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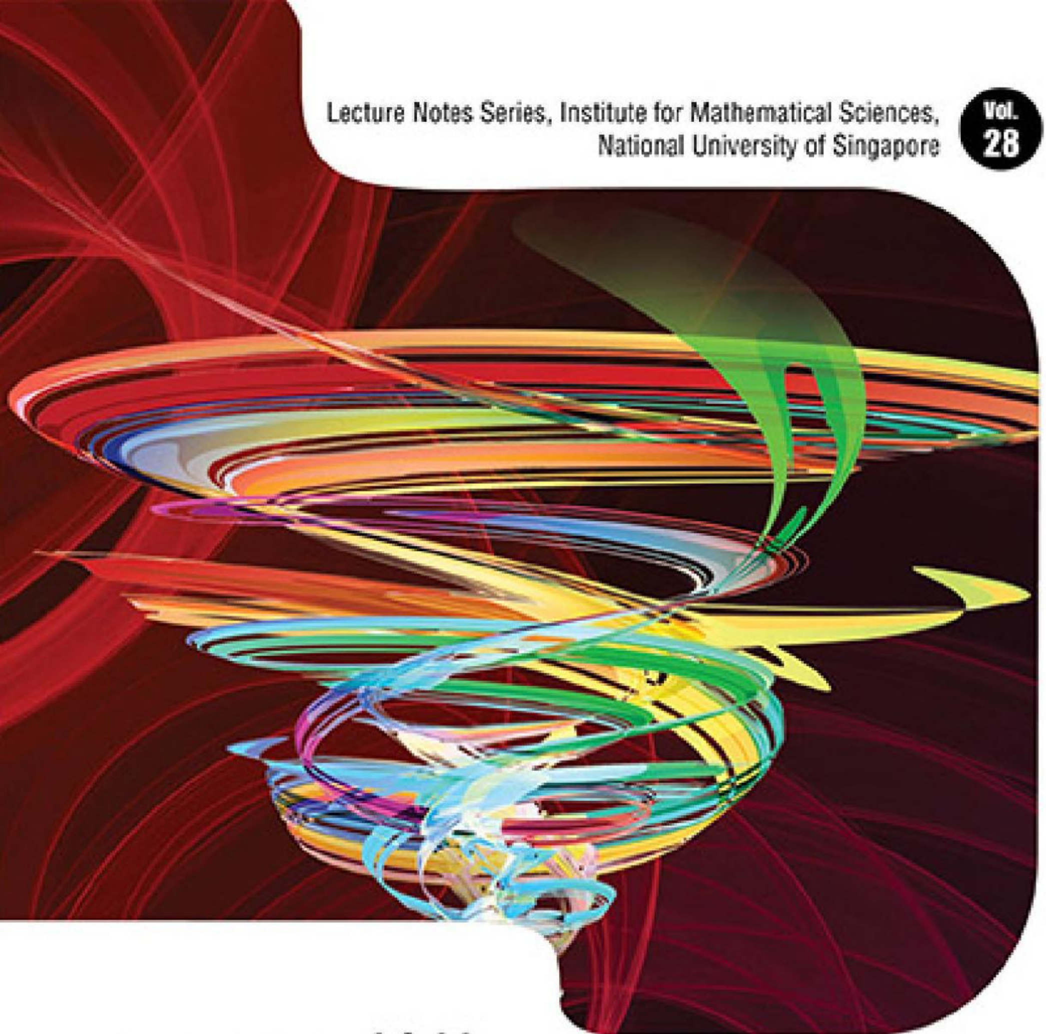

Denis R Hirschfeldt

SLICING THE TRUTH

On the Computable and Reverse Mathematics of Combinatorial Principles

Editors: Chitat Chong • Qi Feng • Theodore A Slaman • W Hugh Woodin • Yue Yang

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Foreword
by Series Editors

The Institute for Mathematical Sciences (IMS) at the National University of Singapore was established on 1 July 2000. Its mission is to foster mathematical research, both fundamental and multidisciplinary, particularly research that links mathematics to other efforts of human endeavor, and to nurture the growth of mathematical talent and expertise in research scientists, as well as to serve as a platform for research interaction between scientists in Singapore and the international scientific community.

The Institute organizes thematic programs of longer duration and mathematical activities including workshops and public lectures. The program or workshop themes are selected from among areas at the forefront of current research in the mathematical sciences and their applications.

Each volume of the *IMS Lecture Notes Series* is a compendium of papers based on lectures or tutorials delivered at a program/workshop. It brings to the international research community original results or expository articles on a subject of current interest. These volumes also serve as a record of activities that took place at the IMS.

We hope that through the regular publication of these *Lecture Notes* the Institute will achieve, in part, its objective of reaching out to the community of scholars in the promotion of research in the mathematical sciences.

July 2014

Chitat Chong
Wing Keung To
Series Editors

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Foreword by Volume Editors

The series of Asian Initiative for Infinity (AII) Graduate Logic Summer School was held annually from 2010 to 2012. The lecturers were Moti Gitik, Denis Hirschfeldt and Menachem Magidor in 2010, Richard Shore, Theodore A. Slaman, John Steel, and W. Hugh Woodin in 2011, and Ilijas Farah, Ronald Jensen, Gerald E. Sacks and Stevo Todorcevic in 2012. In all, more than 150 graduate students from Asia, Europe and North America attended the summer schools. In addition, two postdoctoral fellows were appointed during each of the three summer schools. These volumes of lecture notes serve as a record of the AII activities that took place during this period.

The AII summer schools was funded by a grant from the John Templeton Foundation and partially supported by the National University of Singapore. Their generosity is gratefully acknowledged.

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Preface

When Rod Downey and I finished our book *Algorithmic Randomness and Complexity*, which was almost a decade in the making, I promised myself I would never again write a book. But accidents happen. In 2010, I was invited to give a short course at the Asian Initiative for Infinity Graduate Summer School, organized by the Institute for Mathematical Sciences and the Department of Mathematics of the National University of Singapore, and to write a version of my lecture notes for publication. The topic of the course was the reverse mathematics and computability theory of combinatorial principles, an area of research whose roots reach back several decades, but which has seen a particular surge of activity in the last few years. Much of this work has proceeded along lines that are fairly distinct from the material covered in Simpson's excellent *Subsystems of Second Order Arithmetic*, and there has been little alternative to reading research articles for those interested in understanding it. While reading original papers is highly recommended, it can be a difficult process without appropriate guidance. I wanted my notes to be an entryway into this area, providing both an overview of some fundamental ideas and techniques, and enough context to make it possible for students with at least a basic knowledge of computability theory and proof theory to appreciate the exciting advances currently happening in the area, and perhaps make contributions of their own.

I decided to adopt a case-study approach, using the study of versions of Ramsey's Theorem (for colorings of tuples of natural numbers) and related principles as illustrations of various aspects of computability theoretic and reverse mathematical analysis. Even within this deliberately narrow focus, I felt no need to be encyclopedic. It was not my goal to write a survey, but to tell a story. Nevertheless, when it comes to mathematics, a properly illustrative story needs details. Furthermore, thorough discussions of some of these details were difficult to find in existing sources. And while there may be some who can tell a long story without digressions, I am not among them. Thus the text grew and grew, until it became a book. So be it.

I will give an overview of the book in Chapter 1, but for now, here is the abstract I wrote when still thinking of this text as an article: We discuss two closely related approaches to studying the relative strength of mathematical principles, computable mathematics and reverse mathematics. Drawing our examples from combinatorics and model theory, we explore a variety of phenomena and techniques in these areas. We begin with variations on König's Lemma, and give an introduction to reverse mathematics and related parts of computability theory. We then focus on Ramsey's Theorem as a case study in the computability theoretic and reverse mathematical analysis of combinatorial principles. We study Ramsey's Theorem for Pairs (RT^2_2) in detail, focusing on fundamental tools such as stability, cohesiveness, and Mathias forcing; and on combinatorial and model theoretic consequences of RT^2_2. We also discuss the important theme of conservativity results. In the final section, we explore several topics that reveal various aspects of computable mathematics and reverse mathematics. An appendix contains a proof of Liu's recent result that RT^2_2 does not imply Weak König's Lemma. There are exercises and open questions throughout.

Acknowledgments

I was partially supported during the writing of this book by grants DMS-0801033 and DMS-1101458 from the National Science Foundation of the United States. This book is a version of a short course given at the Asian Initiative for Infinity Graduate Summer School, sponsored by the Institute for Mathematical Sciences and the Department of Mathematics of the National University of Singapore from 28 June to 23 July, 2010, and funded by the John Templeton Foundation and NUS. I thank these organizations; the organizers Ted Slaman and Hugh Woodin; our hosts at NUS Chi Tat Chong, Qi Feng, Frank Stephan, and Yue Yang; the other lecturers Moti Gitik and Menachem Magidor; and all of the participants for a delightful and rewarding experience. I also thank the Einstein Institute of Mathematics of The Hebrew University of Jerusalem for hosting a visit during which much of this book was written, Menachem Magidor for arranging this visit, and the students in a short course I taught there based on a draft version of this book. Finally, I thank Tsvi Benson-Tilsen, Chi Tat Chong, Damir Dzhafarov, Bill Gasarch, Noam Greenberg, Jeff Hirst, Carl Jockusch, Joe Mileti, Joe Miller, Antonio Montalbán, Ludovic Patey, Ted Slaman, Reed Solomon, Wei Wang, and Yue Yang for useful comments and responses to queries.

Chapter 1

Setting Off: An Introduction

Every mathematician knows that if $2 + 2 = 5$ then Bertrand Russell is the pope. Indeed, Russell is credited with having given a proof of that fact in a lecture by arguing as follows: If $2+2 = 5$, then, subtracting 3 from each side, $1 = 2$. The pope and Russell are two, therefore they are one. Of course, from the point of view of classical logic, no such proof is needed, since a false statement implies every statement. Contrapositively, every statement implies a given true statement. But suppose we were to take seriously the task of proving that, say, the Four Color Theorem implies that there are infinitely many primes. What are the chances that any of us could come up with a proof that "really uses" the Four Color Theorem? The exercise may seem as pointless as it is difficult, but of course mathematicians do set and perform tasks of this kind on a regular basis. "Use the Bolzano-Weierstrass Theorem to show that if $f : [0, 1] \to \mathbb{R}$ is continuous, then f is uniformly continuous." is a typical homework problem in analysis, and the question "Can Chaitin's information-theoretic version of Gödel's First Incompleteness Theorem be used to prove Gödel's Second Incompleteness Theorem?" led to a lovely recent paper by Kritchman and Raz [116]. There is also a well-established practice of showing that a given theorem can be proved *without* using certain methods, for instance in the exercise of proving the irrationality of $\sqrt{2}$ without using the fundamental theorem of arithmetic, or in elementary proofs of the prime number theorem. We have all heard our teachers and colleagues say things like "Theorems A and B are equivalent." or "Theorem C does not just follow from Theorem D." or "Using Lemma E in proving Theorem F is convenient but not necessary." These are often crucial things to understand about an area of mathematics.

They are also things that can help us make connections between different areas of mathematics. For example, consider the following theorems: the existence of suprema for continuous real-valued functions on $[0, 1]$, the local existence theorem for solutions of ordinary differential equations, Gödel's completeness theorem, the existence of primes ideals for countable commutative rings, and Brouwer's Fixed Point Theorem. Dissimilar as these theorems might seem, at heart they all involve compactness arguments in an essential way, and can all be seen as reflections in different fields of the same fundamental combinatorial idea, expressed in a principle known as Weak König's Lemma that we will discuss in some detail below. We will be able to make this claim formal in Section 4.4.

In this book, we will discuss two closely related approaches to making mathematically precise sense of this idea of establishing implications and nonimplications between provably true principles: computable mathematics and reverse mathematics. We will focus on combinatorial principles that are easy to state and understand, but exhibit intricate and intriguing behavior from these points of view. This book is not meant as a survey of results in this area, but rather as an introduction to a constellation of ideas and methods, unapologetically biased towards my own interests (particularly the computability theoretic and reverse mathematical analysis of combinatorial and model theoretic principles related to Ramsey's Theorem for pairs), but hopefully with enough breadth and depth to engage and motivate newcomers to the area. In particular, although the program of reverse mathematics has close ties with the foundations of mathematics, I will not say much about that aspect of the field.

I will assume some background in mathematical logic, in particular the basics of computability theory, though a few essential computability theoretic concepts will be reviewed briefly in Section 2.1. Otherwise, this book should be self-contained. There are exercises scattered throughout; working them out is an integral part of using this text. A few open questions will also be mentioned, and readers are encouraged to do battle with them as well. One never knows when a clever idea will solve a long-standing problem.

1.1 A measure of motivation

There are many things that comparing the relative strength of theorems can do for us. The process of revealing the "combinatorial core" of a theorem can give us significant insight. For example, it can tell us when a method is not just useful in proving a theorem, but in fact *necessary*. In other cases,

$T \subset S$ consisting only of finitistically acceptable principles, such as those involving simple manipulations of strings. Mathematicians would then be able to sleep in peace, knowing that the consistency of S is as sure as that of T. This hope was shattered by Gödel's Second Incompleteness Theorem, which showed in particular that not even S itself, let alone any such T, is powerful enough to prove the consistency of S (unless, of course, S is actually inconsistent, in which case it proves everything).

But the ashes of Hilbert's Program have proved a fine fertilizer. Methods of mathematical logic that could have been merely tools to settle a single problem (albeit an exceptionally important one) could now become instruments of fine analysis. Instead of a simple division between unexceptionable methods and doubtful ones in need of justification, work in the foundations of mathematics has revealed subtle gradations, and metamathematical work has provided formal analogs and results about where various theorems, methods, and even whole areas of mathematics fall in this foundational universe. Reverse mathematics in particular has been tied to such concerns from its outset, and its classification of the strength of mathematical principles into various levels has implications for this kind of foundational work. Some discussion of these matters can be found in Simpson [187, 190, 191]; see in particular the table on page 43 of [191]. As the present book is meant as a tutorial on the mathematical practice of reverse mathematics and computable mathematics, and as my own interest in these subjects does not stem primarily from such foundational considerations, but rather from a desire to understand (at a purely mathematical level) some of the complex interactions between "ordinary" mathematics, combinatorial structure, and computability, I will not say more on this subject, except to comment on a line from Borges' "Fragmentos de un Evangelio apócrifo":

> "*Nada se edifica sobre la piedra, todo sobre la arena, pero nuestro deber es edificar como si fuera piedra la arena.*"
> ["Nothing is built on stone, all on sand, but our duty is to build as if the sand were stone."]

The work of Gödel and others has shown that mathematics, like everything else, is built on sand. As Borges reminds us, this fact should not keep us from building, and building boldly. However, it also behooves us to understand the nature of our sand.

We finish this section with an important remark: The approaches to analyzing the strength of theorems we will discuss here are tied to the countably infinite. Finite structures are of course of great interest, but complexity theoretic methods are usually better suited to their analysis than computability theoretic ones. In the other direction, the application of computability theoretic and reverse mathematical methods to essentially uncountable mathematics is still in its infancy. (Here "essentially uncountable" is meant to exclude areas where uncountable objects have reasonable countable approximations, such as countable dense subsets of separable metric spaces.) For a discussion of various approaches to uncountable computable mathematics (and reverse mathematics), see [73].

Simpson [191] makes a distinction between "set-theoretic" and "ordinary", or "non-set-theoretic", mathematics in formulating what he calls the main question of his book: "Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?" In the former camp he places set theory itself, and other branches such as point-set topology and uncountable discrete mathematics, which arose from the development of set theory and involve essentially uncountable structures. In the latter, he places countable algebra, analysis, number theory, and so on, areas in which objects are either countable or have countable approximations. As he puts it, "the set existence axioms which are needed for set-theoretic mathematics are likely to be much stronger than those which are needed for ordinary mathematics. Thus our broad set existence question really consists of two subquestions which have little to do with each other. Furthermore, while nobody doubts the importance of strong set existence axioms in set theory itself and in set-theoretic mathematics generally, the role of set existence axioms in ordinary mathematics is much more problematical and interesting." Because of our focus on countable objects, "infinite" below will mean countably infinite unless otherwise stated.

1.2 Computable mathematics

Computability theory gives us many tools to calibrate the complexity of mathematical principles. Particularly fundamental is the idea of a set of natural numbers Y being computable in another set Z, which means that there is an algorithm that, on input n, decides whether $n \in Y$ while using Z as an *oracle*. That is, the algorithm is allowed to ask as many questions as it wants about whether certain particular numbers are in Z (but only a finite number of questions for each input, of course, since if an algorithm is to terminate, it must do so in finite time). We can formalize this notion using

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