# Symmetries in Science XI

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(Eds.)

Kluwer Academic Publishers

# Symmetries in Science XI

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**KLUWER ACADEMIC PUBLISHERS**

NEW YORK, BOSTON, DORDRECHT, LONDON, MOSCOW

eBook ISBN: 1-4020-2634-X
Print ISBN: 1-4020-2633-1



Created in the United States of America

# Contents

# List of Figures

# List of Tables

# Preface

The symposium “Symmetries in Science XIII” was held at the Mehrerau, Bregenz, Austria, during the period July 20 - 24, 2003. On the occasion of the symposium three outstanding scientists were honored who have contributed significantly to the success of the series of symposia “Symmetries in Science”. The honored scientists are

- **Professor Akito Arima**, The House of Councillors, Japan
- **Professor Francesco Iachello**, J.W. Gibbs Professor of Physics and Chemistry, Yale University
- **Professor Marcos Moshinsky**, Universidad Nacional Autonoma de Mexico

We all, but in particular one of us (B.J. G.), wish to thank the authorities of the Land Vorarlberg and the Landeshauptstadt Bregenz for their generous and continuous support of the symposia series. On part of the Landeshauptstadt Bregenz we wish to thank Mag. Michael Rauth and the mayor of the Landeshauptstadt, Dipl. Ing. Markus Linhart, as well as his predecessor as mayor, Dipl. Vw. Siegfried Gasser. On part of the Land Vorarlberg our thanks go to Mag. Gabriela Duer, Dr. Hubert Regner, and former Landesrat Dr. Guntram Lins.

While the support given by these officials to the symposia series is thankfully appreciated, it was the continuous, consistent, cooperation and support given to the symposia series by Dr. Hubert Regner during a period of some 20 years which was the key to the success of 13 symposia. Again, at this point, one of us (B.J. G.) wishes to express his sincere thanks to Dr. Hubert Regner.

The preparation of the submitted manuscripts for publication was done at Saitama University and the University of Naples, where the totality of articles was assembled. We wish to thank Professor Wolfgang Bentz of Tokai University for helping out with the manuscripts submitted to Saitama University. Our special thanks go to Guido Celentano of the University of Naples who had to deal with the totality of articles.

We also wish to thank Professor Michael Ramek of the Technical University Graz who, as scientific secretary to a number of symposia, has contributed to the smooth day to day operation of the symposia series.

BRUNO J. GRUBER, CHAIR OF THE ORGANIZING COMMITTEE

GIUSEPPE MARMO, MEMBER

NAOTAKA YOSHINAGA, MEMBER

# WHY SYMMETRY?

## *Some personal reflections*

P. Roman

*Boston University, Emeritus*

**Abstract** • What is symmetry? • Why is symmetry important in science? • Historical developments. • Mathematical characterization of Symmetry. • Basic areas where symmetry principles are used. • Some special topics.

Thanks to the enthusiasm and administrative skills of Bruno Gruber, and to the adroitness of his assistants, as well as the generosity of many sponsors, for thirty years now we have been regaled with a periodic sequence of inspiring, exciting, pleasurable encounters centered on the topic "Symmetries in Science". Since this is, unfortunately, the last occasion when this group of colleagues (nay friends) meet, perhaps it will be not amiss to distance ourselves, briefly, from details of our field, and spend some time on contemplating the deeper, perhaps we may say philosophical aspects of Symmetry.

What follows will certainly not be a scholarly, exhaustive, authoritative treatment of the topic. I can only transmit to you very individualistic, almost personal thoughts (or rather sentiments) about the topic in question. There have been, virtually, whole libraries written on symmetry, and I cannot add more wisdom. I shall not even be systematic in my exposition, and won't attempt to give credit to the workers in the field. If I quote opinion of authors, these will be haphazard and far from comprehensive. Further, I'll take the viewpoint of a physicist, neglecting, for instance, crystallography, chemistry, biology. Even within physics, my treatment will be prejudiced by the views of the quantum theory of "fundamental particles" and interactions.

Let us now start with the question: **What is symmetry**? To answer in the broadest sense, it will be well to go back in time as far as the early stage of mankind's awakening. Perplexed and troubled by the apparent diversity, complexity, and unpredictability of nature, man conceived and took solace in the notion of an all-embracing, ultimate harmony of the Universe. "Harmony", that is congruence of parts, balance and unification of elements, is but one of the



*B. Gruber (ed.), Symmetries in Science XI*, 1–12.

synonyms we use for "symmetry". In fact, only the belief in some underlying symmetry makes it possible for us to develop science. We shall come back to this point later; right now I'd like to quote Hermann Weyl, one of the four greatest mathematicians of the 20th century, to succinctly sum up these thoughts. "Symmetry is one of the ideas" - Weyl notes - "by which man throughout the ages has tried to comprehend and create order, beauty, and perfection." This applies to science, art, and human conduct in general.

Well then: **why is symmetry important for science**? Once again, we must delve deeper and ask first *what is science*? Contrary to what is taught in most junior high schools, science is not "the explanation of Nature". Nature, be it even objective reality, just *is*. It cannot be "explained", at least not as far as science is concerned. Existence is a primary category, including, by the way, ourselves, too. (Which would imply that the explainer himself must be explained.) And certainly science is much more than "the description of Nature". That alone would be ad hoc, incidental, utterly unsatisfying. We have gone far beyond such a casual phenomenology and even empiricism. We want to "understand", and we have in part succeeded. Indeed, as Anatole France has put it: "The wonder is not that the field of stars is so vast, - but that man has measured it". And Einstein went even further: "The most incomprehensible thing about our Universe is that it can be comprehended", says he. Comprehended? What do we mean by that? Perhaps surprisingly, several humanists of the past came near to a comprehensive characterization of science. Goethe says: "Herein consists the scientific method: that we show the concept of a single phenomenon in its connection with the rest of the world of ideas." And the twentieth century German writer Hermann Hesse tells us: "Every science is . . . a kind of ordering, simplifying; an attempt to make digestible for the spirit that which is indigestible."

Indeed, we are safe to say: *Science is the attempt to correlate individual phenomena and events into a coherent framework* (or systems of such frameworks). The correlation of part-entities into a coherent framework must satisfy two criteria (at least): 1. It should be systematic, comprehensible, attractive, nay: beautiful; 2. it should have predictive power, that is, the framework should encompass special items that are extensions of the already encompassed ones.

A minute's reflection then tells us: the essential feature of a scientific theory is *structure*. And the framework for studying, analyzing, understanding, enjoying structure is *mathematics*. That is why Wigner spoke of the "unreasonable effectiveness of mathematics in the natural sciences" and came to the conclusion that "mathematics does play a sovereign role in physics", "it is, in a very real sense, the correct language" of science.

Finally, we are back at the concept of symmetry. Mathematics deals with structures in two basic ways: a) by topology, b) by algebra. Topological structures use mostly analysis; algebraic structures are much more varied and use mostly construction and composition. *The concept of symmetry is a central part of algebraic systems in revealing and classification of structures.* This is the answer to our question: why is symmetry so important for science.

Now that we have clarified the basic role of symmetry in scientific thinking, we may come back for a moment to reconsider our earlier casual observation, viz. that symmetry is a crucial element in the perception of beauty. The connection to science goes both ways. We recognize willingly and with ease a structure that, analyzed in terms of symmetry, is beautiful. Conversely, a structure scrutinized by symmetry-analysis becomes acceptable to us only if we find that it is beautiful. It will be well to remember Einstein's dictum: "A theory is acceptable to us only if it is beautiful". A similar statement was made by Dirac when someone, in public, questioned him as to why he chose precisely his experimentally then unsupported equation out of other possible ones. "Because it was beautiful" he answered. And, of course, beauty is created, assessed, enjoyed via symmetry. Thus the circle closes.

We shall now remark on the **historical development of the idea of symmetry**. Already to prehistoric man obvious, natural, concrete, geometric symmetry in Nature was manifest: he recognized the patterns on sea-shells or the multifarious forms of snowflake crystals. Later these geometric symmetries became consciously imitated and applied in art - to be soon abstracted to more sophisticated manifestations (such as balance of colors), and applied to music (both in the guise of harmony as well as rhythmic patterns). These roles of symmetry continued and were amplified up to the present, but a discussion would lead us too far.

Concerning rather the evolution of symmetry-ideas in science, we observe that the development was slower. If we disregard such fancies as the orders of Celestial Spheres and their music, it appears that the implanting of symmetry-ideas into physics begins only with the late renaissance. But Galileo, and later even Newton, relied on symmetry principles only unconsciously and implicitly. Nevertheless, Newton made a giant step forward. He realized that in the study of phenomena one must make a clear distinction between the underlying *dynamical law* on the one hand and the *initial conditions* on the other. The former are rigorously structured; the latter are entirely irrelevant and haphazard, in that they are not encompassed by the law. This separation made analytical science at all possible. The for us at present important thing here is that the set of possible initial conditions is obtained by applying onto the system certain symmetry transformations. For example, subjecting the system to a translation in space, we obtain a (shifted) initial coordinate. Subjecting the system to a Galilei transformation, we obtain an initial state with a changed velocity. Furthermore, if we know the relation between two initial specifications (given by a symmetry transformation), then the resulting dynamical development in the second case can be obtained from that pertaining to the first case by means of a certain code (given by a symmetry transformation) which does not depend on the particular nature of the relevant specifications. Finally, the mathematical form of the dynamical law cannot depend on the specific nature of the actual initial specifications: this means that the dynamical law is covariant (form invariant) under the pertinent symmetry transformations. We now see what fundamental role symmetry considerations play in the very foundations of "doing science" - but of course Newton did not use this language: he relied on his ingenious instinct.

In the century or so following Newton, symmetries of known dynamical laws were noted and described - but just as an interesting afterthought or observation. Also, various *conservation laws* were established (derived sometimes tortuously, from the equations of motion), but without understanding (or even noting) their relation to symmetries.

The turning point came at the beginning of the twentieth century. Two great breakthroughs occurred at that time. The first was the establishment of relativity theory. Einstein was faced with a problem. There was a discrepancy between the symmetries of mechanics and those of electrodynamics. The laws of mechanics possessed Galilean symmetry (as we now call it), those of electromagnetism had Lorentz symmetry. With his unerring insight and intuition, Einstein chose the latter as the *guiding principle of physics*. Thus was the theory of special relativity born. For the first time in history, *a symmetry consideration became a guiding principle*. From then on, it was not acceptable just to propose, by trial and error or otherwise, a law of nature (and then examine its properties, including symmetry), but rather it became obligatory to insist that any law of nature should be formulated so that it exhibits Lorentz covariance. (We now speak more generally of Poincaré covariance.) Thus postulates of symmetry became guiding principles of exploration. That this was a radical change in outlook may be illustrated by the fact that the Galilei symmetry group which Einstein replaced by the Poincaré symmetry, had not ever been consciously explored earlier; only in the 1960's were we treated to a systematic study of the Galilei group - even the name was not common knowledge. (Parenthetically it should be mentioned that the inhomogeneous Galilei and Poincaré groups are not closely related: the latter is purely kinematical, the former dynamical, because it contains the equations of motion, too.)

At about the same time that Einstein radically changed our views on symmetry transformations, Hilbert and his school in Göttingen made great progress in the mathematical handling and application of symmetry. Here the connection between symmetry and *conservation laws* was clarified, and by Noether's theorem the derivation of such laws was rendered almost automatic (for continuous symmetries, at least).

The second great breakthrough in the history of symmetries occurred in the 1920's when modern quantum theory was born. Symmetries play a much more fundamental role in quantum theory than in classical physics. The reason for this is the linear structure of quantum theory: that is, the superposition principle. More explicitly: the set of states of a system corresponds to a set of vectors in a suitable Hilbert space, and so symmetry operations connecting different states give rise to linear operations in Hilbert space. *These linear operations thus carry a representation of the symmetry in the Hilbert space* of the system. Later we shall say a bit more about symmetries in quantum theory, but right now we shall just recall two names: Eugene Wigner and Hermann Weyl. These giants recognized very early the never-before-thought-of power of symmetry principles in the quantum world.

Wigner's 1928 book, applying representations of symmetry groups to atomic and molecular physics, opened a new world and revolutionized spectroscopy as well as chemistry. And when in 1939 he classified the unitary representations of the Poincaré group in Hilbert space, he gave thereby a definition of elementary particles which, essentially, is still valid. In fact, as Heisenberg remarked in 1973: "We will have to give up ... the concept of fundamental elementary particles [and] should rather accept the concept of *fundamental symmetry*".

On the other hand, Weyl, among several achievements in quantum symmetry theory, discovered the basic ideas of gauge symmetry, which eventually grew into the present fundamental theory of fields and particles, the Standard Model.

Next I will turn briefly to the **mathematical characterization of symmetry**. When we talk of a symmetry, we mean thereby an *automorphism of a given system* onto an equally possible description of the system. The details (both of the specification of the system and of the particular features of the mapping) vary enormously from case to case, but the principle is the same. Systems that are related by a symmetry transformation form an equivalence-class. Thus, *a symmetry transformation leads to an equivalent alternative description of the system*, i.e. it is a canonical transformation in classical physics and a unitary transformation in quantum theory.

Most frequently we recognize not an isolated symmetry but a (finite or infinite) set of them which, as an algebraic system, satisfies the axioms of a group. Why such a conglomeration of closely related symmetries occur is not at all clear to me, except for space-time groups.

It is amusing to note that, for continuous space-time symmetries (and only for those) one may take not the active but instead the passive view of description. This consists of the following. Instead of saying that a state with transformed data is a possible state of the same system as seen by a different observer, we could also say that the system has been physically transformed into another one, and the two descriptions are both given by the same observer. It is not clear to me whether this is a triviality or whether it has some deeper meaning.

Often the terms "symmetry" and "invariance" are used interchangeably. This is a mistake. Any symmetry transformation can be performed on any system, and it gives an alternative description of the system. The question is: do some selected *features* of the system remain unchanged or not? In particular, for a given physical system, is the *dynamical behavior* of the transformed system the same as for the original? *If* yes, then we can say: the dynamical law is invariant under the symmetry transformation, we have an invariant law, we have an "invariance". This must be distinguished from covariance of an equation, which simply means that the *form* of *some* equation doesn't change. In addition, the term "an invariant" has also several different meanings. Mostly, a selected quantity whose numerical value does not change under a symmetry transformation is called "an invariant of the system". Also, Casimir operators (see later) of a symmetry group are "invariants". In addition, a state which belongs to the trivial one-dimensional (scalar) representation of some symmetry group, is also often called "an invariant state".

A piece of art with perfect symmetry may appear boring. Nature seems to know this: indeed, very often we encounter **broken symmetries**. There are several mechanisms operative here. First, if the system is not isolated, properties of the environment may break the symmetry (such as the vertical gravitational field on the surface of the Earth). We can safely disregard these as trivial cases. Important (and not fully understood) are what one may call dynamical (or explicit) symmetry breakings. For a set of circumstances a symmetry holds, but for other dynamical circumstances (other forces) only a subset of these are maintained. (Example: Systems governed fully by "strong interaction dynamics" have iso-$SU(2)$ symmetry, but systems governed by electrodynamical dynamics [or which simultaneously are subject to strong and electromagnetic forces] exhibit only $U(1)$ symmetry.) Often a clash of symmetries causes symmetry breaking. These phenomena lead sometimes to uneasy situations which we call "approximate symmetries". But apart from the explicit symmetry breaking, more fundamental and more interesting is the case of spontaneous symmetry breaking. We have here a situation where the equation of motion (the dynamics)is invariant under a certain symmetry, but there are solutions which do not conform with the symmetry. This is a typical quantum phenomenon. The cause of such behavior is that the ground state of the system ("the vacuum state") is not invariant under the symmetry. In fact, it is degenerate. A related spontaneous quantum symmetry breaking occurs in field theory when a so-called "triangular (or similar) anomaly" occurs. Here the associated renormalization procedure causes the effective breaking of symmetry. Spontaneous symmetry breaking has enormous importance both in condensed matter physics as well as in the quantum theory of fundamental particles. In the latter case, for example, it leads to gauge particles' achieving of mass, so that alone with this spontaneous symmetry breaking becomes the gauge theory of the electroweak interactions possible.

Broken symmetries provide again an occasion to point out the difference between symmetry and invariance. Even if a conservation law (invariance) does not hold, symmetry may still be a useful and even powerful computational concept. A good example is provided by current algebras. In the presence of weak interactions the iso-$SU(2)$ vector current is no longer conserved. Yet the $SU(2)$ algebra of the vector current holds, and leads to important physical conclusions. More than this: the $SU(2)$ axial-current is never conserved, but the axial $SU(2)$ symmetry algebra holds and leads to very deep physical results.

As a final remark to the topic of symmetry breaking we note that (in all types of it) the breaking is not haphazard and disorderly, but is subject again to some symmetry argument.

Above we have repeatedly pointed out the fundamental roles of symmetries in physics. As a way of summary and overview, we will now explicitly list the major **areas where symmetry principles are used** in the every day praxis of physics.

1. Symmetry principles provide a most valuable *heuristic guide* in the search for dynamical laws. We believe that all fundamental laws of nature share certain symmetry properties, and specific branches of physics or specific systems may exhibit additional symmetries. (We do not yet have, and may never have, a "theory of everything".) We intuit, from masses of observations, particular symmetry properties, and then formulate laws so as to satisfy these symmetries in a general and unified frame.

2. Once the appropriate fundamental equations have been found, symmetry properties furnish powerful *tools for their solution.* This topic has two major aspects:

2.a) The symmetries restrict the forms the solutions can take. Roughly speaking, all admissible solutions will be classified by their symmetry character. This is why, for example, the solutions of Lorentz covariant equations can be only tensors or spinors. Similarly, the possible state vectors of a quantized system must be and are labelled by appropriate characteristics (viz, eigenvalues of generators and values of Casimir operators) of the symmetry group allowed for by the dynamical equations.

2.b) Invariance of the dynamical law under some symmetry gives rise to *conservation laws*: constants of motion can be constructed. The existence of such constants then leads to *selection rules*: processes that would connect states with different values of the conserved quantity are forbidden.

The treatment is particularly striking in the framework of quantum theory. It is easy to show that the (self-adjoint generator of the) unitary operator realizing a symmetry is a constant of motion if and only if it commutes with the Hamiltonian (or $S$-matrix). This means that then its expectation value is time independent; and if a state is an eigenstate of it with some eigenvalue at a given time, then the same state at a later time will be still an eigenstate belonging to the same eigenvalue. It is important to note here that invariance of the Hamiltonian and dynamical invariance of the system are equivalent statements. In particular, we have a conservation law if and only if there exists a symmetry (which, specifically, leaves the Hamiltonian invariant). In praxis, we encounter mainly symmetry *groups*, and the physically interesting entities (conserved observables) are the self-adjoint generators corresponding to the infinitesimal unitary transformations. Besides these generators (all conserved, but of course not all simultaneously measurable) we have also certain polynomials of these generators, the so-called Casimir invariants in the enveloping algebra, which automatically commute not only with the Hamiltonian (i.e. are conserved) but also with the generators. This explains why eigenvalues of the Casimir operators plus those of a selected set of commuting generators can be used as a complete set of state labelling parameters.

We further note that invariance of the dynamics under a symmetry transformation manifests itself in quantum theory also in the *equality of transition amplitudes* for the original and the transformed pair of systems; a very useful facet, both for establishing a symmetry and for exploiting its consequences.

3. Established symmetry properties greatly facilitate the *computation of specific quantities* that are of interest. For example, a lengthy calculation of matrix elements can be shortened by invoking some symmetry property. Further, matrix elements pertaining to different processes become related by symmetries (cf. branching relations). In particular, if a symmetry holds, transition probabilities between pairs of different states (i.e. rates of different processes) can be expressed in terms of a small number of invariant amplitudes. (In fact, experimentally observed relations between cross sections may be utilized to spot the symmetry that underlies the processes.)

Quantum numbers labelling states of composite systems can be easily computed from those of the constituents, if a symmetry holds.

Perturbational calculations are also facilitated in the presence of a known symmetry. (For example, a symmetry imposes restrictions on admissible trial-functions.)

Finally, symmetry principles often give, without any detailed calculation, the general pattern of a perturbation's effect when applied to an unperturbed state or system. Actually, about 40 years ago attempts were made to reveal the entire level system of an object by using only symmetry arguments, assuming the operation of some very powerful symmetry group – this without explicitly solving the energy eigenvalue equation. These attempts in particle theory went under the name of "dynamical groups" and "spectrum generating algebras".

In the rest of this individualistic survey I would like to pinpoint **some isolated special topics** which, to me, appear interesting and not too outdated.

1. There is, first of all, the topic of superselection rules, discovered by Wigner, Wightman and Wick in the 1950's. Suppose there is a generator of some symmetry which is simultaneously measurable (commutes) not only with the Hamiltonian but also with *all* observables of a system. (A typical example is the operator of electric charge.) Then this observable is not only conserved but has a tremendous structural effect on the system's quantum theory. It forces the Hilbert space to decompose into incoherent subspaces: the superposition principle becomes restricted. Matrix elements of transition operators between these incoherent subspaces are automatically zero – hence the name "super*selection rule*". Even the concept of an observable becomes restricted if a superselection rule operates. Thus, a symmetry property can influence the entire structure of a quantal system. I feel that this topic has not yet been given sufficient consideration.

2. Particle theorists used to distinguish between *space-time symmetries and internal symmetries*. (The Poincaré group, its possible conformal extension, the de Sitter group etc. represent the first category; isospin-$SU(2)$, flavor-$SU(3)$ etc. the second. Of course, there are further symmetries too, e.g. permutation symmetry, which do not really fit in any category.) Somewhat misleadingly, Wigner used to call internal symmetries "dynamical symmetries" because they seem to be connected to specific forces. In the 1960's beautiful attempts were made to find some large symmetry group that nontrivially combines and contains both the space-time and the internal symmetries, and reveals their connection. Unfortunately, it soon turned out that such attempts, however successful they seemed to be in particular aspects, are doomed because of deep lying algebraic reasons pertinent to the Poincaré algebra (O'Raiffertaigh's theorem). So the issue had to be dropped. But then, less than ten years later, *gauge theories* triumphed. The gauge symmetry idea lingered in the minds for decades. The wonderful facet of gauge theories is that the form and even the relative strengths of interactions becomes determined by local gauge invariance. This is an entirely new aspect of invariance, the key being the local character of the transformations. When the vexing problem of how to obtain masses for gauge fields was solved by spontaneous symmetry breaking, and when the difficult formalism of renormalization in gauge theories was mastered, the unification of weak and electromagnetic forces was accomplished. Soon, strong interaction dynamics was also encompassed by an (unbroken) color-$SU(3)$ gauge invariance of the quark-gluon system. The necessary ingredients of confinement and asymptotic freedom are also connected to the specifics of the gauge group. More than that: an important ingredient in the renormalizability of the quantum field theory is precisely gauge invariance. Thus, finally, the relation of space-time to "internal" symmetries acquired a new and satisfying aspect.

In the fiber bundle representation of gauge theories, the fiber built at each base space-time point "contains" the appropriate "internal" gauge symmetry, relative to that point. This is implied by the locality principle underlying the gauge symmetry concept.

In rapid sequel to these breakthroughs, courageous efforts were made to unify the electroweak and strong interactions in some Grand Unified Theory (GUT). Several gauge groups have been considered, primarily an $SU(5)$ system. Notwithstanding the formal attractiveness and some numerical successes of GUT, several serious problems blemish the picture. First, because quarks and leptons appear in the same representation of the Big Gauge Group, baryon and lepton numbers are not conserved. In particular, baryon decays are predicted. The calculated lifetime of the proton, unfortunately, is about two orders of magnitude shorter than the experimentally allowed lower limit. Second, the calculated relations between the masses of the fundamental fermions do not appear to be correct. Then there is the obnoxious hierarchy problem: roughly speaking, why is there a stable enormous gap between the energy scales of symmetry breaking of the GUT group to the Standard Model symmetry, and the breaking of the letter to the $SU(3)$ [color] $\times U(1)$[e.m.] world we live in? This difficulty is connected to uncertainties, ambiguities, and technical problems pertaining to the symmetry breaking Higgs scalars. It is possible that supersymmetry (see below) may alleviate these problems. On the other hand, personally, I also feel uneasy about the subtleties of the various renormalization schemes and regularization methods that must be used to get any numerical results at all from GUTs. In any case, the last word about GUT has not yet been spoken.

It is interesting to observe that the "greater" the symmetry, the more unification of particles and forces we obtain. The earlier the Universe, the hotter and more energetic, the higher is the symmetry. Thus, it may well be that the dreamland of an "ultimate unification" will be achieved by finding the "primeval" (pre-Planck state) symmetry of the Universe.

3. Yet another relatively novel symmetry idea in particle theory is that of *supersymmetry*. This implies quasi a symmetric role between fermions and bosons. According to such schemes, to every fermion there corresponds a boson and vice versa. Mathematically, supersymmetry may be realized by enlarging the Poincaré group Lie algebra to a graded algebra. This is done by adjoining to the tensorial generators of the Poincaré group one (or more) 4-spinor generators, with appropriate commutators and anticommutators. One may reformulate the theory by considering the fields as functions not only of the usual vectorial Minkowski coordinates but also of suitable spinorial "coordinates". If such an attractive symmetry of Nature really exists, it must be badly broken, because the "new" accompanying particles ($s$-lepton and $s$-quark bosons, gauginofermions,

etc.) ought to have an enormous mass compared to their customary partners. So far there is no evidence of $s$-particles, and personally I find it odd to have a basic symmetry which, to manifest itself, must be badly broken to start with. I say this despite the fact that supersymmetry, made into a local gauge theory in the spinor coordinates, may perhaps include quantum gravity. Even this "11-dimensional supergravity theory" seems by now to have been subsumed (together with all super-string theories) into what is fancifully called "$M$-theory". But the discussion would take us into even more unsafe waters.

4. There is one system that, by definition, is unique: this is our Universe. Yet, even in *cosmology* symmetries play a basic role. It is generally accepted (and used even as a guiding principle) that the Universe is endowed by the basic symmetries of spatial isotropy and spatial homogeneity. (The latter may be even crucial for us to do science, since it guarantees that laws of nature are everywhere valid.) These symmetries are often called the "cosmological principle". Any deviation from these symmetries, global or even local, have tremendous cosmological and astrophysical consequences. At one time (the 1950's) the idea was put forward that the Universe possesses also temporal homogeneity (in the sense of stationarity). This was referred to by Hoyle and Bondi as the "perfect cosmological principle". Its consequence was shocking: it lead to the beautifully attractive steady-state-theory of the cosmos. But observation of the thermal cosmic background radiation refuted this convenient model and gave way to the big bang picture. In the latter, a problem is encountered: the traditional dissociation of initial conditions and laws becomes obscure.

Concomitantly, the role and application of symmetries pertaining to the pre-Planck period ought to be thought over more carefully. There is also another area of cosmology where symmetry (more accurately symmetry breaking) plays a substantial role, to which we already hinted above. If some GUT theory of all interactions and matter prevailed at the "earliest time" (when time could be defined at all), then, to end up with the present features of the world, it must have been broken in successive steps. Each step can be described as a phase transition. These, again, are ruled by symmetry considerations. Together with these phase transitions of GUT, some "space-time" symmetries are also affected. In particular, PC is presumably broken - this permits the over-preponderance of matter versus anti-matter. However, the most amazing discrete space-time symmetry, TCP, remains unbroken. This symmetry has deep consequences: it ensures that to every particle there belongs an antiparticle with the same mass and lifetime. Here again we see the world-shaping role of symmetries.

*Instead of summarizing, let us ask one more question: Why do we love symmetries? Because symmetries are such a basic feature of nature, including our mental apparatus, that they enable us to discover, explore, analyze - even,* *to some extent, understand - structure. And structure, recognized and properly contemplated, allows us to enjoy and adore the miracle of the Creation.*

# $J$-PAIRING INTERACTIONS OF FERMIONS IN A SINGLE-$J$ SHELL

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**Abstract** In this talk I shall introduce our recent works on general pairing interactions and pair truncation approximations for fermions in a single-$j$ shell, including the spin zero dominance, and features of eigenvalues of fermion systems in a single-$j$ shell interacting by a $J$-pairing interaction.

**Keywords:** $J$-pairing interaction, sum-rules, spin zero dominance

## 1. Introduction

It is my great pleasure to talk to you here. I would like to thank the organizers, especially Dr. Bruno Gruber. I am extremely glad to see many of my friends again today in this beautiful city Bregenz.

My talk consists of four subjects:

- Spin $0^+$ ground state dominance
- Pair approximations for fermions in a single-$j$ shell
- Regularities of states in the presence of $J_{\max}$-pairing
- Solutions for cases of $n = 3$ and 4 with $H_J$

## 2. $0^+$ ground state dominance

A preponderance of $0^+$ ground states was discovered by Johnson, Bertsch and Dean in 1998 [1] using the two-body random ensemble (TBRE), and was related to a reminiscence of generalized seniority by Johnson, Bertsch, Dean and Talmi in 1999 [2]. These phenomena have been confirmed in different systems [3, 4].

Let us take a simple system consisting of four particles in a single-$j$ shell. The Hamiltonian that we use is as follows:



*B. Gruber (ed.), Symmetries in Science XI*, 13–21.

*Table 1.* The angular momenta which give the lowest eigenvalues for 4 fermions in single-$j$ shells when $G_J = -1$ and all other parameters are 0.

| $2j$ | $G_0$ | $G_2$ | $G_4$ | $G_6$ | $G_8$ | $G_{10}$ | $G_{12}$ | $G_{14}$ | $G_{16}$ | $G_{18}$ | $G_{20}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 0 | 4 | 2 | 8 | | | | | | | |
| 9 | 0 | 4 | 0 | 0 | 12 | | | | | | |
| 11 | 0 | 4 | 0 | 4 | 8 | 16 | | | | | |
| 13 | 0 | 4 | 0 | 2 | 2 | 12 | 20 | | | | |
| 15 | 0 | 4 | 0 | 2 | 0 | 0 | 16 | 24 | | | |
| 17 | 0 | 4 | 6 | 0 | 4 | 2 | 0 | 20 | 28 | | |
| 19 | 0 | 4 | 8 | 0 | 2 | 8 | 2 | 16 | 24 | 32 | |
| 21 | 0 | 4 | 8 | 0 | 2 | 0 | 0 | 0 | 20 | 28 | 36 |

$$H = \sum_J G_J A^{J\dagger} \cdot A^J \equiv \sum_J G_J \sqrt{2J+1} \left[ A^{J\dagger} \times A^J \right]^{(0)} ,$$

$$A_M^{J\dagger} = \frac{1}{\sqrt{2}} \left[ a_j^\dagger \times a_j^\dagger \right]_M^{(J)} , \; A_M^J = -(-1)^M \frac{1}{\sqrt{2}} \left[ \tilde{a}_j \times \tilde{a}_j \right]_M^{(J)} , \quad (1)$$

where $G_J$ is given by

$$G_J = \langle j^2 J | V | j^2 J \rangle .$$

Here $V$ is a two-body interaction.

We have used a two-body random ensemble to confirm the interesting phenomenon of $0^+$ ground state dominance, and discovered an empirical method to predict the probability of the ground state to have a spin $I$ [5]. We keep only one $G_J$ to be $-1$ and all others 0 :

$$G_J = -\delta_{JJ'} .$$

We then diagonalize the Hamiltonian to find the angular momenta which give the lowest eigenvalues. They are shown in Table I. We count how many different $G_J$'s give the lowest eigenvalue to an angular momentum $I$. The number is denoted as $\mathcal{N}_I$. For example for $j = \frac{21}{2}$ and $n = 4$, $\mathcal{N}_0$=5, $\mathcal{N}_2$=$\mathcal{N}_4$=$\mathcal{N}_8$ = $\mathcal{N}_{20}$=$\mathcal{N}_{28}$=$\mathcal{N}_{36}$=1 and all others are equal to 0. The total number of different $G_J$'s is $N = \frac{2j+1}{2}$. Then the $I$ g.s. probability is approximately predicted as

$$P^{\text{pred}}(I) = \mathcal{N}_I / N. \quad (2)$$

Fig. 1 shows a comparison between $P^{\text{pred}}(0)$ and $P^{\text{TBRE}}(0)$, which is obtained by diagonalization of a TBRE Hamiltonian for four fermions in a single-$j$ shell. Fig. 2 shows a comparison between $P^{\text{pred}}(I)$ and $P^{\text{TBRE}}(I)$ for examples of various systems.

One can see that the agreements between the $P^{\text{pred}}(I)$ and $P^{\text{TBRE}}(I)$ are very good. It is therefore important to diagonalize $H$ with $G_J = -\delta_{JJ'}$. For this purpose we introduce the $J$-pair approximation for low-lying states.

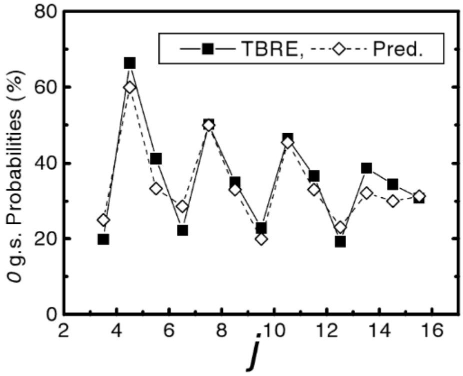


*Figure 1.* Comparison between $P^{\rm pred}(0)$ and $P^{\rm TBRE}(0)$ of four fermions in a single-$j$ shell. The solid squares are obtained by 1000 runs of a TBRE Hamiltonian and the open squares are predicted by Eq. (2).

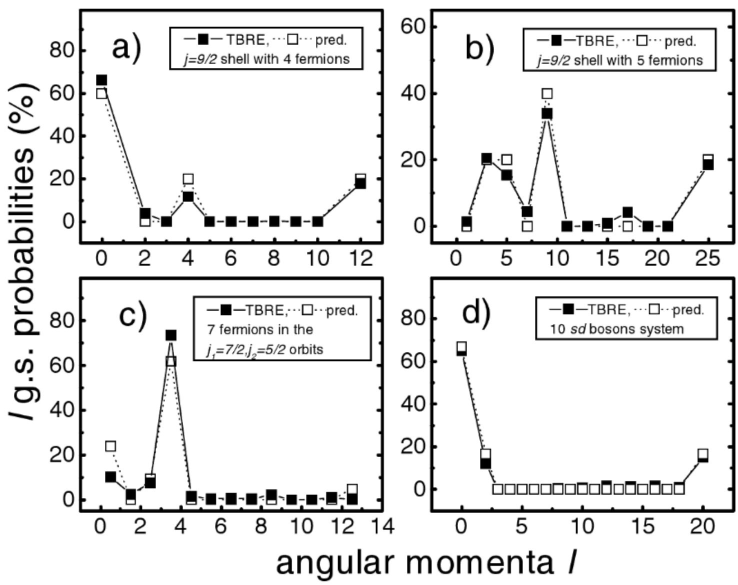


*Figure 2.* Comparison between $P^{\rm pred}(I)$ and $P^{\rm TBRE}(I)$ for more complicated systems. The solid squares are obtained by 1000 runs of a TBRE Hamiltonian and the open squares are predicted by Eq. (2).

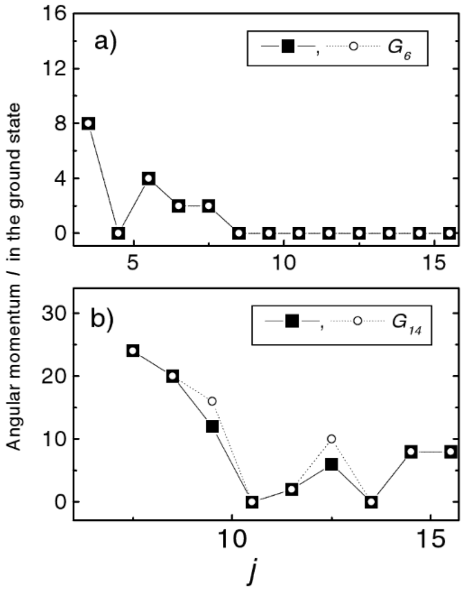


*Figure 3.* Ground state spin $I$ for four fermions in a single-$j$ shell for $J = 6$ in (a) and 14 in (b) as a function of $j$. The solid squares are obtained by diagonalizing $H_J$ in the full shell-model space, and open circles are obtained by truncating the space to two pairs with spin $J$ only.

## 3. Pair Approximation for Fermions in a single-$j$ shell

Our Hamiltonian is defined as

$$H_J = -A^{J\dagger} \cdot A^J \ . \tag{3}$$

We first point out that the low-lying eigenvalues of $H_J$ can be approximated by wavefunctions of pairs with spin $J$:

$$\Phi(I) = \frac{1}{\sqrt{N}} \left[ A^{J\dagger} \times A^{J\dagger} \times \cdots \times A^{J\dagger} \right]^{(I)} |0\rangle \ , \tag{4}$$

where $\frac{1}{\sqrt{N}}$ is the normalization factor. It is very easy to prove that the $J$-pair truncation (with one pair and one particle) describes the low-lying states exactly in three-body systems.

Fig. 3(a) shows the spin of the ground state of $j^4$ configuration for $J = 6$. The ground states with spin 0 are obtained by exact shell-model calculations and by the $J$-pair approximation. Fig. 3 (b) shows the similar thing for $J = 14$. Fig. 4 shows energy levels obtained by the shell-model calculation and by the $J$-pair approximation when $j = 25/2$, $J = 14$ and $n = 4$. For the low-lying states, the pair approximation is very good. Giving the four low-lying states, two of

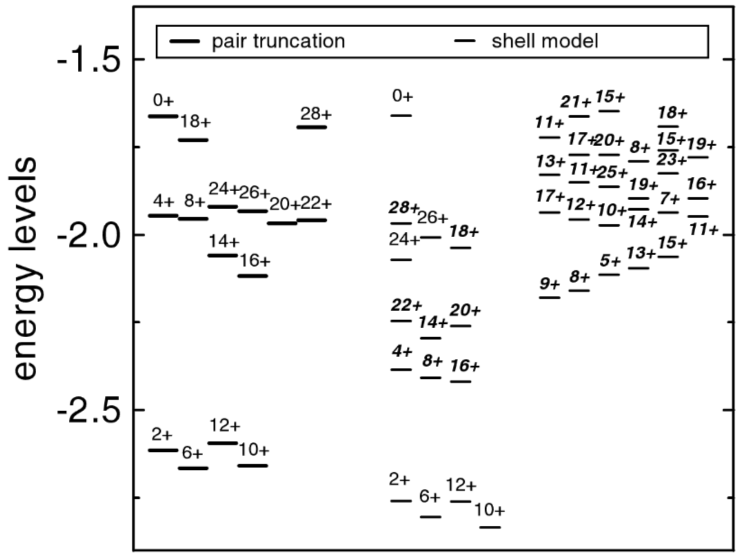


*Figure 4.* A comparison of low-lying spectra, obtained from two pairs with spin $J = 14$ (the column on the left hand side) and by a diagonalization of the full space (the columns in the middle and on the right hand side) for the case of four nucleons in a single-$j$ ($j = 25/2$) shell. The middle column plots the shell-model states which are well reproduced by the two $J = 14$ pairs, and the right column plots the shell-model states which are not well reproduced by two $J = 14$ pairs. All the levels below $0_1^+$ in the full shell-model space are included. One sees that the low-lying states with $I = 2_1^+$, $6_1^+$, $12_1^+$, and $10_1^+$ are well reproduced.

them ($6_1^+$ and $10_1^+$) compete to be the ground state. Their energies are almost the same in both the exact shell-model calculation and the pair approximation. This is why we failed to predict the ground state in this case, see Fig. 3 (b). For the $n$=5 and 6 cases that we have examined, the low-lying states are reasonably well approximated by the $J$-pair truncation.

So far $J$ is general, between 0 and $2j-1$. Now let us take a very special value, $J_{\max} = 2j-1$. For $H = H_{\max} = H_{2j-1}$, the $I = I_{\max} = 4j-6$ corresponds to the lowest state, and $I = I_{\max} - 2$ state to the second lowest. These two states can be constructed by using pairs with angular momenta of either $J_{\max}$ or $J_{\max}-2$. However, pairs with angular momentum $J_{\max}-2$ do not present a good approximation of the other $I$ states, while those with angular momentum $J_{\max}$ do. For example, for $n=4$, $|J_{\max}^2, I=0\rangle$ is exact but $|(J_{\max}-2)^2, I=0\rangle$ is not exact, $|J_{\max}^2, I(\leq j)\rangle$ is almost exact ($\simeq -2$) but $|(J_{\max}-2)^2, I(\leq j)\rangle$ are not.

## 4. Regularities of states in the presence of $H_{J_{\max}}$

We first point out that eigenvalues of low $I$ states ($n=3,4,5$) are approximately integers. This can be proved in terms of six-$j$ symbols for $n=3$ [6]. For $n=4$, one can prove this in terms of nine-$j$ symbols [7].

Another regularity may be exemplified below by $j=21/2$ and $n=3$ and 4. Among many states of $n=4$ with the same $I$, the lowest eigenvalue is expressed as $\mathcal{E}_I$ (obtained by a shell-model diagonalization). The $\mathcal{E}_I$ of four fermions in a single-$j$ ($j=21/2$) shell with $I$ between 18 to 25 are shown in Table II. Note that there is no eigenvalue lower than $-2$ when $I$ is smaller than 18. The eigenvalue of the $I_{\max}^{(3)}(=3j-3)$ state with three fermions in the same $j$ shell is $-\frac{59}{26}$=$-2.26923076923077$ (denoted as $E_{I_{\max}^{(3)}}$). From Table II, one sees that the $\mathcal{E}_I$'s of $n=4$ with 18$\leq I \leq 25$ are very close to $E_{I_{\max}^{(3)}}$ and also very close to that of an $I$ state constructed by $\Psi_I = \left[a_j^\dagger \times \left[a_j^\dagger \times a_j^\dagger \times a_j^\dagger\right]^{(I_{\max}^{(3)})}\right]^{(I)}$ (denoted as $F_I$). This indicates that the last added particle behaves just like a spectator and do not contribute to the total energy of the system.

We have calculated overlaps between the above states of $n=4$ and the $\Psi_I$. They are almost 1 within a precision of $10^{-5}$. This phenomenon has been confirmed for $n$ up to 6 ($j \geq 11/2$).

## Acknowledgments

I would like to thank Drs. Y.M. Zhao, J.N. Ginocchio, and N. Yoshinaga for their collaborations in this work and also thank Dr. W. Bentz for reading the manuscript.

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# SUPERSYMMETRY IN NUCLEI

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**Abstract** The concept of spectrum generating superalgebras and associated dynamic supersymmetries is introduced and placed within the general context of symmetry in physics. Applications of this concept to the study of spectra of atomic nuclei are presented.

## 1. Introduction

In the last 40 years the concept of spectrum generating algebras and dynamic symmetries has been extensively used to study physical systems. In the late 1970's this concept was enlarged to spectrum generating superalgebras and associated supersymmetries. In this article, dynamic symmetries are first placed within the context of symmetries in physics and applications to the structure of atomic nuclei are reviewed. Subsequently, the concept of dynamic supersymmetries is introduced and placed within the context of supersymmetry in physics. Applications to the study of spectra of nuclei are reviewed.

## 2. Symmetries

Symmetry is a wide-reaching concept used in a variety of ways.

### 2.1 Geometric symmetries

These symmetries are the first to be used in physics. They describe the arrangement of constituent particles into a structure. An example of symmetries of this type is the arrangement of atoms in a molecule. The mathematical framework needed to describe these symmetries is finite groups, sometimes called point groups. In Fig.1, the molecule $C_{60}$ is shown as an example. The symmetry of this molecule is $I_h$. Geometric symmetries are used to reduce the complexity of the equations describing the system through the construction of a symmetry adapted basis. The Hamiltonian matrix in this basis is block diagonal.



*B. Gruber (ed.), Symmetries in Science XI*, 23–35.

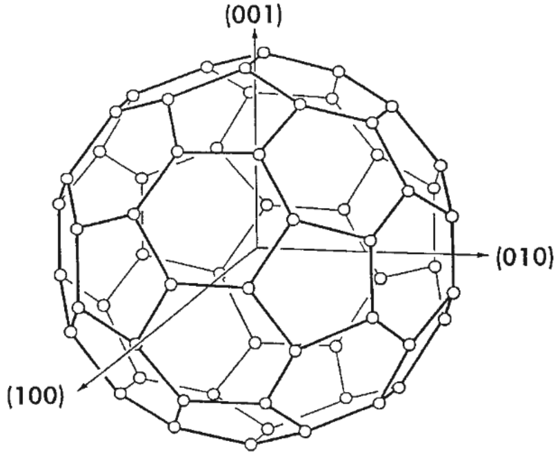


*Figure 1.* The fullerene molecule $C_{60}$ is shown as an example of geometric symmetry, $I_h$.

## 2.2 Space-time symmetries

These symmetries fix the form of the equations governing the motion of the constituent particles. An example is provided by Lorentz invariance that fixes the form of the Dirac equation to be

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0. \tag{1}$$

The mathematical framework needed to describe these symmetries is continuous groups, in particular Lie groups, here the Lorentz group $SO(3, 1)$.

## 2.3 Gauge symmetries

These symmetries fix the form of the interaction between constituent particles and/or with external fields. An example is provided by the Dirac equation in the presence of an external electromagnetic field

$$[\gamma^{\mu}(i\partial_{\mu} - eA_{\mu}) - m]\psi(x) = 0. \tag{2}$$

Electrodynamics is invariant under the gauge transformation $A'_{\mu}(x) \rightarrow A_{\mu}(x) - \partial_{\mu}\Lambda(x)$. Also here the mathematical framework is Lie groups, in the case of electrodynamics being $U(1)$. In view of the fact that the strong and weak forces appear to be guided by gauge principles, gauge symmetries have become in recent years, one of the most important tools in physics.

## 2.4 Dynamic symmetries

These symmetries fix the interaction between constituent particles and/or external fields (hence the name dynamic). They determine the spectral proper-

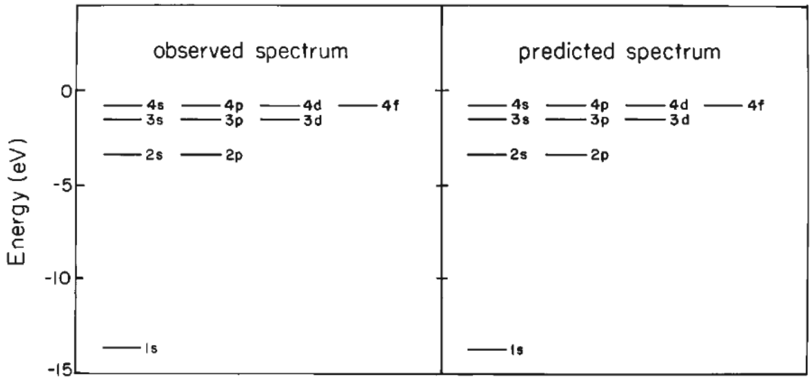


*Figure 2.* The spectrum of the non-relativistic hydrogen atom is shown as an example of dynamic symmetry of the Schrödinger equation, $SO(4)$.

ties of quantum systems (patterns of energy levels). They are described by Lie groups.

The earliest example of this type of symmetry is provided by the non-relativistic hydrogen atom. The Hamiltonian of this system can be written in terms of the quadratic Casimir operator $C_2$ of $SO(4)$ as [1]

$$H = \frac{p^2}{2m} - \frac{e^2}{r} = -\frac{A}{C_2(SO(4)) + 1}, \tag{3}$$

where $A$ is a constant that depends on $m$ and $e$. As a result, the energy eigenvalues can be written down explicitly in terms of quantum numbers

$$E(n, \ell, m) = -\frac{A}{n^2} \tag{4}$$

providing a straightforward description of the spectrum, Fig.2.

Another example is provided by hadrons. These can be classified in terms of a flavor $SU_f(3)$ symmetry [2]. The mass operator for hadrons can be written in terms of the Casimir operators of isospin, $SU(2)$, and hypercharge, $U(1)$, as

$$M = a + b\,[C_1(U(1)] + c\left[C_2(SU(2)) - \frac{1}{4}C_1^2(U(1))\right] \tag{5}$$

leading to the mass formula [4]

$$M(Y, I, I_3) = a + bY + c[I(I+1) - \frac{1}{4}Y^2]. \tag{6}$$

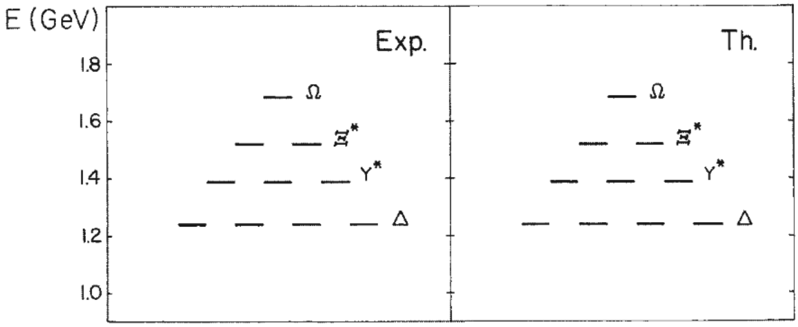


*Figure 3.* The spectrum of the baryon decuplet is shown as an example of dynamic symmetry of the mass operator, $SU_f(3)$.

This mass formula provides a very realistic description of hadron spectra, Fig.3.

The concept of dynamic symmetry was introduced implicitly by Pauli in the above mentioned paper [1], expanded by Fock [5], and, reintroduced in explicit form, by Dothan, Gell-Mann and Ne'emann [6] and Barut and Böhm [7]. It has been used extensively in the last 25 years and has produced many important discoveries. A mathematical definition is given in [8].

A dynamic symmetry is that situation in which:

(i) The Hamiltonian $H$ is written in terms of elements, $G_\alpha$, of an algebra $G$, called spectrum generating algebra (SGA), $G_\alpha \in G$.

(ii) $H$ contains only invariant (Casimir) operators, $C_i$, of a chain of algebras $G \supset G' \supset G'' \supset \ldots$

$$H = f(C_i). \tag{7}$$

When a dynamic symmetry occurs, all observables can be written in explicit analytic form. For example, the energy levels are

$$E = \langle H \rangle = \alpha_1 \langle C_1 \rangle + \alpha_2 \langle C_2 \rangle + \ldots \tag{8}$$

One of the best studied cases is that of atomic nuclei, to be described in the following section.

## 3. Dynamic symmetries of the Interacting Boson Model

Atomic nuclei with an even number of nucleons can be described as a collection of correlated pairs with angular momentum $J = 0$ and $J = 2$. When the pairs are highly correlated they can be treated as bosons, called $s$ and $d$. The corresponding model description is called Interacting Boson Model [9]. The spectrum generating algebra (SGA) of the Interacting Boson Model can be easily constructed by introducing six boson operators

$$b_\alpha(\alpha = 1, \ldots, 6) \equiv s, d_\mu(\mu = 0, \pm 1, \pm 2) \tag{9}$$

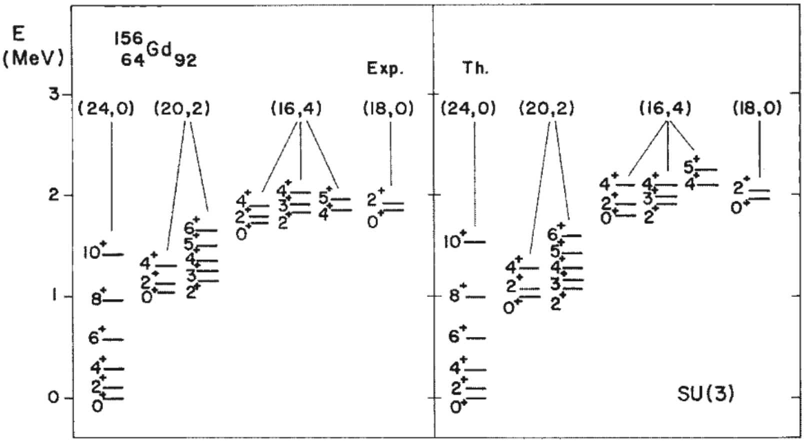


*Figure 5.* An example of $SU(3)$ dynamic symmetry in nuclei: $^{156}Gd$.

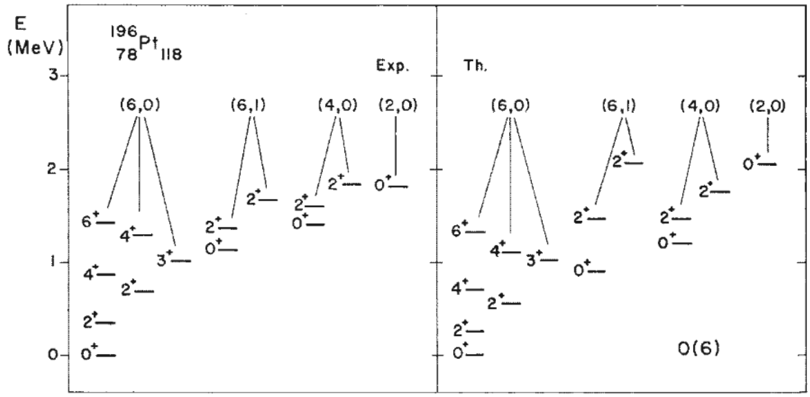


*Figure 6.* An example of $SO(6)$ dynamic symmetry in nuclei: $^{196}Pt$.

mathematical framework to describe it is point supergroups, that is discrete subgroups of supergroups.

## 4.2 Space-time supersymmetries

These supersymmetries fix the form of the equation governing the motion of mixed systems of bosons and fermions. An example is the original Wess-

Zumino Lagrangian [17]

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2}\left(\partial_\mu A(x)\right)^2 - \frac{1}{2}\left(\partial_\mu B(x)\right)^2 - \frac{1}{2} i\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) \\
& -\frac{1}{2}m^2[A^2(x) + B^2(x)] - \frac{1}{2}im\bar{\psi}(x)\psi(x) \\
& -gmA(x)\left[A^2(x) + B^2(x)\right] - \frac{1}{2}g^2\left[A^2(x) + B^2(x)\right] \\
& -ig\bar{\psi}(x)\left[A(x) - \gamma_5 B(x)\right]\psi(x).
\end{aligned}
$$

The mathematical framework here is continuous supergroups, as for example the SuperPoincaré group.

## 4.3 Gauge supersymmetries

These fix the form of interactions. For example in a supersymmetric gauge theory one has the occurrence of both bosonic and fermionic gauge fields with related properties.

## 4.4 Dynamic supersymmetries

These symmetries fix the interaction between constituent particles. They produce patterns of energy levels for mixed systems of bosons and fermions. They are a very ambitious unifying concept. A mathematical definition of dynamic supersymmetries is given in [19].

A dynamic supersymmetry is that situation in which:

(i) The Hamiltonian $H$ is written in terms of elements $G^*_\alpha$ of a graded algebra $G^*$.

(ii) $H$ contains only Casimir operators of a chain of algebras $G^* \supset G^{*\prime} \supset G^{*\prime\prime} \supset \ldots$ The subalgebras can be either graded or not.

One of the best studied cases is again that of atomic nuclei, where supersymmetries were introduced in 1980 [19], as described in the following section.

# 5. Dynamic Supersymmetries of the Interacting Boson-Fermion Model

In nuclei with an odd number of nucleons at least one is unpaired. Furthermore at higher excitation energies, some of the pairs may be broken. A comprehensive description of nuclei requires a simultaneous treatment of correlated pairs (bosons) and of fermions [20]. The corresponding model has been called Interacting Boson-Fermion Model [22]. The building blocks in this model are:

$$\begin{aligned} \text{Bosons} &: \quad s(J=0); d_\mu(J=2; \mu=0,\pm1,\pm2) \\ \text{Fermions} &: \quad a_{j,m}(m=\pm j, \pm(j-1),\ldots,\pm\frac{1}{2} \end{aligned} \tag{16}$$

The model Hamiltonian can be written as

$$H = H_B + H_F + V_{BF} \tag{17}$$

with

$$\begin{aligned} H_B &= E_0 + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} b_\alpha^\dagger b_\beta + \sum_{\alpha\alpha'\beta\beta'} v_{\alpha\alpha'\beta\beta'} b_\alpha^\dagger b_{\alpha'}^\dagger b_\beta b_{\beta'} \\ H_F &= E_0' + \sum_{ik} \eta_{ik} a_i^\dagger a_k + \sum_{ii'kk'} u_{ii'kk'} a_i^\dagger a_{i'}^\dagger a_k a_{k'} \\ V_{BF} &= \sum_{\alpha\beta ik} w_{\alpha\beta ik} b_\alpha^\dagger b_\beta a_i^\dagger a_k. \end{aligned} \tag{18}$$

In order to study the possible symmetries of a mixed system of bosons and fermions, a new mathematical framework is needed, namely that of graded Lie algebras (also called superalgebras).

A set of elements

$$G_\alpha, F_i \tag{19}$$

are said to form a Lie superalgebra if they satisfy the following commutation relations

$$\begin{aligned} [G_\alpha, G_\beta] &= c_{\alpha\beta}^\gamma G_\gamma \\ [G_\alpha, F_i] &= d_{\alpha i}^j F_j \\ \{F_i, F_j\} &= g_{ij}^\alpha G_\alpha \end{aligned} \tag{20}$$

plus super Jacobi identities. [Graded semisimple Lie algebras with $Z_2$ grading have been classified by V. Kac [22]]. By inspection of Eq.(18) one can see that the combined boson-fermion Hamiltonian can be written in terms of elements of the graded superalgebra $G^* \equiv U(n/m)$

$$\begin{aligned} G_{\alpha\beta} &= b_\alpha^\dagger b_\beta \\ G_{ij} &= a_i^\dagger a_j \\ F_{\alpha i}^\dagger &= b_\alpha^\dagger a_i \\ F_{i\alpha} &= a_i^\dagger b_\alpha \end{aligned} \tag{21}$$

These elements can be arranged in matrix form

$$\begin{pmatrix} b_\alpha^\dagger b_\beta & b_\alpha^\dagger a_i \\ a_i^\dagger b_\alpha & a_i^\dagger a_j \end{pmatrix}. \tag{22}$$

The basis upon which the elements act is the totally supersymmetric irrep of $U(n/m)$ with Young supertableau

$$[\mathcal{N}\} \equiv \boxtimes\boxtimes \ldots \boxtimes \,. \qquad (23)$$

For applications to Nuclear Physics, $\mathcal{N} = N_B + N_F$, $n = 6$ and $m = \sum_j (2j + 1) \equiv \Omega$, where $\Omega$ is the total degeneracy of the fermionic shell. A dynamic supersymmetry occurs whenever the Hamiltonian of Eq.(18) can be written in terms only of the Casimir operator of $U(n/m)$ and its subalgebras.

## 5.1 Supersymmetry in nuclei found

One of the consequences of supersymmetry is that if bosonic states are known, one can predict fermionic states. Both are given by the same energy formula. Indeed all properties of the fermionic system can be found from a knowledge of those of the bosonic system. Supersymmetry has thus a predictive power that can be tested by experiment. After its introduction in the 1980’s, several examples of spectra with supersymmetric properties were found, relating spectra of even-even nuclei with those of odd-even nuclei (odd proton or odd neutron). In the first example, $j = 3/2$ fermions were coupled to $s$ and $d$ bosons. States were classified then in terms of the group $U(6/4)$ [23]. An example is shown in Fig.7, referring to the pair of nuclei Os-Ir. Other cases were subsequently found, for example $j = 1/2, 3/2, 5/2$ fermions with $s$ and $d$ bosons, described algebraically by $U(6/12)$ [24].

## 5.2 Supersymmetry in nuclei confirmed

Supersymmetry in nuclei has been recently confirmed in a series of experiments involving several laboratories. The confirmation relates to an improved description of nuclei in which proton and neutron degrees of freedom are explicitly introduced. The model with proton and neutron bosons is called Interacting Boson Model-2. The basic building blocks of this model are boson operators $b_{\alpha\pi}, b_{\alpha\nu} (\alpha = 1, \ldots, 6)$ , where the index $\pi(\nu)$ refers to proton (neutron). The boson operators span a (six+six)-dimensional space with algebraic structure $U_\pi(6) \oplus U_\nu(6)$. Consequently, when going to nuclei with unpaired particles, one has a model with two types of bosons (proton and neutron) and two types of fermions (proton and neutron), called Interacting Boson-Fermion Model-2. If supersymmetry occurs for this very complex systems one expects now to have supersymmetric partners composed of a quartet of nuclei, even-even, even-odd, odd-even and odd-odd, for example

$$\begin{array}{cc} {}^{194}Pt & {}^{195}Pt \\ {}^{195}Au & {}^{196}Au \end{array}$$

Spectra of even-even and even-odd nuclei have been known for some time. However, spectra of odd-odd nuclei are very difficult to measure, since the

Some supersymmetric theories have been constructed in condensed matter physics [28]. Nambu has suggested that supersymmetry may occur in Type II superconductors [29].

Recently, it has been suggested that cuprate materials (high-$T_c$ superconductors) may display supersymmetry. This case is being investigated at the present time [30].

## 7. Conclusions

A form of supersymmetry has been found and confirmed in Nuclei!

## Acknowledgments

This work was supported in part under USDOE Contract No. DE-FG-02-91ER-40608.

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# SEEING SCIENCE THROUGH SYMMETRY

## *An Interdisciplinary Multimedia Course*

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**Abstract** Seeing Through Symmetry is a course that introduces non-science majors to the pervasive influence of symmetry in science. The concept of symmetry is used both as a link between subjects (such as physics, biology, mathematics, music, poetry, and art) and as a method within a subject. This is done through the development and use of interactive multimedia learning environments to stimulate learning. Computer-based labs enable the student to further explore the concept by being gently led from the arts to science. This talk is an update that includes some of the latest changes to the course. Explanations are given on methodology and how a variety of interactive multimedia tools contribute to both the lecture and lab portion of the course (created in 1991 and taught almost every semester since then, including one in Sweden).

## 1. Introduction

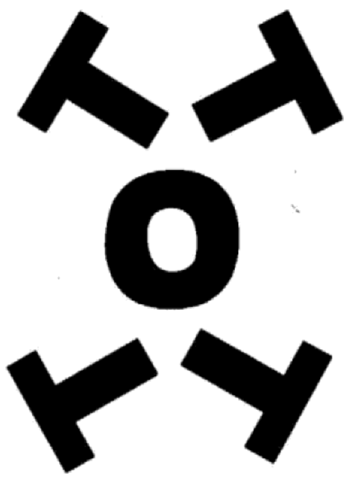
Symmetry is something that we are all probably aware of, for better or for worse, in our everyday lives: A desirable situation can occur in the supermarket, when a shopper attempts to find another tomato in the pile that looks just like the nice one already selected. In the faculty dining room, on the other hand, a colleague sometimes guesses which of two apparently identical metal dispensers contains the hot water for tea; and, failing the determination, releases coffee onto a tea bag!

The subject of symmetry has been written about extensively. There are numerous works on symmetry in science, art, mathematics, philosophy, music, poetry, and information processing. From the World Wide Web, two journals ([1], [3]), articles from conference proceedings [5], and collections of essays [2] can be found many examples covering all of those subjects... and more.



*B. Gruber (ed.), Symmetries in Science XI*, 227–238.

The term "symmetry" used here has no meaning apart from some *operation*. With this in mind one can put forth the following fairly standard definition: *An object is symmetric under a particular operation if it appears unchanged after that operation has been performed.* A very simple example: the square has rotational symmetry because after rotating it about its center through 90°, in its plane, the square appears as it did prior to the rotation. A more complicated example: In the figure below there are four letters "T" and one letter "O". The whole figure has reflectional symmetry about two perpendicular lines passing through the center of the "O", 2-fold rotational symmetry about the same center, and infinite-fold rotational symmetry of just the "O" (if it was a perfect circle) with respect to that center.




**Trial restriction**

Furthermore, if you begin at the center and go counterclockwise through the "T" at the upper right (UR), the "T" at the upper left (UL), and back to the "O", the word "OTTO" is spelled. The identical word is spelled again if you go clockwise (from the "O" to UL to UR and back to the "O"). Hence, one has what can be dubbed a *palindromic symmetry*, usually called a *palindrome* (the same palindrome occurs by going clockwise or counterclockwise using the bottom half of the figure). Finally, starting at the "O", going the counterclockwise route through the top half of the figure, followed by a clockwise route through the bottom half of the figure, will again bring you back to the "O"; thus tracing out a figure eight which, moreover, repeats under those combined operations. The Italian word for "eight" is "otto"!

If one defines an object's "image" as the result of having performed a particular operation on the object, then a more concise definition of symmetry is: *An*

# Topic Index