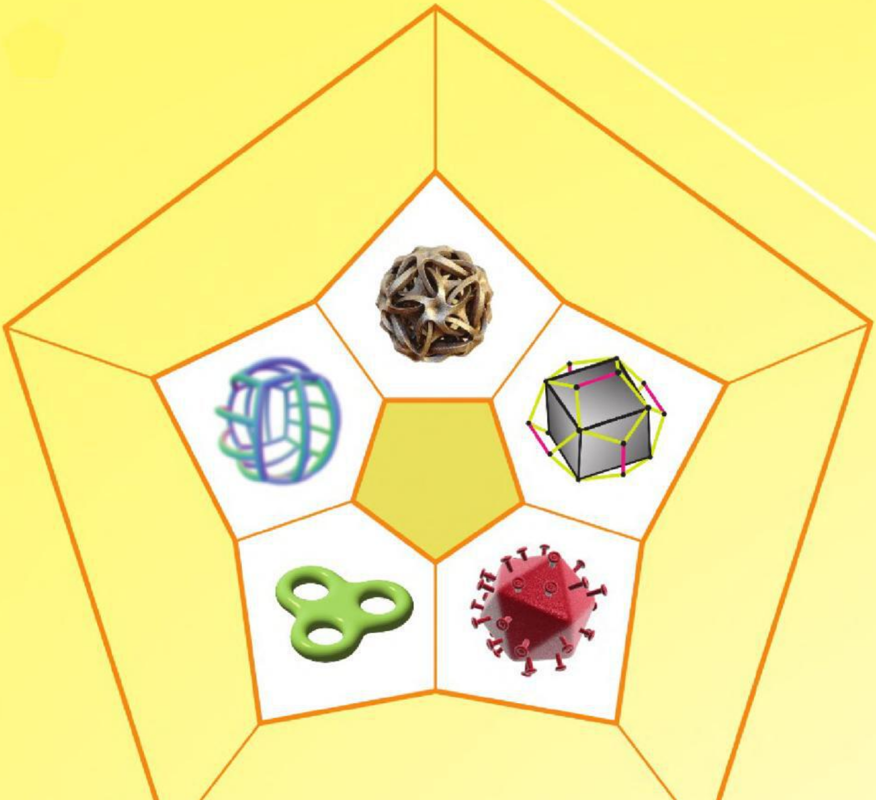
Kristopher Tapp

Symmetry

A Mathematical Exploration



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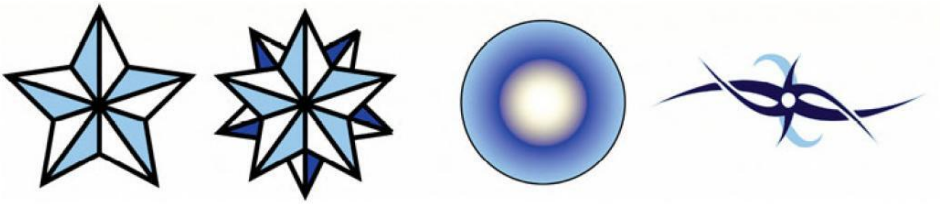
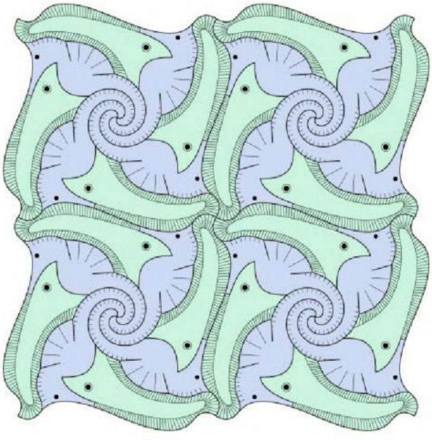
1. Introduction to Symmetry

Our journey starts with the question: what does "symmetry" mean? Look at the following four objects, and rank them from the most symmetric to the least symmetric:



How do you interpret this question in a manner which is precise enough to lead you to a justifiable ranking of the four objects? And how symmetric is the following painting by Robert Fathauer?



Seahorses and Eels **by Robert Fathauer**
http://members.cox.net/fathauerart/

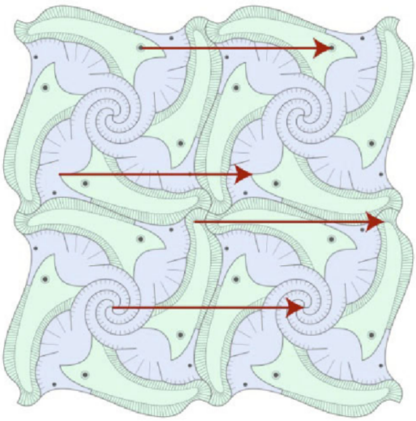
To answer any of these questions, we must first make the questions more precise.

K. Tapp, *Symmetry: A Mathematical Exploration*, DOI 10.1007/978-1-4614-0299-2_1,

A Precise Definition of "Symmetry"

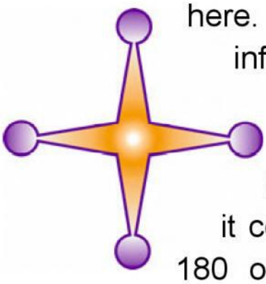
Most people would agree that *Seahorses and Eels* looks symmetric, perhaps because it contains repeated images. But symmetry involves more than just repeated images. A haphazard arrangement of green eels and purple seahorses would not look nearly as symmetric. Fathauer arranged his seahorses and eels together like jigsaw puzzle pieces, so that the pattern of neighbors surrounding one eel is the same as the pattern surrounding any other.

To phrase this idea more precisely, let us imagine that the pattern is painted onto an infinite glass wall that extends indefinitely up, down, right and left. Imagine that the painted pattern extends indefinitely so as to cover the whole infinite wall, which requires infinitely many seahorses and infinitely many eels. This infinite painting looks exactly the same from many different positions. If the viewer is positioned in front of the eye of one right-facing eel, then what she sees is exactly the same as if she were positioned in front of the eye of another right-facing eel.


A translation symmetry

Here is an equivalent way to say the same thing, rephrased in terms of moving the glass wall rather than moving the viewer: there are many ways in which the glass wall could be moved/repositioned so that the painted image looks exactly the same before and after the re-positioning. For example the wall, together with the pattern painted onto it, could be translated (which means slid) so that each right-facing eel moves one position to the right. This translation is called a symmetry of the infinite painting because a viewer who closes her eyes while the wall is moved, could not, after she opens her eyes, detect that any change has occurred. This translation is encoded by the length (about an inch) and direction (right) of the red arrow pictured above. Several copies of the red arrow are included to demonstrate that each composition element (the eye of a right-facing eel, the tail of a down-facing eel, the center of the purple tail spiral, etc.) moves exactly onto an identical element. That is why a viewer would not detect the change.

This way of thinking about symmetry applies equally well to other objects, including the orange and the purple star pictured here. Imagine this star image is painted onto our infinite glass wall. Again, there are several ways in which the wall could be repositioned/moved so that the image looks exactly the same before and after the repositioning. For example, it could be rotated 90° about the star's center, or 180 or 270°. Each of these motions is called a symmetry of the star image.



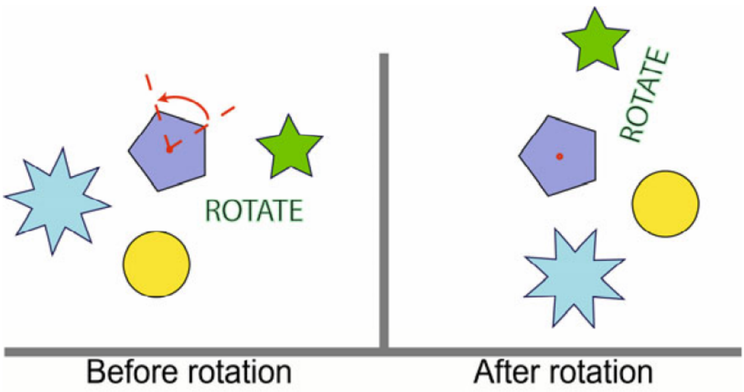
Here is our first attempt at formulating a precise definition of the word "symmetry":

DEFINITION: A symmetry of an object in the plane is a rigid motion of the plane that leaves the object apparently unchanged.

In the above discussions, the "plane" was represented as an infinite glass wall, the "object" was represented as a painting on the wall (of a star or an infinite pattern), and a "rigid motion" meant a moving or repositioning of the glass wall, like a rotation or a translation. What does it mean for a rigid motion to "leave the object apparently unchanged"? It means that, if a viewer was to close her eyes during the repositioning, she would not detect a difference; the object would look exactly the same when she opened her eyes.

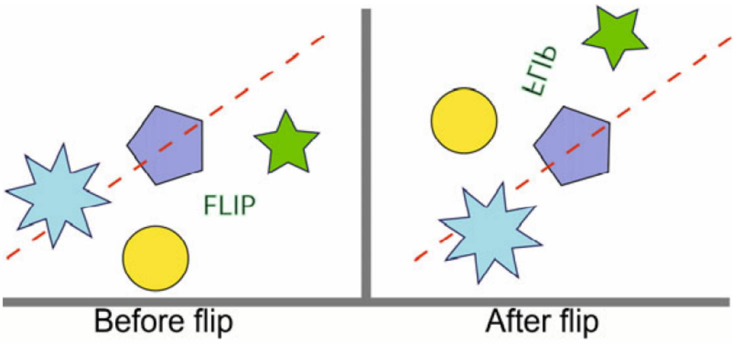
Precise language is crucially important in mathematics! Our above definition of symmetry cuts straight to the underlying reason that objects intuitively appear symmetric (they look the same from many positions and/or contain repeated images), but in a manner which is precise enough to form a foundation for a rigorous mathematical investigation of symmetry. To really pull this off, we will eventually require a more precise definition of the term “rigid motion”. But for now, it will be enough to think of a rigid motion as a moving/repositioning of the glass wall, like a rotation or a translation. A rigid motion may NOT break, bend, stretch, compress or otherwise distort distances on the glass wall (you cannot use a glass cutter or a blow torch).

A rigid motion is always a motion of the whole plane (the whole glass wall); one may then ask whether it is a symmetry of any object in the plane. For example, in the illustration below, the 72° rotation about the red point is a rigid motion of the plane that is a symmetry of the purple pentagon, but is not a symmetry of the other shapes.



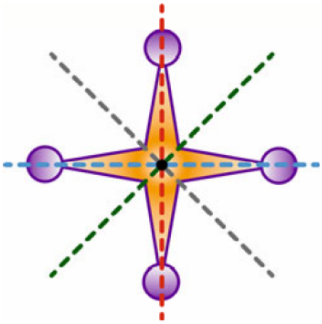
We emphasize that a rigid motion is completely determined by its effect on each point of the plane; that is, two motions that do the same thing to every point are considered the same motion. For example, a rotation by 90° is the same as a rotation by 90+360 = 450° about the same point. When enumerating the symmetries of an object, we would NOT list both a 90 and a 450° rotation, or any other such redundancies. What matters is the effect of a motion, not the motion itself.

One type of rigid motion that we have not yet considered is a flip over a line. Visualize a flip over a line as achieved by flipping the glass wall over to expose its back surface (its underside). Points along the line remain in their original position, while points on one side of the line flip over to the opposite side of the line. Imagine that the plane is a completely transparent glass wall, so that any image in the plane shows through the back surface of the glass, and looks reversed after the flip is performed. For example, in the illustration below, the flip over the red line is a rigid motion of the plane that is a symmetry of the purple pentagon but is not a symmetry of the other shapes.



The words flip and reflection are synonymous; we will henceforth use these terms interchangeably.

To better understand why the previously illustrated star image appears symmetric, we will list all of its symmetries. There are three obvious symmetries; namely rotations about its center point by 90, 180, and 270° *(rotation angles are always counterclockwise in this book)*. A fourth valid symmetry is called the identity. This is the motion that does nothing; it leaves every point of the plane in its original position. The identity can be considered a rotation (by 0°) or a translation (by zero distance). The identity is the only rigid motion that is a symmetry of every object. Thus, the star has four rotation symmetries. The star can also be flipped over any of the four colored lines illustrated here. In summary, the star has four rotation symmetries and four reflection symmetries, giving exactly eight symmetries in total. The number of symmetries which an object has provides a measurement of how symmetric that object is.

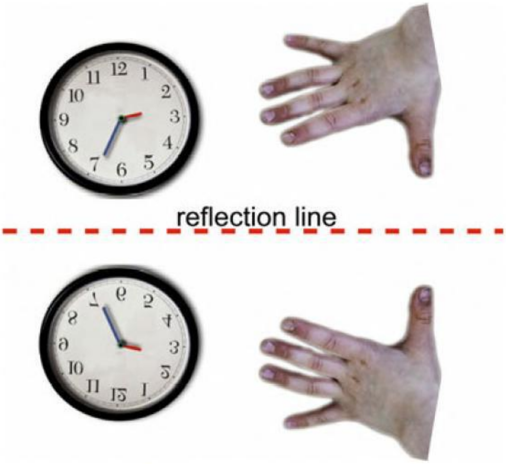


The star's 4 reflection lines

Types of Symmetries and Types of Objects

To describe to me your favorite rotation or translation or flip, what information must you give me? A rotation is specified by its center point and its (counterclockwise) angle. A translation is specified by the length and direction of a single arrow. A flip is specified by its reflection line.

Rotations and translations are called “proper” rigid motions. Flips are called “improper” rigid motions. The intuitive difference is that improper motions leave the plane’s underside facing the viewer. A more precise explanation of this difference is obtained by comparing how proper and improper motions affect a right hand or a clock. In the illustration on the right, flipping the top image over the red line transforms the right hand into a left hand and the clock into a "counterclock" (a clock that turns counterclockwise).



A flip reverses orientation

DEFINITION: A rigid motion is called <u>proper</u> if it preserves orientation, which means that after the motion is applied, an image of a right hand still looks like a right hand and a clock still looks like a clock. It is called <u>improper</u> if it reverses orientation, which means that it turns a right hand into a left hand and a clock into a counterclock.

An even more precise definition of proper/improper will be discussed later when we learn about matrices. For now, let us turn our attention to another intuitive concept which needs to be described more precisely. We previously imagined that the pattern in the Seahorses and Eels painting was extended infinitely up, down, right and left, so the resulting object is "unbounded". On the other hand, the star image did not extend infinitely in any direction; we could fit the entire star image into a frame, so it is called "bounded". We make this distinction precise by focusing, not on the imprecise "extended infinitely" verbiage, but instead on more precise issue of whether the object can be framed (say by a square frame):

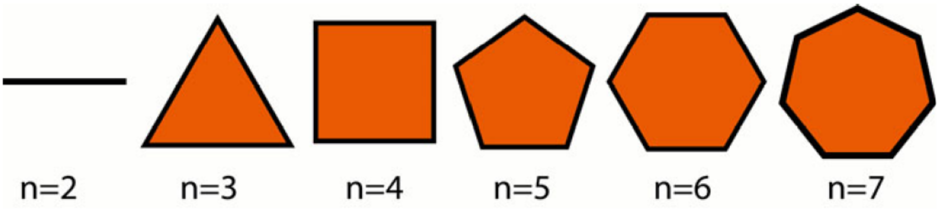
DEFINITION: An object in the plane is called bounded if it is fully contained in some square in the plane. Otherwise it is called unbounded.

The meaning would remain unaltered if the word "square" were replaced by "circle" or "pentagon" or many other possibilities. If an image can be framed by one of these frame shapes, then it can be framed by all of them.

In the study of symmetry, the most important bounded objects in the plane are the "regular polygons."

DEFINITION: The regular *n*-sided polygon (also called the regular *n*-gon) is the shape in the plane enclosed by *n* equal length straight sides, assembled so that all *n* of its angles are equal.

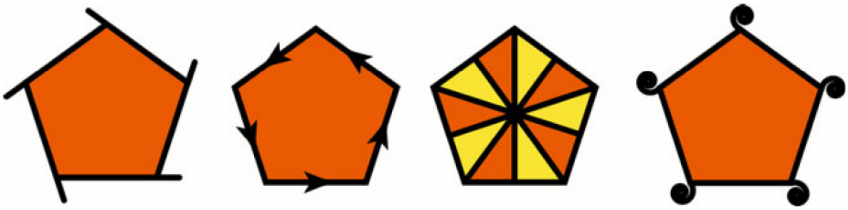
Thus, a regular 3-gon means an equilateral triangle, a regular 4-gon means a square, a regular 5-gon means a pentagon, a regular 6-gon means a hexagon, and so on.



Regular polygons

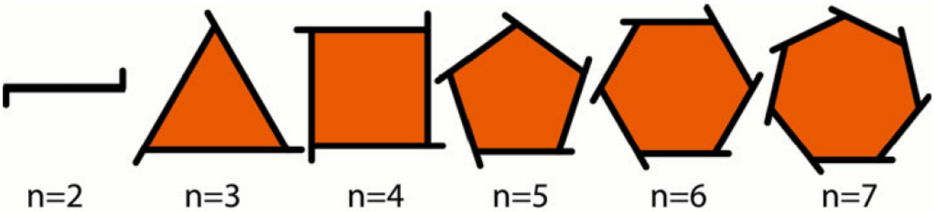
The two sides of the 2-gon lie on top of each other (because they meet at angles of 0°), so the 2-gon looks like a line segment.

Notice that each of these regular polygons has both rotation and reflection symmetries. Can you think of a way to orient each of these polygons, which means to alter it in such a way that its rotation symmetries are preserved but it no longer has any reflection symmetries? For example, here are a few artistic ways to orient the pentagon.



Oriented pentagons

There are many other possibilities; whichever you choose, the result is called an oriented pentagon. Each oriented pentagon pictured above has five rotation symmetries but NO reflection symmetries. Do you see why? A reflection would reverse the issue of whether it appears to spin clockwise or counterclockwise, and would therefore not be a symmetry. Since the first method is the simplest, we will use it to orient the other regular polygons:



Oriented regular polygons

Here is the general definition:

DEFINITION: An object in the plane is called oriented if it has NO improper symmetries.

You can always detect that an oriented object has been flipped. A clock is oriented because flipping it would make it look like a counterclock. Similarly, each oriented regular polygon above appears to spin counterclockwise, but would appear to spin clockwise after being flipped. The knotted blue object pictured here is oriented; hold it up to a mirror, and notice how its over/under crossing pattern differs from that of its mirror image. A flip would make it look like its mirror image, so a flip could not be a symmetry.

An oriented object

There are two types of unbounded objects that are classically important within the study of symmetry. First, a wallpaper pattern intuitively means an unbounded pattern that extends infinitely in all directions (left, right, up, and down) according to some organized scheme. The Seahorse and Eels painting is a wallpaper pattern (after being indefinitely extended). Second, a border pattern (also called a Frieze pattern) means an unbounded pattern that only extends infinitely along one line (usually the *x*-axis). For example, if the following pattern is extended infinitely to the right and left, then the result is a border pattern:

A border pattern

Border patterns are usually drawn horizontally as above, so they extend infinitely to the right and left, but not up or down. When positioned like this, all of the pattern's translation symmetries are in directions parallel to a horizontal line. This observation helps us to formulate a more precise definition.

DEFINITION: An unbounded pattern in the plane that has at least one translation symmetry (besides the identity) is called

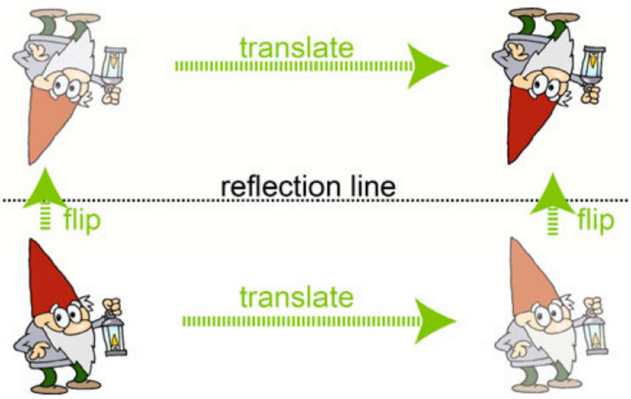
a border pattern if all its translations are parallel to a single line.

a wallpaper pattern otherwise.

The previously illustrated border pattern has many types of symmetries. You can translate it any number of positions to the right or left. You can reflect it over the horizontal center line. There are also vertical lines over which you can reflect it. If you perform any pair of the above-mentioned symmetries, one after the other, the result will also be a symmetry. For example, if you translate it any number of positions to the right or left and then reflect it over the horizontal center line, then the result is aptly called a glide reflection.

DEFINITION: A glide reflection means the result of performing a translation (other than the identity) followed by a reflection over a line that is parallel to the direction of the translation.

It does not matter which you do first: translate or reflect. In the illustration below, either order has the same effect of moving the bottom-left gnome to the top-right position.



A glide reflection translates and flips

Gnome image (used here and elsewhere) created by Paul Söderholm, www.gnurf.net.

Can you invent a border pattern with a glide reflection symmetry that has the peculiar property that the reflection and translation out of which it is built are not themselves symmetries of the border pattern? An answer is found in an exercise at the end of this chapter.

The Classification of Plane Rigid Motions

We began with vague intuitive notions of the word “symmetry.” A symmetric object often contains repeated images, and often looks the same from many positions. Based on these, we formulated a mathematically precise definition: a symmetry of an object is a rigid motion of the plane that leaves the object apparently unchanged. We then formulated precise definitions of other terms: “bounded”, “proper”, “oriented”, “border pattern”, and “wallpaper pattern”. This provides us with a vocabulary for more precisely discussing symmetry. In the remainder of the book, this precision will serve us well. It will allow us to ask and answer many precise questions, and eventually to prove beautiful theorems about the possible types of symmetries that objects may have. In this book, definitions are placed in green boxes and theorems are placed in blue boxes.

What is still missing? Well, we defined a “symmetry” using the term “rigid motion” but we have not yet precisely defined the term “rigid motion”. Rather, we have relied on an intuitive feeling for this concept. When it becomes necessary, we will eventually give a more precise definition of “rigid motion.” To help us get by for now, we mention the following (which our eventual precise definition will allow us to prove):

CLASSIFICATION OF PLANE RIGID MOTIONS (VERSION 1):

Every proper rigid motion of the plane is a translation, a rotation, or a rotation followed by a translation.

Every improper rigid motion of the plane is a flip or a flip followed by a translation.

In other words, there are no rigid motions other than the types that we have already considered (and combinations thereof). You may take this classification as your definition of rigid motion for now, if you like.

The story is even simpler for rigid motions that are symmetries of a *bounded* object. The symmetries of a bounded object include only flips and rotations (no translations). In fact:

THE CENTER POINT THEOREM: Any bounded object in the plane has a "center point" such that:
(1) Every proper symmetry is a rotation about this center point.
(2) Every improper symmetry is a flip over a line through this center point.

You might think of an object's center point as a balancing point; if you cut the object out of cardboard and wish to balance it on your finger tip, this is the correct place to position your finger.

[redacted]

Exercises

Challenge problems are designated ⟐.

(1) Which are proper and which are improper:

1. A proper symmetry followed by an proper symmetry
2. A proper symmetry followed by an improper symmetry
3. An improper symmetry followed by an improper symmetry

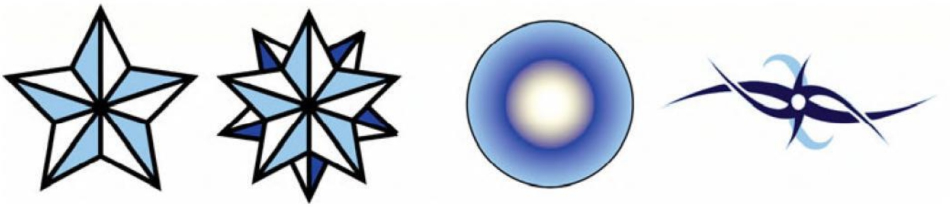
(2) How many symmetries does each capital letter in the English language have:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z.

(3) How many symmetries does the *n*-gon have for each of the values $n = 2, 3, 4, 5, 6, 7$ (these polygons are illustrated in the chapter)? Guess a general formula for the number of symmetries of an *n*-gon. What about an oriented *n*-gon?

(4) Draw a wallpaper pattern with NO improper symmetries.

(5) How many symmetries does each object have? Which object has the most? The least?



(6) Any capital letter in the English language can be used to create a border pattern like this:

...A A A A A A A A....
...B B B B B B B B ...
...C C C C C C C C ...

For each of the 26 letters, decide whether the resulting border pattern (a) contains reflections across any horizontal lines, (b) contains reflections across any vertical lines, and (c) contains rotations (other than by 0°).

(7) Consider the following border pattern to be extended infinitely to the right and left.

Characterize all of its translation, rotation, reflection, and glide reflection symmetries. What if the pattern was built from As rather than Rs?

(8) Draw two different objects that have exactly the same collection of symmetries.

(9) How many bounded objects can you think of that have infinitely many symmetries?

(10) What do you think is the most symmetric object in the plane?

(11) Make sketches of several bounded objects that have interesting collections of symmetries. Try to sketch a bounded object whose collection of symmetries is significantly different from that of an oriented or non-oriented polygon.

(12) "If an object has any translation symmetries (other than the identity), then it must have infinitely many translation symmetries." Explain why this statement is true.

(✪13) "If an object has any translation symmetries (other than the identity), then it must be an unbounded object." Prove this statement from scratch (without using The Center Point Theorem). *HINT: Visualize the object as painted on the (glass)* the answer. Think of "A*B" as meaning "A following B" or "A performed after B." It is important to keep the order straight – it is the opposite of what you might have expected.

Cayley Tables

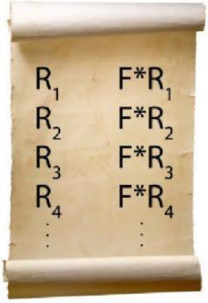
As a child, you became familiar with the algebraic operations of addition and multiplication by memorizing tables. Similarly, you will now study the algebraic operation of composition by building a table that exhibits the result of composing any pair of symmetries.

Let us start with a square. Its eight symmetries are:

$$\{\mathbf{I}, \mathbf{R}_{90}, \mathbf{R}_{180}, \mathbf{R}_{270}, \mathbf{H}, \mathbf{V}, \mathbf{D}, \mathbf{D'}\}$$

where **H** means horizontal flip, **V** means vertical flip, **D** and **D′** mean the two diagonal flips, and **R** means a counterclockwise rotation by the subscripted angle. The illustrations below show the effect of these eight symmetries on a square whose corners are labeled A, B, C, D and whose center is decorated with a picture of a gnome. The front of the square is green and the back is yellow.

These illustrations are followed by a table which exhibits the composition of any pair of these symmetries. This table is called a Cayley table for the square (or a Cayley table for the symmetries of the square). You find a composition, **A*B**, in a Cayley table like this one, by locating **A** along the left edge and **B** along the top edge.



illustrated). Your left column is now a list of all of the proper symmetries. Your right column is a list of all of the improper symmetries. The left and right columns have the same sizes; thus, there are equal numbers of proper and improper symmetries. Why are the symmetries in the right column all improper? Because an improper symmetry composed with a proper symmetry is always improper. Why does every improper symmetry appear somewhere in the right column, with no repetitions? Because every symmetry appears exactly once in **F**'s row of the Cayley table……………………………………………………………………□

Our final application of the existence of inverses has to do with objects that lack symmetry.

DEFINITION: An object is called <u>asymmetric</u> if it has no symmetries other than the identity.

A haphazard doodle will almost certainly be asymmetric. The next theorem says that asymmetric objects are very useful as rigid motion detectors:

RIGID MOTION DETECTOR THEOREM: If an object is asymmetric, then any rigid motion of the plane is uniquely determined by knowing the object's appearance after that motion is applied.

To understand this theorem, let us first think about why it is NOT true for a symmetric object, like a square. Suppose you close your eyes and then reopen them to discover that the square has moved 3 in. to the right. From this, you can NOT tell what motion I performed while your eyes were closed. I might have translated 3 in. to the right or I might have rotated 90° and then translated 3 in. to the right. You have no way of knowing. The rotation part

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