Peter Lynch
THAT'S MATHS
The mathematical magic in everyday life
Based on the popular *Irish Times* column

CONTENTS

PREFACE

This book is a collection of articles covering all major aspects of mathematics. It is written for people who have a keen interest in science and mathematics but who may not have the technical knowledge required to study mathematical texts and journals. The articles are accessible to anyone who has studied mathematics at secondary school.

Mathematics can be enormously interesting and inspiring, but its beauty and utility are often hidden. Many of us did not enjoy mathematics at school and have negative memories of slogging away, trying to solve pointless and abstruse problems. Yet we realise that mathematics is essential for modern society and plays a key role in our economic welfare, health and recreation.

Mathematics can be demanding on the reader because it requires active mental effort. Recognising this, the present book is modular in format. Each article can be read as a self-contained unit. I have resisted the temptation to organise the articles into themes, presenting them instead in roughly the order in which they were written. Each article tells its own story, whether it is a biography of some famous mathematician, a major problem (solved or unsolved), an application of maths to technology or a cultural connection to music or the visual arts.

I have attempted to maintain a reasonably uniform mathematical level throughout the book. You may have forgotten the details of what you learned at school, but what remains should be sufficient to enable you to understand the articles. If you find a particular article abstruse or difficult to understand, just skip to the next one, which will be easier. You can always return later if you wish.

The byline of my blog, thatsmaths.com, is 'Beautiful, Useful and Fun'. I have tried to bring out these three aspects of mathematics in the articles. Beauty can be subjective, but, as you learn more, you cannot fail to be impressed by the majesty and splendour of the intellectual creations of some of the world's most brilliant minds. The usefulness of maths is shown by its many applications to modern technology, and its growing role in medicine, biology and the social sciences. The fun aspect will be seen in the field known as recreational mathematics, aspects of maths that no longer attract active professional research but that still hold fascination.

About half the articles have appeared in *The Irish Times* over the past four years. The remainder are newly written pieces and postings from thatsmaths.co m. If you have a general interest in scientific matters and wish to be inspired by the beauty and power of mathematics, this book should serve you well.

INTRODUCTION

BEAUTIFUL, USEFUL AND FUN: THAT'S MATHS

Type a word into Google: a billion links come back in a flash. Tap a destination into your satnav: distances, times and highlights of the route appear. Get cash from an ATM, safe from prying eyes. Choose a tune from among thousands squeezed onto a tiny chip. How are these miracles of modern technology possible? What is the common basis underpinning them? The answer is mathematics.

Maths now reaches into every corner of our lives. Our technological world would be impossible without it. Electronic devices like smartphones and iPods, which we use daily, depend on the application of maths, as do computers, communications and the internet. International trade and the financial markets rely critically on secure communications, using encryption methods that spring directly from number theory, once thought to be a field of pure mathematics without 'useful' applications.

We are living longer and healthier lives, partly due to the application of maths to medical imaging, automatic diagnosis and modelling the cardiovascular system. The pharmaceuticals that cure us and control disease are made possible through applied mathematics. Agricultural production is more efficient thanks to maths; forensic medicine and crime detection depend on it. Control and operation of air transport would be impossible without maths. Sporting records are broken by studying and modelling performance and designing equipment mathematically. Maths is everywhere.

THE LANGUAGE OF NATURE

Galileo is credited with quantifying the study of the physical world, and his philosophy is encapsulated in the oft-quoted aphorism, 'The Book of Nature is written in the language of mathematics.' This development flourished with Isaac Newton, who unified terrestrial and celestial mechanics in a grand theory of universal gravitation, showing that the behaviour of a projectile like a cannonball and the trajectory of the moon are governed by the same dynamics.

Mechanics and astronomy were the first subjects to be 'mathematicised', but over the past century the influence of quantitative methods has spread to many other fields. Statistical analysis now pervades the social sciences. Computers enable us to simulate complex systems and predict their behaviour. Modern weather forecasting is an enormous arithmetical calculation, underpinned by mathematical and physical principles. With the recent untangling of the human genome, mathematical biology is a hot topic.

The mathematics that we learned at school was developed centuries ago, so it is easy to get the idea that maths is static, frozen in the seventeenth century or fossilised since ancient Greece. In fact, the vast bulk of mathematics has emerged in the past hundred years, and the subject continues to blossom. It is a vibrant and dynamic field of study. The future health of our technological society depends on this continuing development.

While a deep understanding of advanced mathematics requires intensive study over a long period, we can appreciate some of the beauty of maths without detailed technical knowledge, just as we can enjoy music without being performers or composers. It is a goal of this book to assist readers in this appreciation. It is hoped that, through this collection of articles, you may come to realise that mathematics is beautiful, useful and fun.

THE TWO CULTURES

'Of course I've heard of Beethoven, but who is this guy Gauss?'

The 'Two Cultures', introduced by the British scientist and novelist C. P. Snow in an influential Rede Lecture in 1959, are still relevant today.

Ludwig van Beethoven and Carl Friedrich Gauss were at the height of their creativity in the early nineteenth century. Beethoven's music, often of great subtlety and intricacy, is accessible even to those of us with limited knowledge and understanding of it. Gauss, the master of mathematicians, produced results of singular genius, great utility and deep aesthetic appeal. But, although the brilliance and beauty of his work is recognised and admired by experts, it is hidden from most of us, requiring much background knowledge and technical facility for a true appreciation of it.

There is a stark contrast here. There are many parallels between music and mathematics: both are concerned with structure, symmetry and pattern; but while music is accessible to all, maths presents greater obstacles. Perhaps it's a left versus right brain issue. Music gets into the soul on a high-speed emotional autobahn, while maths has to follow a rational, step-by-step route. Music has instant appeal; maths takes time.

It is regrettable that public attitudes to mathematics are predominantly unsympathetic. The beauty of maths can be difficult to appreciate, and its significance in our lives is often underestimated. But mathematics is an essential thread in the fabric of modern society. We all benefit from the power of maths to model our world and facilitate technological advances. It is arguable that the work of Gauss has a greater impact on our daily lives than the magnificent creations of Beethoven.

In addition to utility and aesthetic appeal, maths has great recreational value, with many surprising and paradoxical results that are a source of amusement and delight. The goal of this book is to elucidate the beauty, utility and fun of mathematics by examining some of its many uses in modern society and to illustrate how it benefits our lives in so many ways.

YOU CAN DO MATHS

Can we all do maths? Yes, we can! Everyone thinks mathematically all the time, even if they are not aware of it. We use simple arithmetic every day when we buy a newspaper, a cinema ticket or a pint of beer. But we also do more high-level mathematical reasoning all the time, unaware of the complexity of our thinking.

The central concerns of mathematics are not numbers, but patterns, structures, symmetries and connections. Take, for example, the Sudoku puzzles that appear daily in newspapers. The objective is to complete a 9 × 9 grid, starting from a few given numbers or clues, while ensuring that each row, each column and each 3 × 3 block contains all the digits from 1 to 9 once and only once. But the numerical values of the digits are irrelevant; what is important is that there are nine distinct symbols. They could be nine letters or nine shapes. It's the pattern that matters.

One Irish daily paper publishes these puzzles with the subscript *'There's no maths involved, simply use reasoning and logic!'* It seems that even the *idea* that something might be tainted by mathematics is enough to scare off potential solvers! Could you imagine the promotion of an exhibition in the National Gallery with the slogan *'No art* *involved, just painting and sculpture*'? If you can do Sudoku, you can do maths!

Whether you are discussing climate averages, studying graphs of house prices, worrying about inflation rates or working out the odds on the horses, you are thinking in mathematical mode. On a daily basis, you seek the best deal, the shortest route, the highest interest rate or the fastest way to get the job done with least effort. The principle of least action encapsulates the fundamental laws of nature in a simple rule. You are using similar reasoning in everyday life. Maximising, minimising, optimising: that's maths.

Maps and charts are ubiquitous in mathematics. They provide a means of representing complex reality in a simple, symbolic way. Subway maps are drastically simplified and deliberately distorted to emphasise what matters for travellers: continuity and connectivity. When you use a map of the London Underground, you are doing topology: that's maths.

Crossing a road, you observe oncoming traffic, estimate its speed and time to arrive, reckon the time needed to cross, compare the two and decide whether to walk or to wait. Estimating, reckoning, comparing: that's maths. Driving demands even more mathematical reasoning. You must constantly gauge closing speeds, accelerations, distances and times. Driverless cars are on the way: they use advanced mathematical algorithms and intensive computation. You can do that yourself in a flash.

Suppose you have the misfortune to fall ill. The doctor spells it out: the most effective treatment has severe side-effects; the alternative therapy is gentler but less efficacious; doing nothing has grave implications. A difficult choice must be made. You weigh up the risks and consequences of each course of action, rank them and choose the least-worst option. Weighing, balancing, ranking: that's maths.

Professional athletes can run 100 metres in ten seconds thanks to sustained, intensive training. Composers create symphonies after years of diligent study and practice. And professional mathematicians derive profound results through arduous application to their trade. You cannot solve technically intricate mathematical problems or prove arcane and abstruse theorems, but you can use logic and reasoning, and think like a mathematician. It is just a matter of degree.

INSTANT INFORMATION

Type a word into Google and a billion links appear in a flash. How is this done? How do computer search engines work, and why are they so good? *PageRank* (the name is a trademark of Google) is a method of measuring the popularity or importance of web pages. PageRank is a mathematical *algorithm*, or systematic procedure, at the heart of Google's search software. Named after Larry Page, a co-founder with Sergey Brin of Google, the PageRank of a web page estimates the probability that a person surfing at random will arrive at that page. Gary Trudeau, of *Doonesbury* fame, has described it as 'the Swiss Army knife of information retrieval'.

At school we solve simple problems like this: 6 apples and 3 pears cost €6; 3 apples and 4 pears cost €5; how much for an apple? This seems remote from practical use, and students may be forgiven for regarding it as pointless. Yet it is a simple example of simultaneous equations, a classical problem in linear algebra, which is at the heart of many modern technological developments. One of the most exciting recent applications is PageRank.

The PageRank computations form an enormous linear algebra problem, like the apples and pears problem but with billions of different kinds of fruit. The array of numbers that arises is called the 'Google matrix' and the task is to find a special string of numbers related to it, called the 'dominant eigenvector'. The solution can be implemented using a beautifully simple but subtle mathematical method that gives the PageRank scores of all the pages on the web.

The web can be represented as a huge network, with web pages indicated by dots and links drawn as lines joining the dots. Brin and Page used hyperlinks between web documents as the basis of PageRank. A link to a page is regarded as an indicator of popularity and importance, with the value of this link increasing with the popularity of the page linking to it. The key idea is that a web page is important if other important pages link to it.

Thus, PageRank is a popularity contest: it assigns a score to each page according to the number of links to that page and the score of each page linking to it. So it is *recursive*: the PageRank score depends on PageRank scores of other pages, so it must be calculated by an iterative process, cycling repeatedly through all the pages. At the beginning, all pages are given equal scores. After a few cycles, the scores converge rapidly to fixed values, which are the final PageRank values.

Google's computers or 'googlebots' are ceaselessly crawling the web and calculating the scores for billions of pages. Special programs called spiders are constantly updating indexes of page contents and links. When you enter a search word, these indexes are used to find the most relevant websites. Since these may number in the billions, they are ranked based on popularity and content. It is this ranking that uses ingenious mathematical techniques.

Efforts to manipulate or distort PageRank are becoming ever more subtle, and there is an ongoing cat-and-mouse game between search engine designers and spammers. Google penalises web operators who use schemes designed to artificially inflate their ranking. Thus, PageRank is just one of many factors that determine the search result you see on your screen. Still, it is a key factor, so those techniques you learned in school to find the price of apples and pears have a real-world application of great significance and value.

(The answer to the puzzle: apples cost 60 cents and pears cost 80 cents.)

NAPIER'S NIFTY RULES

Spherical trigonometry is not in vogue. A century ago, a Tripos student at Cambridge might resolve half a dozen spherical triangles before breakfast. Today, even the basics of the subject are unknown to many students of mathematics. That is a pity, because there are many elegant and surprising results in spherical trigonometry. For example, two spherical triangles that are similar – having corresponding angles equal – have the same area. This contrasts sharply with the situation for plane geometry.

There is no denying the crucial importance of spherical trigonometry in astronomy and in the geosciences. A good memory is required to master all the fundamental results: the sine law, the cosine law for angles, the cosine law for sides, Gauss's formulae and many more. But we can get a long way with a few simple and easily remembered rules formulated by the inventor of logarithms.

The equation for a great circle involves the intersection of a plane and a sphere, an easy problem in three-dimensional Cartesian geometry. It is

$$\tan \phi = \tan \varepsilon \sin (\lambda - \lambda_0)$$

where λ and ϕ are longitude and latitude and the great circle crosses the equator through $\lambda0$ at an angle ε. A more direct approach of showing this is possible: the formula for a great circle turns out to be one of Napier's Rules.

These rules are easy to state. Every spherical triangle has three angles and three sides. The sides are also expressed as angles, the angles they subtend at the centre of the sphere. For a sphere of unit radius, these angles (in radians) equal the lengths of the sides. *Napier's Rules apply to right-angled triangles*. Omitting the right angle, we write the remaining five angles in order on a pie diagram, but replace the three angles *not* adjacent to the right angle by their complements (their values subtracted from 90 degrees). If we select any three angles, we will always have a middle one and either two angles adjacent to it or two angles opposite to it. Then Napier's Rules are:

SIne of m**I**ddle=Product of t**A**ngents of **A**djacent angles
angle

SIne of m**I**ddle=Product of c**O**sines of **O**pposite angles
angle

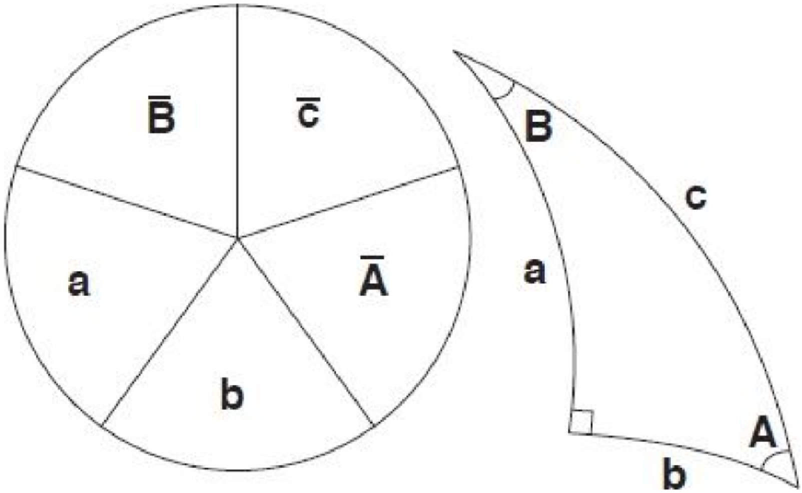
With five choices for the middle angle and adjacent and opposite cases for each, there are ten rules in all. As a mnemonic, note the correspondences of the first vowels in key words, indicated in bold.

Napier's Rules apply only to right triangles, but we can often handle a general spherical triangle by dividing or extending it. Suppose we want to find out the great circle distance from Paris to Cairo, and we know the latitude and longitude of each city. The meridians from the North

Pole to these cities, together with the great circle between them, form a spherical triangle for which we know two sides and the included angle. We can apply the cosine law for sides to get the great circle distance. But what if we have forgotten the cosine law? We can drop a perpendicular from Paris to the meridian through Cairo and apply Napier's Rules twice to find the inter-city distance (it turns out to be about 3,200 km).

John Napier (1550–1617), formulator of the rules, is best remembered as the inventor of logarithms. Also out of vogue today, his tables of logs enabled Johannes Kepler to analyse Tycho Brahe's observations and deduce the orbits of the planets. Napier also popularised the use of decimal fractions in arithmetic. But his work in mathematics was essentially recreational, for Napier was foremost a theologian. An ardent, even fanatical, Protestant, he regarded his commentary on the Book of Revelation as his best work. In *A Plaine Discovery of the Whole Revelation of St John*, he predicted that the apocalypse and the end of the world would occur in 1700.

Napier's book on logarithms contained his 'Rules of Circular Parts' of right spherical triangles. As we have seen, they are easily remembered and simple to apply. If you are ever marooned on a desert island and know the location, you can use them to work out how far you will have to swim home. I hope you make it.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SPROUTOLOGY

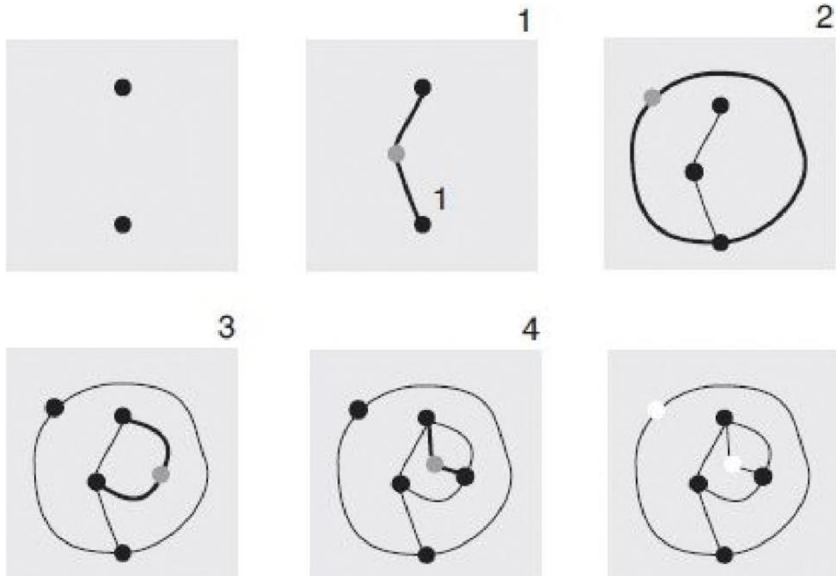
Sprouts is a simple and delightfully subtle pencil and paper game for two players. The game is set up by marking a number of spots on a page. Each player makes a move by drawing a curve that joins two spots, or that loops from a spot back to itself, without crossing any lines drawn earlier, and then marking a new spot on the curve. A maximum of three lines may link to a spot, and any spot with three lines is considered dead, since it plays no further role in the game. Sooner or later, no further moves are possible and the player who draws the last line wins the game.

Sprouts was devised by two Cambridge mathematicians, John Horton Conway and Michael Stewart Paterson, in 1967. It has an addictive appeal, and it immediately became a craze, being played in mathematics departments around the world. Despite the simple rules, the analysis of the game presents some challenges, and no general winning strategy is known. It is fairly easy to show that if there are n spots to start, the game will have at least $2n$ moves, and must end in at most $3n-1$. Thus, with 8 spots to start, there will be between 16 and 23 moves.

The mathematics of Sprouts, which we might call sproutology, involves topology, a form of geometry that considers continuity and connectedness but disregards distances and shapes. Topology is often called rubber-sheet geometry since a figure drawn on an elastic sheet retains its topological properties when the sheet is stretched but not torn. Sprouts is topological, since the precise positions of the spots is unimportant; it is only the pattern of connections between them that counts. The game exploits the Jordan curve theorem, which states that simple closed curves divide the plane into two regions. This apparently obvious result is actually quite difficult to prove.

The one-spot game of Sprouts is trivial: the first player must join the spot to itself and draw another spot; the second player then joins the two spots, winning the game. Games with a small number of starting spots have been fully investigated, and a pattern is evident: if the remainder when n is divided by 6 is 3, 4 or 5, the first player can force a win (assuming perfect play); otherwise, the second player has a winning strategy. This 'Sprouts conjecture' remains unproven.

For up to seven spots to start, Sprouts can be checked by hand, but for larger numbers of spots it rapidly becomes too complex and a computer analysis is required. Recently, Julien Lemoine and Simon Viennot analysed games with up to 47 spots, and their findings support the Sprouts conjecture. Of course, the existence of a winning strategy does not guarantee a win. Despite its elementary rules, Sprouts is surprisingly subtle, and prowess comes only with practice. You should start with a small number of spots, between five and 10, and gradually build up skill. But beware the addictive appeal of the game: you may well become a sproutaholic.



A sample game of Sprouts in which the second player wins after four moves.

WHY DON'T CLOUDS FALL DOWN?

A stone memorial was unveiled in 1995 in the tiny Sligo townland of Skreen to honour a great nineteenth-century mathematician and physicist who hailed from there. George Gabriel Stokes was born in Skreen in 1819, the youngest of seven children of Reverend Gabriel Stokes, Rector of the Church of Ireland.

George showed clear signs of brilliance from an early age, excelling at mathematics. After education in Skreen, Dublin and Bristol, he matriculated to Pembroke College, Cambridge, graduating in 1841 as Senior Wrangler; that is, gaining first place in the entire University of Cambridge in Part II of the Mathematical Tripos, the final mathematics examinations. Just eight years later he was appointed Lucasian Professor of Mathematics, a position that he held for over fifty years. This prestigious chair had earlier been held by Isaac Newton and more recently by Stephen Hawking.

Stokes's scientific interests were very broad, and he corresponded on a wide range of subjects with another giant of Victorian science, Belfast-born Lord Kelvin. A particular focus of his work was wave phenomena in various media. Some of his best-known research was on the theory of light waves. In this work, he obtained some major advances in the mathematical theory of diffraction and elucidated the phenomenon of fluorescence, the emission of light by a substance that has absorbed electromagnetic radiation. We benefit from this work through fluorescent lamps; these use electricity to excite mercury atoms, which then cause a phosphor coating to fluoresce, producing visible light.

Stokes investigated the internal friction of fluids, explaining how small droplets are suspended in the air and giving an answer to the age-old question asked by children: Why don't clouds fall down? His description of fluid viscosity was incorporated into the equations of fluid motion, now called the Navier–Stokes equations. These equations are of fundamental importance in all studies of fluid motion and are central to the study of turbulence, for modelling the oceans and for weather prediction and climate modelling.

In 1859 Stokes married Mary Susanna, daughter of Thomas Romney Robinson, Astronomer at Armagh Observatory. Robinson had an interest in the atmosphere and had invented the spinning cup anemometer for measuring wind speed. This interest must have influenced Stokes, who later developed an instrument called the Campbell–Stokes sunshine recorder.

In 1851, Stokes was elected a Fellow of the Royal Society. For thirty years he was secretary of the society and later served as its president. He was also an MP for a time, representing the University of Cambridge. But he never forgot his origins in Skreen, and returned to Sligo regularly for summer vacations. And in one of his heavily mathematical papers he wrote of 'the surf which breaks upon the western coasts as a result of storms out in the Atlantic', recalling the majestic rollers thundering in as he strolled as a boy along Dunmoran Strand near Skreen.

Stokes won many honours during his life, and his name is preserved in a large number of scientific contexts, including Stokes' Law (in fluid dynamics), Stokes' Theorem (in vector calculus), the Stokes shift (fluorescence), the Stokes phenomenon (in asymptotics) and many more.

PACKING ORANGES AND STACKING CANNONBALLS

Packing problems are concerned with storing objects as densely as possible in a container. Usually the goods and the container are of fixed shape and size. Many packing problems arise in the context of industrial packaging, storage and transport, in biological systems, in crystal structures and in carbon nanotubes, tiny molecular-scale pipes.

Packing problems illustrate the interplay between pure and applied mathematics. They arise in practical situations but are then generalised and studied in an abstract mathematical context. The general results then find application in new practical situations. A specific example of this interplay is the sphere-packing problem.

In 1600, the adventurer Walter Raleigh asked his mathematical adviser Thomas Harriot about the most efficient way of stacking cannonballs on a ship's deck. Harriot wrote to the famous astronomer Johannes Kepler, who formulated a conjecture that a so-called 'face-centred cubic' was the optimal arrangement.

Let's start with a simpler problem: How much of a table-top can you cover with non-overlapping €1 coins? Circular discs can be arranged quite densely in a plane. If they are set in a square formation, they cover about 79% of the surface. But a hexagonal arrangement, like a honeycomb, with each coin touching six others, covers over 90%; that's pretty good. Joseph-Louis Lagrange showed in 1773 that no regular arrangement of discs does better than this. But what about irregular arrangements? It took until 1940 to rule them out.

In three dimensions, we could start with a layer of spheres arranged in a hexagonal pattern like the coins, and then build up successive layers, placing spheres in the gaps left in the layer below. This is how grocers instinctively pile oranges, and gunners stack cannonballs. The geometry is a bit trickier than in two dimensions, but it is not too difficult to show that this arrangement gives a packing density of about 74%. The great Gauss showed that this is the best that can be done for a regular or lattice arrangement of spheres.

But again we ask: what about irregular arrangements? Is it not possible to find some exotic method of packing the spheres more densely? Kepler's Conjecture says 'No', and the problem has interested many great mathematicians in the intervening four hundred years. In 1900 David Hilbert listed 23 key problems for twentieth-century mathematicians, and the sphere-packing puzzle was part of his 18th problem.

In 1998 Thomas Hales announced a proof of Kepler's Conjecture. He broke the problem into a large number of special cases and attacked each one separately. But there were some 100,000 cases, each requiring heavy calculation, far beyond human capacity, so his proof depended in an essential way upon using a computer. After detailed review, Hales' work was finally published in 2005 in a 120-page paper in *Annals of Mathematics*. Thus, Kepler's Conjecture has become Hales' Theorem! Most mathematicians accept that the matter is resolved, but there remains some discomfort about reliance on computers to establish mathematical truth.

Why should we concern ourselves with a problem for which grocers and cannoneers knew the solution long ago? Well, in higher dimensions the corresponding problem has more intriguing aspects. It is a key result in data communication: to minimise transmission errors, we design codes that are based on maximising the packing density of hyper-spheres in high-dimensional spaces. So the apparently abstruse conjecture of Kepler has some eminently practical implications for our technological world.

MODELLING EPIDEMICS

The film *Contagion* painted a terrifying picture of the breakdown of society following a viral pandemic. The movie identified a key parameter, the basic reproduction number R-nought (R_0). This number measures how many new people catch the virus from each infected person, and is crucial in determining how fast an infection spreads.

In March 2003, an epidemic of severe acute respiratory syndrome (SARS) spread rapidly across the globe. The World Health Organisation issued a global alert after SARS had been detected in several countries. Since the spread of infections is greatly facilitated by international air travel, controls on movement can certainly be effective: with appropriate travel restrictions, the SARS epidemic was brought under control within a few months.

Epidemiological analysis and mathematical models are now essential tools in understanding and responding to infectious diseases such as SARS. Models range from simple systems of a few variables and equations to highly complex simulations with many millions of variables. A broad range of mathematics, both conventional techniques and methods emerging from current research, are involved. These include dynamical systems theory, statistics, network theory and computational science.

Public health authorities are faced with crucial questions: How many people will become infected? How many do we need to vaccinate to prevent an epidemic? How should we design programmes for prevention, control and treatment of outbreaks? The models allow us to quantify mortality rates, incubation periods, levels of threat and the timescale of epidemics. They can also predict the effectiveness of vaccination programmes and control policies, such as travel restrictions.

Parameters like transmission rates and basic reproduction numbers cannot be accurately estimated for a new infection until an outbreak actually occurs. But models can be used to study 'what if' scenarios to estimate the likely consequences of future epidemics or pandemics.

In a paper published in 1927, 'A Contribution to the Mathematical Theory of Epidemics', two scientists in Edinburgh, William Kermack and Anderson McKendrick, described a simple model with three variables, and three 'ordinary differential equations' that describe how infection levels change with time, which was successful in predicting the behaviour of some epidemics. Their model divided the population into three groups: susceptible, infected and recovered people, denoted S, I and R respectively. This SIR model simulates the growth and decline of an epidemic and can be used to predict level of infection, timescale and the total percentage of the population afflicted by the infection.

However, many important factors are omitted from the simple SIR model. The swine flu epidemic in Britain reached a peak in July 2009 and then declined rapidly and unexpectedly. The key factor not included in the model was the effect on the transmission rate of the school holidays, with contacts between children greatly reduced. The growth of the outbreak was interrupted, but an even larger peak was reached in October, after school had resumed. When these social mixing patterns were included, the model produced two peaks, in agreement with the observed development.

The statistician George Box, a pioneer in time series analysis, design of experiments and Bayesian inference, once remarked: ‘All models are wrong, but some are useful.’ All models of epidemics have limitations, and those using them must bear these in mind. Given the vagaries of human behaviour, prediction of the exact development of an infectious outbreak is never possible. Nevertheless, models provide valuable insights not available through any other means.

Future influenza pandemics are a matter of ‘when’ rather than ‘if’. In planning for these, mathematical models will play an indispensable role.

A FALLING SLINKY

If you drop a slinky from a hanging position, something very surprising happens. *The bottom remains completely motionless* until the top, collapsing downwards, coil upon coil, crashes into it.

How can this be so? We all know that anything with mass is subject to gravity, and this is certainly true of the lower coils of the slinky. But there's another force acting on them, the tension due to the stretching of the slinky. When hanging in an equilibrium position, these two forces, gravity and tension, balance exactly, so there is no movement.

When we let go of the top, the tension in the uppermost coils is relaxed and, since there is nothing to balance gravity, they start to fall. But this relaxation has to be transmitted or communicated down the slinky before gravity can pull the bottom downwards. This transmission takes time: the time for the 'message' to travel the length of the slinky depends on the ratio of the mass to the stiffness.

A slinky has large mass and small stiffness, so this time is relatively long, typically about half a second. But a freely falling object falls five metres in the first second. Moreover, the top coils of the slinky initially accelerate downwards even faster than in free fall, because the downward tension augments gravity. Thus, the slinky reaches a crunch point, where the top crashes into the bottom, before the signal of the release can reach it. You might say that *the bottom doesn't know what hit it*!

It is worthwhile playing with a real slinky to study this curious behaviour. If you put the slinky on a table, stretch it, hold one end steady and jerk the other end, you will see the signal propagating along the spring. But the best way to view the falling slinky is in slow motion.

There are several videos on YouTube illustrating falling slinkies, for example http://www.youtube.com/watch?v=uiyMuHuCFo4.

A 'MERSENNERY' QUEST

Prime numbers are of central importance in pure mathematics and also in a wide range of applications, most notably cryptography. The security of modern communication systems depends on their properties. Recall that a prime number is one that cannot be evenly divided by a smaller number. Thus, 2, 3 and 5 are primes, but 4 and 6 are not, since 4 = 2 x 2 and 6 = 2 x 3. Primes are the atoms of the number system: every whole number is a product of primes.

The search for patterns in the distribution of primes has occupied mathematicians for centuries. They appear to be randomly strewn among the whole numbers, but there are tantalising indications of structure. Often, a hint of a pattern emerges, only to evaporate upon further study. Thus, 31 is prime, as are 331, 3331, 33331 and the next three members of this sequence. But 333,333,331 is divisible by 17, and the pattern is broken.

In elementary algebra, we learn to solve quadratic equations. This corresponds to finding the zeros of a simple polynomial equation. The zeros of a more complicated function, called the zeta function, are intimately connected with the distribution of the prime numbers, but the location of all these zeros is enormously difficult. They are believed to satisfy a pattern first proposed in 1859 by Bernhard Riemann, but this has never been proved. The Riemann Hypothesis is widely regarded as the most important unsolved problem in mathematics. A proof would have far-reaching implications and whoever proves it will win lasting fame. They will also collect a $1 million prize from the Clay Mathematics Institute.

The frantic dash to find ever-larger prime numbers has been dominated in recent years by the Great Internet Mersenne Prime Search (GIMPS), a voluntary collaborative project involving a large number of personal computers. The record for the largest prime is broken on a regular basis. Almost all the recent examples have been found by GIMPS, and are numbers of a particular form called *Mersenne primes*, which are one less than a power of 2. As of June 2016, the largest known prime is obtained by multiplying 2 by itself 74,207,281 times and subtracting 1. With more than 22 million digits, it would fill many thousands of printed pages.

Mersenne numbers take their name from a seventeenth-century friar called Marin Mersenne. Born in France in 1588, Mersenne was a strong apologist for Galileo, whose scientific ideas challenged religious orthodoxy. Mersenne's main scientific work was in acoustics, but he is remembered today for his association with the Mersenne primes. He had contact with many mathematical luminaries, and provided a vital communication channel, corresponding with mathematicians, including René Descartes and Étienne Pascal, in many countries. Mersenne was, in essence, a one-man internet hub.

GIMPS has found the ten largest known prime numbers, and regularly smashes its own record. The project uses a search algorithm called the Lucas–Lehmer primality test, which is particularly suitable for finding Mersenne primes and is very efficient on binary computers. The test was originally developed by Édouard Lucas in the nineteenth century, and improved by Derrick Henry Lehmer in the 1930s.

For discovering a prime with more than 10 million decimal digits, GIMPS won a $100,000 prize and a Cooperative Computing Award from the Electronic Frontier Foundation (EFF). A prize of $150,000 is on offer from EFF for the first prime number found with at least 100 million decimal digits, and a further $250,000 for one with at least a billion digits. What are you waiting for?

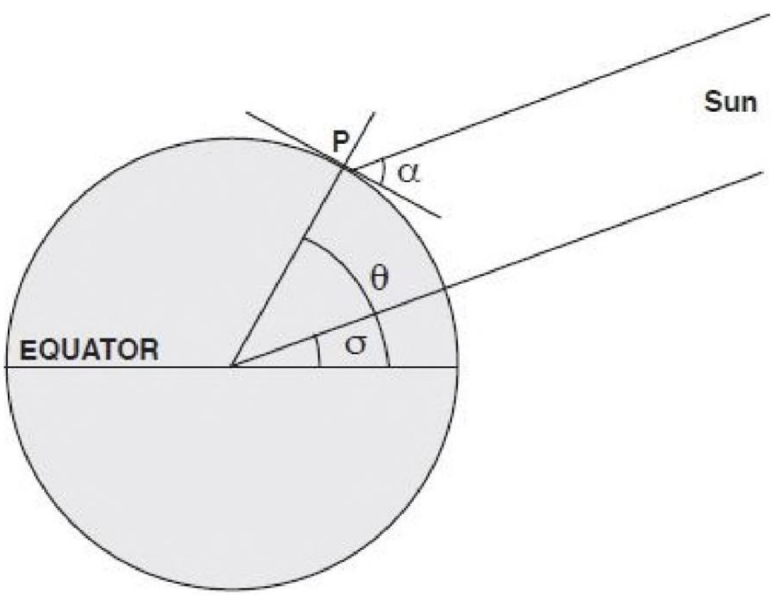
SHACKLETON'S SPECTACULAR BOAT JOURNEY

A little mathematics goes a long, long way. Elementary geometry brought a small team of heroes 800 sea miles across the treacherous Southern Ocean, and resulted in 28 lives being saved.

For eight months, Ernest Shackleton's expedition ship *Endurance* had been carried along, ice-bound, until it was finally crushed and sank in October 1915. This put an end to the plans of the Irish-born explorer and his team of 28 men to cross the Antarctic continent. They salvaged three boats and made their way to Elephant Island, at the tip of the Antarctic Peninsula.

With five companions, Shackleton set out in one of the boats, a whaler called the *James Caird*, setting a course for South Georgia, some 800 nautical miles distant. With unceasing gales, the sea was tempestuous. Navigation depended on sightings taken with a sextant during rare appearances of the sun. Heavy rollers tossed the boat about, making it difficult to sight the horizon. The process was described by navigator Frank Worsley as ‘a merry jest of guesswork’.

The strategy was to reach the latitude of South Georgia and let the westerly winds and currents carry the boat to the island. *Latitude* is measured by ‘shooting the sun’ with a sextant. The horizon and the lower limb of the sun are aligned in a split mirror, viewed through a telescope. The altitude of the sun can then be read from an indicator on the sextant arc. The geometry is straightforward: looking at the diagram below, we can see that the latitude θ is given by $\theta = 90° + \sigma - \alpha$ where α is the sun’s altitude read from the sextant and σ is the latitude of the sun. This last depends on the date and time, and is given in the *Nautical Almanac*.



Angles used to calculate the latitude. Alpha (α) is the altitude of the sun measured with the sextant; sigma (σ) is the latitude of the sun, obtained from the Nautical Almanac*; and theta (θ) is the latitude of point P.*

To get the *longitude*, a clear shot of the sun at local noon is required. The navigator tracks the solar altitude to determine the exact time when the sun reaches its highest point. This is local apparent noon. The chronometer is set to Coordinated Universal Time (UTC or GMT). Since the earth rotates in 24 hours, the sun appears to move westwards 15 degrees in each hour. Thus, if the chronometer reads 15:00 GMT, local noon is three hours behind Greenwich and the longitude is 45° west.

After 17 days, Shackleton and his companions landed on the west coast of South Georgia, at about 54°. The voyage was a marvel of navigation, one of the greatest boat journeys ever accomplished. But the trouble was not over yet. Shackleton still had to cross the mountainous interior of the island to reach the whaling station at Stromness and arrange a rescue mission to relieve the men left behind on Elephant Island.

Ultimately, the entire party reached the safety of Punta Arenas, Chile in September 1916. The survival of Shackleton and all his companions was 'a triumph of hope and inspired leadership'.

WHERE IN THE WORLD?

Most hill-walkers can recall an anxious time when, caught on a ridge between steep slopes, they were suddenly enshrouded by dense fog. A carefree ramble becomes a terrifying test of survival. The immediate question is 'Where exactly am I?' Map and compass are vital aids, but they cannot answer that question. A hand-held device about the size of a mobile phone can. How does it do that?

The Global Positioning System is a satellite-based navigation system, owned and operated by the US government, that provides information on location in all weathers, anywhere in the world. It is freely available to anyone with a GPS receiver, costing perhaps €100. The system comprises a constellation of between 24 and 32 satellites, orbiting at about 20,000 km above the earth. Each satellite carries a high-precision atomic clock, accurate to about one nanosecond. A nanosecond (ns) is one billionth of a second, the time it takes light to travel one foot.

To compute the position, the GPS receiver uses signals from several satellites, each including the precise time and location of the satellite. The satellites are synchronised so that the signals are transmitted at precisely the same instant. But they arrive at the GPS receiver at slightly different times. Using the known signal speed, the speed of light, the distance to each satellite is determined. These distances are then used to calculate the position of the receiver, using *trilateration*.

Trilateration determines position by using distances to known locations. This is in contrast to triangulation, which uses angles. For example, if you are 110 km from Athlone, you are somewhere on a circle of this radius centred at Athlone. If you are also 140 km from Belfast, you must be in Dublin or in Garrison, Fermanagh, the points where two circles intersect. Finally, if you are also 220 km from Cork, you can only be in Dublin. Three distances suffice for a unique location.

In three-dimensional space, spheres replace circles and four are needed, so the GPS receiver uses signals from four satellites. This provides distances from four known locations, sufficient to pin down the position of the receiver. GPS receivers available today give location to an accuracy of about ten metres. This position may be plotted on a background map or given as latitude and longitude or a National Grid reference.

Navigation is just one of the many civilian and military applications of GPS. The system is vital for search and rescue, for vehicle tracking, for map-making and surveying and for detecting movements in the earth's crust. Monitoring the movements of elephants in Africa is one among many other applications. Satnav is considered so essential that the European Union is developing a GPS system called Galileo. As of June 2016 there were 14 of 30 satellites in orbit and the system should be fully operational by 2019.

GPS is a striking example of the practical importance of Einstein's relativity theory. Special relativity implies that a moving clock ticks slowly relative to a stationary one, so for an observer on earth, the satellite clocks lose about 7,000 ns (7 microseconds) each day. But general relativity says that these clocks should go about 45,000 ns *faster*, because the earth's gravitational pull is weaker higher up. The net effect is a speed-up of about 38,000 ns per day. To avoid cumbersome corrections, the clocks are reset before launch to compensate for relativistic effects. Without this, GPS would be useless for navigation!

The Global Positioning System is a remarkable synthesis of old and new. It involves high-tech engineering and complex relativistic physics to enable it to function, but the mathematics used to determine location is simple, being a straightforward application of the geometry of circles and spheres developed in ancient Greece.

SRINIVASA RAMANUJAN

Srinivasa Ramanujan, one of the greatest mathematical geniuses ever to emerge from India, was born in 1887 into a poor Brahmin family. Ramanujan had limited formal education but was consumed by his passion for mathematics. He neglected all other subjects and failed the entrance exam for the University of Madras. However, he continued his mathematical research with intensity.

In 1913, Ramanujan wrote to G. H. Hardy, the leading mathematician in Britain, enclosing some of his results. Hardy examined them and concluded that they 'could only be written down by a mathematician of the highest class'. Thus began one of the most successful mathematical collaborations of all time. For five years, Ramanujan worked with Hardy in Cambridge, publishing many papers of great richness and originality. In 1918 he was elected a Fellow of the Royal Society.

Ramanujan returned to India in 1919, but lived for only one more year. Shortly before his death, aged only 32, Ramanujan wrote a last letter to Hardy in which he introduced 17 completely new and strange power series that he called 'mock theta functions'.

In 1976 the American mathematician George Andrews was looking through some papers in the Wren Library in Cambridge and recognised Ramanujan's handwriting. What he found, now known as the 'lost notebook', contains many remarkable results, including Ramanujan's results on the mysterious mock theta functions.

Andrews' discovery opened up a vast new landscape. The results were of stunning novelty, representing what many regard as Ramanujan's deepest work. The finding of the lost notebook has been compared to finding a manuscript of Beethoven's Tenth Symphony. The consequences have been profound, for both pure mathematics and theoretical physics.

Ramanujan gave no clue as to how he had discovered the mock theta functions. An intrinsic meaning of them has eluded mathematicians until very recently. Sander Zwegers, a lecturer at UCD until he moved to Cologne in 2011, finally explained how they fit into a broader context. Zwegers' 2002 PhD thesis was groundbreaking, and has led to numerous publications and international conferences.

The breakthrough in our understanding is having an impact on many aspects of mathematics and physics. In pure mathematics, the results have been applied to graph theory, group theory and differential topology. In physics, there are applications in particle physics, statistical mechanics and cosmology. In particular, Ramanujan's functions have proved valuable for calculating the entropy of black holes.

Ramanujan's startlingly brilliant and innovative research paved the way for many major breakthroughs in number theory over the past century. Mathematician and theoretical physicist Freeman Dyson spoke of 'a grand synthesis still to be discovered', and he speculated about applications of the mock theta functions to string theory. This is an indication of the prescience and genius of Ramanujan's work, confirming Hardy's description of him as having 'profound and invincible originality'.

SHARING A PINT

Four friends, exhausted after a long hike, stagger into a pub to slake their thirst. But, pooling their funds, they have enough money for only one pint.

Annie drinks first, until the surface of the beer is halfway down the side (Fig. 1 (A)). Then *Barry* drinks until the surface touches the bottom corner (B). *Cathy* then takes a sup, leaving the level as in (C), with the surface through the centre of the bottom. Finally, *Danny* empties the glass.

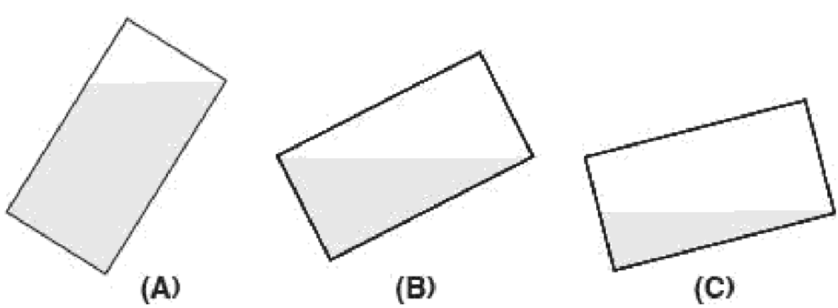


Figure 1

Question: Do all four friends drink the same amount? If not, who gets most and who gets least?

By symmetry, Annie has drunk half of the top half of the glass. So she has consumed 25% of the beer. Again by symmetry, Barry has left exactly 50% of the beer in the glass, so he has swallowed 25%. So far so good.

But Cathy has left beer forming a less regular shape: the liquid remaining in (C) is in the shape of an ungula, the volume formed by a plane slicing a cylinder and passing through the centre of the base. We have to calculate the volume of the ungula to see how much beer is left for Danny.

Ungula means hoof, and a section of a cylinder or cone cut off by a plane oblique to the base is so called because it resembles a horse's hoof. The shape is shown in Figure 2 below. Its volume can be calculated by a mathematical operation known as integration.

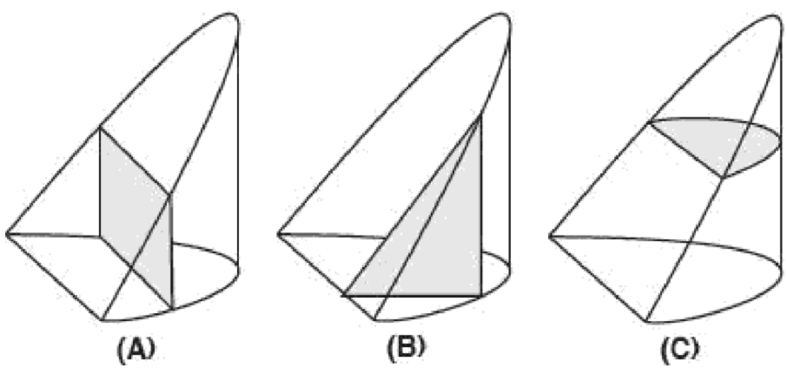


Figure 2 Cross-sections of an ungula perpendicular to (A) the x *axis, (B) the* y *axis and (C) the* z *axis.*

The three panels in Figure 2 show cross-sections of the ungula perpendicular to the x, y and z axes. They are respectively a rectangle, a triangle and a segment, and the volume is obtained by integration along the relevant axis. So, schematically, we can write the volume as

$$V = \int (\text{Rectangle})\, \mathrm{d}x = \int (\text{Triangle})\, \mathrm{d}y = \int (\text{Segment})\, \mathrm{d}z$$

where the symbol ∫ denotes an integral, or sum over all the relevant shapes. Naturally, all three yield the same result, $V = (2/3)r^2h$ where r is the radius and h the height.

Now, the volume of the cylinder is $\pi r^2 h$, so the fraction left for Danny is $2/(3\pi)$ or about 21%. Thus, while Annie and Barry drank 25% of the beer, Cathy must have drunk about 29%, leaving Danny short.

There is something remarkable about the volume of the ungula: it does not involve π, even though one of the surfaces is curved. **Where has π gone?**

More remarkable still is that Archimedes showed that the volume of the ungula is one-sixth that of the surrounding cube or block. The volume is $(2/3)r^2h$, and the volume of the rectangular box containing the cylinder is $2r \times 2r \times h = 4r^2h$, so indeed the ratio is 1/6.

Archimedes used the triangular cross-section, integrating in the y direction. His reasoning was not some crude approximation, but a true application of the method of integral calculus.

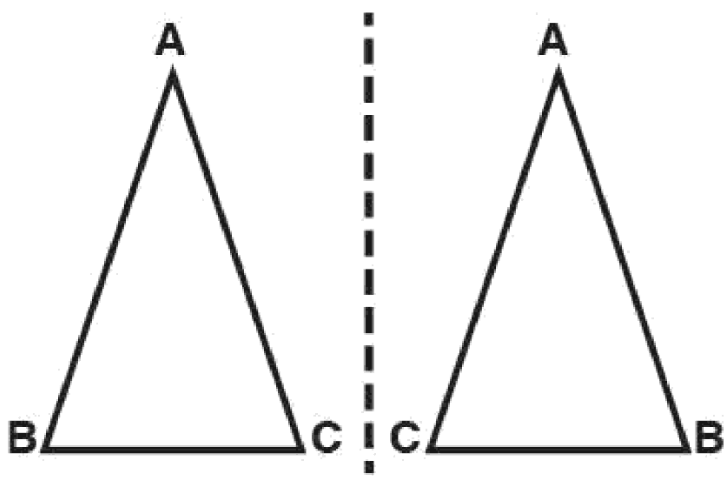
PONS ASINORUM

The fifth proposition in Book I of Euclid's *Elements* states that the two base angles of an isosceles triangle are equal (in the figure here, angles B and C; an isosceles triangle is one having two equal sides). For centuries, this result has been known as the Pons Asinorum, or Bridge of Asses, apparently a metaphor for a problem that separates bright sparks from dunces.

Euclid proves the proposition by extending the sides AB and AC and drawing lines to form additional triangles. His proof is quite complicated. A simpler approach, popular for a hundred years or so, is to draw the line that bisects the apex angle A, splitting the triangle into two parts, which are then shown to be congruent, or equal in all respects. This requires use of an earlier result, Euclid's proposition I.4, which says that two triangles are congruent if they have two sides and the included angle equal.

Around 1960 another proof appeared, allegedly discovered by a computer. It ingeniously compared the triangle ABC to its mirror image ACB (see figure) and used Proposition I.4 to show that they are congruent, whence angle B equals angle C. This 'new' proof is intriguing in that it treats the triangle and its mirror image as separate for purposes of deduction but identical for purposes of conclusion.

When first published, this proof was considered to provide a convincing demonstration that computers can be creative. It was frequently cited as evidence of artificial intelligence (AI), for example in Douglas Hofstadter's remarkable book *Gödel, Escher, Bach: An Eternal Golden Braid*. But Michael Deakin of Monash University, Australia has investigated the matter. He reports an interview in 1981 in the *New Yorker*, in which AI guru Marvin Minsky of MIT stated that he produced the proof himself 'by hand simulation of what a machine might do'.



Amazingly, the ingenious proof was first discovered by the last great Greek geometer, Pappus of Alexandria, working around AD 320. It was derided by the nineteenth-century Oxford mathematician C. L. Dodgson, who imagined the reaction of Euclid: 'Surely that has too much of the Irish bull about it.' Dodgson was none other than Lewis Carroll, author of *Alice's Adventures in Wonderland*.

But what of the 'standard proof' using the bisector of the apex angle? Deakin points out that the reasoning in *A School Geometry*, a book by Hall and Stevens that some of us slaved over long ago, is circular. The proof of Pappus, rediscovered by Minsky and wrongly attributed to a computer, is certainly elegant. But perhaps it is safest to stick with Euclid's original proof. At least one child produced the Pappus proof in an examination and was marked wrong for it.

LOST AND FOUND: THE SECRETS OF ARCHIMEDES

Archimedes of Syracuse was the greatest mathematician of antiquity. He was also a brilliant physicist, engineer and astronomer, famed for founding hydrostatics, for formulating the law of the lever, for designing the helical pump that bears his name, for designing engines of war, and for much more. Generations of children have learned how, upon discovering a way to assay King Hieron's crown, Archimedes ran naked through the streets crying 'Eureka!'

Archimedes estimated the value of π, the ratio of the circumference to the diameter of a circle, to remarkable accuracy, using polygons of 96 sides within and around a circle. And he found the volume of a sphere, showing that it is two-thirds of the volume of the smallest cylinder in which it is contained. He asked that an image of a sphere within a cylinder be inscribed on his tombstone. Centuries later, the Roman orator Cicero found such a carving on a grave in Syracuse.

Many of Archimedes' writings are lost, known to us only through references made to them by later writers. Other works have reached us by a circuitous route: they were translated into Arabic in the ninth century, and from Arabic into Latin during the Renaissance. But some of Archimedes' most important work remained hidden from us until the remarkable discovery of the Archimedes Palimpsest.

Palimpsests were works written on parchment that had been scraped clean of earlier writing. This was common practice in the Middle Ages because vellum was very expensive. In 1906 a prayer book written in the thirteenth century came to light in Constantinople. Upon close examination by the Danish philologist Johan Heiberg, the incompletely erased work underlying the text was recognised as a tenth-century copy of several works of Archimedes, which had been thought to have been lost for ever.

The palimpsest is the only source we have of *The Method of Mechanical Theorems*, in which Archimedes uses infinitesimal quantities to calculate the volumes of various bodies. This method foreshadowed integral calculus, invented independently by Newton and Leibniz nearly two thousand years later.

The Archimedes Palimpsest is the earliest extant manuscript of Archimedes' work; it includes copies of the geometric diagrams that he drew in the sand in the third century BC. It contains several treatises by Archimedes, including *The Method*. The palimpsest was bought at auction in New York in 1998 for $2 million. The German magazine *Der Spiegel* reported that the purchaser was Jeff Bezos, founder of Amazon.com.

The palimpsest has been intensively analysed over the past ten years, using advanced imaging methods. A recent exhibition at the Walters Art Museum in Baltimore, 'Lost and Found: The Secrets of Archimedes', was devoted to the analysis of the palimpsest and to the outcome of the project to study it. All the images and translations are freely available on the Archimedes Palimpsest website (www.archimedespalimpsest.org), providing a treasure trove for scholars of Greek mathematics.

A page of the palimpsest showing older and more recent writing in orthogonal directions (from www.archimedespalimpsest.org).

SUBTERRANEAN TOPOLOGY

The London Underground map is a paragon of design excellence. If you know where you are and where you want to go, it shows you how to get there. But as a map of London it is inaccurate in almost all respects. The beauty of the design, originated by Harry Beck in 1931, is that the key information is kept, and everything else is stripped away.

The Tube map is what mathematicians call a graph. The stations are the *vertices* and the train lines joining them are the *edges*. Interchanges are shown where different lines connect. Distances and directions are distorted in the interests of clarity and simplicity. One of the earliest such graphs was drawn by the renowned Swiss mathematician Leonhard Euler. Euler solved a puzzle called 'The Seven Bridges of Königsberg' by drastically simplifying a map of that city. This made it clear that it is impossible to find a route crossing all seven bridges without recrossing any of them.

Graph theory is a branch of *topology*, the branch of mathematics dealing with continuity and connectivity.

Topology is concerned with properties that remain unchanged under continuous deformations, such as stretching or bending, but not cutting or gluing.

Topology is often called rubber sheet geometry. If a figure such as a triangle is drawn on a sheet of rubber and the sheet is stretched, certain things change but others remain unaltered. For example, the lengths of the sides are changed, but points inside the figure remain inside and points outside remain outside.

In three dimensions, a cube made of plasticine may be distorted continuously into a ball without tearing it, so a cube and a ball are topologically equivalent. In contrast, to make a bagel, or a doughnut with a hole, a ball of plasticine must be torn at some point. So a ball and a bagel are not equivalent.

The formal way of showing that two sets are topologically equivalent is to establish a correspondence or mapping between the two sets, such that nearby points in one are mapped to nearby points in the other. If such a correspondence – called a *homeomorphism* – exists, the two sets are topologically equivalent.

In the familiar school geometry of Euclid, we have straight lines, fixed distances between points and rigid shapes such as triangles. Since topological deformations sacrifice all these, is there anything useful left? Yes: while the London Tube map distorts distances, it preserves the order of stations and the connections between lines, so the traveller knows where to get on and off and where to change trains. It is this topological information that is critical; precise distances are of secondary importance.

The Tube map might be 'corrected' by drawing it on a sheet of rubber and delicately stretching it in places, gradually but continuously, until the stations are all in the correct positions. Or it might be further distorted until the Circle Line became a true circle. But the remarkable success and longevity of the map proves that Harry Beck got it just right all those years ago.

THE EARTH'S VAST ORB

The shape of the earth has been a topic of great interest to savants for millennia. It is an over-simplification to say that the ancients believed the world to be flat. Just to watch a ship appear or disappear over the horizon, or to climb a mountain and notice the changing perspective of a distant island, is enough to provide a hint about the curved nature of the planet. But the prevalent view of earth was one of a vast flat plane surface; only a few people had greater insight.

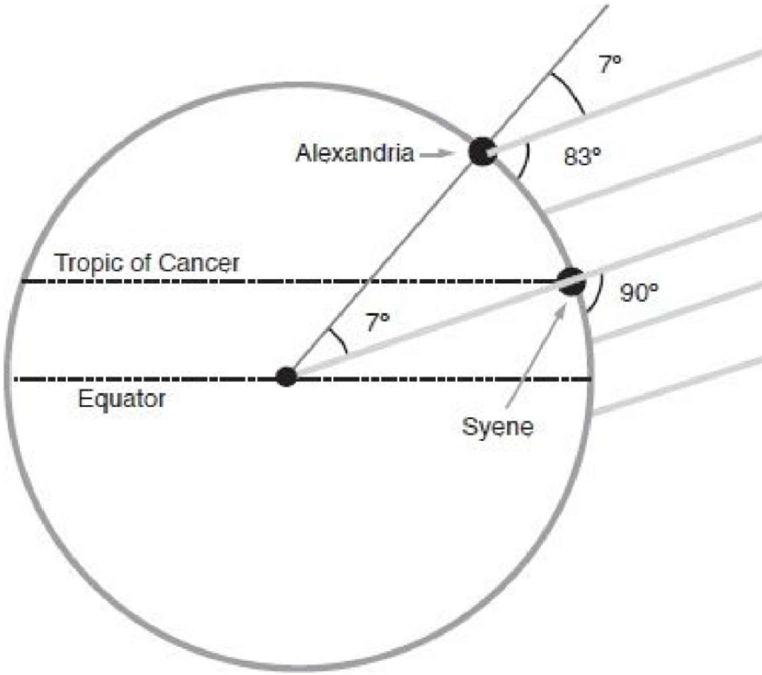
Eratosthenes, a Greek mathematician, astronomer and geographer, went further than others and made an estimate of the earth's circumference that is close to the true value. His method, simple but clever, demonstrates the power of geometric reasoning. Eratosthenes knew that in midsummer the noonday sun was overhead in the city of Syene, modern-day Aswan, on the Tropic of Cancer. Observers there had noticed that at midday the sun's rays reached the bottom of a deep well.

But in Alexandria, when Eratosthenes measured the sun's angle relative to the zenith he obtained a value of about

7 degrees, or 1/50th of a circle. So, if the earth were a sphere, its circumference must be 50 times the distance from Aswan to Alexandria. Eratosthenes knew this distance to be about 800 km (in modern units), so he deduced a value close to 40,000 km for the earth's circumference, a remarkably accurate result.

Eratosthenes was librarian at the Great Library in Alexandria. He devised a system of latitude and longitude and made the first map of the known world with parallels and meridians. He is sometimes called the Father of Geography because he in effect invented the discipline and was the first person to use the term 'geography'.

Eratosthenes was a contemporary and friend of Archimedes. Eratosthenes himself was no mean mathematician. In addition to his use of geometry for map-making and measuring the size of the earth, he devised a simple method, or algorithm, for finding prime numbers, those numbers that cannot be evenly divided into smaller parts. List the natural numbers, 1, 2, 3, …, up to some limit, say 500; then strike out every second number following 2, every third number after 3, every fifth after 5 and so on. Finally, only the prime numbers, 2, 3, 5, 7, 11, …, remain. The procedure, the simplest way of listing small prime numbers, is known as the 'sieve of Eratosthenes'.



The angle of the noonday sun measured in midsummer at Alexandria and Syene (Aswan).

MORE EQUAL THAN OTHERS

In his scientific bestseller, *A Brief History of Time*, Stephen Hawking remarked that every equation he included would halve sales of the book, so he put only one in it, Einstein's equation relating mass and energy, $E = mc^2$. There is no doubt that mathematical equations strike terror in the hearts of many readers. This is regrettable because equations are really just concise expressions of precise statements. They are actually quite user-friendly and more to be loved than feared.

An equation indicates that whatever is to the left side of the 'equals' sign has the same value as whatever is to the right. For Einstein's equation, E is the energy and it is equal to the mass multiplied by the square of the speed of light. In ancient times, equalities were expressed in verbal terms like this. It was Robert Recorde, a Welsh-born mathematician, who introduced the symbol = for 'equals'. In his book *The Whetstone of Witte*, written in 1557, Recorde wrote that he chose this symbol consisting of two parallel lines 'bicause no 2 thynges can be moare equalle'.

The first equation to appear in symbolic form was in Recorde's book, and was

$$14x + 15 = 71.$$

The quantity x is called the unknown (although early writers on mathematics called it 'the thing') and the equation states that if we take 14 times x and add 15 we get 71. How might such an equation arise? Suppose you need a hammer and some nails. A hammer costs €15 and a packet of nails is €14. If you buy a hammer and x packets of nails, the total cost is $14x + 15$, the left side of Recorde's equation. If you have just €71 to spend, how many packets of nails can you buy? The answer is the solution x of the equation.

Recorde explained the transformations that can be made to an equation to 'solve' it – that is, to find the unknown quantity x. In the present case, you can subtract 15 from each side to get a new equation $14x = 56$ and then divide both sides by 14 to get another one, $x = 4$. This is the solution, and you can afford four packets of nails.

Recorde is credited with introducing algebra into England with his book. But he had other talents in addition to mathematics. He was physician to Edward VI and Mary I, and in 1551 was appointed Surveyor of the Mines and Monies of Ireland. Alas, he ended his days in prison, for reasons that are unclear. Perhaps he became embroiled in a religious controversy, or ensnared in some political intrigue. Or perhaps some of the 'Monies of Ireland' went astray.

MATHS AND CAT SCANS

Many lives are saved each year through a synergistic combination of engineering, computing, physics, medical science and mathematics. This combination is CT imaging, or 'computed tomography', which is now an essential tool for medical diagnosis.

The story began in 1895, when Wilhelm Röntgen made the first radiograph using what he called X-rays. These high-energy electromagnetic beams can penetrate body tissues where light cannot reach. Internal organs can be examined non-invasively and abnormalities located with precision. For his trailblazing work, Röntgen was awarded the first Nobel Prize in Physics in 1901.

The power and utility of X-ray imaging has been greatly expanded by combining X-rays with computer systems to generate three-dimensional images of organs of the body. The diagnostic equipment used to do this is called a CT scanner (or CAT scanner). The word 'tomography' comes from the Greek *tomos*, meaning slice, and a CT scan is made by combining X-ray images of cross-sections or slices through the body. From these, a 3-D representation of internal organs can be built up.

Radiologists can use CT scans to examine all the major parts of the body, including the abdomen, chest, heart and head. In a CT scan, multiple X-ray images are taken from different directions. The X-ray data are then fed into a tomographic reconstruction program to be processed by a computer. The image reconstruction problem is essentially a mathematical procedure.

The tissue structure is deduced from the X-rays using a technique first devised by an Austrian mathematician, Johann Radon. He was motivated by purely theoretical interests when, in 1917, he developed the operation now known as the Radon transform. He could not have anticipated the great utility of his work in the practical context of CT. Reconstruction techniques have grown in complexity, but are still founded on Radon's work.

As they pass through the body, X-rays are absorbed to different degrees by body tissues of different optical density. The total attenuation, or dampening, is expressed as a 'line integral', the sum of the absorptions along the path of the X-ray beam. The more tissue along the path, and the denser that tissue, the less intense the beam becomes. The challenge is to determine the patterns of normal and abnormal tissue from the outgoing X-rays.

If the X-ray patterns were uncorrupted, the mathematical conversion to 3-D images would be straightforward. In reality, there is always noise present, and this introduces difficulties: Radon's 'inverse transform' is very unstable and error-prone, so a stable modification of the method, known as 'filtered back-projection', is used. More accurate algorithms have been developed in recent years, and research in this field is continuing.

Applications of tomography are not confined to medicine. The technique is also used in non-destructive materials testing, both in large-scale engineering and in the manufacture of microchips. It is also used to compute ozone concentrations in the atmosphere from satellite data. In addition to CT, there are numerous other volume-imaging techniques. Electron tomography uses a beam of electrons in place of the X-rays, ocean acoustic tomography uses sound waves, and seismic tomography analyses waves generated by earth movements to understand geological structures. All involve intricate mathematical processing to produce useful images from raw data.

BAYES RULES OK

In May 2009, en route from Rio de Janeiro to Paris, Air France flight AF447 crashed into the Atlantic Ocean. Bayesian analysis played a crucial role in the location of the flight recorders and the recovery of the bodies of passengers and crew. What is Bayesian analysis?

Classical and Bayesian statistics interpret probability in different ways. To a classical statistician, or frequentist, probability is the relative frequency of an event. If it occurs on average 3 out of 4 times, he or she will assign to it a probability of 3/4.

For Bayesians, probability is a subjective way to quantify belief, or degree of certainty, based on incomplete information. All probability is conditional and subject to change when more data emerge. If a Bayesian assigns a probability of 3/4, he or she should be willing to offer odds of 3 to 1 on a bet.

Frequentists find it impossible to draw conclusions about once-off events. By using prior knowledge, Bayesian analysis can deal with individual incidents. It can answer questions about events that have never occurred, such as the risk of an asteroid smashing into the earth or the chance of a major international war breaking out over the next ten years.

The danger of a major accident for the Challenger space shuttle was estimated by a Bayesian analysis in 1983 as 1 in 35. The official NASA estimate at the time was an incredible 1 in 100,000. In January 1986, during the 25th launch, the Challenger exploded, killing all seven crew members.

Bayes' Rule transformed probability from a statement of relative frequency into a measure of informed belief. In its simplest form, the rule, devised in the 1740s by the Reverend Thomas Bayes, tells us how to calculate an updated assessment of probability in the light of new evidence. We start with a prior degree of certainty. New data then make this more or less likely.

An advantage of Bayesian analysis is that it answers the questions that scientists are likely to ask. But, despite spectacular successes, Bayesian methods have been the focus of major controversy and their acceptance has been slow and tortuous. Through most of the twentieth century, the academic community eschewed Bayesian ideas and derided practitioners who applied them.

The nub of the controversy was that the probability using Bayesian methods depends on prior opinion. When data are scarce, this yields a subjective rather than an objective assessment. When information is in short supply, subjective opinions may differ widely.

The controversy was long and often bitter, with aspects of a religious war. The opponents vilified each other, generating great hostility between the two camps. The war is now over: frequentists and Bayesians both recognise that the two approaches have value in different circumstances.

Today, Bayesian analysis plays a crucial role in computer science, artificial intelligence, machine learning and language translation. Applications include search and rescue operations, like the recovery of the Air France flight AF447 black box, risk analysis, image enhancement, face recognition, medical diagnosis, setting insurance rates, filtering spam email and much more. Bayesian inference is likely to find many new applications over the coming decades.

PYTHAGORAS GOES GLOBAL

About binomial theorem I'm teeming with a lot o' news,
With many cheerful facts about the square of the hypotenuse.

('I Am the Very Model of a Modern Major-General', from Gilbert and Sullivan, *The Pirates of Penzance*)

Spherical trigonometry has all the qualities we expect of the best mathematics: it is beautiful, useful and fun. It played an enormously important role in scientific development for thousands of years, from ancient Greece through India, the Islamic Enlightenment and the Renaissance, to more modern times. It was crucial for astronomy, and essential for global navigation. Yet it has fallen out of fashion, and is almost completely ignored in modern education.

PYTHAGORAS ON THE SPHERE

Napier's Nifty Rules (here) give ten relationships between the angles and sides of right-angled triangles on the sphere. (Recall that the sides of the triangle are expressed in terms of the angles they subtend at the centre; the radius of the sphere is taken to be unity.) One – and only one – of Napier's Rules relates the three sides of the right-angled triangle. We take the right angle to be *C*, and the side opposite to it to be *c*. Then the rule is

$$\cos c = \cos a \cos b$$

This beautifully simple equation may not appear familiar but, believe it or not, this is just the spherical form of Pythagoras' Theorem!

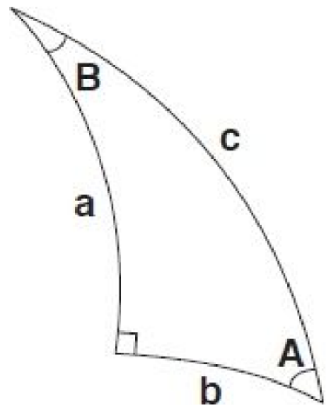
Let us consider a small triangle, so all the sides *a*, *b* and *c* are small quantities. Then we may replace the cosine functions by their first few terms:

$$\cos a \approx (1 - ½\, a^2), \cos b \approx (1 - ½\, b^2), \cos c \approx (1 - ½\, c^2).$$

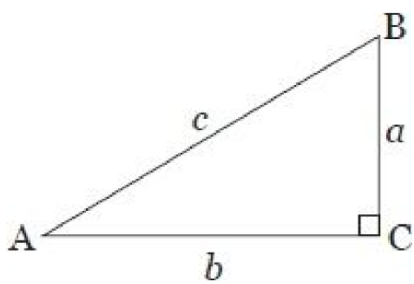
To second-order accuracy, we can write the equation $\cos c = \cos a \cos b$ as

$$c^2 = a^2 + b^2,$$

the usual form of Pythagoras' Theorem that we all know and love.



A spherical right-angled triangle with cos c = *cos* a *cos* b



A plane right-angled triangle with $c^2 = a^2 + b^2$

DOZENAL DIGITS: FROM DIX TO DOUZE

How many fingers has Mickey Mouse? He has three fingers and a thumb on each hand, so eight in all. Thus we may expect Mickey to reckon in octal numbers, with base eight. We use decimals, with ten symbols from 0 to 9 for the smallest numbers and larger numbers denoted by several digits, whose position is significant. Thus, 47 means four tens plus seven units.

But the base ten is divisible only by 2 and 5. There are advantages to having a highly composite base – one with many divisors. The Sumerians and Babylonians used base 60, divisible by 2, 3, 4, 5, 6, 10, 12, 15, 20 and 30. We still use remnants of this sexagesimal system in reckoning time and measuring angles, but base 60 is uncomfortably large for general use.

The duodecimal, or *dozenal*, system with base twelve has been proposed because 12 is divisible by 2, 3, 4 and 6. We need two extra symbols for the numbers ten and eleven, which are less than the base. Let's write them as X and E and call them *dek* and *el*. Then twelve is written 10 and called do (pronounced *doh* and short for a dozen). The system continues with 11, 12, … 1X, 1E and 20 or do one, do two, … do dek, do el and twodo.

Then, jumping in twelves, threedo, fourdo, up to eldo and gro. This gro is short for gross or twelve twelves, written 100. Twelve gro is one mo (twelve cubed or 1728 in decimal). So the decimal number 47 becomes 3E, threedo el or three twelves and el units. And we are currently (in 2016) in the year 1200, or mo twogro.

What advantages has the dozenal system? For one thing, multiplication tables are substantially simpler in dozenal. And many small fractions (one-quarter, one-third, three-quarters, etc.) have a simpler form in this system. So why don't we move from dix to douze? The Dozenal Societies of America and of Britain would favour this. We already have twelve months in a year and twice twelve hours in a day. But a number base change would be seriously disruptive, causing unimaginable confusion.

Computers convert numbers to binary form, using only zeros and ones, and convert the answer back to decimal before presenting it. We are generally oblivious to what goes on under the bonnet, and unconcerned about it.

The chance of the Dozenal Societies persuading us to change to base twelve is about the same as the likelihood of Mickey Mouse converting us to octal. But hold on: How many toes has Mickey got? In the notorious phrase beloved of maths book writers, this is left as an exercise for the student.

HOW LEOPARDS GET THEIR SPOTS

Mathematical models enable us to understand many features of a growing embryo. For example, the patterns of hair colour that give leopards their spots and tigers their stripes can be produced by solving a mathematical equation with different inputs.

The information to form a fully grown animal is encoded in its DNA, so there is a lot of data in a single cell. But there are only about three billion base pairs in DNA and tens of trillions of cells in the body. So minute details like the twists and whorls of a fingerprint cannot be predetermined. Rather, they emerge during embryonic growth as a result of conditions determined by the DNA, following the basic laws of physics and chemistry.

Alan Turing is famous for cracking the Enigma code during World War II, but he was a polymath and worked on many other problems. In 1952, Turing published a paper, 'The chemical basis of morphogenesis', presenting a mechanism of pattern formation. He developed a theory of how the chemistry in the cell influences factors like hair colour.

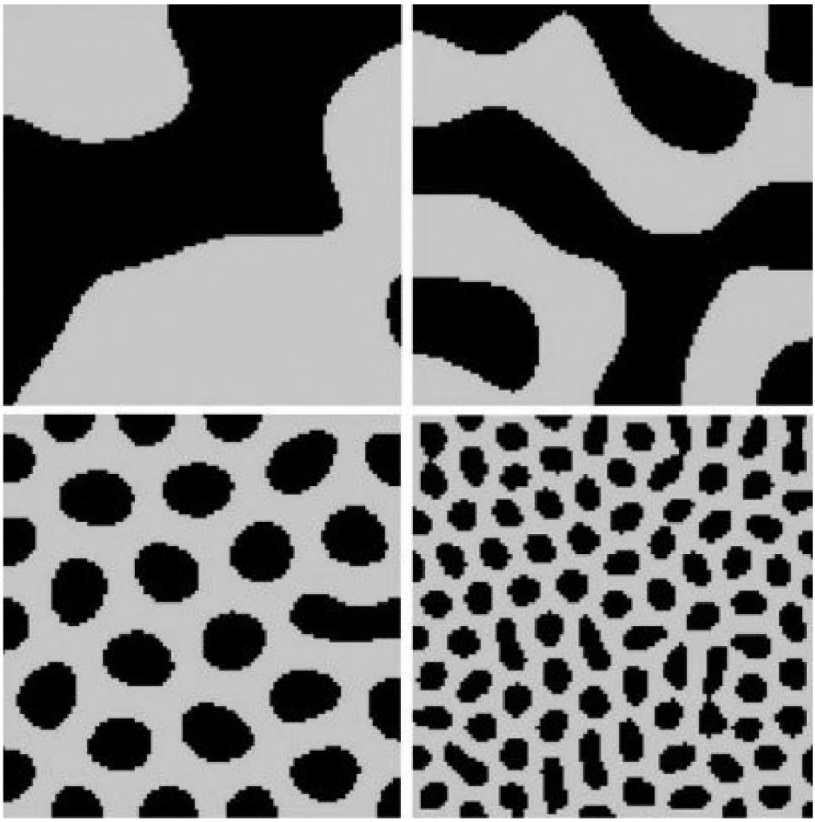
Turing's model included two chemical processes: *reaction*, in which chemicals interact to produce different substances; and *diffusion*, in which local concentrations spread out over time. Suppose we have two chemicals, *A* and *B*, called morphogens, with *A* triggering hair colouring and *B* not doing so. Now suppose that *A* is a catalyst, stimulating production of further morphogens, whereas *B* suppresses production of them. Thus, *A* is called an activator and *B* an inhibitor.

Where *A* is abundant, the hair is black; where *B* is dominant, it is white. Now comes Turing's crucial assumption: the inhibitor *B* spreads out faster than the activator *A*. So *B* fills a circular region surrounding the initial concentration, forming a barrier where concentration of *A* is reduced. The result is an isolated spot of black hair where *A* is plentiful.

What is going on here is a competition between the reaction and diffusion processes. Many reaction–diffusion models have been proposed. The resulting patterns depend on the reaction rates and diffusion coefficients, and a wide range of geometrical patterns of hair colouring can result from this mechanism.

The figure opposite shows the concentration of chemical *A* for varying strengths of reaction and diffusion. High values of *A* are shaded black since hair colouring in these regions is expected to be black. For strong diffusion, the regions are large and striped like a tiger. For weak diffusion, the black hair is confined to spots like the coat of a cheetah.

Many other patterns can be generated by varying the parameters. Thin stripes, like those on an angel fish, or thick stripes, like those of a zebra, can be generated, and clusters of spots found on a leopard's coat can be produced. Biological systems are hugely complex, and simple mathematical models are valuable for elucidating key factors and predicting specific aspects of behaviour.



Concentration of constituent A at equilibrium predicted by the Schnakenberg equations. Black indicates high concentrations. Upper left: γ = 100. Upper right: γ = 400. Lower left: γ = 1600. Lower right: γ = 6400.