at sea, gets religion, becomes an Anglican priest, writes "Amazing Grace" (which has a very different prescription for what book you should study when you're depressed), and finally renounces the slave trade and becomes a major player in the movement to abolish slavery in the British Empire. But back on Plantain Island, yes—he had one book along, Isaac Barrow's edition of Euclid, and in his dark moments he retreated into its abstract comforts. "Thus I often beguiled my sorrows," he writes, "and almost forgot my feeling."

Wordsworth's appropriation of Newton's geometry-in-the-sand story wasn't his only flirtation with the subject. Thomas De Quincey, a contemporary of Wordsworth, wrote in his *Literary Reminiscences*: "Wordsworth was a profound admirer of the sublimer mathematics; at least of the higher geometry. The secret of this admiration for geometry lay in the antagonism between this world of bodiless abstraction and the world of passion." Wordsworth had done poorly in mathematics at school but formed a mutually admiring friendship with the young Irish mathematician William Rowan Hamilton, who some believe inspired Wordsworth to add to *The Prelude* the famous description of Newton (Isaac, not John): "A mind forever / Voyaging through strange seas of Thought, alone."

Hamilton was fascinated with all forms of scholastic knowledge—mathematics, ancient languages, poetry—from his earliest youth, but found his interest in math hyperactivated by a childhood encounter with Zerah Colburn, the "American Calculating Boy." Colburn, as a six-year-old boy from a Vermont farm family of modest means, was discovered by his father, Abia, sitting on the floor reciting multiplication tables he had never been taught. The boy proved to have immense powers of mental calculation, unlike anything before seen in New England. (He also, like all the men in his family, had six fingers on each hand and six toes on each foot.) Zerah's father brought him to meet with various local dignitaries, including the governor of Massachusetts, Elbridge Gerry (we'll come back to this guy later in a very different context), who advised Abia that only in Europe were there people capable of understanding and nurturing the boy's peculiar skills. They crossed the Atlantic in 1812, at which point Zerah was alternately educated and exhibited for money across Europe. In Dublin he appeared alongside a giant, an albino, and Miss Honeywell, an American woman who performed feats of dexterity with her toes. And in 1818, now aged fourteen, he engaged in a calculation competition with Hamilton, his Irish teen math counterpart, in which Hamilton "came off with honor, though his antagonist was generally the victor." But Colburn did not go on in mathematics; his interest was purely in mental computation. When Colburn studied Euclid, he found it easy, but "dry and devoid of interest." And when Hamilton met the Calculating Boy two years later and quizzed him about his methods ("He has lost every trace of his sixth finger," Hamilton recalls; Colburn had had it cut off by a London surgeon), he found that Colburn had little insight into the reasons his arithmetic methods worked. After abandoning his education, Colburn tried his hand on the English stage, didn't succeed there, moved back to Vermont, and lived out his life as a preacher.

When Hamilton met Wordsworth in 1827, he was just twenty-two, and had already been appointed a professor at the University of Dublin and the royal astronomer of Ireland. Wordsworth was fifty-seven. Hamilton wrote a letter to his sister describing their meeting: the young mathematician and the old poet took "a *midnight walk* together for a long, long time, *without any companion* except the stars and our own burning thoughts and words." As his style here suggests, Hamilton had not wholly given up his poetic ambitions. He immediately began to send his poems to Wordsworth, who responded warmly but critically. Shortly afterward, Hamilton renounced poetry; in fact, he did so in verse, directly addressing the Muse in a poem called "To Poetry," which he sent on to Wordsworth. Then, in 1831, he changed his mind, marking his decision by writing *another* poem called "To Poetry." He sent that one to Wordsworth, too. Wordsworth's response is one of the all-time classic gentle letdowns: "You send me showers of verses, which I receive with much pleasure, as do we all; yet have we fears that this employment may seduce you from the path of Science which you seem destined to tread with so much honour to yourself and profit to others."

Not everyone in Wordsworth's circle appreciated the interplay between passion and cold strange lonely reason as much as he and Hamilton did. At a dinner party at the painter Benjamin Robert Haydon's house at the end of 1817, Wordsworth's friend Charles Lamb got drunk and began to tease Wordsworth by abusing Newton, calling the scientist "a fellow who believed nothing unless it was as clear as the three sides of the triangle." John Keats joined in to accuse Newton of having stripped the rainbow of all its romance by showing that a prism exhibits the same optical effect. Wordsworth laughed along, his jaw set shut, one imagines, to avoid a quarrel.

De Quincey's portrait of Wordsworth goes on to advertise yet another math scene in *The Prelude*, still unpublished at the time. Poems had trailers in those days! In this scene, which De Quincey excitedly promises "reaches the very *ne plus ultra* of sublimity in my opinion," Wordsworth falls asleep while reading *Don Quixote* and dreams of meeting a Bedouin riding a camel across the empty desert. The Arab has two books in his hand, except one of the books, in the way of dreams, is not just a book but a heavy stone, too, and the other book is also a glowing seashell. (A few pages later, the Bedouin himself turns out to be Don Quixote.) The seashell-book issues apocalyptic prophecies when you hold it up to your ear. And the stone-book? That's Euclid's *Elements* again, here appearing not as a humble instrument of self-help but as a means of connection with the uncaring and unchanging cosmos: the book "wedded soul to soul in purest bond / Of reason, undisturbed by space or time." It makes sense De Quincey would be pretty into this psychedelic stuff; he was a former child prodigy who picked up a tenacious laudanum habit and wrote up his dizzying visions in *Confessions of an English Opium-Eater*, a sensational bestseller of the early nineteenth century.

Wordsworth's take is typical of geometry as viewed from a distance. Admiration, yes, but the way we admire an Olympic gymnast, executing flips and contortions that seem impossible for ordinary humans. It's what you get, too, in the most famous geometry poem of all, Edna St. Vincent Millay's sonnet "Euclid alone has looked on Beauty bare."* Millay's Euclid is a singular, unearthly figure, blasted with enlightenment by a shaft of insight on a "holy, terrible day." Not like the rest of us, who Millay says might, *if we're lucky*, get to hear Beauty's footsteps hurrying off down a faraway hallway.

That's not the geometry this book is about. Don't get me wrong—as a mathematician, I get a lot of benefit from geometry's prestige. It feels good when people think the work you do is mysterious, eternal, elevated above the common plane. "How was your day?" "Oh, holy and terrible, the usual."

But the harder you push that point of view, the more you incline people to see the study of geometry as an obligation. It acquires the slightly musty smell of something admired because it is good for one. Like opera. And that kind of admiration isn't enough to sustain the enterprise. There are plenty of new operas—but can you name them? No: you hear the word "opera" and you think of a mezzo-soprano in furs bellowing Puccini, probably in black and white.

There's plenty of new geometry, too, and, like new opera, it's not as well publicized as it could be. Geometry isn't Euclid, and it hasn't been for a long time. It's not a cultural relic, trailing an odor of the schoolroom, but a living subject, moving faster now than it ever has before. In the chapters to come we'll encounter the new geometry of pandemic spread, of the messy U.S. political process, of professional-level checkers, of artificial intelligence, of the English language, of finance, of physics, even of poetry. (A *lot* of geometers secretly dreamed, like William Rowan Hamilton, of being poets.)

We are living in a wild geometric boomtown, global in scope. Geometry isn't out there beyond space and time, it's right here with us, mixed in with the reasoning of everyday life. Is it beautiful? Yes, but not bare. Geometers see Beauty with its work clothes on.

Chapter 1

"I Vote for Euclid"

In 1864, the Reverend J. P. Gulliver, of Norwich, Connecticut, recalled a conversation with Abraham Lincoln about how the president had acquired his famously persuasive rhetorical skill. The source, Lincoln said, was geometry.

> In the course of my law-reading I constantly came upon the word *demonstrate*. I thought, at first, that I understood its meaning, but soon became satisfied that I did not. . . . I consulted Webster's Dictionary. That told of "certain proof," "proof beyond the possibility of doubt;" but I could form no idea what sort of proof that was. I thought a great many things were proved beyond a possibility of doubt, without recourse to any such extraordinary process of reasoning as I understood "demonstration" to be. I consulted all the dictionaries and books of reference I could find, but with no better results. You might as well have defined *blue* to a blind man. At last I said, "Lincoln, you can never make a lawyer if you do not understand what *demonstrate* means;" and I left my situation in Springfield, went home to my father's house, and staid there till I could give any propositions in the six books of Euclid at sight. I then found out what "demonstrate" means, and went back to my law studies.

Gulliver was fully on board, replying, "No man can talk well unless he is able first of all to define to himself what he is talking about. Euclid, well studied, would free the world of half its calamities, by banishing half the nonsense which now deludes and curses it. I have often thought that Euclid would be one of the best books to put on the catalogue of the Tract Society, if they could only get people to read it. It would be a means of grace." Lincoln, Gulliver tells us, laughed and agreed: "I vote for Euclid."

Lincoln, like the shipwrecked John Newton, had taken up Euclid as a source of solace at a rough time in his life; in the 1850s, after a single term in the House of Representatives, he seemed finished in politics and was trying to make a living as an ordinary traveling lawyer. He had learned the rudiments of geometry in his earlier job as a surveyor and now aimed to fill the gaps. His law partner William Herndon, who often had to share a bed with Lincoln at small country inns in their sojourns around the circuit, recalls Lincoln's method of study; Herndon would fall asleep, while Lincoln, his long legs hanging over the edge of the bed, would stay up late into the night with a candle lit, deep in Euclid.

One morning, Herndon came upon Lincoln in their offices in a state of mental disarray:

He was sitting at the table and spread out before him lay a quantity of blank paper, large heavy sheets, a compass, a rule, numerous pencils, several bottles of ink of various colors, and a profusion of stationery and writing appliances generally. He had evidently been struggling with a calculation of some magnitude, for scattered about were sheet after sheet of paper covered with an unusual array of figures. He was so deeply absorbed in study he scarcely looked up when I entered.

Only later in the day did Lincoln finally get up from his desk and tell Herndon that he had been trying to square the circle. That is, he was trying to construct a square with the same area as a given circle, where to "construct" something, in proper Euclidean style, is to draw it on the page using just two tools, a straightedge and a compass. He worked at the problem for two straight days, Herndon remembers, "almost to the point of exhaustion."

I have been told that the so-called squaring of the circle is a practical impossibility, but I was not aware of it then, and I doubt if Lincoln was. His attempt to establish the proposition having ended in failure, we, in the office, suspected that he was more or less sensitive about it and were therefore discreet enough to avoid referring to it.

Squaring the circle is a very old problem, whose fearsome reputation I suspect Lincoln might actually have known; "squaring the circle" has been a metaphor for a difficult or impossible task for a long time. Dante name-checks it in the *Paradiso*: "Like the geometer who gives his all trying to square the circle, and still can't find the idea he needs, *that's* how I was." In Greece, where it all started, a standard exasperated comment when someone is making a task harder than necessary is to say, "I wasn't asking you to square the circle!"

There is no *reason* one needs to square a circle—the problem's difficulty and fame is its own motivation. People with a conquering mentality tried to square circles from antiquity until 1882, when Ferdinand von Lindemann proved it couldn't be done (and even then a few die-hards persisted; okay, even *now*). The seventeenth-century political philosopher Thomas Hobbes, a man whose confidence in his own mental powers is not fully captured by the prefix "over," thought he'd cracked it. Per his biographer John Aubrey, Hobbes discovered geometry in middle age and quite by accident:

Being in a Gentleman's Library, Euclid's *Elements* lay open, and 'twas the 47 *El. Libri* 1. He read the Proposition. By G_, saydd he (he would now and then sweare an emphaticall Oath by way of emphasis) *this is impossible!* So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. *Et sic deinceps* that at last he was demonstratively convinced of that trueth. This made him in love with Geometry.

Hobbes was constantly publishing new attempts and getting in petty feuds with the major British mathematicians of the time. At one point, a correspondent pointed out that one of his constructions was not quite correct because two points P and Q he claimed to be equal were actually at very slightly different distances from a third point R; 41 and about 41.012 respectively. Hobbes retorted that his points were big enough in extent to cover such a minor difference. He went to his grave still telling people he'd squared the circle.*

An anonymous commentator in 1833, reviewing a geometry textbook, described the typical circle-squarer in a way that quite precisely depicts both Hobbes, two centuries prior, and intellectual pathologies still hanging around here in the twenty-first:

> [A]ll they know of geometry is, that there are in it some things which those who have studied it most have long confessed themselves unable to do. Hearing that the authority of knowledge bears too great a sway over the minds of men, they propose to counterbalance it by that of ignorance: and if it should chance that any person acquainted with the subject has better employment than hearing them unfold hidden truths, he is a bigot, a smotherer of the light of truth, and so forth.

In Lincoln, we find a more appealing character: enough ambition to try, enough humility to accept that he hadn't succeeded.

What Lincoln took from Euclid was the idea that, if you were careful, you could erect a tall, rock-solid building of belief and agreement by rigorous deductive steps, story by story, on a foundation of axioms no one could doubt: or, if you like, truths one holds to be self-evident. Whoever *doesn't* hold those truths to be self-evident is excluded from discussion. I hear the echoes of Euclid in Lincoln's most famous speech, the Gettysburg Address, where he characterizes the United States as "dedicated to the proposition that all men are created equal." A "proposition" is the term Euclid uses for a fact that follows logically from the self-evident axioms, one you simply cannot rationally deny.

Lincoln wasn't the first American president to look for a basis of democratic politics in Euclidean terms; that was the math-loving Thomas Jefferson. Lincoln wrote, in a letter read at an 1859 Jefferson commemoration in Boston he was unable to attend:

> One would start with great confidence that he could convince any sane child that the simpler propositions of Euclid are true; but, nevertheless, he would fail, utterly, with one who should deny the definitions and axioms. The principles of Jefferson are the definitions and axioms of free society.

Jefferson had studied Euclid at William and Mary as a young man, and esteemed geometry highly ever afterward.* While vice president, Jefferson took the time to answer a letter from a Virginia student about his proposed plan of academic study, saying: "Trigonometry, so far as this, is most valuable to every man, there is scarcely a day in which he will not resort to it for some of the purposes of common life" (though he describes much of higher mathematics as "but a luxury; a delicious luxury indeed; but not to be indulged in by one who is to have a profession to follow for his subsistence").

In 1812, retired from politics, Jefferson wrote to his predecessor in the presidency, John Adams:

> I have given up newspapers in exchange for Tacitus and Thucydides, for Newton and Euclid; and I find myself much the happier.

Here we see a real difference between the two geometer-presidents. For Jefferson, Euclid was part of the classical education required of a cultivated patrician, of a piece with the Greek and Roman historians and the scientists of the Enlightenment. Not so for Lincoln, the self-educated rustic. Here's the Reverend Gulliver again, recalling Lincoln recalling his childhood:

> I can remember going to my little bedroom, after hearing the neighbors talk of an evening with my father, and spending no small part of the night walking up and down, and trying to make out what was the exact meaning of some of their, to me, dark sayings. I could not sleep, though I often tried to, when I got on such a hunt after an idea, until I had caught it; and when I thought I had got it, I was not satisfied until I had repeated it over and over, until I had put it in language plain enough, as I thought, for any boy I knew to comprehend. This was a kind of passion with me, and it has stuck by me, for I am never easy now, when I am handling a thought, till I have bounded it north and bounded it south, and bounded it east and bounded it west. Perhaps that accounts for the characteristic you observe in my speeches.

This is not geometry, but it's the mental habit of the geometer. You don't settle for leaving things half-understood; you boil down your thoughts and trace back their steps of reason, just as Hobbes had amazedly watched Euclid do. This kind of systematic self-perception, Lincoln thought, was the only way out of confusion and darkness.

For Lincoln, unlike Jefferson, the Euclidean style isn't something belonging to the gentleman or the possessor of a formal education, because Lincoln was neither. It's a hand-hewn log cabin of the mind. Built properly, it can withstand any challenge. And anybody, in the country Lincoln conceived, can have one.

FROZEN FORMALITY

The Lincolnian vision of geometry for the American masses, like a lot of his good ideas, was only incompletely realized. By the middle of the nineteenth century, geometry had moved from college to the public high school; but the typical course used Euclid as a kind of museum piece, whose proofs were to be memorized, recited, and to some extent appreciated. How anyone might have *come up* with those proofs was not to be spoken of. The proof-maker himself almost disappeared: one writer of the time remarked that "many a youth reads six books of the *Elements* before he happens to be informed that Euclid is not the name of a science, but of a man who wrote upon it." The paradox of education: what we most admire we put in a box and make dull.

To be fair, there is not much to say about the historical Euclid, because there is not much we know about the historical Euclid. He lived and worked in the great city of Alexandria, in North Africa, sometime around 300 BCE. That's it—that's what we know. His *Elements* collects the knowledge of geometry possessed by Greek mathematics at the time, and lays the foundations of number theory for dessert. Much of the material was known to mathematicians prior to Euclid's time, but what's radically new, and was instantly revolutionary, is the *organization* of that huge body of knowledge. From a small set of axioms, which were almost impossible to doubt,* one derives step by step the whole apparatus of theorems about triangles, lines, angles, and circles. Before Euclid—if there actually was a Euclid, and not a shadowy collective of geometry-minded Alexandrians writing under that name—such a structure would have been unimaginable. Afterward, it was a model for everything admirable about knowledge and thought.

There is, of course, another way to teach geometry, which emphasizes invention and tries to put the student in the Euclidean cockpit, with the power to make their own definitions and see what comes of them. One such textbook, *Inventional Geometry*, starts from the premise that "the only true education is self-education." Don't look at other people's constructions, the book counsels, "at least until you have discovered a construction of your own," and avoid anxiety and comparing yourself with other students, because everyone learns at their own pace and you're more likely to master the material if you're enjoying yourself. The book itself is no more than a series of puzzles and problems, 446 in all. Some of these are straightforward: "Can you make three angles with two lines? Can you make four angles with two lines? Can you make more than four angles with two lines?" Some of them, the author warns, are not actually solvable, the better to put yourself in the position of a *true* scientist. And some of them, like the very first one, have no clear "right answer" at all: "Place a cube with one face flat on a table, and with another face towards you, and say which dimension you consider to be the thickness, which the breadth, and which the length." Altogether, it is just the kind of "child-centered," exploratory approach that traditionalists deride as what's wrong with education nowadays. It came out in 1860.

A few years ago, the mathematics library at the University of Wisconsin came into possession of a huge trove of old math textbooks, books that had actually been used by Wisconsin schoolchildren over the last hundred years or so* and eventually discarded in favor of newer models. Looking at the weathered books, you see that every controversy in education has been waged before, multiple times, and everything we think of as new and strange—math books like *Inventional Geometry* that ask students to come up with proofs on their own, math books that make problems "relevant" by relating them to students' everyday lives, math books designed to advance social causes, progressive or otherwise—is also old, and was thought of as strange at the time, and no doubt will be new and strange again in the future.

A note in passing: the introduction to *Inventional Geometry* mentions that geometry has "a place in the education of all, not excepting that of women"—the book's author, William George Spencer, was an early advocate of coeducation. A more common nineteenth-century attitude toward women and geometry is conveyed in (but not endorsed by) *The Mill on the Floss*, by George Eliot*, published the same year as Spencer's textbook: "Girls can't do Euclid, can they, sir?" one character asks the schoolmaster Mr. Stelling, who responds, "They've a great deal of superficial cleverness; but they couldn't go far into anything." Stelling represents, in satirically exaggerated form, the traditional mode of British pedagogy Spencer was rebelling against: a long march through memorization of the masters, in which the slow messy process of building understanding is not just neglected but actively guarded against. "Mr. Stelling was not the man to enfeeble and emasculate his pupil's mind by simplifying and explaining." Euclid, a kind of tonic of manliness, was to be suffered straight, like a strong drink or an ice-cold shower.

Even in the highest reaches of mathematical research, dissatisfaction with Stellingism had begun to build. The British mathematician James Joseph Sylvester, whose geometry and algebra (and distaste for the stultified deadness of British academia) we'll be talking about later, thought Euclid should be hidden "far out of the schoolboy's reach," and geometry taught through its relation to physical science, with an emphasis on the geometry of *motion* supplementing Euclid's static forms. "It is this living interest in the subject," Sylvester wrote, "which is so wanting in our traditional and mediaeval modes of teaching. In France, Germany, and Italy, everywhere where I have been on the Continent, mind acts direct on mind in a manner unknown to the frozen formality of our academic institutions."

BEHOLD!

We don't make students memorize and recite Euclid anymore. In the late nineteenth century, textbooks started including exercises, asking students to construct their own proofs of geometric propositions. In 1893, the Committee of Ten, an educational plenum convened by Harvard president Charles Eliot and charged with rationalizing and standardizing American high school education, codified this shift. The point of geometry in high school, they said, was to train up the student's mind in the habits of strict deductive reasoning. This idea has stuck. A survey conducted in 1950 asked five hundred American high school teachers about their objectives in teaching geometry: the most popular answer by far was "To develop the habit of clear thinking and precise expression," which got almost twice as many votes as "To give a knowledge of the facts and principles of geometry." In other words, we are not here to stuff our students with every known fact about triangles, but to develop in them the mental discipline to build up those facts from first principles. A school for little Lincolns.

And what is that mental discipline for? Is it because, at some point in the student's later life, they will be called upon to demonstrate, finally and incontrovertibly, that the sum of the exterior angles of a polygon is 360 degrees?

I keep waiting for that to happen to me and it never has.

The ultimate reason for teaching kids to write a proof is not that the world is full of proofs. It's that the world is full of *non-proofs*, and grown-ups need to know the difference. It's hard to settle for a non-proof once you've really familiarized yourself with the genuine article.

Lincoln knew the difference. His friend and fellow lawyer Henry Clay Whitney recalled: "[M]any a time have I seen him tear the mask off from a fallacy and shame both the fallacy and its author." We encounter non-proofs in proofy clothing all the time, and unless we've made ourselves especially attentive, they often get by our defenses. There are tells you can look for. In math, when an author starts a sentence with "Clearly," what they are really saying is "This seems clear to me and I probably should have checked it, but I got a little confused, so I settled for just asserting that it was clear." The newspaper pundit's analogue is the sentence starting "Surely, we can all agree." Whenever you see this, you should at all costs *not* be sure that all agree on what follows. You are being asked to treat something as an axiom, and if there's one thing we can learn from the history of geometry, it's that you shouldn't admit a new axiom into your book until it really proves its worth.

Always be skeptical when someone tells you they're "just being logical." If they are talking about an economic policy or a culture figure whose behavior they deplore or a relationship concession they want you to make, and not a congruence of triangles, they are not "just being logical," because they're operating in a context where logical deduction—if it applies at all—can't be untangled from everything else. They want you to mistake an assertively expressed chain of opinions as the proof of a theorem. But once you've experienced the sharp *click* of an honest-to-goodness proof, you'll never fall for this again. Tell your "logical" opponent to go square a circle.

What was distinctive about Lincoln, Whitney says, wasn't that he possessed a superpowered intellect. Lots of people in public life, Whitney writes ruefully, are very smart, and among them one finds both the good and the bad. No: what made Lincoln special was that "it was morally impossible for Lincoln to argue dishonestly; he could no more do it than he could steal; it was the same thing to him in essence, to despoil a man of his property by larceny, or by illogical or flagitious reasoning." What Lincoln had taken from Euclid (or what, already existing in Lincoln, harmonized with what he found in Euclid) was *integrity*, the principle that one does not say a thing unless one has justified, fair and square, that one has the right to say it. Geometry is a form of honesty. They might have called him Geometrical Abe.

The one place I'll part ways with Lincoln is in his shaming the author of the fallacy. Because the hardest person to be honest with is yourself, and it's our self-authored fallacies we need to spend the most time and effort unmasking. You should always be prodding your beliefs as you would a loose tooth, or, better, a tooth whose looseness you're not quite sure about. And if something's not solid, shame is not required, just a calm retreat to safer ground, and a reassessment of where you can get to from there.

That, ideally, is what geometry has to teach us. But the "frozen formality" Sylvester complained about is far from gone. In practice, the lesson we often teach kids in geometry class is, as math writer–cartoonist–raconteur Ben Orlin puts it:

> A proof is an incomprehensible demonstration of a fact that you already knew.

Orlin's example of such a proof is the "right angle congruence theorem," the assertion that any two right angles are congruent to each other. What might be asked of a ninth grader presented with this assertion? The most typical format is the *two-column proof*, a mainstay of geometry education for more than a century, which in this case would look something like this:

angle 1 and angle 2 are both right angles	**given**
the measure of angle 1 equals 90 degrees	**definition of right angle**

the measure of angle 2 equals 90 degrees	**definition of right angle**
the measure of angle 1 equals the measure of angle 2	**transitivity of equality**
angle 1 is congruent to angle 2	**definition of congruence**

"Transitivity of equality" is one of Euclid's "common notions," arithmetic principles he states at the beginning of the *Elements* and treats as prior even to the geometric axioms. It is the principle that two things which are equal to the same thing are thereby equal to each other.*

I don't want to deny that there's a certain satisfaction in reducing everything to such tiny, precise steps. They snap together so satisfyingly, like Lego! That feeling is something a teacher truly wants to convey.

And yet . . . isn't it *obvious* that two right angles are the same thing, just placed on the page in a different place and pointing in a different direction? Indeed, Euclid makes the equality of any two right angles the fourth of his axioms, the basic rules of the game that are taken to be true without proof and from which all else is derived. So why would a modern high school require students to manufacture a proof of this fact when even Euclid said, "Come on, that's obvious?" Because there are many different sets of starting axioms from which one can derive plane geometry, and proceeding exactly as Euclid did is generally no longer considered the most rigorous or the most pedagogically beneficial choice. David Hilbert rewrote the whole foundation from scratch in 1899, and the axioms used in American schools today typically owe more to those laid down by George Birkhoff in 1932.

Whether it's an axiom or not, the fact that two right angles are equal is something the student just plain knows. You can't blame someone for being frustrated when you tell them, "You *think* you knew that, but you didn't *really* know it until you followed the steps in the two-column proof." It's a little insulting!

Too much of geometry class is devoted to proving the obvious. I remember well a course in topology I took my first year of college. The professor, a very distinguished elder researcher, spent two weeks proving the following fact: if you draw a closed curve in the plane, no matter how squiggly and weird it may be, the curve cuts the plane into two pieces; the part outside the curve and the part inside.

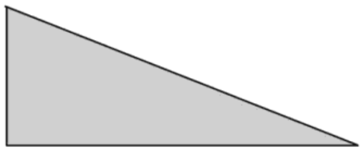
Now, on the one hand it's quite difficult, it turns out, to write a formal proof of this fact, known as the Jordan Curve Theorem.* On the other hand, I spent those two weeks in a state of barely controlled irritation. Was *this* what math was truly about? Making the obvious laborious? Reader, I zoned out. So did my classmates, among them many future mathematicians and scientists. Two kids who sat right in front of me, very serious students who would go on to earn PhDs in math at top-five universities, would start vigorously making out every time Distinguished Elder Researcher turned back to the board to chalk out yet another delicate argument on a perturbation of a polygon. I mean just really going at it, as if the force of their teen hunger for each other could somehow rip them into another part of the continuum where this proof was not still taking place.

A highly trained mathematician such as my current self might say, standing up a little straighter: well, young people, you are simply not sophisticated enough to know which statements are truly obvious and which conceal subtleties. Perhaps I would bring up the feared Alexander Horned Sphere, which shows that the analogous question in three-dimensional space is not as simple as one might imagine.

But pedagogically, I think that's a pretty bad answer. If we take our time in class to prove things that seem obvious, and insist that those statements are *not* obvious, our students will stew in resentment, just like I did, or find something more interesting to do while the teacher isn't looking.

I like the way master teacher Ben Blum-Smith describes the problem: for students to really feel the fire of math, they have to experience the *gradient of confidence*—the feeling of moving from something obvious to something not-obvious, pushed uphill by the motor of formal logic. Otherwise, we're saying, "Here is a list of axioms that seem pretty obviously correct; put these together until you have another statement that seems pretty obviously correct." It's like teaching somebody about Lego by showing them how you can make two little bricks into one big brick. You can do that, and sometimes you need to, but it is definitely not the point of Lego.

The gradient of confidence is perhaps better experienced than just talked about. If you want to *feel* it, think for a moment about a right triangle.



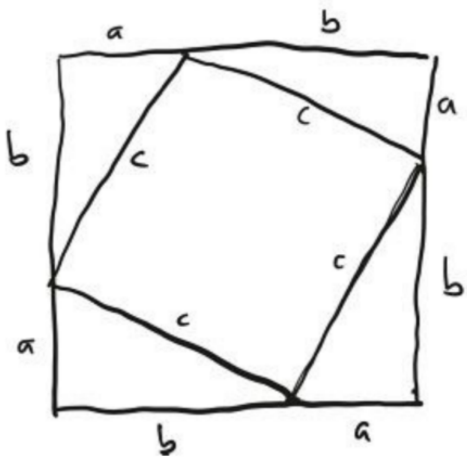
One starts with an intuition: if the vertical and horizontal sides are determined, so is the diagonal side. Walking 3 km south and then 4 km east leaves you a certain distance from your starting point; there is no ambiguity about it.

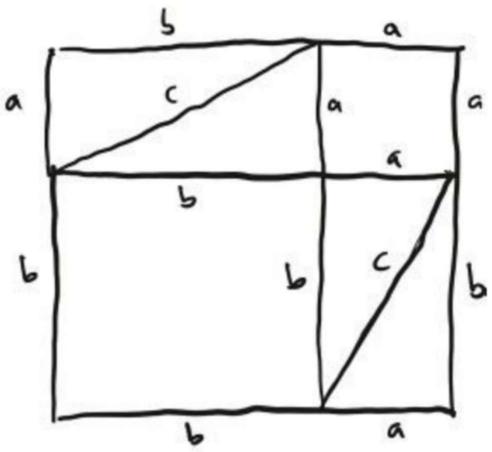
But what *is* the distance? That's what the Pythagorean Theorem, the first real theorem ever proved in geometry, is for. It tells you that if a and b are the vertical and horizontal sides of a right triangle, and c is the diagonal side, the so-called hypotenuse, then

$$a^2 + b^2 = c^2$$

In case a is 3 and b is 4, this tells us that c^2 is $3^2 + 4^2$, or 9 + 16, or 25. And we know what number, when squared, yields 25; it is 5. That's the length of the hypotenuse.

Why would such a formula be true? You could start climbing the gradient of confidence by literally drawing a triangle with sides 3 and 4 and measuring its hypotenuse—it would look really close to 5. Then draw a triangle with sides 1 and 3 and measure *its* hypotenuse; if you were careful enough with the ruler, you'd get a length really close to 3.16 . . . whose square is 1 + 9 = 10. Increased confidence derived from examples isn't a proof. But this is:





The big square is the same in both pictures. But it's cut up in two different ways. In the first picture, you have four copies of our right triangle, and a square whose side has length c. In the second picture, you also have four copies of the triangle, but they're arranged differently; what's left of the square is now two smaller squares, one whose side has length a and one whose side has length b. The area that remains when you take four copies of the triangle out of the big square has to be the same in both pictures, which means that c^2 (the area left over in the first picture) has to be the same as $a^2 + b^2$ (the area left over in the second).

If we are to be persnickety, we might complain that we have not exactly *proved* that the figure in the first picture is actually a square (that its sides are all the same length is not enough; squeeze opposite corners of a square between your thumb and forefinger and you get a diamond shape called a *rhombus* that's definitely not a square but still has all four sides the same length). But come on. Before you see the picture, you have no reason to think the Pythagorean Theorem is true; after you see it, you know *why* it's true. Proofs like this, where a geometric figure is cut up and rearranged, are called *dissection proofs*, and are prized for their clarity and ingenuity. The twelfth-century mathematician-astronomer Bhāskara* presents a proof of Pythagoras in this form, and finds the picture a demonstration so convincing as not to require verbal explanation, merely a caption that reads "Behold!"* The amateur mathematician Henry Perigal came up with his own dissection proof of Pythagoras in 1830, while trying to square the circle, like Lincoln; he esteemed his diagram so highly that he had it carved into his tombstone some sixty years later.

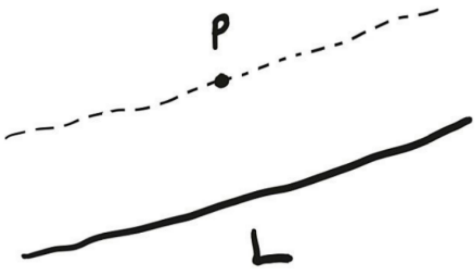
ACROSS THE BRIDGE OF ASSES

We need to know how to do geometry by purely formal deduction; but geometry isn't *merely* a sequence of purely formal deductions. If it were, it would be no better a way to teach the art of systematic reasoning than a thousand other things. We could teach chess problems, or Sudoku. Or we could make up a system of axioms with no relation to any known human practice at all and force our students to derive their consequences. We teach geometry instead of any of those things because geometry is a formal system that's not *just* a formal system. It's built into the way we think about space, location, and motion. We can't help being geometric. We have, in other words, intuition.

The geometer Henri Poincaré, in a 1905 essay, identifies intuition and logic as the two indispensable pillars of mathematical thought. Every mathematician leans in one direction or the other, and it is the intuition-leaners, Poincaré says, that we tend to call "geometers." We need both pillars. Without logic, we'd be helpless to say anything about a thousand-sided polygon, an object we cannot in any meaningful sense imagine. But without intuition, the subject loses all its savor. Euclid, Poincaré explains, is a dead sponge:

> You have doubtless seen those delicate assemblages of silicious needles which form the skeleton of certain sponges. When the organic matter has disappeared, there remains only a frail and elegant lace-work. True, nothing is there except silica, but what is interesting is the form this silica has taken, and we could not understand it if we did not know the living sponge which has given it precisely this form. Thus it is that the old intuitive notions of our fathers, even when we have abandoned them, still imprint their form upon the logical constructions we have put in their place.

Somehow we need to train students to deduce without denying the presence of the intuitive faculty, the living spongy tissue. And yet we don't want to let our intuition completely drive the bus. The story of the parallel postulate is instructive here. Euclid, as one of his five axioms, listed this one: "Given any line L and any point P not on L, there is one and only one line through P parallel to L."*



This is complicated and chunky compared to his other axioms, which are sleeker things like "any two points are connected by a line." It would be nicer, people thought, if the fifth axiom could be proven from the other four, which felt somehow more primal.

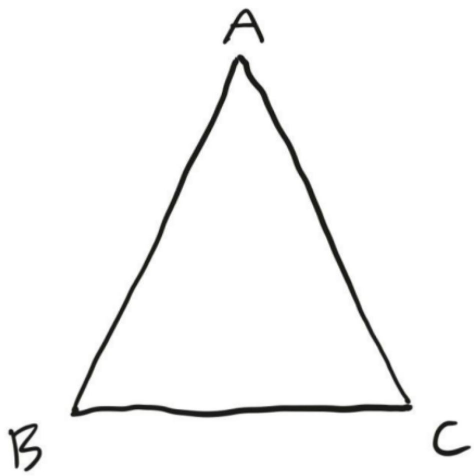
But why? Our intuition, after all, shouts out loud that the fifth axiom is true. What could possibly be more useless than trying to prove it? It's like asking whether we can really prove that 2 + 2 = 4. We *know* that!

And yet mathematicians persisted, trying and failing and trying and failing to show that the fifth axiom followed from the others. And finally they showed that they'd been doomed to fail from the start; because there were *other* geometries, in which "line" and "point" and "plane" meant something other than what Euclid (and probably you) mean by those words, but that nonetheless satisfied the first four axioms while failing the last. In some of these geometries, there were infinitely many lines through P parallel to L. In some, there were none.

Isn't that cheating? We weren't asking about *other* bizarro-world geometric entities which we perversely refer to as "lines." We were talking about *actual lines*, for which Euclid's fifth is certainly true.

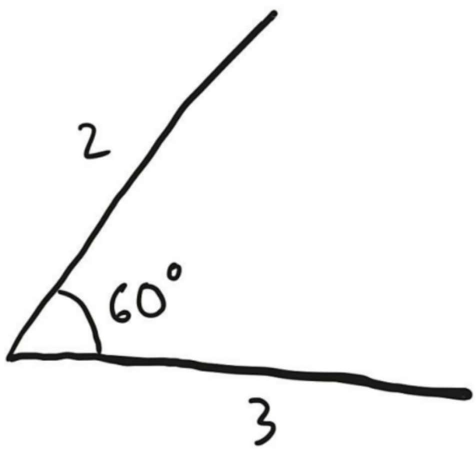
Sure, that's a tack you're free to take. But by doing so, you're willfully closing off access to a whole world of geometries, just because they're not the geometry you're used to. Non-Euclidean geometry turns out to be fundamental to huge regions of math, including the math that describes the physical space we actually inhabit. (We'll come back to that in a few pages.) We *could* have refused to discover it on uptight Euclid-purist grounds. But it would be our loss.

Here's another place where a careful balance between formal logic and intuition is called for. Suppose a triangle is isosceles

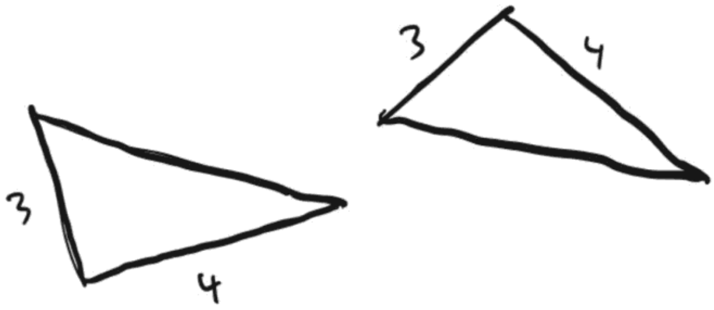


which is to say the sides AB* and AC are equal in length. Here's a theorem: the angles at B and C are equal as well.

This statement is called the *pons asinorum*, the "bridge of asses," because it's something almost all of us have to be carefully led across. Euclid's proof has somewhat more to it than the business with the right angles above. We're a little in medias res here, since in a real geometry class we'd arrive at the ass-bridge only after several weeks of prep; so let's take for granted Euclid's Proposition 4 of book I, which says that if you know two side lengths of a triangle and you know the angle between those two sides, then you know the remaining side length and the remaining two angles, too. That is, if I draw this:



there's only one way to "fill in" the rest of the triangle. Another way to say the same thing: if I have two different triangles that have two side lengths and the angle between them in common, then the two triangles have *all* their angles and *all* their side lengths in common; they are, as the geometer's lingo has it, "congruent."



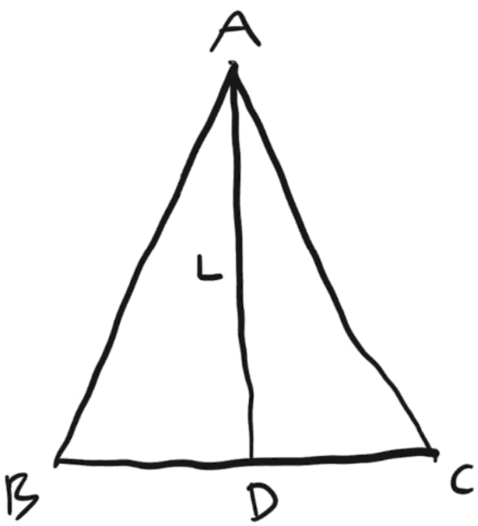
We invoked this fact already in the case where the angle between the two sides is a right angle, and I think the fact feels just as clear to the mind whatever the angle is.

(It's also true, by the way, that if the three side lengths of two triangles match up, the two triangles must be congruent; if the lengths are 3, 4, and 5, for instance, the triangle *must* be the right triangle I drew above. But this is less obvious, and Euclid proves it only a bit later, as Proposition I.8. If you think it *is* obvious, consider this: What about a four-sided figure? Remember the rhombus we just encountered; same four side lengths as a square, but definitely not a square.)

Now for the pons asinorum. Here's how a two-column proof might look.

Let L be a line through A which cuts angle BAC in half	**okay, I'll let you**
Let D be the point where L intersects BC	**still no objection**

Hey, me again, I know we're in midproof here, but we made a new point and invoked a new line segment AD, so we'd better update our picture! By the way, remember our hypothesis that our triangle is isosceles, so AB and AC have the same length; we're about to use that.



AD and AD have the same length	**a segment is equal to itself**

AB and AC have the same length	**given**
angles BAD and CAD are congruent	**We chose AD to cut angle BAC in half**
triangles ABD and ACD are congruent	**Euclid I.4, told you we'd need this**
angle B and angle C are equal	**corresponding angles in congruent triangles are equal**

QED.*

This proof has more to it than the first one we saw, because you actually have to *make* something; you made up a new line L and gave the name D to the point where L hits BC. That allows you to identify B and C with edges of two newborn triangles ABD and ACD, which we then show are congruent.

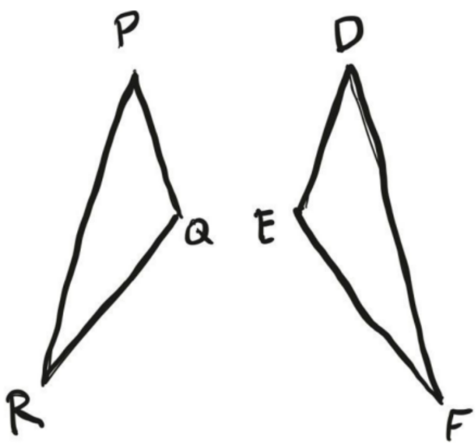
But there's a slicker way, written down about six hundred years after Euclid by Pappus of Alexandria, another North African geometer, in his compendium *Synagogue* (which in the ancient world could refer to a collection of geometrical propositions, not just a collection of Jews at prayer).

AB and AC have the same length	**given**
the angle at A equals the angle at A	**an angle is equal to itself**
AC and AB have the same length	**you already said that, what are you up to, Pappus?**
Triangles BAC and CAB are congruent	**Euclid I.4 again**
angle B and angle C are equal	**corresponding angles in congruent triangles are equal**

Wait, what happened? It seemed like we were doing nothing, and then all at once the desired conclusion appeared out of that nothing, like a rabbit jumping out of the absence of a hat. It creates a certain unease. It was not the sort of thing Euclid himself liked to do. But it is, by my lights at any rate, a true proof.

The key to Pappus's insight is that penultimate line: triangles BAC and CAB are congruent. It seems as if we're merely saying that a triangle is the same as itself, which looks like a triviality. But look more carefully.

What, really, are we saying when we say that two different triangles, PQR and DEF, are congruent?



We're saying six things in one: the length of PQ is the same as the length of DE, the length of PR is the length of DF, the length of QR is the length of EF, the angle at P is the same as the angle at D, the angle at Q is the angle at E, and the angle at R is the angle at F.

Is PQR congruent to DFE? Not in this picture, no, because PQ does *not* have the same length as the corresponding side DF.

If we take the definition of congruence seriously—and we're being geometers, so taking definitions seriously is kind of our thing—then DEF and DFE are not congruent to each other, *despite being the same triangle*. Because DE and DF don't have the same length.

But in the proof of the pons asinorum, we're saying that our isosceles triangle, when you think of it as triangle BAC, is the same as the triangle when thought of as CAB. That is *not* an empty statement. If I tell you the name "ANNA" is the same backward and forward, I'm really telling you something about the name: that it's a palindrome. To object to the very concept of a palindrome by saying, "Of course they're the same, it consists of two *A*'s and two *N*'s whichever order you write it in," would be pure perverseness.

In fact, "palindromic" would be a good name for a triangle like BAC, which is congruent to the triangle CAB you get when you write the vertices in the opposite order. And it was by thinking this way that Pappus was able to give his faster path across the pons, without having to invoke any extra lines or points at all.

And yet even Pappus's proof doesn't quite capture *why* an isosceles triangle has two equal angles. It does come closer. This notion that the isosceles triangle is a palindrome, that it stays the same when written backward, records something I'll bet your intuition also tells you—that the triangle is unchanged when you pick it up, flip it over, and lay it back down again in the same spot. Like a palindromic word, it has a *symmetry*. That, one feels, is why the angles have to be the same.

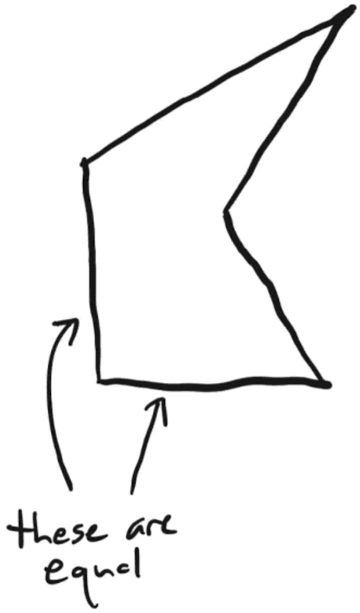
In geometry class we are usually not allowed to talk about picking up shapes and turning them over.* But we ought to be. As abstract as we may try to make it, math is something we do with our body. Geometry most of all. Sometimes literally; every working mathematician has found themself drawing invisible figures with hand gestures, and at least one study has found that children asked to act out a geometric question with their body become more likely to arrive at the correct conclusion.* Poincaré himself was said to rely on his sense of motion when reasoning geometrically. He was not a visualizer, and his recollection for faces and figures was poor; when he needed to draw a picture from memory, he said, he remembered not what it looked like but how his eyes had moved along it.

EQUAL ARMS

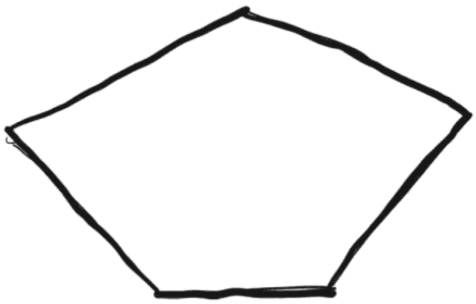
What does the word "isosceles" really mean? Well, it means two sides of the triangle are equal. Literally, in Greek, it refers to the two σκέλη *(skeli)*, or "legs." In Chinese, 等腰 means "equal waists"; in Hebrew an isosceles triangle is one with "equal calves," in Russian "equal arms." In every case, we seem to agree that what it means to be isosceles is to have two sides equal. But why? Why not define an isosceles triangle to be one that has two angles equal? You can probably see (and indeed the whole point of the pons asinorum is to prove!) that two sides being equal means two angles are equal, and vice versa. In other words, the two definitions are equivalent; they pick out the same collection of triangles. But I wouldn't say they're the *same* definition.

Nor are they the only option. It would be more modern in flavor to define an isosceles triangle as a palindromic one: a triangle you can pick up, flip over, and place back down, only to find it unchanged. That such a triangle has two equal sides and two equal angles is just about automatic. In this geometric world, Pappus's proof would be the means of showing that a triangle with two equal sides was isosceles; that the triangles BAC and CAB are the same.

A good definition is one that sheds light on situations beyond the ones it was devised for. The idea that "isosceles" means "unchanged when flipped over" gives us a good idea of what we should mean by an isosceles trapezoid, or an isosceles pentagon. You *could* say that an isosceles pentagon is one that has two sides equal; then you'll be admitting saggy, lopsided pentagons like this one into the fold:



But do you want to? Surely a pentagon like this handsome figure



is more what one means by isosceles. Indeed, in your schoolbook, an "isosceles trapezoid" isn't one with two equal sides, or with two equal angles; it is one that can be flipped without changing it. The post-Euclidean notion of symmetry has crept in, and it's there because our minds are built to find it. More and more geometry classes are placing the idea of symmetry at the center, and building structures of proof starting from there. It's not Euclid, but it's where geometry is now.

Chapter 2

How Many Holes Does a Straw Have?

It is always a pleasure, for those of us in the mathematical professions, when the internet spends a day or two tying itself in a knot over a math problem. We get to watch other people discovering and enjoying the mode of thought we spend our whole lives taking pleasure in. When you have a really nice house, you like it when people unexpectedly come over.

The problems that catch on in this way are usually good ones, though they might appear frivolous at first. What hooks and holds the attention is the sensation of encountering an *actual mathematical issue.*

For example: How many holes does a straw have?

Most people I've asked this question see the answer as obvious. And they are extremely surprised, sometimes even a little aggrieved, to learn that there are people whose obvious answer differs from their own. It's the math version of "You've got another think coming" vs. "You've got another thing coming."*

As far as I can tell, the hole-in-the-straw question first appears in a 1970 paper in the *Australasian Journal of Philosophy* by the husband-and-wife team of Stephanie and David Lewis, where the tubular object under discussion is a paper towel roll. The question then reappears in 2014 as a poll on a bodybuilding forum. The arguments presented on the bodybuilding forum are different in tone from those that appear in the *Australasian Journal of Philosophy*, but the outline of the controversy is quite consistent; the answers "zero holes," "one hole," and "two holes" all command substantial support.

Then a Snapchat clip of two college friends getting angrier and angrier over two holes vs. one hole appeared, and started spreading, eventually drawing more than a million and a half views. The straw question showed up all over Reddit and Twitter and in *The New York Times*. A group of young, attractive, extremely-confused-about-holes BuzzFeed staffers shot a video, and that too racked up hundreds of thousands of hits.

You've probably already started formulating the main arguments in your mind. Let's recount them here:

> **Zero holes:** A straw is what you get when you take a rectangle of plastic and roll it up and glue it closed. A rectangle doesn't have any holes in it. You didn't *put* any holes in it when you wrapped it up—so it still has no holes.
>
> **One hole:** The hole is the empty space in the center of the straw. It extends all the way from the top to the bottom.

Two holes: Just look at it! There's a hole at the top and a hole at the bottom!

My first goal is to convince you you're confused about the holes, even if you think you're not. Each of these views has serious problems.

I'll dispense with the zero-holers first. Something can have a hole in it without any substance being removed. You don't make a bagel by first baking a bialy and then punching out the center. No—you roll out a snake of dough and join the ends together to form the bagel. If you denied that a bagel has a hole, you'd be laughed out of New York City, Montreal, and any self-respecting deli worldwide. I consider this final.

What about the two-hole theory? Here's a question to think about: If there are two holes in the straw, where does one hole stop and the other begin? If that doesn't bug you, consider a slice of swiss cheese. Someone asks you to count the holes. Do you count the holes in the top of the slice and the bottom of the slice separately?

Or this: fill in the bottom of the straw, thus eliminating what you, two-holer, refer to as the bottom hole. Now the straw is basically a tall, thin cup. Does a cup have a hole in it? Yes, you say—the opening at the top is a hole. Okay, what if the cup gets stubbier and stubbier, until it's an ashtray? Surely we wouldn't call the upper rim of the ashtray a "hole." But if the hole is lost in the passage from cup to ashtray, *when* exactly is it lost?

You might say, here, that the ashtray still has a hole, because it has a depression in it, a negative space where material that might be there actually isn't. A hole doesn't have to "go all the way through," you insist—think about what we mean by a hole in the ground! This is a fair objection, but I think if we're going to be so laid back about what counts as a hole that *any* concavity or divot counts, we are going to expand the concept beyond usefulness. When you say a bucket has a hole in it, you don't mean it has a dent in the bottom, you mean it won't hold water. When you take a bite out of a bialy it doesn't turn into a bagel.

This leaves us with "one hole." That's the most popular of the three choices. Now let me ruin it for you. When I asked my friend Kellie about the straw, she rejected the one-hole theory very simply: "Does that mean the mouth and the anus are the same hole?" (Kellie is a yoga teacher, so she tends to see things anatomically.) It's a fair question.

But let's say you're one of those bold enough to accept the "mouth = anus" equation. There are still challenges. Here's a scene from the college dudes' Snapchat (but seriously, go watch this yourself, I can't fully capture the beautifully mounting frustration in words and stage directions). Bro 1 is an advocate of the one-hole theory, while Bro 2 is a two-holer.

Bro 2 [holds up a vase]: "How many holes does this have? So this has one hole, right?"

[Bro 1 assents nonverbally.]

Bro 2 [holds up a paper towel roll]: "So how many holes does this have?"
Bro 1: "One."
Bro 2: "How?" [Holds up the vase again] "Do these look the same?

Bro 1: "Because if I put a hole right here" [gestures at bottom of vase] "it's still gonna be one hole!"
Bro 2 [exasperated]: "You just said, *if I put a hole right there*."

[emits a kind of frustrated keening noise]

Bro 1: "If I put another hole in here it's gonna be—"
Bro 2: "Right—*another* hole including this hole! Two! Period!"

The two-hole bro in this scene is expressing a very plausible principle: making a new hole in something should increase the number of holes in it.

Let's make this harder still: How many holes are there in a pair of pants? Most people would say three: the waist and the two leg holes. But if you sew the waist shut, what you've got left is a very big denim straw with a bend in it. If you started with three holes, and you closed one up, you should have two holes left, not one—right?

If you're committed to one hole in the straw, maybe you then say the pants have just two holes, so that when you close up the waist you're down to one. This is an answer I hear a lot. But this answer suffers from the same problem as the two-hole straw theory: If there are two holes in the pants, where are they, and where does one stop and the other begin?

Or perhaps your take is that the pair of pants has only *one* hole, because what you mean by a hole is the region of negative space inside the pants. So what if I rip my jeans at the knee and make a new hole? Does that not count? No, you insist, there's still just one hole; all you've done with your artful rip is create a new *opening* to the hole. And when you sew up the bottom of your pants, or plug the bottom of a straw, you're not removing the hole, just closing off an entrance to or exit from the hole.

But this brings us back to the problem of having to say there's a hole in an ashtray. Or even worse: suppose I have a blown-up balloon. According to you, that balloon has a hole in it—the hole is the zone of pressurized air inside the balloon. Now I take a pin and make a hole in the balloon, so it pops. What's left is a round disc of rubber, maybe with a knot tied into it. A circular piece of rubber obviously doesn't have a hole. So you took something that had a hole, you poked a hole in it, and now it *doesn't* have a hole.

Are you confused now? I hope so!

Math doesn't answer this question, not exactly. It can't tell you what you should mean by the word "hole"—that's between you and your idiolect. But it can tell you something you *could* mean, which at least keeps you from tripping over your own assumptions.

Let me start with an annoyingly philosophical slogan. The straw has two holes; but they are the *same* hole.

REASONING WELL FROM BADLY DRAWN FIGURES

The style of geometry we're adopting here is called topology, and it's characterized by the fact that we don't really care about how big things are or how far apart they are or how they might be bent and deformed, which first of all may seem like a wrenching departure from the themes of this book and second of all may leave you wondering whether I am proposing some kind of geometric nihilism where we don't care about *anything*.

No! A lot of math is figuring out what we can, temporarily or for all time, get away with not caring about. This kind of selective attention is a basic part of our reason. You're crossing the street and a car blows through a red light and comes straight at you—there are all kinds of things you *might* consider as you plan your next move. Can you get a good enough look through the windshield to see if the driver seems incapacitated? What model of car is it? Did you put on clean underwear today in case you end up splayed in the street? These are all questions you *do not* ask—you license yourself not to care about them and you devote your whole consciousness to the task of gauging the car's path and jumping out of its way as fast and as far as you can.

The problems of mathematics are usually less dramatic, but they invoke in us the same process of abstraction, willful ignorance of every feature that doesn't touch the question immediately before us. Newton was able to do celestial mechanics when he understood that heavenly bodies weren't driven by their idiosyncratic whims but by universal laws that applied to every chunk of matter in the universe. To do this he had to steel himself to fail to care about what a thing was made of and what its shape was; all that mattered was its mass and its location relative to other bodies. Or go back even further, to the very beginning of math. The very idea of *number* is that for purposes of reckoning you can treat seven cows or seven rocks or seven people using exactly the same rules of enumeration and combination—and from there it's a short step to seven nations, or seven ideas. It doesn't matter (for those purposes) *what* things are—only how many there are.

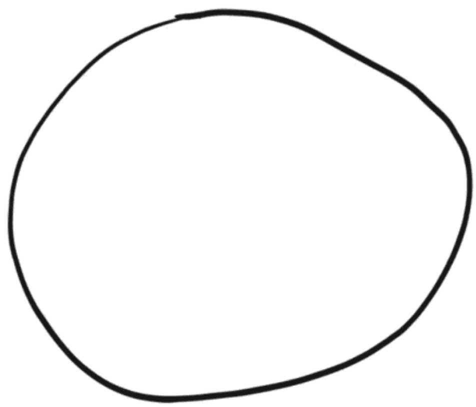
Topology is like that but for shapes. In its modern form it comes to us from the French mathematician Henri Poincaré. Him again! It's a name we'll be hearing a lot, because Poincaré had a hand in an astonishingly broad range of geometric developments, from special relativity to chaos to the theory of card shuffling. (Yes, there's a theory, and that's geometry, too; we'll get to it.) Poincaré was born in 1854 into a well-off academic family in Nancy, the son of a professor of medicine. At the age of five he fell seriously ill with diphtheria and was, for several months, completely unable to speak; he recovered fully, but remained physically weak throughout his childhood. Even as an adult, one student described him: "I recall above all his unusual eyes: myopic, yet luminous and penetrating. Otherwise, my memory is that of a man small in stature, stooped and ill at ease, as it were, in limb and joint." When Poincaré was a teenager, the Germans conquered Alsace and Lorraine, though Nancy stayed under French rule. France's unexpected and thorough defeat in the Franco-Prussian War was a national trauma; not only did France resolve to win back the territory it had lost, it undertook to imitate the bureaucratic efficiency and technological adeptness it believed had given Germany the advantage. Just as the surprise launch of Sputnik created a huge wave of funding for scientific education in the United States in the late 1950s, the loss of Alsace and Lorraine (or Elsass-Lothringen, as it now had to be called) spurred France to catch up with Germany's more fully realized scientific institutions. Poincaré, who had learned to read German during the occupation, became one of the new vanguard of French mathematicians trained in the modern way who would make Paris one of the mathematical centers of the world, with Poincaré at the center of the center. Poincaré was an excellent student, but not a prodigy; his first work of importance began to appear in his mid-twenties, and it was only in the late 1880s that he became an internationally famous figure. In 1889 he won the prize offered by King Oscar of Sweden for the best essay on the "three-body problem," concerning the motion of three celestial bodies subject only to the forces of each other's gravity. That problem remains only incompletely understood in the twenty-first century, but the theory of dynamical systems, the method by which modern mathematicians study the three-body problem and a thousand other problems like it, was launched by Poincaré in his prize paper.

Poincaré was a man of precise habits, who worked on research mathematics exactly four hours per day, from ten in the morning until noon and from five to seven in the evening. He was a believer in the critical importance of intuition and unconscious work, but his career was in some sense quite methodical, characterized not so much by blazing moments of insight as by a systematic and steady expansion of the realm of the understood against the territory of darkness, four hours each weekday and never during holidays. On the other hand, Poincaré had famously terrible handwriting. He was ambidextrous, and the joke in Paris circles was to say he could write equally well—that is, poorly—with either hand.

He was not only the most distinguished mathematician of his time and place, but a popular writer about science and philosophy for the general public; his books popularizing au courant topics like non-Euclidean geometry, the phenomena of radium, and novel theories of infinity sold tens of thousands of copies, and were translated into English, German, Spanish, Hungarian, and Japanese. He was a skilled writer, especially able at capturing a mathematical idea in a finely tuned epigram. Here's one that's very relevant to the question before us:

> Geometry is the art of reasoning well from badly drawn figures.

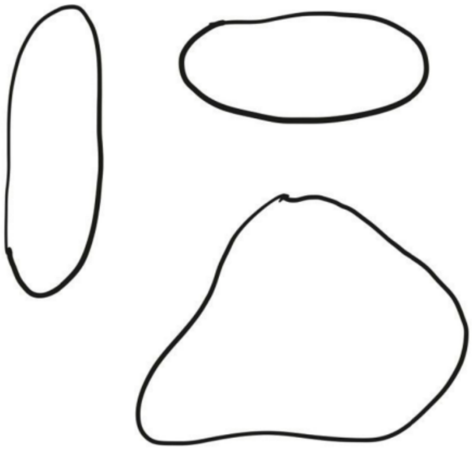
That is: if you and I are going to talk about a circle, I need us to have something to look at, so I'm going to take out a piece of paper and draw one:



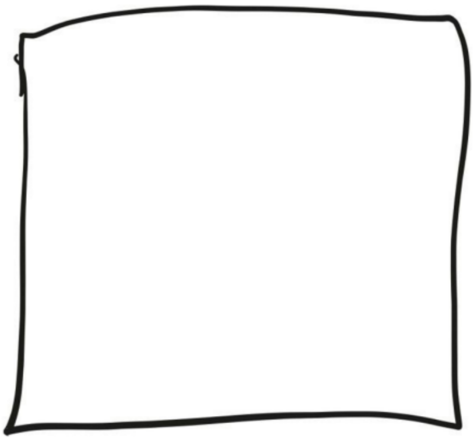
And you might, if you are in a pedantic mood, complain that this is *not* a circle; maybe you have your ruler on you and you check that the distance from my purported center isn't exactly the same for every point on the purported circle. Fine, I say, but if what we're talking about is how many holes there are in the circle, that doesn't matter. In this respect I'm following the example of Poincaré himself, who, true to his epigram and his crappy handwriting, was terrible at drawing figures. His student Tobias

Dantzig remembers: “The circles he drew on the board were purely formal, resembling the normal variety only in that they were closed and convex.”*

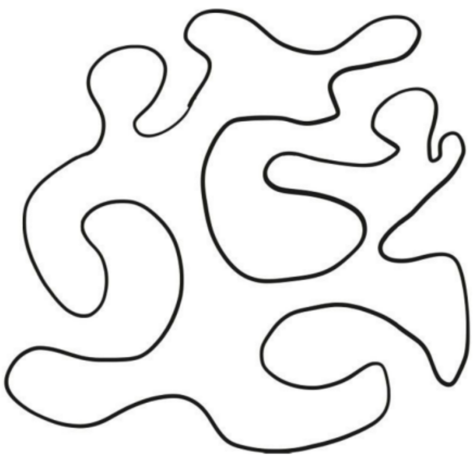
For Poincaré, and for us, these are all circles:



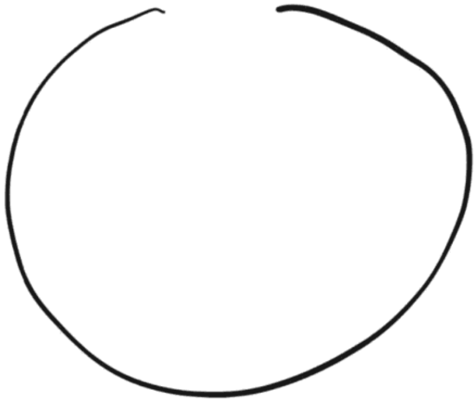
Even a square is a circle!*



This goofy squiggle, too:



But this isn’t a circle—

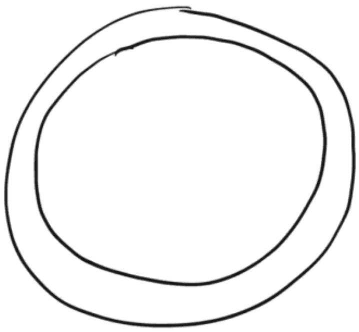


—because it's broken. And by breaking it, I've done something more irrevocably violent than mashing it or bending it or even kinking some corners in it; I have *truly* changed its shape, making it a badly drawn line segment instead of a badly drawn circle. And I have changed it from a thing with a hole in it to a thing without one.

The question of the hole in the straw *feels* like a topological question. Do the two mathematical bros, presented with it, demand to know the precise dimensions of the straw, or whether it's exactly straight, or whether its cross-section is a perfect circle of the kind Euclid would endorse? They do not. On some level they understand these are questions that, for the purpose at hand, can be safely laid aside.

And once you've laid those things aside, what's left? Poincaré counsels us to take the straw and shorten it, shorten it, shorten it. It is the same straw, as far as Henri P. is concerned. Pretty soon it's just a narrow band of plastic:

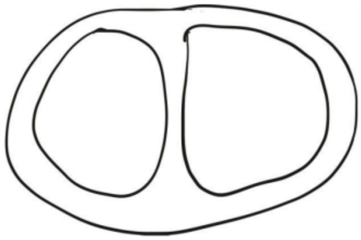
You could go further still, and bend the walls of the band outward in order to flatten the shape onto the page of a book.



Now it's a shape bounded between two circles, whose official geometry name is an *annulus* but which you might also know as a seven-inch single or an Aerobie or, if you can imagine it with a razor-sharp outer edge and being flung at you in combat in sixteenth-century India, a chakram.

Whatever you call it, it's still a badly drawn figure of a straw, and it has just one hole in it.

If topology insists we say a straw has just one hole, what can we say about pants? We can make those pants shorter, just as we did the straw. First they become shorts, then short-shorts, finally a thong. And when I press that thong flat against the pages of the book you're reading, you see a double annulus:



which visibly has two holes. So that's where we end up, for now: a straw has one hole, and pants have two.

NOETHER'S PANTS

But our problems aren't quite over. If the pants have two holes, *which holes are they?* The shortening process we described seems to identify the two holes in the pants as the legs, while the waist has become the outer rim. But as you might have noticed folding laundry, you could just as well have flattened the thong in a different way, with a leg hole on the outside and the other leg and waist making up the two "holes."

My daughter, without benefit of formal education in Poincaré's work, says pants have two holes, her argument being that a pair of pants is really just two straws. The hole in the waist, she says, is the combination of the two leg holes. She's right! And the best way to grasp this is to take the analogy between pants and straw seriously. Imagine, if you will, a straw shaped like a pair of pants, through which one endeavors to drink a malted milkshake. You might dip one leg in the shake and sip; then the same amount of milkshake is passing into the leg as passes out of the waist and into your mouth. Or you could do the same with the other leg; or you could let both legs penetrate the shake. But whatever you do, by the law of conservation of milkshake, the amount of milkshake coming out of the waist hole is the *sum* of the amount coming in through each leg. If three milliliters of milkshake per second is coming into the left leg and five milliliters into the right, then eight milliliters of milkshake is flowing out the top.* This is why my daughter is right to say that the waist hole isn't really a new hole at all, but the combination of the two leg holes.

So does that mean the two leg holes are the "real" holes? Not so fast. Just a second ago, when we were folding the just-laundered thong, it seemed like there was no true difference between the waist and the leg. But now the waist seems to be playing a special role again; 3 + 5 = 8 but not 5 + 8 = 3 or 8 + 3 = 5.

This is a matter of being careful about positives and negatives. Outflow is the opposite of inflow, so we should keep track of it with a negative sign; rather than saying 8 milliliters of milkshake is flowing out the waist of the straw, we say that -8 milliliters is flowing in! And now we have a beautifully symmetric description; the sum of the milkshake flow through all three openings is zero. In order to give a complete picture of the flow of milkshake through the pants, I just have to tell you two of those three numbers; but it doesn't matter *which* two numbers. Any pair would do.

Now we're ready to correct the lie we told earlier. It is not quite right to say the hole in the top of a straw (a straw-shaped straw, that is) is the same hole as the one at the bottom. But it's not really a brand-new hole, either. The hole at the top is the *negative* of the hole in the bottom. What flows into one must flow out the other.

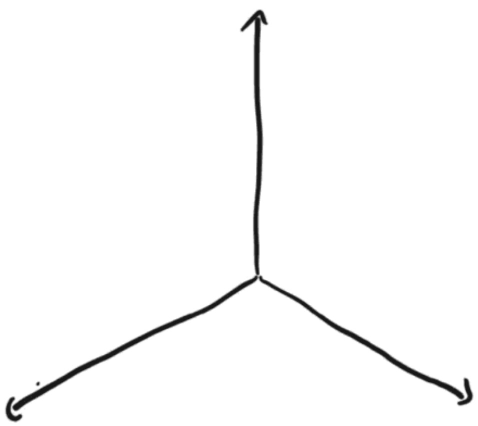
Mathematicians before Poincaré, especially the Tuscan geometer and politician Enrico Betti, had wrestled with the question of assigning a shape a number of holes, but Poincaré was the first to grasp the issue that some holes could be combinations of others. And even Poincaré didn't really think about holes the way mathematicians do today; that would have to wait for the work of the German mathematician Emmy Noether in the mid-1920s. Noether introduced the notion of the *homology group* into topology, and it is her notion of "holes" we've been using ever since.

Noether expressed her ideas in the language of "chain complexes" and "homomorphisms," not pants and milkshakes, but I'll stick with our current notation to avoid a wrenching stylistic shift. Noether's innovation was to see that it wasn't right to think of holes as discrete objects, but rather as something more like directions in space.

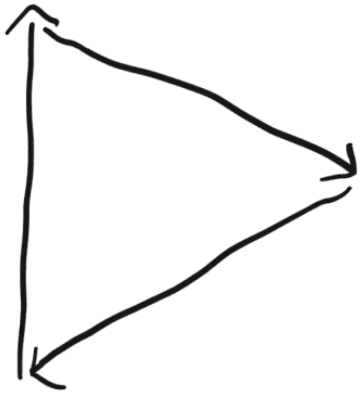
How many directions can you move on a map? In some sense, you can move in infinitely many different directions: you can go north, south, east, or west, you can go southwest or northeast by east, you can travel at an angle precisely 43.28 degrees eastward from due south, whatever. The point is, for all this infinitude of choice, there are only two *dimensions* in which you can travel; you can get anywhere you want to go by combining just two directions, north and east (as long as you're willing to express a ten-mile journey west as a negative-ten-mile journey east, that is).

But it doesn't make sense to ask which two directions are *the* fundamental ones from which all others derive. Any pair would be as good as any other; you can choose north and east, you can choose south and west, you can choose NW and NNE. The only thing you can't do is choose two directions that are either the same or directly opposite each other; try that and you're confined to a single line on the map.

The top and bottom of a straw are like that: exact opposites, one north, the other south. There's only one dimension to be found here. The waist and two legs of a pair of pants, by contrast, fill out two dimensions, like so:



Traveling a mile in one of these directions, then in the second, then in the third, brings you back to your starting point:



The three directions cancel each other out, combining into a zero.

"Nowadays this tendency is taken as self-evident," Paul Alexandroff and Heinz Hopf wrote in their foundational topology textbook of 1935, "but it wasn't so eight years ago. It required the energy and personality of Emmy Noether to make it common knowledge among topologists. Because of her it came to play the role in the problems and methods of topology that it does today."

"NOBODY DOUBTS NOWADAYS THAT THE GEOMETRY OF N DIMENSIONS IS A REAL OBJECT"

Poincaré created modern topology, but he didn't call it "topology"—he used the more cumbersome "analysis situs" (analysis of position). Good thing that didn't catch on! The word "topology" is actually sixty years older, a word made up by Johann Benedict Listing, a scientific jack-of-all-trades who also invented the word "micron" to mean a millionth of a meter, made major developments in the physiology of sight, dabbled in geology, and studied the sugar content of diabetic patients' urine. He traveled the world measuring the Earth's magnetic field with the magnetometer his PhD advisor, Carl Friedrich Gauss, had invented. He was a convivial and well-liked companion, maybe a little too convivial, because he was constantly running just ahead of his debts. The physicist Ernst Breitenberger called him "one of the many minor universalists who lend so much colour to the history of 19th century science."

Listing accompanied his well-heeled friend Wolfgang Sartorius von Waltershausen on a surveying trip to the volcanic Mount Etna, in Sicily, in the summer of 1834, and during his downtime, while the volcano slumbered, he thought about shapes and their properties, and gave topology its name. His approach wasn't systematic, like Poincaré's or Noether's. In topology, as in science and as in life, he was a sort of magpie, alighting where his interest carried him. He drew lots of pictures of knots, and he drew the Möbius strip before August Ferdinand Möbius did (though there's no evidence that Listing understood, as Möbius did, its curious property of being a surface with only one side). Late in his life he constructed an elaborate "Census of Spatial Aggregates," a

Noether provide a general theory of holes of any dimension, and in their setup, the number of zero-dimensional holes in a space is just the number of pieces it breaks into. A balloon, like a straw, is a single connected piece, so it has just one zero-dimensional hole. But *two* balloons have two zero-dimensional holes.

This might seem like a weird definition, but it makes everything work. The balloon has

(1 zero-dimensional hole + 1 two-dimensional hole) – (0 one-dimensional holes)

for an* Euler characteristic of 2.

A capital *B* has one zero-dimensional hole and two one-dimensional holes, so it has Euler characteristic −1.* Snip the bottom loop of the *B* and it becomes an *R*, which has Euler characteristic 0; there's one fewer (one-dimensional) hole, so the Euler characteristic goes up. Snip the loop of the *R* and you get a *K*, which has Euler characteristic 1. Or you could have used your snip to sever the *R*'s lower leg, leaving you with a *P* and an *I*; now there are two separate pieces, so two zero-dimensional holes, and the lone one-dimensional hole in the *P*, which gives you an Euler characteristic of 2 – 1 = 1 again. Each time you snip, you increase the Euler characteristic by 1, and this persists even when what you're doing isn't in any way cutting open a one-dimensional hole anymore. An *I* has Euler characteristic 1; snip it and you get two *I*'s, which has Euler characteristic 2; another snip makes the Euler characteristic 3, and so on.

What if you sew the two leg holes of your pants together, ankle to ankle? It's a bit too grainy to explain in this space, but in Poincaré's system the resulting shape has one zero-dimensional hole and two one-dimensional holes, for an Euler characteristic of –1; in other words, the vandalized pants have the same number of holes as the original ones. You got rid of one when you sewed the two ankle holes together, but created a new one encircled by the two conjoined legs. Is that convincing? It's a Snapchat argument I'd love to see.

Chapter 3

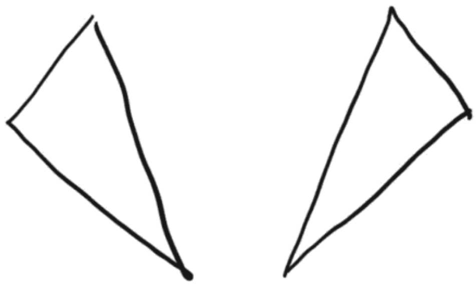
Giving the Same Name to Different Things

Symmetry is the basis of geometry as geometers now see it. More than that: what we decide to count as a symmetry is what determines what kind of geometry we're doing.

In Euclidean geometry, the symmetries are the *rigid motions*: any combination of sliding things around (translations), picking them up and flipping them over (reflections), and rotating them. The language of symmetry provides us a more modern way of talking about congruence. Rather than saying two triangles are congruent if all their sides and all their angles agree, we say they are congruent if you can apply a rigid motion to one that makes it coincide with the other. Isn't that more natural? Indeed, reading Euclid, you can feel the man himself straining (not always successfully) to avoid expressing himself this way.

Why take rigid motions as the fundamental symmetries? One good reason for this choice is that (though this is not so easy to prove!) the rigid motions are exactly those things you can do to the plane that keep every line segment the same length—thus, *symmetry*, from the Greek for "with measure." A better Greekism would be to use the phrase for "equal measure," or *isometry*, and that is indeed what we call a rigid motion in modern math.

The two triangles below are congruent:



and so we're inclined to do as Euclid did and declare them to be equal, even though they're not *really* the same; they're two different triangles about three inches apart.

This brings us to another slogan of the always-quotable Poincaré: "Mathematics is the art of giving the same name to different things." Definitional collapses like this are part of our everyday thinking and speaking. Imagine if, when someone asked you if you were from Chicago, you said, "No, I'm from the Chicago of twenty-five years ago"—

that would be absurdly pedantic, because when we talk about cities we implicitly invoke a symmetry under translation in time. In Poincaréan fashion, we call Chicago-then and Chicago-now by the same name.

Of course, we could be stricter than Euclid about what counts as a symmetry; we could, for instance, forbid reflections and rotations, allowing only sliding around the plane without spinning. Then those two triangles above wouldn't be the same anymore, because they're pointing in different directions.

What if we allow rotations but not reflections? You might think of that as the class of transformations we're allowed to carry out if we're stuck in the plane with the triangle, able to slide things and spin them but never to *pick them up and flip them over*, because that would involve making use of the three-dimensional space we're now barred from exploring. Under these rules, we still can't call those two triangles by the same name. In the triangle on the left, ordering the edges from shortest to longest takes us on a counterclockwise path. No matter how we may slide and spin the figure, that fact never changes; which means it can never be made to coincide with the triangle on the right, in which shortest-middle-longest is a clockwise path. Reflection switches clockwise and counterclockwise; rotations and translations don't. Without reflections, the clock-direction of the shortest-middle-longest path is a feature of a triangle that cannot be changed by any symmetry. It is what we call an *invariant.*

Every class of symmetries has its particular invariants. Rigid motions can never change the area of a triangle, or of any figure; in physical terms, we might say there's a "law of conservation of area" for rigid motions. There's a "law of conservation of length," too, for a rigid motion can't change the length of a line segment.*

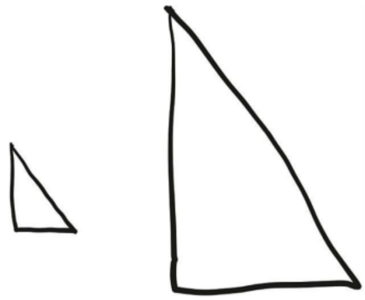
Rotations of the plane are easy to understand, but going up to three-dimensional space elevates the challenge considerably. It was understood as far back as the eighteenth century (Leonhard Euler again!) that any rotation of three-dimensional space can be thought of as rotation about some fixed line, or axis. So far, so good: but that leaves a lot of questions unanswered. Suppose I rotate 20 degrees about a vertical line and then 30 degrees about a line pointing horizontally to the north. The resulting rotation must be a rotation by some number of degrees around some axis, but what? It's approximately 36 degrees around an axis that points up and off somewhere to the north-northwest. But that's not easy to see! The person who developed a much handier way of thinking about these rotations—thinking of a rotation as a kind of *number* called a *quaternion*—was Wordsworth's young friend William Rowan Hamilton. As the famous story goes, on October 16, 1843, Hamilton and his wife were on a walk along the Royal Canal in Dublin when—well, I'll let Hamilton tell it:

> [A]lthough she talked with me now and then, yet an *under-current* of thought was going on in my mind, which gave at last a *result*, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth. . . . Nor could I resist the impulse—unphilosophical as it may have been—to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula . . .

Hamilton spent much of the rest of his life working through the consequences of his discovery. Needless to say, he also wrote a poem about it. ("Of high Mathesis, with her charm severe / Of line and number, was our theme; and we / Sought to behold her unborn progeny . . .” You get the idea.)

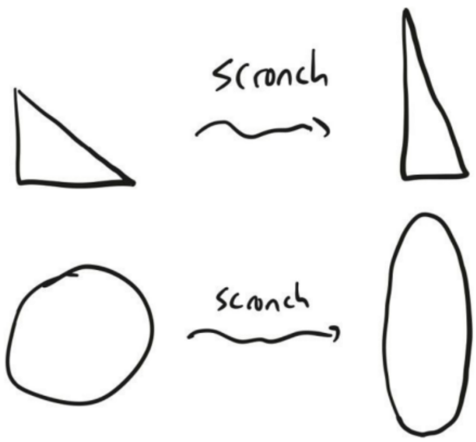
SCRONCHOMETRY

We can also turn the knob toward loosey-goosey and consider a wider range of transformations. We could allow magnification and shrinking, so that these two figures are the same:



Things about triangles that were invariants before, like area, are no longer invariant under this more forgiving notion of sameness. Other things, like the three angles, do remain invariant. In your high school geometry class, shapes that are the same in this looser sense were called *similar.*

Or we can invent entirely new notions, never seen in the classroom. We might, for instance, allow a kind of transformation we’ll call a *scronch*, which stretches a figure vertically by some factor, and compensates by shrinking it horizontally the same amount:*

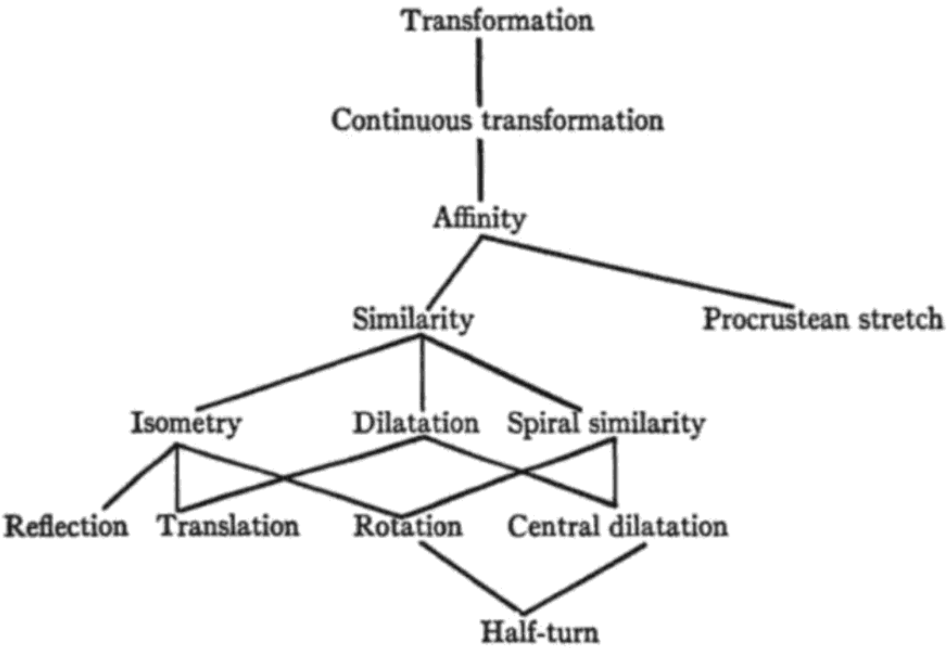


When I scronch a figure, its area doesn’t change. This is straightforward for rectangles oriented to have vertical and horizontal sides, since their area is given by width times height; the scronch multiplies the height by something and divides the width by the same thing, so their product, the area, remains the same. See if you can prove the same fact for a triangle, which is a little harder!

In scronch geometry, we call two figures the same if you can get from one to the other by translating and scronching. Two scronch-same triangles have the same area, but two triangles with the same area need not be scronch-same; for instance, any horizontal line segment is still horizontal after scronching, so a triangle with a horizontal side cannot be the scronch-same as a triangle that does not.

The possible types of symmetry, even in the plane, are too numerous to cover here anything like exhaustively. To give a modest idea of this menagerie, the next page shows a diagram from H. S. M. Coxeter and Samuel Greitzer's magisterial textbook *Geometry Revisited*.

This is a tree, much like a family tree, where each "child" is a special case of its "parent"—so an isometry, what we've called a "rigid motion," is a special kind of a similarity, and reflections and rotations are special kinds of isometries. A "Procrustean stretch" is Coxeter and Greitzer's vivid term for a scronch. The "affinities" are what you get if you allow scronches and similarities. The language of symmetry gives us a natural way of organizing the many definitions in plane geometry. Exercise: satisfy yourself that an ellipse is any figure with an affinity to a circle. Harder exercise: show that a parallelogram is any figure with an affinity to a square.



There's no right answer to the question of which pairs of figures are "really" the same. It depends what we're interested in. If we're interested in area, similarity is not good enough, because area isn't invariant under similarity. But if we're only interested in angles, there's no reason to insist on congruence; maybe that's just too demanding. Similarity would be good enough. Each notion of symmetry induces its own geometry, its own way of deciding which things are different enough that we'd better not give them the same name.

Euclid didn't write much about symmetry directly, but his disciples couldn't help thinking about it, even in contexts very far from plane figures. The idea that quantities of importance should be preserved by symmetries sits naturally in the mind. Here, for instance, is Lincoln, writing in his own private notes in 1854, in a very geometrical style:

ether, or to measure the speed of Earth's passage through it, had all failed. Attempts to account for these failures had taken the form of unpleasantly ad hoc extra postulates, like Hendrik Lorentz's "contraction"—the idea that all moving objects grow foreshortened in their direction of velocity. Fundamental physics was in an unsound condition. Poincaré closed his lecture with an attempt to envision a way past the danger:

> Perhaps too we shall have to construct an entirely new mechanics, which we can only just get a glimpse of, where, the inertia increasing with the velocity, the velocity of light would be a limit beyond which it would be impossible to go. The ordinary, simpler mechanics would remain a first approximation since it would be valid for velocities that are not too great, so that the old dynamics would be found in the new. We should have no reason to regret that we believed in the older principles, and indeed since the velocities that are too great for the old formulas will always be exceptional, the safest thing to do in practice would be to act as though we continued to believe in them. They are so useful that a place should be saved for them. To wish to banish them altogether would be to deprive oneself of a valuable weapon. I hasten to say, in closing, that we are not yet at that pass, and that nothing proves as yet that they will not come out of the fray victorious and intact.

Just as Poincaré had predicted, the patient did not die. On the contrary: it was about to bolt up from the table in bizarrely altered form. In 1905, less than a year after the St. Louis conference, Poincaré would show that Maxwell's equations were symmetric after all. But the symmetries involved, the so-called Lorentz transformations, were of a novel kind, which intermingled space and time in a much more subtle way than "I've been on this bus two hours so I'm forty kilometers north of where I was." (The difference is especially noticeable when the bus is moving at 90% of the speed of light.) From this new point of view, the Lorentz contraction was not a weird, inelegant kluge, but a natural symmetry; that the same object can change its length when hit with a Lorentz symmetry is no stranger than the fact that the same triangle can change its shape when you scronch it. Once you know the symmetries, you know the whole story of how different two things called "the same" are allowed to be. Poincaré was well prepared to make this leap, because he was already one of the innovators in pure mathematics developing forms of plane geometry distinct from Euclid's, having in particular a different group of symmetries. And Poincaré's "fourth geometry," which he had formulated back in 1887, was none other than the scronch plane.

Scronch geometry has laws of "conservation of horizontal and vertical"; if two points are joined by a horizontal or vertical line segment, so are their respective scronches. Lorentzian spacetime is much the same. A point in spacetime is a location *and* a moment; the special line segments conserved by Lorentz symmetries are those joining two location-moments whose two locations are separated by the exact distance light would cover in the amount of time between the two moments. The speed of light, in other words, is built into the geometry. The question of whether light can reach location-moment A to location-moment B has a definite answer, which is the same whether you get on the bus or not.

The scronch plane is like a baby version of Lorentz spacetime. You might think of it as what relativistic physics would look like if, instead of three dimensions of space, there were only one, joining with the one dimension of time to make a two-dimensional spacetime.

But Poincaré did not invent the theory of relativity. The very last sentence of his St. Louis lecture shows why. Poincaré hoped *not* to fundamentally change physics. He had discovered, by mathematical inspection, the strange geometry that Maxwell's equations pointed to, but he was not quite bold enough to follow the finger all the way to the strange point at the horizon it indicated. He was willing to accept that physics might not be what he and Newton had thought, but not that *the geometry of the universe itself* might not be what he and Euclid had thought.

What Poincaré saw in Maxwell's equations, Albert Einstein saw, too, in that same year of 1905. The younger scientist was bolder. And it was Einstein, out-geometrizing the world's foremost geometer, who remade physics as symmetry instructed.

Mathematicians were quick to understand the importance of the new developments. Hermann Minkowski was the first to work out Einstein's theory of spacetime all the way to its geometrical bottom (thus, what we call here the "scronch plane" is actually called the Minkowski plane, if you need to look it up). And in 1915, Emmy Noether established the fundamental relation between symmetries and conservation laws. Noether lived for abstraction; as a senior mathematician, she would describe her 1907 PhD thesis, a computational tour de force involving the determination of 331 invariant features of polynomials of degree four in three variables, as "crap" and "Formelngestrupp" ("formula-thicket"). Too messy and specific! Modernizing Poincaré's theory of "holes" to make it refer to the space of holes instead of just counting how many was very much in her line, and so was cleaning up the welter of conservation laws in mathematical physics. Finding quantities that are conserved by the symmetries of interest is almost always an important physical question; Noether proved that *every* flavor of symmetry comes with an associated conservation law, tying up what had been a messy bundle of computations into a neatly finished mathematical theory, and solving a mystery that had baffled Einstein himself.

Noether was expelled from the mathematics department at Göttingen in 1933, along with all the other Jewish researchers; she made it to the United States and joined the faculty at Bryn Mawr, but not long afterward died, at only fifty-three, of an infection following an apparently successful surgery to remove a tumor. Einstein wrote a letter to *The New York Times*, honoring her work in words the great abstractionist would surely have appreciated:

> [S]he discovered methods which have proved of enormous importance in the development of the present-day younger generation of mathematicians. Pure mathematics is, in its way, the poetry of logical ideas. One seeks the most general ideas of operation which will bring together in simple, logical, and unified form the largest possible circle of formal relationships. In this effort toward logical beauty spiritual formulas are discovered necessary for the deeper penetration into the laws of nature.

Chapter 4

A Fragment of the Sphinx

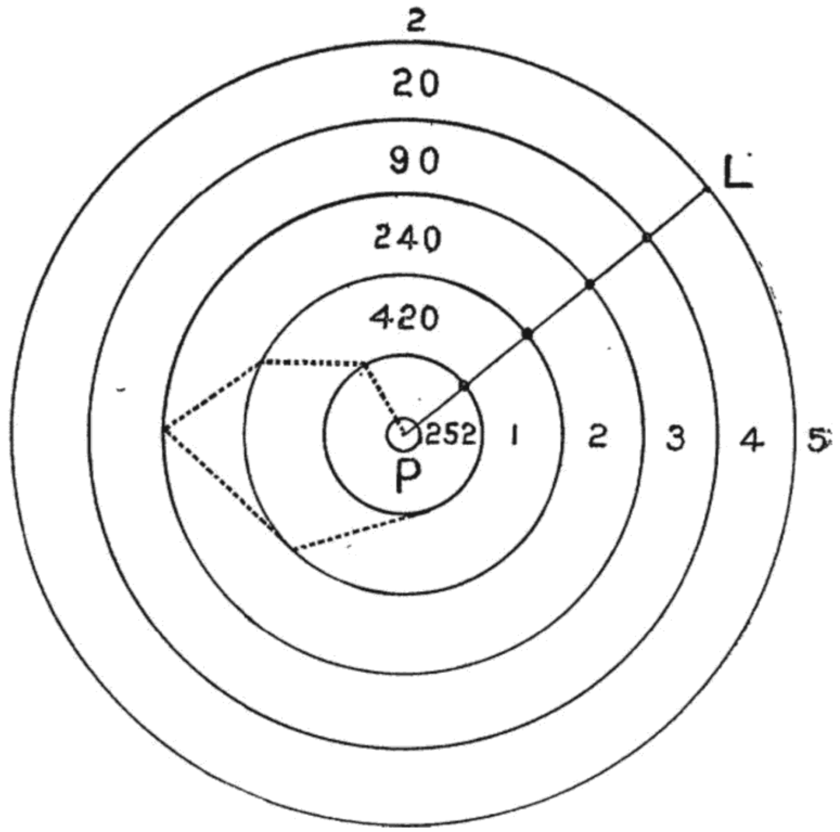
Back to the St. Louis exposition. Among the scientific bigshots present, remember, was Sir Ronald Ross, who in 1897 had discovered that malaria was carried by the bite of the anopheles mosquito. By 1904 he was a global celebrity, and getting him to Missouri for a public lecture was a coup. “Mosquito Man Coming,” read a headline in the *St. Louis Post-Dispatch*.

Ross’s lecture was titled “The Logical Basis of the Sanitary Policy of Mosquito Reduction,” which does not, I’ll concede, sound like a barn burner. But in fact the talk was the first glimmer of a new geometric theory that was about to explode into physics, finance, and even the study of poetic style: the theory of the random walk.

Ross spoke on the afternoon of September 21, while elsewhere at the exposition Governor Richard Yates of Illinois reviewed a parade of award-winning livestock. Suppose, Ross began, you eliminate propagation of mosquitoes in a circular region by draining the pools where they breed. That doesn’t eliminate all potentially malarial mosquitoes from the region, because mosquitoes can be born outside the circle and fly in. But a mosquito’s life is brief and it lacks focused ambition; it won’t set a course straight for the center and stick to it, and the odds seem against its meandering far into the interior in the short time it has to fly. So some region around the center would hopefully be malaria-free, as long as the circle is large enough.

How large is large enough? That depends how far a mosquito is likely to wander. Ross said:

> Suppose that a mosquito is born at a given point, and that during its life it wanders about, to or fro, to left or to right, where it wills. . . . After a time it will die. What are the probabilities that its dead body will be found at a given distance from its birthplace?



Here's the diagram Ross provided. The dotted line is the wandering mosquito; the straight line is the path a more goal-directed mosquito would take, covering a much greater distance before its demise. "The full mathematical analysis determining the question is of some complexity," Ross said, "and I cannot here deal with it in its entirety."

In the twenty-first century, you can easily simulate a mosquito moving on a Rossian path, so you can improve Ross's diagram to see what happens when the mosquito flits ten thousand times instead of five:

This is typical—sometimes the mosquito sticks around one area for a while, its path crisscrossing itself so much it almost fills up space; sometimes the mosquito appears to acquire a brief sense of purpose and manages to cover some distance. Watching animation of this process, I have to tell you, is unreasonably captivating.

Ross was only able to handle the much simpler case where the mosquito is fixed to a straight line, choosing merely whether to flit northeast or southwest. We can handle this, too! Suppose the mosquito lives for ten days, choosing each day whether to fly a kilometer to the northeast or a kilometer to the southwest. Each day it makes one of two choices, so the total number of career paths a mosquito can have is 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 = 1,024, and—assuming an unbiased mosquito—each of these paths is equally likely. In order for the mosquito to expire 10 km to the northeast of its hatching ground, it would have to make the choice to fly northeast ten times in a row, which only 1 in 1,024 mosquitoes will manage to do. The same tiny proportion ends up 10 km southwest; so, in all, 2 out of 1,024 make it 10 km from home. How many travel 8 km? That requires the mosquito to make a sequence of choices like

NE, NE, NE, SW, NE, NE, NE, NE, NE, NE

with nine of one choice and one of the other. The lone “SW” can be in any one of the ten spaces, so there are 10 out of 1024 paths which end 8 km northeast, and 10 that end 8 km southwest, for a total of 20. If you squint, you can see that Ross has written a little “20” and a “2” on the outer two rings of his circle. If you want to, you can write down the 45 paths that end up 6 km northeast of home, or the 210 that end up 2 km northeast, or the 252 that lead the mosquito right back to the fetid pond it spawned from. The mosquito’s starting point is its most likely grave. Which makes sense, because this random mosquito problem is really the same thing as flipping ten coins, counting heads as northeast and tails as southwest. Ending up 8 km away amounts to getting nine heads and one tail; ending up at home means getting five of each, which untrustworthy when you put it that way. It's like trying to figure out what kind of soup is in your bowl by tasting a single spoonful.

But in fact you can totally do that! Because you have every reason to think what's in your spoon is a random sample of the soup. You'll never dip into a bowl of clam chowder and get one sip of minestrone.

The soup principle is what makes polls so effective. But it doesn't tell you how closely you can expect the poll to reflect the city, state, or country being surveyed. The answer to that lies in the slow, disorderly progress of the mosquito from its pond. Take a state like the one I live in, Wisconsin, whose population is just about exactly evenly distributed between Democrats and Republicans. And now imagine a mosquito whose motion is determined as follows: I call a random Wisconsinite on the phone, ask them their political leanings, and instruct the mosquito to fly northeast if my respondent is a Democrat or southwest if they vote GOP. That's exactly Ross's model; the mosquito moves randomly in one direction or its opposite, two hundred times. How do we know we won't just happen to call two hundred Democrats and get a totally skewed view of how Wisconsin votes? Sure, that could happen—and the mosquito *could* have just gone for it and flown single-mindedly northeast from its birth to its demise. But it probably won't. We've already seen that the mosquito's distance from home after two hundred days, which in kilometers is exactly the difference between the number of Democrats and the number of Republicans in our poll, is about 11 km on average. So to find 106 Republicans and 94 Democrats in our poll wouldn't be at all strange. Something as far from political reality as a 120–80 split is another story. That's like dipping into a bowl of Wisconsin and getting a spoonful of Missouri. Finding 40 more Republicans than Democrats is the equivalent of the mosquito wandering 40 km from home, and we've already computed the chance of that to be just 3 in 1000.

In other words, it's quite unlikely the two hundred poll respondents will differ substantially from Wisconsinites as a whole. The sip tastes like the soup. There's about a 95% chance the proportion of Republicans in our sample will wind up between 43% and 57%, which is why a poll like this would be reported as having a margin of error of ± 7%.

But: that's assuming there's no bias lurking in our choice of whom to poll. Ross understood very well that bias could confound his mosquito model; before he gets down to calculating and circle drawing, he stipulates a landscape so homogeneous that "every point of it is equally attractive to them [the mosquitoes] as regards food supply, and that there is nothing—such, for instance, as steady winds or local enemies—which tends to drive them into certain parts of the country."

Ross insists on this assumption for a really good reason: without it, everything goes to hell. Suppose it *is* windy. Mosquitoes are little and even a light breeze can sway them in their course. Maybe a northward wind gives the mosquito a 53% chance instead of 50% of flying northeast. That's like an unnoticed bias in our poll that causes each random voter I call to have a 53% chance of being a Republican; maybe because Republicans are more likely than Democrats to agree to answer our survey questions, or to answer their phone in the first place, or to have a phone at all. That makes it a lot more likely that our poll deviates from the truth about the electorate. With an unbiased poll, the chance of finding 120 Republicans and 80 Democrats was only 3 in 1000. With this Republican wind, that chance jumps to 2.7%, almost ten times larger.

In real life, we never know a poll is perfectly unbiased. So we probably ought to be at University College, London, a position he had attained in his late twenties after reading law, abandoning that, studying medieval German folklore in Heidelberg, being offered a professorship in Cambridge in the subject, and then abandoning that, too. He was in love with Germany, which compared to England seemed a paradise of fiery intellectual life unencumbered by social convention in general and religion in particular. A fan of Goethe, Pearson wrote a romantic novel called *The New Werther* under the pen name "Loki." The University of Heidelberg misspelled "Carl" as "Karl" in his paperwork and he found he preferred that spelling to the one he'd been born with. Impressed that German had a gender-neutral word *Geschwister*, meaning "brother or sister," he invented the word "sibling."

Back in England, he advocated for irreligious rationalism and women's liberation and gave scandalous lectures on such topics as "Socialism and Sex." *The Glasgow Herald* wrote of one of his talks: "Mr. Pearson would nationalise land and nationalise capital: he at present stands alone in proposing to nationalise women also." His charisma enabled him to get away with moderate outrages of this kind; he was remembered by one former student as a "typical Greek athlete, with finely cut features, crisp curly hair and a magnificent physique." Photographs from the early 1880s show a man with a towering forehead, an intense gaze, and a jaw set in a way that suggests he's about to set you straight about something.

In his adulthood, he returned to mathematics, the subject he'd excelled at in college. He wrote that he "longed to be working with symbols rather than words." He applied to two professorships of mathematics and was rejected; when he finally got the appointment in London, his friend Robert Parker wrote to Pearson's mother:

> Knowing Karl as I do, I always felt sure that he would some day make his value felt and get something which would *really* suit him, however disheartened his friends might be at momentary failure. And now we can realise too what a great thing it has been for him to have three or four years quite free and occupied in other studies than mathematics; I do not mean that it has at all conduced to his present success, but no doubt it will make him a happier and more useful man and enable him to avoid any taint of that narrowness which one sees so often in, and fears so much for, men who have devoted themselves exclusively to one absorbing pursuit. And besides, great ideas are often suggested outside the range of the special subjects to which they relate, and Karl is returning to science with a fund of such ideas to be worked out and to make him some day as famous as Clifford* or any of his predecessors.

Pearson himself wasn't so sure: he wrote to Parker, in November of his first semester, "[I]f I only had a spark of originality or was a genius, I would *never* have settled down to the life of a teacher, but instead would have wandered through life* in the hope of producing something that might survive me." But Parker had the better side of the argument. Pearson became one of the founders of the new discipline of mathematical statistics, not because he proved theorems as magnificent as his physique, but because he understood how to bring the wider world in contact with the language of mathematics.

It was with that end in mind that Pearson, in 1891, took up the Gresham Professorship in Geometry, a position whose sole duty since its founding in 1597 has

> The mathematical statement of the simplest case of your mosquito problem is not difficult, but the solution is another thing! I have spent more than a whole day over it & only succeeded in getting the distribution after two flights. . . . It is, I fear, beyond my powers of analysis & wants a strong mathematical analyst. If you set such men the thing as a mosquito problem, however, they will not look at it. I must restate it as a chessboard problem or something of that sort in order to get mathematicians to work at it!

A contemporary mathematician trying to arouse interest in an unfamiliar problem might post a question to social media, or to a public Q&A website like MathOverflow. The 1905 analogue was the letters column of *Nature*, which is where Pearson posed the question, removing all mention of mosquitoes, as promised, but also, to Ross's irritation, all mention of Ross. On the same page of the July 27 edition we find a letter from the physicist James Jeans vainly attempting to beat back Max Planck's newfangled theory of quanta. Between Jeans and Pearson comes a notice from one John Butler Burke, who believed he had observed spontaneous generation of microscopic life in a vat of beef bouillon by exposure to the recently discovered element radium. This is not, perhaps, where you'd expect to find the very beginnings of a mathematical field that flourishes to the present day.

Ross's question was answered very quickly. In fact, it took about negative twenty-five years. The very next issue of *Nature* included a letter from Lord Rayleigh, the previous year's Nobel Prize winner in physics, who informed Pearson that he had solved the random walk problem in 1880, in the course of some investigations of the mathematical theory of sound waves. Pearson responded, rather defensively I think, "Lord Rayleigh's solution . . . is most valuable, and may very probably suffice for the purposes I have immediately in view. I ought to have known it, but my reading of late years has drifted into other channels, and one does not expect to find the first stage in a biometric problem provided in a memoir on sound." (You'll note that, despite Pearson's concession that the problem's origin is in biology, Ronald Ross has still been entirely erased.)

What Rayleigh had shown was that a mosquito that could fly in any direction wasn't so different from Ross's simpler one-dimensional model. It's still true that the mosquito tends to wander only very slowly from its starting point, its typical distance from home being proportional to the square root of the number of days it's been in flight. And it's still true that the very most likely location for the mosquito to be is at the place where it started. This led Pearson to remark, "The lesson of Lord Rayleigh's solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!"*

It's from this offhand comment of Pearson that we get the customary metaphor of the random walk as the path of an intoxicated human instead of a disease-carrying insect. It was once often called a "drunkard's walk," though in the kinder present era most people no longer think of a life-ruining addiction as an amusing peg to hang a mathematical concept on.

A RANDOM WALK TO THE BOURSE

Ross and Pearson weren't the only people thinking about random walks as the new century rolled in. In Paris, Louis Bachelier, a young man from Normandy, was working at the Bourse, the great stock exchange at the financial center of France. He began studying mathematics at the Sorbonne in the 1890s, taking great interest in the probability courses, which were taught by Henri Poincaré. Bachelier was not a typical student; an orphan, he had to work for his living, and hadn't received the lycée training that had molded most of his peers in the styles and mores of French mathematics. He struggled to get through his exams, scraping by with close to the lowest passing score. And his interests were just plain weird. High-status math at the time was celestial mechanics and physics, like the three-body problem Poincaré had wrestled with to win King Oscar's prize. What Bachelier wanted to study was the fluctuations of bond prices he'd observed at the Bourse; he proposed to treat those motions mathematically, just as his professors were treating the motions of the heavenly bodies.

Poincaré was deeply skeptical about applying mathematical analysis to human actions, dating back at least as far as his reluctant participation in the Dreyfus affair, the fiery controversy over a Jewish soldier accused of spying for the Germans. Poincaré had little taste for political battles and had somehow managed to stay largely neutral as the conflict engulfed French society. But his colleague Paul Painlevé, a fervent Dreyfusard (also the second Frenchman to fly in an airplane and, much later, briefly prime minister of France under the presidency of Poincaré's cousin Raymond), was able to convince him to wade in. Police Chief Alphonse Bertillon, the founder of "scientific policing," had presented a case against Dreyfus arguing that Dreyfus's innocence was ruled out by the laws of probability. The most distinguished mathematician in France, Painlevé argued, could not be silent now that the matter had become a question of numbers. Poincaré, won over, wrote a letter assessing Bertillon's calculations, to be read to the jury at Dreyfus's 1899 retrial at Rennes. Just as Painlevé had hoped, when Poincaré read the police chief's analysis he found crimes against mathematics. Bertillon had found many "coincidences" that he believed pointed irrefutably to Dreyfus's guilt. Poincaré observed that Bertillon's methods allowed him so many opportunities to locate coincidences that it would have been unusual for him *not* to find some. Bertillon's case, Poincaré concluded, was "absolutely devoid of scientific value." But Poincaré went further, declaring that "the application of the calculus of probability to the moral sciences"—what we would now call the *social* sciences—"is the scandal of mathematics. To wish to eliminate moral elements and replace them with numbers is as dangerous as it is pointless. In short, the calculus of probabilities is not, as people seem to believe, a marvelous science which excuses those who have mastered it from having common sense."

Dreyfus was convicted anyway.

Poincaré's student Bachelier set out in his thesis, a year later, to establish the appropriate price for an option, a financial instrument that allows you to purchase a bond at a specified price at some fixed time in the future. Of course, the option has value only if the market price of the bond exceeds the price you've locked in. So to understand the worth of the option you need to have some sense of *how likely* it is the bond's price will end up above or below that crucial line. Bachelier's idea for analyzing

The explanation of Brownian motion was hotly fought over. One popular theory was that the pieces of pollen or Sphinx were being kicked around by innumerable even smaller particles, the molecules of the fluid, too small to be seen in a nineteenth-century microscope. The molecules were constantly buffeting the pollen at random, forcing it into its lifelike Brownian dance. But remember, not everyone believed that matter was made of tiny invisible particles! This was the substance of a great dispute, with Ludwig Boltzmann on the "tiny particles" side and Wilhelm Ostwald on the other. To the Ostwaldians, "explaining" a physical phenomenon by postulating tiny undetectable molecules doing the work was little better than invoking invisible demons to push the pollen around. Karl Pearson himself had written, in his 1892 book *The Grammar of Science*, "No physicist ever saw or felt an individual atom." But Pearson was an atomist, in his way; whether atoms could ever be detected by instruments or not, he wrote, the hypothesis of their existence could bring clarity and unity to physics and generate experiments that could be tested. In 1902, Einstein hosted an occasional scholarly discussion society and dinner club, "The Olympia Academy," in his apartment in Bern. The frugal dinner typically consisted of "one slice of bologna, a piece of Gruyere cheese, a fruit, a small container of honey and one or two cups of tea." (Einstein, who had not yet gotten his position at the Swiss patent office, was scraping out a living tutoring physics at three francs an hour, and was contemplating a side hustle as a street violinist to keep himself fed.) The Academy read Spinoza, they read Hume, they read Dedekind's *What Are Numbers and What Should They Be?*, and they read Poincaré's *Science and Hypothesis*. But the very first book they studied was Pearson's *The Grammar of Science*. And Einstein's breakthrough, three years later, was very much in the spirit Pearson had imagined.

Invisible demons are unpredictable; there's no mathematical model for what those rascals will do next. Molecules, on the other hand, are subject to the laws of probability. If a particle is struck by a tiny water molecule moving in a random direction, the particle is moved by the impact to travel a tiny distance in that direction. If there are a trillion such impacts every second, then the pollen moves a small fixed distance in a randomly chosen direction every one-trillionth of a second. What does the pollen do in the long term? That might be predictable, even if the individual impacts can't be seen.

This is exactly the question Ross had asked. Instead of a pollen particle, Ross had a mosquito, and instead of a trillion motions a second, he had one per day, but the mathematical idea is the same. Just as Rayleigh had done, Einstein worked out mathematically how particles would tend to behave under a sequence of motions in random directions. This made the molecular theory something you could test experimentally, as Jean Perrin subsequently did, with complete success; this was the decisive blow for Boltzmann's side of the battle. Molecules were invisible, but the accumulated effect of a trillion randomly jostling molecules was not.

To analyze Brownian motion and the stock market and mosquito all at once, with the mathematics of the random walk, is to follow Poincaré's slogan and give the same name to different things. Poincaré formulated his famous advice in his 1908 address to the International Congress of Mathematicians in Rome. He spoke movingly of the way doing complex computations can feel like "blind groping," until that moment when you encounter something more: a common mathematical understructure shared by two separate problems, illuminating each in the light of the other. "[In] a word,"