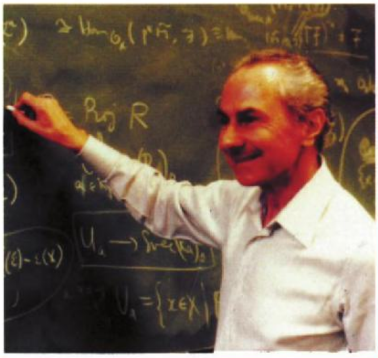

Serge Lang

The Beauty of Doing Mathematics

Three Public Dialogues

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Three Public Dialogues

With 91 Illustrations

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Contents

Who is Serge Lang?

Serge Lang was born in Paris in 1927. He went to school until the 10th grade in the suburbs of Paris, where he lived. Then he moved to the United States. He did two years of high school in California, then entered the California Institute of Technology (Caltech), from which he graduated in 1946. After a year and a half in the American army, he went to Princeton in the Philosophy Department where he spent a year. He then switched to mathematics, also at Princeton, and received his PhD in 1951. He taught at the university and spent a year at the Institute for Advanced Study, which is also in Princeton.

Then he got into more regular positions: Instructor at the University of Chicago, 1953–1955; Professor at Columbia University, 1955–1970. In between, he spent a year as a Fulbright scholar in Paris in 1958.

He left Columbia in 1970. He was Visiting Professor at Princeton in 1970–1971, and Harvard in 1971–1972. Since 1972 he has been a professor at Yale.

Besides math, he mostly likes music. During different periods of his life, he played the piano and the lute.

From 1966 to 1969, Serge Lang was politically and socially active, during a period when the United States faced numerous problems which affected the universities very deeply.

He has also been concerned with the problems of financing the universities, and of their intellectual freedom, threatened by political and bureaucratic interference. As he says, such problems are invariant under ism transformations: socialism, communism, capitalism, or any other ism in the ology.

However, his principal interest has always been for mathematics. He has published 28 books and more than 60 research articles. He received the Cole Prize in the U.S. and Prix Carriere in France.

What does a mathematician do and why?

Prime numbers

16 May 1981

Summary: *The conference started with why, for ten minutes. I do mathematics because I like it. We discussed briefly the distinction between pure mathematics and applied mathematics, which actually intermingle in such a way that it is impossible to define the boundary between one and the other precisely; and also the aesthetic side of mathematics. Then we did mathematics together. I started by defining prime numbers, and I recalled Euclid's proof that there are infinitely many. Then I defined twin primes, (3, 5), (5, 7), (11, 13), (17, 19), etc. which differ by 2. Is there an infinite number of those? No one knows, even though one conjectures that the answer is yes. I gave heuristic arguments describing the expected density of such primes. Why don't you try to prove it? The question is one of the big unsolved problems of mathematics.*

Almost every Saturday, from October to June, the Palais de la Découverte (Science Museum in Paris) traditionally welcomes and presents to the public eminent lecturers in all disciplines.

Thus we were honored to welcome Professor Serge Lang who is a world renowned mathematician, during a brief stay in Paris. He is the author of more than twenty seven mathematical books devoted both to teaching and research. When I invited Serge Lang, I knew him by reputation, and by some of his works. Therefore I had no worry on a purely mathematical plane.

I had only one apprehension, however: would he be a good lecturer? Would he know how to relate to a large non-mathematical public? As I expressed these thoughts to him shortly before the conference, sitting at a café, he told me that a good teacher is not only a specialist in his discipline, but also an actor, sensitive to the public's reactions. He also explained that he was very happy to have this new experience: talk and do mathematics with people who are not students, and show them what mathematics is by "doing mathematics" with them.

"And," he added laughing, "you will see what happens!" I saw! And the conference was a success! Of course, people were surprised to have to participate, and not only to listen; however, after a few minutes, Serge Lang's fiery spirit won them over, and the dialogue went on.

There remained the question of publication. Rather than a technical version, it seemed better to publish the whole conference and the questions that followed it, except for minor changes. Indeed, besides the actual mathematical content, it seemed to me useful to preserve, as far as was possible, the dynamic quality of this conference; the liveliness of the dialogue; and, why not, the actor's performance. Neither those present that day nor popularizers and pedagogues will complain about this.

So I submitted to him a first version, transcribed from the tapes of the lecture. Being very careful and concerned with precision, (which is also one of his personal characteristics), Serge Lang not only checked, but also retyped the whole text. In so doing, he had to familiarize himself with our computer terminals, the only ones which had an American keyboard! So he came every day for more than a week, conciliatory for certain changes, uncompromising as far as the style was concerned, choosing the most appropriate words, and adding a few items, especially on the Riemann Hypothesis, as well as a short bibliography on the topics which were discussed.

Jean Brette
Responsible for the Mathematics
Section of the Palais de la
Découverte

The conference

So, the conference, I think I'm going to talk about things in general for ten minutes, and after that, we'll try to do mathematics together. The talk will be as the title says: "What does a pure mathematician do and why?"

It's very difficult to explain "why" in a general way, and also what we do, in a general way. For example, "mathematics" is a word which is used for a lot of activities which don't have much to do with each other. I am sure that the word means very different things for different people. For instance, you, Madam [*Serge Lang points to a lady in the audience*], what does "mathematics" mean to you?

LADY. The abstraction of numbers, the manipulation of numbers.

SERGE LANG. In fact, one can do mathematics without using numbers at all; as in geometry, or spatial mathematics. It's true that to give you an example of mathematics, as I shall do a little later, I shall use numbers, but in a context which, I think, will be different from the one you are thinking about. And you, sir, what does it mean, "mathematics"?

GENTLEMAN. The manipulation of structures.

SERGE LANG. Yes, but which ones? There are lots of structures which are not mathematical. Mathematics is not just a question of structures. For example, when you do physics, you also manipulate certain structures. In fact, the word "mathematics" is used in many different contexts. You have mathematics as they are done in elementary or high school. You have computer mathematics, applied to problems of communications. If you are into physics or chemistry, you use mathematics to describe the empirical world. But what I want to talk about today is what I will call "pure mathematics", those which are done from a purely aesthetic point of view. To do mathematics like that is very different from studying the empirical world. It's different from describing or classifying the empirical world by means of mathematical models. An experimental scientist makes a choice among many possible models, to find those which fit the empirical world, the world of experiments, in trying to find a system for the world. There are lots of pure mathematics which are not used in studying the empirical world, and which are considered solely for their beauty. And this has been the case forever, for centuries, since there have been civilizations–Arabic, Hindu, whatever. The Greeks did mathematics for the beauty of it.[1]

It is true that some parts of mathematics have their source in the empirical world, but much mathematics is done independently of these sources. This point of view has been expressed by other mathematicians,

[1] Which does not exclude that they also did mathematics which had practical applications. Everyone agrees to include physics, chemistry, biology, under the general heading of "science". To decide whether "pure mathematics" as I have described them should also be placed under this heading is a question of terminology which I don't want to get into now.

and I want to read you something written by other mathematicians, for instance on the relation between doing mathematics as they relate to applied math.

Jacobi, who was a mathematician of the 19th century, wrote in a letter to Legendre:[2]

> I read with pleasure Mr. Poisson's report on my work, and think I can be very satisfied by it . . . but Mr. Poisson should not have reproduced a rather clumsy phrase by Mr. Fourier, who reproached Abel and me for not having preferred to work on heat flow. It is true that Mr. Fourier thought that the principal goal of mathematics was their public utility and their use in explaining natural phenomena. A philosopher like him should have known that the only goal of Science is the honor of the human spirit, and that as such, a question in number theory is worth a question concerning the system of the world.

In an article which appeared in the collection "Great Currents of Mathematical Thought", directed by F. Le Lionnais in 1948, Andre Weil (who is one of the great mathematicians of this century), quoted Jacobi in the following context:

> But if, like Panurge, we ask the oracle questions which are too indiscreet, then the oracle will answer as to Panurge: "Drink!" Advice which the mathematician is only too glad to follow, satisfied that he is to quench his thirst at the very sources of knowledge, satisfied that these sources always gush pure and abundant, while others must have recourse to the muddy paths of a sordid actuality. That if one reproaches him for his arrogant attitude, if one challenges him to engage himself in the actual world, if one asks why he persists on these high glaciers where none but others like him can follow him, he answers with Jacobi: "For the honor of the human spirit!"[3]

OK, that's literature. It's also a pompous style, which does not reflect accurately Jacobi's thoughts. To refer to others, who "must have recourse to the muddy paths of a sordid actuality" is not exactly the same thing as to say that "a question of number theory is worth a question concerning the system of the world". Weil, elsewhere, described in another way his

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[2] No date, stamped 2 July 1830, *Collected Works of Jacobi*, Vol. 1, p. 454.

[3] The original is in French, and very literary French at that:

> Mais si, comme Panurge, nous posons a l'oracle des questions trop indiscretes, l'Oracle nous répondra comme à Panurge: Trinck! Conseil auquel le mathématicien obéit volontiers, satisfait qu'il est de croire étancher sa soif aux sources memes du savoir, satisfait qu'elles jaillissent toujours aussi pures et abondantes, alors que d'autres doivent recourir aux sentiers boueux d'une actualité sordide. Que si on lui fait reproche de la superbe de son attitude, si on le somme de s'engager, si on demande pourquoi il s'obstine en ces hauts glaciers ou nul que ses congénères ne peut le suivre, il répond avec Jacobi: "Pour l'honneur de l'esprit humain!"

own reasons to do mathematics. In an interview published in "Pour la Science" (November 1979, the French version of "Scientific American") he says:

> According to Plutarch, it is a noble ideal to work to make one's name immortal. Ever since I was young, I hoped that my work would have a certain place in the history of mathematics. Is that not a motivation as noble as to try to get a Nobel prize?[4]

So, it's not so much for the honor of the human spirit, it's for the honor of his own spirit. I think rather that one does mathematics because one likes to do this sort of thing, and also, much more naturally, because when you have a talent for something, usually you don't have any talent for something else, and you do whatever you have talent for, if you are lucky enough to have it. I must also add that I do mathematics also because it is difficult, and it is a very beautiful challenge for the mind. I do mathematics to prove to myself that I am capable of meeting this challenge, and win it.

So one does mathematics, but that does not mean people are unhappy if the mathematics they do is sufficiently good to make it in the history books. Of course, all the mathematicians that I know are perfectly happy when they do mathematics at this level. They are happy with the possible honors they may get from it, and they are happy to leave a name in mathematics. But I would not say that they do mathematics specifically for this purpose, that they give themselves to mathematics, whether they be pure or applied.

If I ask you what music means to you, would you answer: "The manipulation of notes"? When one does pure mathematics, one does something quite different from "manipulating". To make clear the reasons behind people doing pure mathematics, from an aesthetic point of view, I have to give you an example. But to show you what mathematics is, if you are not yourself in mathematics, I have difficulties which are analogous to those which I would have if I tried to tell an ancient Japanese, or a Hindu who never had contact with Western civilization, what a Beethoven symphony or a Chopin ballade is like. If you take someone totally foreign to Western culture, and deaf besides, how can you make that person realize what a Beethoven symphony or a Chopin ballade is like? It's impossible. Even if the person is not deaf, and is able to listen, it is still almost impossible if the person has no connection with Western culture, if the person has not heard these pieces several times. Western music is too different from Japanese music, or Hindu music; it is played on different instru-

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[4] In a conference at the International Congress of Mathematics in Helsinki, 1976, reproduced in his *Collected Works* Vol. III, Weil had already touched this theme: "That mankind should be spurred on by the prospect of eternal fame to ever higher achievements is of course a classical theme, inherited from antiquity; we seem to have become less sensitive to it than our forefathers were, although it has perhaps not quite spent its force."

ments, with different orchestrations, with different rhythms, etc. So there is a great difficulty in making somebody understand what it's about. And conversely, Koto or Sitar concerts here in Paris don't happen so often, and affect only a small number of people.

Besides, there is a difficulty which occurs in all aesthetic situations: somebody may like one thing and not another. There are people who like Brahms and don't like Bach; who like Bach and don't like Chopin; who like Chopin and don't like Dowland (an English composer of lute pieces and lute songs at the time of Shakespeare).

How are you going to make somebody understand what a song by Dowland is like, or a Chopin ballade, without making them listen? It's impossible! And it's much easier to make you listen to some music than to make you do mathematics, because to listen to music you are in a passive state. You are taken in by the musical aesthetic, and you let the composer and interpreter take the active part. But to do mathematics, you need a much higher degree of concentration, and a personal effort. Furthermore, to make you do mathematics, I have to find a topic which is sufficiently deep, which is a real topic of mathematics, recognized as such by mathematicians. I can't cheat, but still I have to be able to explain things with words which everybody will understand. There are only very few such topics; and since I have to make a choice, maybe some people will like it and some others won't like it.

The topic has to be sufficiently deep to make you understand why some people will do mathematics all their life, and perhaps will neglect their wives, or husbands, or children, or God knows what. By the way, let me read you two sentences from a letter by Legendre to Jacobi[5] who had just gotten married rather late in life:

> Congratulations for having met a young wife who, after a *rather long* experience, you decided will make you happy forever. You were of a suitable age to get married. A man destined to spend a lot of time working in his office needs a companion who will deal with all the details of housework, and saves her husband from having to worry about those small day to day items which a man is not able to handle.

The sentence has a funny ring, especially in our "liberated" age.

Well, I have been talking in generalities for about ten minutes, that's enough. Now let's do mathematics. In the choice of the subject, I am very restricted, and it was almost necessary to pick a topic having to do with numbers. It concerns prime numbers.

Who has heard of prime numbers? [*Varied reactions and response in the audience.*] Almost everybody, or nobody? Raise your hand. Who has never heard of prime numbers? [*Almost everybody in the audience has heard about prime numbers and knows approximately what the word means.*] For instance, you, Madam, what are the prime numbers?

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[5] Written 30 June 1832, loc. cit p. 460.

LADY. 1, 3, 5, 7 . . .

SERGE LANG. No! These are the odd numbers. I mean the prime numbers, that is 2, 3, 5, 7, 11, 13. What's the next one?

LADY. 17, 19 . . .

SERGE LANG. Very good, you have understood what a prime number is.

LADY. I forgot 2.

SERGE LANG. Yes, you are right. I misunderstood. But it is a general convention that 1 is not called a prime number. So to say that a number is prime means that it is at least equal to 2, and that it is divisible only by itself and by 1.

The number 4 is not prime because $4 = 2 \times 2$.
6 is not prime because $6 = 2 \times 3$.
8 is not prime because $8 = 2 \times 4$.
9 is not prime because $9 = 3 \times 3$.

And so on. As for the prime numbers, we have already listed them up to 19. After that, we find 23, 29, 31, 37 . . .

Now here is a question about prime numbers. Are there infinitely many of them or is there only a finite number of them?

LADY. Yes, infinitely many.

SERGE LANG. Very good. How do you prove it?

LADY. I don't know.

SERGE LANG. [*Pointing to a young man.*] You, do you know how to prove it?

YOUNG MAN. Mathematicians have found millions of them.

SERGE LANG. No, I don't mean finding millions of them, I mean prove that the sequence of prime numbers does not stop.

[*Brouhaha in the audience, various proofs are suggested by some people.*]

SERGE LANG. Are you a mathematician? Yes? OK, I ask the mathematicians in the audience not to say anything. I am not talking here for them. [*Laughter.*] Otherwise, it's cheating.

I say that there are infinitely many prime numbers. This means that the sequence of prime numbers does not stop. And I am going to prove it, because there is a very simple proof, which is also very old, and is attributed to Euclid. Here is how the Greeks did it.

Let's start with a remark. Take any integer, that is a whole number, for instance 38, which I can write as 2×19 where 2 and 19 are prime numbers. Then 38 is a product of these two prime numbers. If I take 144, then I can write

$$144 = 12 \times 12 = 3 \cdot 4 \cdot 3 \cdot 4 = 3 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 2.$$

Again, it's a product of prime numbers, and I have written some of them several times. In any case, I can always express an integer as a product of

AUDIENCE. They are multiples of . . .

SERGE LANG. Shhh! The gentleman over there. [*Hesitations. No answer from the gentleman.*] They have a property, those numbers: they are all divisible by 3. That's a very easy exercise, to show that in every triplet of odd numbers, there is always a multiple of 3. Hence there cannot be a triplet of prime numbers.

AUDIENCE. Except the first one, 3, 5, 7.

SERGE LANG. Except the first, of course, which also has a multiple of 3, but 3 is prime, and there won't be any other.

Let's go back to the twin primes, the couples of primes if you want. Let's try to understand why there should be an infinite number of them. But before, let's go back to the question: how many primes are there less than or equal to x? An approximate formula.

OK, let's take all integers up to x:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, \dots, x.$$

Among these numbers, you have the even numbers and the odd numbers. What does it mean that a number is prime? It means that it is divisible only by itself and 1. Therefore, if a number if prime, it is certainly not even.

AUDIENCE. Except 2.

SERGE LANG. Of course, except 2. Now, if I go up to x, how many odd numbers are there?

SEVERAL VOICES IN THE AUDIENCE. Half of them.

SERGE LANG. Approximately half. That's right, $x/2$. It's a certain fraction of x. The number of primes less than or equal to x will be a certain fraction times x. And this fraction will depend on x. It is this fraction which we are trying to determine.

All right, so among all the integers 1, 2, 3 up to x, there will be approximately half of them which will be odd, so not divisible by 2. Among the odd numbers, how many will not be divisible by 3?

AUDIENCE. One third.

SERGE LANG. No, one third is divisible by 3 and two thirds won't be divisible by 3. OK? Let's write 2/3 in the form $(1 - 1/3)$. Now among the remaining ones, how many will not be divisible by 5?

A VOICE IN THE AUDIENCE. $1 - 1/5$.

SERGE LANG. Are you a mathematician? Yes? Then shut up! It's cheating. It's not nice. Among the remaining ones, how many are there which are not divisible by the next prime number?

AUDIENCE. $1 - 1/7$.

SERGE LANG. Good, and then finally to find the numbers which are prime, what do we need? We need that they should not be divisible by any prime number, from 2 to . . . somewhere. Thus we are led to take the product

$$\frac{1}{2}(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})\cdots$$

which must go up to where?

AUDIENCE. Up to the last prime number before x.

SERGE LANG. Yes, but one can do better than that. Anyhow, at worst, it will be the product

$$\text{product of all factors } (1 - \frac{1}{p})$$

where p goes up to x. This will be approximately the fraction of x which gives the fraction of all numbers which are prime.

Now, in fact, I don't need to go up to x. I need to go only up to the square root of x, which is denoted by $\sqrt{x}$. Because suppose that a number which is smaller than x and is not prime, is divisible by some prime bigger than $\sqrt{x}$. Then it is necessarily divisible by a prime number smaller than $\sqrt{x}$.[6] Hence we can eliminate such a number when we have met the smallest of its prime factors. But when x is large, and when p is between $\sqrt{x}$ and x, the term $(1 - 1/p)$ is very close to 1. One can show that the product taken over all p with $\sqrt{x} \leqq p \leqq x$ is close to $1/2$. To simplify the formulas, I shall continue to write the product with $(1 - 1/p)$ for all $p \leqq x$. To have a better approximation, or the best possible approximation, I would anyhow have to multiply the product by a constant which is hard to determine, and which reflects relations which are more hidden than the relation which we have just described.

Here I count approximately, and I am led to consider that product. It gives approximately the fraction of x giving the number of primes less than or equal to x. This fraction of x is rather mysterious, but still, it gives some idea of what's happening. For instance, is this fraction constant? Clearly not. The further we go, the smaller it becomes. If I take x very large, the fraction will be small. How fast it becomes small is not clear. It's not at all clear how this product behaves. And now, I am stuck. I will give you some answer later, but I won't be able to prove it because it would get too technical.

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[6] I give the details of this assertion. Let N be less than or equal to x. Suppose that N is a product, $N = pN'$ with p prime greater than $\sqrt{x}$. Then $N' = N/p$, and N' is smaller than $\sqrt{x}$. If q is a prime factor of N', then q is smaller than $\sqrt{x}$ and is also a factor of N.