The BEST WRITING on MATHEMATICS

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Mircea Pitici, Editor

FOREWORD BY
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Foreword: Recreational Mathematics

Freeman Dyson

Hobbies are the spice of life. Recreational mathematics is a splendid hobby which young and old can equally enjoy. The popularity of Sudoku shows that an aptitude for recreational mathematics is widespread in the population. From Sudoku it is easy to ascend to mathematical pursuits that offer more scope for imagination and originality. To enjoy recreational mathematics you do not need to be an expert. You do not need to know the modern abstract mathematical jargon. You do not need to know the difference between homology and homotopy. You need only the good old nineteenth-century mathematics that is taught in high schools, arithmetic and algebra and a little geometry. Luckily for me, the same nineteenth-century mathematics was all that I needed to do useful calculations in theoretical physics. So, when I decided to become a professional physicist, I remained a recreational mathematician. This foreword gives me a chance to share a few of my adventures in recreational mathematics.

The articles in this collection, *The Best Writing on Mathematics 2011*, do not say much about recreational mathematics. Many of them describe the interactions of mathematics with the serious worlds of education and finance and politics and history and philosophy. They are mostly looking at mathematics from the outside rather than from the inside. Three of the articles, Doris Schattschneider's piece about Maurits Escher, the Fergusons' piece on mathematical sculpture, and Dana Mackenzie's piece about packing a circle with circles, come closest to being recreational. I particularly enjoyed those pieces, but I recommend the others too, whether they are recreational or not. I hope they will get you interested and excited about mathematics. I hope they will tempt a few of you to take up recreational mathematics as a hobby.

I began my addiction to recreational mathematics in high school with the fifty-nine icosahedra. *The Fifty-Nine Icosahedra* is a little book published in 1938 by the University of Toronto Press with four authors, H.S.M. Coxeter, P. DuVal, H.T. Flather, and J. F. Petrie. I saw the title in a catalog and ordered the book from a local bookstore. Coxeter was the world expert on polyhedra, and Flather was the amateur who made models of them. The book contains a complete description of the fifty-nine stellations of the icosahedron. The icosahedron is the Platonic solid with twenty equilateral triangular faces. A stellation is a symmetrical solid figure obtained by continuing the planes of the twenty faces outside the original triangles. I joined my school friends Christopher and Michael Longuet-Higgins in a campaign to build as many as we could of the fifty-nine icosahedra out of cardboard and glue, with brightly colored coats of enamel paint to enhance their beauty. Christopher and Michael both went on later to become distinguished scientists. Christopher, now deceased, was a theoretical chemist. Michael is an oceanographer. In 1952, Michael took a holiday from oceanography and wrote a paper with Coxeter giving a complete enumeration of higher-dimensional polytopes. Today, if you visit the senior mathematics classroom at our old school in England, you will see the fruits of our teenage labors grandly displayed in a glass case, looking as bright and new as they did seventy years ago.

My favorites among the stellations are the twin figures consisting of five regular tetrahedra with the twenty vertices of a regular dodecahedron. The twins are mirror images of each other, one right-handed and the other left-handed. These models give to anyone who looks at them a vivid introduction to symmetry groups. They show in a dramatic fashion how the symmetry group of the icosahedron is the same as the group of 120 permutations of the five tetrahedra, and the subgroup of rotations without reflections is the same as the subgroup of 60 even permutations of the tetrahedra. Each of the twins has the symmetry of the even permutation subgroup, and any odd permutation changes one twin into the other.

Another book which I acquired in high-school was *An Introduction to the Theory of Numbers* by G. H. Hardy and E. M. Wright, a wonderful cornucopia of recreational mathematics published in 1938. Chapter 2 contains the history of the Fermat numbers, $F_n = 2^{2^n} + 1$, which Fermat conjectured to be all prime. Fermat was famously wrong. The first four Fermat numbers are prime, but Euler discovered in 1732 that $F_5 = 2^{32} + 1$ is divisible by 641, and Landry discovered in 1880 that lenge to readers of this volume to find it. To find it requires no expert knowledge. All that you have to do is to study the partitions and smallest parts for a few small values of n, and make an inspired guess at the property that divides them equally. A second challenge is to prove that the guess actually works. To succeed with the second challenge probably requires some expert knowledge, since I am asking you to beat George Andrews at his own game.

My most recent adventure in recreational mathematics is concerned with the hypothesis of Decadactylic Divinity. Decadactylic is Greek for ten-fingered. In days gone by, serious mathematicians were seriously concerned with theology. Famous examples were Pythagoras and Descartes. Each of them applied his analytical abilities to the elucidation of the attributes of God. I recently found myself unexpectedly following in their footsteps, applying elementary number theory to answer a theological question. The question is whether God has ten fingers. The evidence in favor of a ten-fingered God was brought to my attention by Norman Frankel and Lawrence Glasser. I hasten to add that Frankel and Glasser were only concerned with the mathematics, and I am solely responsible for the theological interpretation. Frankel and Glasser were studying a sequence of rational approximations to π discovered by Derek Lehmer. For each integer k, there is a rational approximation $[R_1(k)/R_2(k)]$ to π, with numerator and denominator defined by the identity

$$R_1(k) + R_2(k)\pi = \sum_{m=1}^{\infty} [(m!)^2/(2m)!]2^m m^k.$$

The right-hand side of this identity has interesting analytic properties which Frankel and Glasser explored. The approximations to π that it generates are remarkably accurate, beginning with 3, 22/7, 22/7, 335/113, for $k = 1, 2, 3, 4$. Frankel and Glasser calculated the first hundred approximations to high accuracy, and found to their astonishment that the kth approximation agrees with the exact value of π to roughly k places of decimals. I deduced from this discovery that God must calculate as we do, using arithmetic to base ten, and it was then easy to conclude that He has ten fingers. It seemed obvious that no other theological hypothesis could account for the appearance of powers of ten in the approximations to such a transcendental quantity as π.

Unfortunately, I soon found out that my theological breakthrough was illusory. I calculated precisely the magnitude of the error of the *k*th Lehmer approximation for large *k*, and it turned out that the error does not go like 10^{-k} but like Q^{-k}, where $Q = 9.1197\ldots$ is a little smaller than ten. For large *k*, the approximation is in fact only accurate to $0.96k$ places of decimals, where 0.96 is the logarithm of *Q* to base ten. *Q* is defined as the absolute value of the complex number $q = 1 + (2\pi i/\ln(2))$. When we are dealing with complex numbers, the logarithm is a many-valued function. The logarithm of 2 to the base 2 has many values, beginning with the trivial value 1. The first nontrivial value of $\log_2(2)$ is q. This is the reason why q determines the accuracy of the Lehmer approximations. This calculation demonstrates that God does not use arithmetic to base ten. He uses only fundamental units such as π and $\ln(2)$ in the design of His mathematical sensorium. The number of His fingers remains an open question.

Two of these recreational adventures were from my boyhood and two from my old age. In between, I was doing mathematics in a more professional style, finding problems in the understanding of nature where elegant nineteenth-century mathematics could be usefully applied. Mathematics can be highly enjoyable even when it is not recreational. I hope that the articles in this volume will spark readers' interest in digging deeper into some aspect of mathematics, whether it is puzzles and games, history of mathematics, mathematics education, or perhaps studying for a professional degree in mathematics. The joys of mathematics are to be found at all levels of the game.

Introduction

Mircea Pitici

This new volume in the series of *The Best Writing on Mathematics* brings together a collection of remarkable texts, originally printed during 2010 in publications from several countries. A few exceptions from the strict timeframe are inevitable, due to the time required for the distribution, surveying, reading, and selection of a vast literature, part of it coming from afar.

Over the past decade or so, writing about mathematics has become a genre, with its own professional practitioners—some highly talented, some struggling to be relevant, some well established, some newcomers. Every year these authors, considered together, publish many books. This abundance is welcome, since writing on mathematics realizes the semantic component of a mental activity too often identified to its syntactic-procedural mode of operation. The appropriation to the natural language of meaningful intricacies latent in symbolic formulas opens up paths toward comprehending the abstraction that characterizes mathematical thinking and some mathematical notions; it also offers unlimited expressive, imaginative, and cognitive possibilities. In the second part of the introduction, I mention the books on mathematics that came to my attention last year; the selection in this volume concerns mostly pieces that are not yet available in book form—either articles from academic journals or good writing in the media that goes unobserved or is forgotten after a little while. *The Best Writing on Mathematics* reflects the literature on mathematics available out there in myriad publications, some difficult to consult even for people who have access to exceptional academic resources. In editing this series I see my task as restitution to the public, in convenient form, of excellent writing on mathematics that deserves enhanced reception beyond the initial publication. By editing this series I also want to make widely available good texts about mathematics that otherwise would be lost in the deluge of information that surrounds us. The content of each volume builds itself up to a point; I only give it a coherent structure and present it to the reader. That means that every year some prevailing themes will be new, others will reappear.

Most readers of this book are likely to be engaged with mathematics in some way, at least by being curious about it. But most of them are inevitably engaged with only a (small) part of mathematics. That is true even for professional mathematicians, with rare exceptions. Mathematics has far-reaching tentacles, in pure research branches as well as in mundane applications and in instructional contexts. No wonder the stakeholders in the metamorphosis of mathematics as a social phenomenon can hardly be well informed about the main ideas and developments in all the different aspects connected to mathematics. Solipsism among mathematicians is surely not as common as the general public assumes it is; yet specialization is widespread, with many professionals finding it difficult to keep abreast of developments beyond their strict areas of interest. By making this volume intentionally eclectic, I aim to break some of the barriers laid by intense specialization. I hope that the enterprise makes it easier for readers, insiders and outsiders, to identify the main trends in thinking about mathematics in areas unfamiliar to them.

Anthologies of writings on mathematics have a long—if sparse and irregular—history. Countless volumes of contributed collections in particular fields of mathematics exist but, to my knowledge, only a handful of anthologies that include panoramic selections across multiple fields. Soon after the Second World War, William Schaaf edited *Mathematics, Our Great Heritage*, which included writings by G. H. Hardy, George Sarton, D. J. Struik, Carl G. Hempel, and others. A few years later James Newman edited *The World of Mathematics* in four massive tomes, a collection widely read for decades by many mathematicians still active today. In parallel, in francophone countries circulated Le Lionnaise's *Les Grand Courants de la Pensée Mathématique*, translated into English only several years ago. During the 1950s and 1960s a synthesis in three volumes edited by A. D. Aleksandrov, A. N. Kolmogorov, and M. A. Lavrentiev, including contributions by Soviet authors, was translated and widely circulated. Very few similar books appeared during the last three decades of the twentieth century; notable was *Mathematics Today,* edited by Lynn Arthur Steen in the late 1970s. In the present century the pace quickened; several excellent volumes were published, starting with

Mathematics: Frontiers and Perspectives edited in 2000 by Vladimir Arnold, Michael Atiyah, Peter Lax, and Barry Mazur, followed by the voluminous tomes edited by Björn Engquist and Wilfried Schmid in 2001, Timothy Gowers in 2008, as well as the smaller collections edited by Raymond Ayoub in 2004 and Reuben Hersh in 2006 (for complete references, see the list of works mentioned at the end of this introduction). *The Best Writing on Mathematics* builds on this illustrious tradition, capitalizing on an ever more interconnected world of ideas and benefiting from the regularity of yearly serialization.

Overview of the Volume

The texts included in this volume touch on many topics related to mathematics. I gave up the thematic organization adopted in the first volume, since some of the texts are not easy to categorize and some themes would have been represented by only one or very few pieces.

Underwood Dudley argues that mathematics beyond the elementary notions is the best preparation for reasoning in general and that most people value it primarily for that purpose, not for its immediate practicality.

Dana Mackenzie describes the overt and the hidden properties of the Apollonian gasket, a configuration of infinitely nested tangent circles akin to a fractal.

Rik van Grol tells the story of finding the optimal number of steps that solve scrambled Rubik's cubes of different sizes—starting with the easy cases and going to the still unsolved ones.

Andrew Schultz writes on the friendly professional interactions that shape the career of a mathematician, from learning the ropes as a graduate student to becoming an accomplished academic.

In a polemical reply to a text we selected in last year's volume of this series (Gowers and Nielsen), Melvyn Nathanson argues that the most original mathematical achievements are distinctively individual, rather than results of collaboration.

Martin Campbell-Kelly meditates on the long flourishing popularity and recent demise of mathematical tables.

Reuben Hersh ponders on the post–World War II abundance of Jewish mathematicians at American universities, in contrast to the pale prewar representation of Jews among American mathematicians.

A captivating first-person account of research in pure mathematics, with insights into the collaborative work of mathematicians, is given by Shing-Tung Yau (with Steve Nadis) in *The Shape of Inner Space*. Alexandre Borovik gives a rich exploration of cognitive aspects of mathematical intuition, based on many biographical accounts, in *Mathematics under the Microscope*. A book difficult to categorize, touching on psychological, cognitive, and historical aspects of mathematical thinking, in an original manner, is *Mathematical Reasoning* by Raymond Nickerson. Also interdisciplinary is *Grammar, Geometry, and Brain* by Jens Erik Fenstad. Two recent books of interviews also offer glimpses into the rich life experiences of mathematicians: *Recountings: Conversations with MIT Mathematicians* edited by Joel Segel and *Creative Minds, Charmed Lives* edited by Yu Kiang Leong—as does Reuben Hersh and Vera John-Steiner's *Loving and Hating Mathematics*.

Among several books that combine excellent expository writing with mathematics proper are *Charming Proofs* by Claudi Alsina and Roger Nelsen, *Creative Mathematics* by Alan Beardon, *Making Mathematics Come to Life* by O. A. Ivanov, *A Mathematical Medley* by George Szpiro, and the latest volume (number 8) of *What's Happening in the Mathematical Sciences* by Dana Mackenzie.

Many notable books on the history of mathematics are being published these days; some of the most recent titles and their authors are the following. Benjamin Wardhaugh has written a brief but badly needed guide called *How to Read Historical Mathematics*. Two excellent books on the history of geometry are *Revolutions of Geometry* by Michael O'Leary and *Geometry from Euclid to Knots* by Saul Stahl. Other accounts of particular mathematical branches, times, or personalities are *The Birth of Numerical Analysis* edited by Adhemar Bultheel and Ronald Cools, *The Babylonian Theorem* by Peter Rudman, *The Pythagorean Theorem: The Story of Its Power and Beauty* by Alfred Posamentier, *Hidden Harmonies: The Lives and Times of the Pythagorean Theorem* by Robert and Ellen Kaplan, *Galileo* by J. L. Heilbron, *Defending Hypatia* by Robert Goulding, *Voltaire's Riddle* by Andrew Simoson, *The Scientific Legacy of Poincare* edited by Éric Charpentier, Étienne Ghys, and Annick Lesne, *Emmy Noether's Wonderful Theorem* by Dwight Neuenschwander, and *Studies in the History of Indian Mathematics* edited by C. S. Seshadri. Thematically more encompassing are *Mathematics and Its History* by John Stillwell, *An Episodic History of Mathematics* by Steven Krantz, *Duel at Dawn* by Amir Alexander, and *History of Mathematics: High-*

ways and Byways by Amy Dahan-Dalmédico and Jeanne Peiffer. The mathematical avatars of representative politics since Ancient Greece are wonderfully narrated by George Szpiro in *Numbers Rule*. Joseph Mazur, in *What's Luck Got to Do with It?* also takes a historical perspective, by telling the story of the mathematics involved in gambling. Alex Bellos gives a refreshingly informal tour of the history of elementary mathematics in a new edition of *Here's Looking at Euclid*. Two histories of sciences with substantial material dedicated to mathematics are *Kinematics: The Lost Origins of Einstein's Relativity* by Alberto Martinez and *Technology and Science in Ancient Civilizations* by Richard Olson. Among editions of old mathematical writings are *Pappus of Alexandria: Book 4 of the Collection* edited by Heike Sefrin-Weis and the fourth volume of Lewis Carroll's pamphlets, containing *The Logic Pamphlets*, an edition by Francine Abeles.

Among recent books of philosophical issues in mathematics I note *There's Something about Gödel* by Francesco Berto, *Roads to Infinity* by John Stillwell, the reprinting of the *Philosophy of Mathematics* by Charles S. Peirce, and several new volumes on logic: *The Evolution of Logic* by W. D. Hart, *Logic and Philosophy of Mathematics in the Early Husserl* by Stefania Centrone, *Logic and How It Gets that Way* by Dale Jacquette, and the collection of classic essays *Thinking about Logic* edited by Steven Cahn, Robert Talisse, and Scott Aikin. In *Mathematics and Reality*, Mary Leng is concerned with the ontology of mathematical objects and its implications for the status of mathematical practice.

Several recent books on mathematics education that came to my attention are worth mentioning. *Theories of Mathematics Education* edited by Bharath Sriraman and Lyn English is a synthesis of contributions by foremost researchers in the field. Others, concerned with particular themes, are *Proof in Mathematics Education* by David Reid and Christine Knipping, *Mathematical Action and Structures of Noticing* edited by Stephen Lerman and Brent Davis (honoring the work of John Mason, a contributor to this volume), *Culturally Responsive Mathematics Education* edited by Brian Greer and collaborators, and *Winning the Math Wars* by Martin Abbott and collaborators. A detailed and authoritative presentation of inquiry-based teaching of mathematics is *The Moore Method* by Charles Coppin, Ted Mahavier, Lee May, and Edgar Parker. A new volume (number VII) in the series of *Research in Collegiate Mathematics* was edited recently by Fernando Hitt, Derek Holton, and Patrick Thompson. Volumes concerned with education generally but with excellent chapters on mathematical aspects are *Instructional Explanations in the Disciplines* edited by Mary Kay Stein and Linda Kucan, *Beauty and Education* by Joe Winston, and *Visualization in Mathematics, Reading, and Science Education* edited by Linda Phillips, Stephen Norris, and John Macnab. The National Council of Teachers of Mathematics has continued the publication of several series, listed below under the authors Marian Small, Michael Shaughnessy, Mark Saul, Jane Schielack, and Frank Lester.

In *Numbers Rule Your World*, Kaiser Fung (who also maintains the Junk Charts blog: http://junkcharts.typepad.com/junk_charts/) presents in detail several cases of sensible statistical thinking in engineering, health, finance, sports, and other walks of life. Related recent titles are *The Pleasures of Statistics: The Autobiography of Frederick Mosteller*, *What Is p-Value anyway?* by Andrew Vickers, and *Probabilities* by Peter Olofsson.

In applied mathematics several remarkable volumes became available or were reissued lately. Colin Clark's *Mathematical Bioeconomics* is now in print in its third edition, offering a wide range of examples in resource management and environmental studies, all in an easy-to-read presentation. Mohammed Farid edited a massive volume covering 37 major topics concerning properties of food, under the title *Mathematical Modeling of Food Processing*. A group including Warren Hare and collaborators wrote *Modelling in Healthcare*, a substantial report on data collection, mathematical modeling, and interpretation in healthcare institutions. Robert Keidel authored a refreshing visual-conceptual view of strategic management in *The Geometry of Strategy*. Closer to mathematics proper is *Mrs. Perkins's Electric Quilt* by Paul Nahin, an introduction to mathematical physics written with much verve. *Once Before Time* by Martin Bojowald and *What If the Earth Had Two Moons?* by Neil Comins underlie the crucial role of mathematics in understanding the meaning of physical laws governing the observable universe. Highly readable, appealing to basic notions of randomness and complexity, is *Biology's First Law* by Daniel McShea and Robert Brandon. In *The Mathematics of Sex*, Stephen Ceci and Wendy Williams zoom in on gender aspects of education, social inequality, and public policy. Two books with a wide span of applications are *Critical Transitions in Nature and Society* by Marten Scheffer and *Disrupted Networks* by Bruce West and Nicola Scafetta. David Easley and Jon Kleinberg take an interdisciplinary approach in *Networks, Crowds, and Markets*. And a lively account of the use, overuse, and abuse of mathematical methods in the financial industry is given by Scott Patterson in *The Quants*.

A splendid illustration of mathematical methods in charting complex data sets is the *Atlas of Science* by Katy Börner. Also visual, at the intersection between mathematics, arts, and philosophy, is *Quadrivium*, edited by John Martineau and others.

Popularizing mathematics is well represented by several titles. In *Our Days Are Numbered,* Jason Brown writes on common occurrences of many elementary (and a few less-than-elementary) mathematical notions—as do Ian Stewart in *Cows in the Maze and Other Mathematical Explorations* (pointing to a wealth of online resources), John D. Barrow in *100 Essential Things You Didn't Know You Didn't Know*, Marcus du Sautoy in *Symmetry: A Journey into the Patterns of Nature* (an attractive blend of history, games, and storytelling), and Jamie Buchan in *Easy as Pi*. A sort of dictionary of number properties is *Number Freak* by Derrick Niederman. First-hand accounts of learning mathematics are *Hot X: Algebra Exposed* by Danica McKellar, *The Calculus Diaries* by Jennifer Ouellette, and *Dude, Can You Count?* by Christian Constanda. Finally in this category, a must-read antidote to blindly taking for granted numerical arguments in public discourse is *Proofiness* by Charles Seife.

In a separate register, it is worth mentioning a moving novel of love and loneliness on a mathematical metaphor, *The Solitude of Prime Numbers* by the young Italian physicist Paolo Giordano.

I conclude this summary overview of the vast number of books on mathematics published last year by mentioning the recent publication of the first issue (January 2011) of a new periodical, the *Journal of Humanistic Mathematics* (thanks to Fernando Gouvêa for drawing my attention to this journal).

This is not a critical review of the recent literature on mathematics and surely not a comprehensive list. Other books, not mentioned here, can be found on the list of notable texts at the end of the volume; perhaps still others have escaped my survey. Authors and publishers can make sure I know about a certain title by contacting me at the address provided just before the list of works mentioned in this introduction.

A Few Internet Resources

The sheer number of excellent websites on mathematics (including those hosted by educational institutions and individual mathematicians) makes any attempt to compile a comprehensive inventory look quixotic. Here

I only suggest a handful of online mathematical resources that caught my attention over the past year that I did not mention in the introduction to the previous volume.

I begin with a few specialized sites. The *Why Do Math* website (http://www.whydomath.org/) is particularly original in highlighting the success stories of applied mathematics in science, society, and everyday life. A multilanguage site profiled on number sequences is the On-Line Encyclopedia of Integer Sequences (http://oeis.org/). The Geometric Dissections site (http://home.btconnect.com/GavinTheobald/Index.html) obviously needs no further description. Neither does the World Federation of National Mathematics Competitions (http://www.amt.edu.au/wfnmc/). And an excellent website for questions and answers is the Math Overflow site (http://mathoverflow.net/).

Among the many good blogs on mathematics, I mention a few very active ones: the Math-blog (http://math-blog.com/), the Computational Complexity blog (http://blog.computationalcomplexity.org/), Wild about Math! blog (http://wildaboutmath.com/), and the Thinkfinity blog (http://www.thinkfinity.org/). A number of prestigious mathematicians maintain active blogs, with a rich network of links to other blogs—including Timothy Gowers (http://gowers.wordpress.com/), Terence Tao (http://terrytao.wordpress.com/), and Richard Lipton (http://rjlipton.wordpress.com/), to mention just a few.

Some good instructional/educational sites, among many, are the Khan Academy (http://www.khanacademy.org/), Mathematically Sane (http://mathematicallysane.com/), and the Reasoning Mind (http://www.reasoningmind.org/).

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What Is Mathematics For?

UNDERWOOD DUDLEY

A more accurate title is "What is mathematics education for?" but the shorter one is more attention-getting and allows me more generality. My answer will become apparent soon, as will my answer to the subquestion of why the public supports mathematics education as much as it does.

So that there is no confusion, let me say that by "mathematics" I mean algebra, trigonometry, calculus, linear algebra, and so on: all those subjects beyond arithmetic. There is no question about what arithmetic is for or why it is supported. Society cannot proceed without it. Addition, subtraction, multiplication, division, percentages: though not all citizens can deal fluently with all of them, we make the assumption that they can when necessary. Those who cannot are sometimes at a disadvantage.

Algebra, though, is another matter. Almost all citizens can and do get through life very well without it, after their schooling is over. Nevertheless it becomes more and more pervasive, seeping down into more and more eighth-grade classrooms and being required by more and more states for graduation from high school. There is unspoken agreement that everyone should be exposed to algebra. We live in an era of universal mathematical education.

This is something new in the world. Mathematics has not always loomed so large in the education of the rising generation. There is no telling how many children in ancient Egypt and Babylon received training in numbers, but there were not many. Of course, in ancient civilizations education was not for everyone, much less mathematical education. Literacy was not universal, and I suspect that many who could read and write could not subtract or multiply numbers. The ancient Greeks, to their glory, originated real mathematics, but they did not do it to fill classrooms with students learning how to prove theorems. Compared to them, the ancient Romans were a mathematical blank. The Arab scholars who started to develop algebra after the fall of Rome were doing it for their own pleasure and not as something intended for the masses. When Brahmagupta was solving Pell's equation a millennium before Pell was born, he did not have students in mind.

Of course, you may think, those were the ancients; in modern times we have learned better, and arithmetic at least has always been part of everyone's schooling. Not so. It may come as a surprise to you, as it did to me, that arithmetic was not part of elementary education in the United States in the colonial period. In *A History of Mathematics Education in the United States and Canada* (National Council of Teachers of Mathematics, 1970), we read

> Until within a few years no studies have been permitted in the day school but spelling, reading, and writing. Arithmetic was taught by a few instructors one or two evenings a week. But in spite of the most determined opposition, arithmetic is now being permitted in the day school.

Opposition to arithmetic! *Determined* opposition! How could such a thing be? How could society function without a population competent in arithmetic? Well, it did, and it even thrived. Arithmetic was indeed needed in many occupations, but those who needed it learned it on the job. It was a system that worked with arithmetic then and that can work with algebra today.

Arithmetic did make it into the curriculum, but, then as now, employers were not happy with what the schools were turning out. Patricia Cline Cohen, in her estimable *A Calculating People: The Spread of Numeracy in Early America* (U. of Chicago Press, 1983; Routledge paperback, 1999) tells us that

> Prior to this act [1789] arithmetic had not been required in the Boston schools at all. Within a few years a group of Boston businessmen protested to the School committee that the pupils taught by the method of arithmetic instruction then in use were totally unprepared for business. Unfortunately, the educators in this case insisted that they were doing an adequate job and refused to make changes in the program.

Both sides were right. It is impossible to prepare everyone for every possible occupation, and it is foolish to try. Hence many school leavers will be unprepared for many businesses. But mathematics teachers, then as now, were doing an adequate job.

A few years ago I was at a meeting that had on its program a talk on the mathematics used by the Florida Department of Transportation. There is quite a bit. For example, the Florida DoT uses Riemann sums to determine the area of irregular plots of land, though it does not call the sums that. After the talk I asked the speaker what mathematical preparation the DoT expects in its new hires. The answer was, none at all. The DoT has determined that it is best for all concerned to assume that the background of its employees includes nothing beyond elementary arithmetic. What employees need, they can learn on the job.

There seems to be abroad in the land the delusion that skill in algebra is necessary in the world of work and in everyday life. In *Moving Beyond Myths* (National Academy of Sciences, 1991) we see

> *Myth:* Most jobs require little mathematics.
> *Reality:* The truth is just the opposite.

I looked very hard in the publication for evidence for that assertion, but found none. Perhaps the NAS was equating mathematics with arithmetic. Many people do this, as I have found in asking them about how, or if, they use mathematics. Almost always, the "mathematics" they tell me about is material that appears in the first eight grades of school.

Algebra, though, is mentioned explicitly in *Everybody Counts* (National Research Council, 1989):

> Over 75 percent of all jobs require proficiency in simple algebra and geometry, either as a prerequisite to a training program or as part of a licensure examination.

I find that statement extraordinary. I will take my telephone Yellow Pages, open it at random, and list in order the first eight categories that I see:

> Janitor service, Janitors' equipment and supplies, Jewelers, Karate and other martial arts, Kennels, Labeling, Labor organizations, Lamps and lamp shades.

In which six is algebra required, even for training or license? I again looked very hard for evidence in the NRC's publication but couldn't find any.

1,000 gallons of ice cream, so you will need 526.3/10 = 52.6 ounces of flavor.

The employee adding the flavor will not need algebra, nor will he or she need to think through this calculation. There will be a formula, or rule, that gives the result, and that is what happens on the job. Problems that arise on the job will be for the most part problems that have been solved before, so new solutions by workers will not be needed.

I am glad that we do not have to depend on workers' ability to solve algebra problems to get through the day because, as every teacher of mathematics knows, students don't always get problems right. The chair of the department of a Big Ten university once observed, probably after a bad day, that it was possible for a student to graduate with a mathematics major without ever having solved a single problem correctly. Partial credit can go a long way. This was in the 1950s, looked on by many as a golden age of mathematics education.

In one of those international tests of mathematical achievement appeared the problem of finding which of two magazine subscriptions was cheaper: 24 issues with (a) the first four issues free and $3 each for the remainder or (b) the first six issues free and $3.50 each for the remainder. This is not a tough problem, so I leave its solution to you. As easy as it is, only 26% of United States eighth-graders could do it correctly. That percentage was above the international average of 24%. Even the Japanese eighth-graders could manage only 39%. No doubt when the eighth-graders become adults they will be better at solving such problems, but even so I do not want them having to solve problems that when solved incorrectly can do me harm.

Though people know that they do not use algebra every day, or even every month, many seem to think that there are hosts of others who do. Perhaps they have absorbed the textbook writers' insistence on the "real world" uses of algebra, even though the texts actually demonstrate that there are none. Were uses of algebra widespread in the world of work, all textbook writers would have to do is to ask a few people about their last applications of algebra, turn them into problems, and put them in their texts. If 75% of all jobs required algebra, they could get a problem from three of every four people they ask. However, such problems do not appear in the texts. We get instead the endlessly repeated problems about investment clubs losing two members and all of the other chestnuts, about cars going from A to B and farmers fencing fields and so on, that I lack the space to display. The reason that problems drawn from everyday life do not appear in the texts is not that textbook authors lack energy and initiative; it is that they do not exist.

Though they may not use algebra themselves, people are solidly behind having everyone learn algebra. Tom and Ray Magliozzi, the brothers who are hosts of National Public Radio's popular "Car Talk" program, like to pose as vulgarians when they are actually nothing of the kind. On one program, brother Tom made some remarks against teaching geometry and trigonometry in high school. I doubt very much that he was serious. Whether he was serious or not does not affect the content of his remarks or the reaction of listeners. The reaction was unanimous endorsement of mathematics. When mathematics is attacked, people leap to its defense.

In his piece Tom alleged that he had an octagonal fountain in his backyard that he wanted to surround with a border and that he needed to calculate the length of the side of the concentric octagon. After succeeding, using, he said, the Pythagorean theorem, he reflected

> That this was maybe the second time in my life—maybe the first—that I had occasion to use the geometry and trigonometry that I had learned in high school. Furthermore, I had never had occasion to use the higher mathematics that the high school math had prepared me for.
>
> *Never!* Why did I—and millions of other students—spend valuable educational hours learning something that we would never use? Is this education? Learning skills that we will never need?

After some real or pretended populism ("The people who run the education business are money-grubbing, self-serving morons"), he concluded that

> The purpose of learning math, which most of us will never use, is only to prepare us for further math courses—which we will use even less frequently than never.

There were answers, quite a few of them, posted at the "Car Talk" website. All disagreed with Tom's conclusion, which actually has elements of truth. (A reply that started with "I agree" might be thought to be a counterexample, but the irony that followed was at least as heavy as lead.) One response included

> Perhaps you've had only one opportunity to use geometry in your life, but there are a number of occupations in which it's a must. Myself, I'm pleased that my house was designed and built by people who were capable of calculating the correct rise of a roof for proper drainage or the number of cubic feet of concrete needed for a strong foundation.

Here is the common error of supposing that problems once solved must be solved anew every time they are encountered. House builders have handbooks and tables, and use them. Indeed, houses, as well as pyramids and cathedrals, were being built long before algebra was taught in the schools and, in fact, before algebra.

Another common misconception occurs in another response:

> You sure laid a big oblate spheroid shaped one when you went on your tirade against having to learn geometry, trigonometry and other things mathematical.
>
> Who uses this stuff? Geologists, aircraft designers, road builders, building contractors, surgeons and, yes, even radio broadcast technicians (amplitude modulation and frequency modulation are both based on manipulating wave forms described by trig functions—don't get me started on alternating current).
>
> So, Tommy, get a life. The only people who don't use these principles every day are those who can't do and can't teach, and thus are suited only for lives as politicians or talk show hosts.

People seem to think that because something involves mathematics it is necessary to know mathematics to use it. Radio does indeed involve sines and cosines, but the person adjusting the dials needs no trigonometry. Geologists searching for oil do not have to solve differential equations, though differential equations may have been involved in the creation of the tools that geologists use.

I am not saying that mathematics is never required in the workplace. Of course it is, and it has helped to make our technology what it is. However, it is needed very, very seldom, and we do not need to train millions of students in it to keep businesses going. Once, when I was an employee of the Metropolitan Life Insurance Company, I was given an annuity rate to calculate. Back then, insurance companies had rate books, but now and then there was need for a rate not in the book. Using my knowledge of the mathematics of life contingencies, I calculated the rate. When I gave it to my supervisor he said, "No, no, that's not right. You have to do it *this* way." "But," I said, "that's three times as much work." Yes, I was told, but that's the way that we calculate rates. My knowledge of life contingencies got in the way of the proper calculation, done the way it had been done before, which any minimally competent employee could have carried out.

It may be that there could arise, say, a partial differential equation that some company needed to solve, the likes of which it had never seen before. If so, there are plenty of mathematicians available to do the job. They'd work cheap, too.

Jobs do not require algebra. I have expressed this truth many times in talks to any group who would listen, and it was not uncommon for a member of the audience to tell me, after the talk or during it, that I was wrong and that he used algebra or calculus in his job all the time. It always turned out that he used the mathematics because he wanted to, not because he had to.

Even those who are not burdened with the error that algebra is necessary to hold many jobs support the teaching of algebra. Everyone supports the teaching of algebra. The public wants more mathematics taught, to more students. The requirements keep going up, never down.

The reason for this, I am convinced, is that the public knows, or senses, that mathematics develops the power to reason. It shows, better than any other subject, how reason can lead to truth. Of course, other sciences exhibit the power of reason, but there's all that overhead—ferrous and ferric, dynes and ergs—that has to be dealt with. In mathematics, there is nothing standing between the problem and the reasoning.

Economists reason as well, but sometimes two economists reason to two different conclusions. Philosophers reason, but never come to *any* conclusion. In mathematics, problems can be solved using reason, and the solutions can be checked and shown to be correct. Reasoning needs to be learned, and mathematics is the best way to learn it.

People grasp this, perhaps not consciously, and hence want their children to undergo mathematics. Many times people have told me that they liked mathematics (though they call it "math") because it was so definite and it was satisfying to get the right answer. Have you not heard the same thing? They liked being able to reason correctly. They knew that the practice was good for them. No one has ever said to me, "I liked math because it got me a good job."

We no longer have the confidence in our subject that allows us to say that. We justify mathematics on its utility in the world of getting and spending. Our forebears were not so diffident. In 1906 J. D. Fitch wrote

> Our future lawyers, clergy, and statesmen are expected at the University to learn a good deal about curves, and angles, and number and proportions; not because these subjects have the smallest relation to the needs of their lives, but because in the very act of learning them they are likely to acquire that habit of steadfast and accurate thinking, which is indispensable in all the pursuits of life.

I do not know who J. D. Fitch was, but he was correct. Thomas Jefferson said

> Mathematics and natural philosophy are so peculiarly engaging and delightful as would induce everyone to wish an acquaintance with them. Besides this, the faculties of the mind, like the members of a body, are strengthened and improved by exercise. Mathematical reasoning and deductions are, therefore, a fine preparation for investigating the abstruse speculations of the law.

In 1834, the Congressional Committee on Military Affairs reported

> Mathematics is the study which forms the foundation of the course [at West Point]. This is necessary, both to impart to the mind that combined strength and versatility, the peculiar vigor and rapidity of comparison necessary for military action, and to pave the way for progress in the higher military sciences.

Here is testimony from a contemporary student:

> The summer after my freshman year I decided to teach myself algebra. At school next year my grades improved from a 2.6 gpa to a 3.5 gpa. Tests were easier and I was much more efficient when taking them and this held true in all other facets of my life. To sum this up: algebra is not only mathematical principles, it is a philosophy or way of thinking, it trains your mind and makes otherwise

A Tisket, a Tasket, an Apollonian Gasket

Dana Mackenzie

In the spring of 2007 I had the good fortune to spend a semester at the Mathematical Sciences Research Institute in Berkeley, an institution of higher learning that takes "higher" to a whole new extreme. Perched precariously on a ridge far above the University of California at Berkeley campus, the building offers postcard-perfect vistas of the San Francisco Bay, 1,200 feet below. That's on the west side. Rather sensibly, the institute assigned me an office on the east side, with a view of nothing much but my computer screen. Otherwise I might not have gotten any work done.

However, there was one flaw in the plan: Someone installed a screen-saver program on the computer. Of course, it had to be mathematical. The program drew an endless assortment of fractals of varying shapes and ingenuity. Every couple minutes the screen would go blank and refresh itself with a completely different fractal. I have to confess that I spent a few idle minutes watching the fractals instead of writing.

One day, a new design popped up on the screen *(see the first figure)*. It was different from all the other fractals. It was made up of simple shapes—circles, in fact—and unlike all the other screen savers, it had numbers! My attention was immediately drawn to the sequence of numbers running along the bottom edge: 1, 4, 9, 16 . . . They were the perfect squares! The sequence was 1 squared, 2 squared, 3 squared, and so on.

Before I became a full-time writer, I used to be a mathematician. Seeing those numbers awakened the math geek in me. What did they mean? And what did they have to do with the fractal on the screen? Quickly, before the screen-saver image vanished into the ether, I sketched it on my notepad, making a resolution to find out someday.

As it turned out, the picture on the screen was a special case of a more general construction. Start with three circles of any size, with each one touching the other two. Draw a new circle that fits snugly into

Figure 1. Numbers in an Apollonian gasket correspond to the curvatures or "bends" of the circles, with larger bends corresponding to smaller circles. The entire gasket is determined by the first four mutually tangent circles; in this case, two circles with bend 1 and two "circles" with bend 0 (and therefore infinite radius). The circles with a bend of zero look, of course, like straight lines. Image courtesy of Alex Kontorovich.

the space between them, and another around the outside enclosing all the circles. Now you have four roughly triangular spaces between the circles. In each of those spaces, draw a new circle that just touches each side. This creates 12 triangular pores; insert a new circle into each one of them, just touching each side. Keep on going forever, or at least until the circles become too small to see. The resulting foamlike structure is called an Apollonian gasket *(see the second figure)*.

Something about the Apollonian gasket makes ordinary, sensible mathematicians get a little bit giddy. It inspired a Nobel laureate to write a poem and publish it in the journal *Nature*. An 18th-century Japanese samurai painted a similar picture on a tablet and hung it in front

Figure 2. An Apollonian gasket is built up through successive "generations." For instance, in generation 1 *(top left)*, each of the lighter circles is inscribed in one of the four triangular pores formed by the dark circles. The complete gasket, whimsically named "bugeye" by Katherine Bellafiore Sanden, an undergraduate student of Peter Sarnak at Princeton University, has circles with bends −1 (for the largest circle that encloses the rest), 2, 2, and 3. The list of bends that appears in a given gasket (here, 2, 3, 6, 11, etc.) form a number sequence whose properties Sarnak would like to explain—but, he says, "the necessary mathematics has not been invented yet." Image courtesy of Katherine Bellafiore Sanden.

of a Buddhist temple. Researchers at AT&T Labs printed it onto T-shirts. And in a book about fractals with the lovely title *Indra's Pearls*, mathematician David Wright compared the gasket to Dr. Seuss's *The Cat in the Hat*:

> The cat takes off his hat to reveal Little Cat A, who then removes his hat and releases Little Cat B, who then uncovers Little Cat C, and so on. Now imagine there are not one but three cats inside each cat's hat. That gives a good impression of the explosive proliferation of these tiny ideal triangles.

Getting the Bends

Even the first step of drawing an Apollonian gasket is far from straightforward. Given three circles, how do you draw a fourth circle that is exactly tangent to all three?

Apparently the first mathematician to seriously consider this question was Apollonius of Perga, a Greek geometer who lived in the third century B.C. He has been somewhat overshadowed by his predecessor Euclid, in part because most of his books have been lost. However, Apollonius's surviving book *Conic Sections* was the first to systematically study ellipses, hyperbolas, and parabolas—curves that have remained central to mathematics ever since.

One of Apollonius' lost manuscripts was called *Tangencies*. According to later commentators, Apollonius apparently solved the problem of drawing circles that are simultaneously tangent to three lines, or two lines and a circle, or two circles and a line, or three circles. The hardest case of all was the case where the three circles are tangent.

No one knows, of course, what Apollonius' solution was, or whether it was correct. After many of the writings of the ancient Greeks became available again to European scholars of the Renaissance, the unsolved "problem of Apollonius" became a great challenge. In 1643, in a letter to Princess Elizabeth of Bohemia, the French philosopher and mathematician René Descartes correctly stated (but incorrectly proved) a beautiful formula concerning the radii of four mutually touching circles. If the radii are r, s, t, and u, then Descartes's formula looks like this:

$$1/r^2 + 1/s^2 + 1/t^2 + 1/u^2 = 1/2\,(1/r + 1/s + 1/t + 1/u)^2.$$

All of these reciprocals look a little bit extravagant, so the formula is usually simplified by writing it in terms of the *curvatures* or the *bends* of the circles. The curvature is simply defined as the reciprocal of the radius. Thus, if the curvatures are denoted by a, b, c, and d, then Descartes's formula reads as follows:

$$a^2 + b^2 + c^2 + d^2 = (a + b + c + d)^2/2.$$

As the third figure shows, Descartes's formula greatly simplifies the task of finding the *size* of the fourth circle, assuming the sizes of the first three are known. It is much less obvious that the very same equation can be used to compute the *location* of the fourth circle as well, and thus completely solve the drawing problem. This fact was discovered in the late 1990s by Allan Wilks and Colin Mallows of AT&T Labs, and Wilks used it to write a very efficient computer program for plotting Apollonian gaskets. One such plot went on his office door and eventually got made into the aforementioned T-shirt.

Descartes himself could not have discovered this procedure, because it involves treating the coordinates of the circle centers as complex numbers. Imaginary and complex numbers were not widely accepted by mathematicians until a century and a half after Descartes died.

In spite of its relative simplicity, Descartes's formula has never become widely known, even among mathematicians. Thus, it has been rediscovered over and over through the years. In Japan, during the Edo period, a delightful tradition arose of posting beautiful mathematics problems on tablets that were hung in Buddhist or Shinto temples, perhaps as an offering to the gods. One of these "Japanese temple problems," or *sangaku*, is to find the radius of a circle that just touches two circles and a line, which are themselves mutually tangent. This is a restricted version of the Apollonian problem, where one circle has infinite radius (or zero bend). The anonymous author shows that, in this case, $\sqrt{a} + \sqrt{b} = \sqrt{c}$, a sort of demented version of the Pythagorean theorem. This formula, by the way, explains the pattern I saw in the screensaver. If the first two circles have bends 1 and 1, then the circle between them will have bend 4, because $\sqrt{1} + \sqrt{1} = \sqrt{4}$. The next circle will have bend 9, because $\sqrt{1} + \sqrt{4} = \sqrt{9}$. Needless to say, the pattern continues forever. (This also explains what the numbers in the first figure mean. Each circle is labeled with its own bend.)

The first thing to notice is the foamlike structure that remains after you cut out all of the discs in the gasket. Clearly the disks themselves take up an area that approaches 100 percent of the area within the outer disk, and so the area of the foam (known as the "residual set") must be zero. On the other hand, the foam also has infinite length. Thus, in fact, it was one of the first known examples of a *fractal*—a curve of dimension between 1 and 2. Even today its dimension (denoted δ) is not known exactly; the best-proven estimate is 1.30568.

The concept of fractional dimension was popularized by Benoît Mandelbrot in his enormously influential book *The Fractal Geometry of Nature*. Although the meaning of dimension 1.30568 is somewhat opaque, this number is related to other properties of the foam that have direct physical meaning. For instance, if you pick any cutoff radius r, how many bubbles in the foam have radius larger than r? The answer, denoted $N(r)$, is roughly proportional to r^{δ}. Or if you pick the n largest bubbles, what is the remaining pore space between those bubbles? The answer is roughly proportional to $n^{1-2/\delta}$.

Physicists are very familiar with this sort of rule, which is called a *power law*. As I read the literature on Apollonian packings, an interesting cultural difference emerged between physicists and mathematicians. In the physics literature, a fractional dimension δ is *de facto* equivalent to a power law r^{δ}. However, mathematicians look at things through a sharper lens, and they realize that there can be additional, slowly increasing or slowly decreasing terms. For instance, $N(r)$ could be proportional to $r^{\delta}\log(r)$ or $r^{\delta}/\log(r)$. For physicists, who study foams empirically (or semiempirically, via computer simulation), the logarithm terms are absolutely undetectable. The discrepancy they introduce will always be swamped by the noise in any simulation. But for mathematicians, who deal in logical rigor, the logarithm terms are where most of the action is. In 2008, mathematicians Alex Kontorovich and Hee Oh of Brown University showed that there are in fact no logarithm terms in $N(r)$. The number of circles of radius greater than r obeys a strict power law, $N(r) \sim Cr^{\delta}$, where C is a constant that depends on the first three circles of the packing. For the "bugeye" packing illustrated in the second figure, C is about 0.201. (The tilde ($\sim$) means that this is not an *equation* but an *estimate* that becomes more and more accurate as the radius r decreases to 0.) For mathematicians, this was a major advance. For physicists, the likely reaction would be, "Didn't we know that already?"

Figure 4. Physicists study random Apollonian packings as a model for foams or powders. In these simulations, new bubbles or grains nucleate in a random place and grow, either with rotation or without, until they encounter another bubble or grain. Different geometries for the bubbles or grains, and different growth rules, lead to different values for the dimension of the residual set—a way of measuring the efficiency of the packing. Image courtesy of Stefan Hutzler and Gary Delaney. First published in Delaney GW, Hutzler S and Aste T (2008), Relation Between Grain Shape and Fractal Properties in Random Apollonian Packing with Grain Rotation http://dx.doi.org/10.1103/PhysRevLett.101.120602, Phys. Rev. Lett., 101, 120602.

Random Packing

For many physical problems, the classical definition of the Apollonian gasket is too restrictive, and a random model may be more appropriate. A bubble may start growing in a randomly chosen location and expand until it hits an existing bubble, and then stop. Or a tree in a forest may grow until its canopy touches another tree, and then stop. In this case, the new circles do not touch three circles at a time, but only one. Computer simulations show that these "random Apollonian packings" still behave like a fractal, but with a different dimension. The empirically observed dimension is 1.56. (This means the residual set is larger, and the packing is less efficient, than in a deterministic Apollonian gasket.) More recently, Stefan Hutzler of Trinity College Dublin, along with Gary Delaney and Tomaso Aste of the University of Canberra, studied the effect of bubbles with different shapes in a random Apollonian packing. They found, for example, that squares become much more efficient packers than circles if they are allowed to rotate as they grow, but surprisingly, triangles become only slightly more efficient. As far as I know, all of these results are begging for a theoretical explanation.

For mathematicians, however, the classical, deterministic Apollonian gasket still offers more than enough challenging problems. Perhaps the most astounding fact about the Apollonian gasket is that if the first four circles have integer bends, then *every other circle* in the packing does too. If you are given the first three circles of an Apollonian gasket, the bend of the fourth is found (as explained above) by solving a quadratic equation. However, every *subsequent* bend can be found by solving a *linear* equation:

$$d + d' = 2(a + b + c)$$

For instance, in the bugeye gasket, the three circles with bends $a = 2$, $b = 3$, and $c = 15$ are mutually tangent to two other circles. One of them, with bend $d = 2$, is already given in the first generation. The other has bend $d' = 38$, as predicted by the formula, $2 + 38 = 2(2 + 3 + 15)$. More importantly, even if we did not know d', we would still be guaranteed that it was an integer, because a, b, c, and d are.

Hidden behind this "baby Descartes equation" is an important fact about Apollonian gaskets: They have a very high degree of symmetry. Circles a, b, and c actually form a sort of curved mirror that reflects

Figure 5."Bugeye" gasket.

circle d to circle d' and vice versa. Thus the whole gasket is like a kaleidoscopic image of the first four circles, reflected again and again through an infinite collection of curved mirrors.

Kontorovich and Oh exploited this symmetry in an extraordinary and amusing way to prove their estimate of the function $N(r)$. Remember that $N(r)$ simply counts how many circles in the gasket have radius larger than r. Kontorovich and Oh modified the function $N(r)$ by introducing an extra variable of position—roughly equivalent to putting a lightbulb at a point x and asking how many circles illuminated by that lightbulb have radius larger than r. The count will fluctuate, depending on exactly where the bulb is placed. But it fluctuates in a very predictable way. For instance, the count is unchanged if you move the bulb to the location of any of its kaleidoscopic reflections.

This property makes the "lightbulb counting function" a very special kind of function, one which is invariant under the same symmetries as the Apollonian gasket itself. It can be broken down into a spectrum of similarly symmetric functions, just as a sound wave can be decomposed into a fundamental frequency and a series of overtones. From this spectrum, you can in theory find out everything you want to know about the lightbulb counting function, including its value at any particular location of the lightbulb.

For a musical instrument, the fundamental frequency or lowest overtone is the most important one. Similarly, it turned out that the first

Figure 6. A favorite example of Sarnak's is the "coins" gasket, so called because three of the four generating circles are in proportion to the sizes of a quarter, nickel, and dime, respectively. Image courtesy of Alex Kontorovich.

symmetric function was all that Kontorovich and Oh needed to figure out what happens to $N(r)$ as r approaches 0.

In this way, a simple problem in geometry connects up with some of the most fundamental concepts of modern mathematics. Functions that have a kaleidoscopic set of symmetries are rare and wonderful. Kontorovich calls them "the Holy Grail of number theory." Such functions were, for instance, used by Andrew Wiles in his proof of Fermat's last theorem. An interesting new kaleidoscope is enough to keep mathematicians happy for years.

Gaskets Galore

Kontorovich learned about the Apollonian kaleidoscope from his mentor, Peter Sarnak of Princeton University, who learned about it from Lagarias, who learned about it from Wilks and Mallows. For Sarnak, the Apollonian gasket is wonderful because it has neither too few nor too many mirrors. If there were too few, you would not get enough information from the spectral decomposition. If there were too many, then previously known methods, such as the ones Wiles used, would already answer all your questions.

The Quest for God's Number

Rik van Grol

The Rubik's cube triggered one of the largest puzzle crazes in the world. The small mechanical puzzle, invented by Ernő Rubik in Hungary, has sold more than 350 million copies. Although it has existed since 1974, the popularity of the cube skyrocketed around 1980, when the cube was introduced outside of Hungary. In the early days, simply solving the puzzle was the main issue, especially because no solution books were available and there was no Internet. But solving the puzzle in the shortest amount of time was also hot news. In the early 1980s the best times were on the order of 24 seconds.

By the end of the 1980s, the craze was starting to ebb, but in certain groups the puzzle remained very much alive, and now the Rubik's cube is making a comeback. Solving the cube the quickest—*speed cubing*—is currently a major activity. The fastest times are under 8 seconds with the average around 11 seconds. The foundation for a fast solving time is a good algorithm, and so the search for efficient algorithms has been an important area of study since the early 1980s. The ultimate goal is to discover the best method of all—the optimal solution algorithm—which has been dubbed *God's algorithm*.

> *God's algorithm is the procedure to bring back Rubik's Cube from any random position to its solved state in the minimum number of steps.*

The maximum of all minimally needed number of steps is referred to as *God's number*. This number can be defined in several ways. The most common is in terms of the number of *face turns* required, but it can also be measured as the number of *quarter turns.* Whereas a quarter turn is either a positive or negative 90° turn, a face turn can be either of these or a 180° turn. A 180° face turn is equal to two quarter turns. Earlier this year, after decades of gradual progress, it was determined that God's number is 20 face turns. Thus, if God's algorithm were used to solve the cube, no starting position would ever require more than 20 face turns.

Apart from determining God's number, another major question has been to find out whether God's algorithm is an elegant sequence of moves that can be "easily" memorized, or if, instead, God's algorithm amounts to a short procedure with giant lookup tables. If it's the latter case, then no one will ever be able to learn how to solve the Rubik's cube in the minimal number of moves (read on to learn why this is true). Still, even if God's algorithm has no practical purpose, it is interesting to know what God's number is.

Playing God (with Small Numbers)

If you start with a solved cube and ask someone to make a few turns, you will (after some practice) be able to return it to the solved state in a minimum number of moves as long as the initial number of scrambling moves is not too large. With fewer than four scrambling moves, it is easy; with four, it becomes tricky. With five, it is simply hard. Some experts can handle six or even seven scrambling moves, but any more and it is essentially impossible to solve the cube in the minimal number of moves.

Generally speaking, most algorithms take between 50 and 100 moves. What if you were to randomly turn the cube 1,000 times? Will it take 1,000 moves to get it back? No, it still takes most algorithms 50 to 100 moves because the algorithms are designed to work from any starting position.

Starting Small: The 2 × 2 × 2 Cube

Unlike the classic 3 × 3 × 3 Rubik's Cube, the 2 × 2 × 2 cube has been completely analyzed. God's number is 11 in face turns and 14 in quarter turns. To find these values, all possible configurations of the 2 × 2 × 2 cube were cataloged, and for each of these configurations, the minimal number of turns needed to reach the solved state was determined. This brute force approach was possible because there are "only" about 3.7 million configurations to study.

To calculate the number of configurations for the 2 × 2 × 2 cube, we start with the observation that the eight corner *cubies* (as they are called) can be permuted in 8! ways. For any such permutation, each corner cubie can be oriented in three ways, leading to 3^8 possible orientations. However, given the orientation of seven corners, the orientation of the eighth is determined by the puzzle mechanism, so the corners really have 3^7 orientations. As the orientation of the whole cube is not fixed in space (any one of the eight corners can be placed in, say, the top-front-right position, and once it is placed there, the entire cube can be rotated so that any one of three faces is on top), the total number of permutations needs to be divided by $8 \times 3 = 24$. Hence, the total number of positions is

$$\frac{8! \times 3^7}{8 \times 3} = 7! \times 3^6 = 3{,}674{,}160.$$

There are only 2,644 positions for which 11 face turns are required to solve the puzzle. Assuming all configurations have the same likelihood of being a starting position, the average number of face turns required to solve the puzzle is 9. Likewise, there are only 276 positions from which 14 quarter turns are required, and on average, 11 quarter turns are required to solve the puzzle.

A Leap in Complexity: The 3 × 3 × 3 Cube

Until recently, God's number for the $3 \times 3 \times 3$ cube was not known. From the late 1970s until now the search area has been limited by two numbers: the lower bound and the upper bound. The lower bound G^{low} is determined by proving that there are positions that require at least G^{low} turns. The upper bound G^{high} is determined by proving that no position requires more than G^{high} turns.

So, how many configurations are there? With 8 corner cubies and 12 edge cubies, there are $8! \times 12! \times 3^8 \times 2^{12}$ different patterns, but not all patterns are possible:

- *With 8 corners there are 8! corner permutations, and with 12 edges there are 12! edge permutations. However, because it is impossible to interchange two edge cubies without also interchanging 2 corner cubies, the total number of permutations should be divided by 2.*
- *Turning of corner cubies (keeping their position, but cycling the colors on their three faces) needs to be done in pairs—only 7 corner cubies can be turned freely.*

- *Flipping of edge cubies (keeping their position, but switching the colors of their two faces) needs to be done in pairs—only 11 edge cubies can be flipped freely.*

Because of the six center pieces, the orientation of the cube is fixed in space, so the number of permutations should not be divided by 24 as with the $2 \times 2 \times 2$ cube. Hence, the number of positions of the $3 \times 3 \times 3$ cube is

$$\frac{8! \times 12!}{2} \times \frac{3^8}{3} \times \frac{2^{12}}{2} = 43{,}252{,}003{,}274{,}489{,}856{,}000 \approx 4.3 \times 10^{19}.$$

This is *astronomically* bigger than 3,674,160 for the $2 \times 2 \times 2$ cube, and it made searching the entire space computationally impossible. For instance, if every one of the 350 million cubes ever sold were put in a new position every second, it would take more than 3,900 years for them to collectively hit every possible position (with no pair of cubes ever sharing a common configuration). Or to put it another way: if a computer were capable of determining the fewest number of moves required to solve the cube for 1,000 different starting positions each second, it would take more than a billion years of computing time to get through every configuration.

Determining God's number by independently improving the upper and lower bounds was a quest that lasted for three decades—but it has finally come to an end. In July 2010 the upper and lower bounds met at the number 20.

Raising the Lower Bound

Using counting arguments, it can be shown that there exist positions requiring at least 18 moves to solve. To see this, one counts the number of distinct positions achievable from the solved state using at most 17 moves. It turns out that this number is smaller than 4.3×10^{19}. This simple argument was made in the late 1970s (see Singmaster's book in the Further Reading section), and the result was not improved upon for many years. Note that this is not a constructive proof; it does not specify a concrete position that requires 18 moves. At some point, it was suggested that the so-called *superflip* would be such a position. The superflip is a state of the cube where all the cubies are in their correct position with the corner cubies oriented correctly, but where each edge cubie is flipped (oriented the wrong way).

It took until 1992 for a solution for the superflip with 20 face turns to be found, by Dik Winter. In 1995 Michael Reid proved that this solution was minimal, and thus a new lower bound for God's number was found. Also in 1995, a solution for the superflip in 24 quarter turns was found by Michael Reid, and it was later proved to be minimal by Jerry Bryan. In 1998 Michael Reid found a new position requiring more than 24 quarter turns to solve. The position, which he calls the *superflip composed with four spots*, requires 26 quarter turns. This put the lower bound at 20 face turns or 26 quarter turns.

Lowering the Upper Bound

Finding an upper bound requires a different kind of reasoning. Perhaps the first concrete value for an upper bound was the 277 moves mentioned by David Singmaster in early 1979. He simply counted the maximum number of moves required by his cube-solving algorithm. Later, Singmaster reported that Elwyn Berlekamp, John Conway, and Richard Guy had come up with a different algorithm that took at most 160 moves. Soon after, Conway's Cambridge Cubists reported that the cube could be restored in at most 94 moves. Again, this reflected the maximum number of moves required by a specific algorithm.

A significant breakthrough was made by Morwen Thistlethwaite. Whereas algorithms up to that point attacked the problem by putting various cubies in place and performing subsequent moves that left them in place, he approached the problem by gradually restricting the types of moves that could be executed. Understanding this method requires a brief introduction to the *cube group*.

As we work with the cube, let's agree to keep the overall orientation of the cube fixed in space. This means that the center cubies on each face will never change. We may then label the faces Left, Right, Front, Back, Up, and Down. The cube group is composed of all possible combinations of successive face turns, where two such combinations are equal if and only if they result in the same cube configuration. We denote the clockwise quarter turns of the faces by *L, R, F, B, U,* and *D,* and use concatenation as the group operation. For instance, the product *FR* denotes a quarter turn of the front face followed by a quarter turn of the right sold more than 1.5 million copies. Another classic, complete with the requisite group theory, is *Inside Rubik's Cube and Beyond*, by Christoph Bandelow (Birkhäuser, 1982). On the web, an excellent how-to guide with several links to other sources can be found at Jaap's Puzzle Page: http://www.jaapsch.net/puzzles/. A brief history of the quest for God's number can be found on Tom Rokicki's site, http://www.cube20.org.

Meta-morphism:
From Graduate Student to Networked Mathematician

Andrew Schultz

While the stereotypical mathematician is a hermit locked alone in his office, the typical mathematician is far from a solitary explorer. A great amount of the mathematics produced today is created collaboratively, spurred into existence during those quintessentially mathematical social interactions: on chalkboards following a seminar talk, on napkins during a coffee break at a conference, on the back of a coaster at a pub. Though it often isn't clear to those wading through graduate programs, one of the key metamathematical skills one should develop while working on a master's or Ph.D. is the ability to participate in this social network. What follows is a rough guide to how you can use graduate school to build the professional relationships that will shape your career.

The Hungry Caterpillar

Stepping into the mathematical social network begins by getting to know your graduate student cohort. It's likely that some of the friendships you form during graduate school will be among the closest in your life, and even those fellow students who aren't your best friends are likely to be professional colleagues long after you've received your degree. It's worth the investment of time and energy to foster these relationships as your first semester begins.

When arriving on campus to start your graduate career, you'll likely convene with the new graduate students in your department and a handful of the faculty for a kind of informal orientation. Ph.D. programs often draw students from a wide variety of backgrounds, so don't be surprised to find people whose professional experience, familial status, or country