The BEST WRITING on MATHEMATICS

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For my parents

Contents

Foreword: The Synergy of Pure and Applied Mathematics, of the Abstract and the Concrete

David Mumford

All of us mathematicians have discovered a sad truth about our passion: It is pretty hard to tell anyone outside your field what you are so excited about! We all know the sinking feeling you get at a party when an attractive person of the opposite sex looks you in the eyes and asks—"What is it you do?" Oh, for a simple answer that moves the conversation along.

Now Mircea Pitici has stepped up to the plate and for the third year running has assembled a terrific collection of answers to this query. He ranges over many aspects of mathematics, including interesting pieces on the history of mathematics, the philosophy of mathematics, mathematics education, recreational mathematics, and even *actual presentations of mathematical ideas*! This volume, for example, has accessible discussions of *n*-dimensional balls, the intricacies of the distribution of prime numbers, and even of octonions (a strange type of algebra in which the "numbers" are 8-tuples of the ordinary sort of number)—none of which are easy to convey to the layperson. In addition—and I am equally pleased with this—several pieces explain in depth how mathematics can be used in science and in our lives—in dancing, for the traveling salesman, in search of marriage, and for full-surround photography, for instance.

To the average layperson, mathematics is a mass of abstruse formulae and bizarre technical terms (e.g., perverse sheaves, the monster group, barreled spaces, inaccessible cardinals), usually discussed by academics in white coats in front of a blackboard covered with peculiar symbols. The distinction between mathematics and physics is blurred and that between pure and applied mathematics is unknown.

But to the professional, these are three different worlds, different sets of colleagues, with different goals, different standards, and different customs.

The layperson has a point, though. Throughout history many practitioners have crossed seamlessly between one or another of these fields. Archimedes not only calculated the volume of a ball of radius r (a pure mathematics problem with the answer $4\pi r^3/3$) but also studied the lever (a physics problem) and used it both in warfare (applied mathematics: hurling fiery balls at Roman ships) and in mind experiments ("Give me a place to stand and I will move the earth"). Newton was both a brilliant mathematician (inventing calculus) and physicist (discovering the law of gravity).

Today it is different: The three fields no longer form a single space in which scientists can move easily back and forth. Starting in the mid-twentieth century, mathematicians were blindsided by the creation of quantum field theory and even more by string theory. Here physicists, combining their physical intuition with all the latest and fanciest mathematical theories, began to use mathematics in ways mathematicians could not understand. They abandoned rigorous reasoning in favor of physical intuition and played wildly with heuristics and extrapolations from well-known mathematics to "explain" the world of high energy. At about the same time (during the '50s and '60s), mathematics split into pure and applied camps. One group fell in love with the dream of a mathematics that lived in and for itself, in a Platonic world of blinding beauty. The English mathematician G. H. Hardy even boasted that his work could never be used for practical purposes. On the other side, another group wanted a mathematics that could solve real-world problems, such as defeating the Nazis. John von Neumann went to Los Alamos and devised a radical new type of mathematics based on gambling, the Monte Carlo technique, for designing the atom bomb. A few years later, this applied group developed a marvelous new tool, the computer—and with it applied mathematics was off and running in its own directions.

I have been deeply involved with both pure mathematics and applied mathematics. My first contact with real mathematical problems was during a summer job in 1953, when I used an analog computer to simulate the neutron flux in the core of an atomic reactor. I was learning the basics of calculus at the time, just getting used to writing Greek letters for numbers and operations—and the idea of connecting resistors in a grid to simulate Δ (technically, the Laplace differential operator) struck me as profoundly beautiful. I was struggling to get my mind around the abstract notions, but luckily I was well acquainted with the use of a soldering iron. I was delighted that I could construct simple electrical circuits that made calculus so tangible.

Later, in college, I found that I could not understand what quantum field theory was all about; *ergo*, I was not a physicist but a mathematician! I went all the way and immersed myself in one of the purest areas of pure mathematics. (One can get carried away: At one time the math department at Cambridge University advertised an opening and a misprint stated that the position was in the Department of "Purer" Mathematics!) I "constructed" something called "moduli schemes." I do not expect the reader to have ever heard of moduli schemes or have a clue what they are. But here is the remarkable thing: To mathematicians who study them, moduli schemes are just as real as the regular objects in the world.

I can explain at least the first steps of the mental gymnastics that led to moduli schemes. The key idea is that an ordinary object can be studied using *the set of functions on the object*. For example, if you have a pot of water, the water at each precise location, at each spatial point inside the pot, has a temperature. So temperature defines a function, a rule that associates to each point in the pot the real number that is the temperature at that exact point. Or you can measure the coordinates of each point, for instance, how many centimeters the point is above the stove. Secondly, you can do algebra with these functions—that is, you can add or multiply two such functions and get a third function. This step makes the set of these functions into a *ring*. I have no idea why, but when you have any set of things that can be added and multiplied, consistent with the usual rules (for instance, the distributive law a [times] $(b + c) = a$ [times] $b + a$ [times] c), mathematicians call this set a ring. You see, ordinary words are used in specialized ways. In our case, the ring contains all the information needed to describe the geometry of the pot because the points in the pot can be described by the map carrying each function to its value at that point.

Then the big leap comes: If you start with any ring—that is, any set of entities that can be added and multiplied subject to the usual rules, you simply and brashly declare that this creates a new kind of geometric approximately 1,000,000,000,000,000,000,000,000 neutrons that are really whizzing about—and it turned out to work well, unless you regret the legacy von Neumann's inspiration left the world.

But the drifting apart of pure and applied mathematics is not the whole story. The two worlds are tied more closely than you might imagine. Each contributes many ideas to the other, often in unexpected ways. Perhaps the most famous example is Einstein's need of new mathematical tools to push to deeper levels the ideas of special relativity. He found that Italian mathematicians, dealing with abstract *n*-dimensional space, had discovered tools for describing higher dimensional versions of curvature and the equations for shortest paths, called geodesics. Adapting these ideas, Einstein turned them into the foundations of general relativity (without which your global positioning system [GPS] wouldn't work). In the other direction, almost a century after Einstein discovered general relativity, working out the implications of Einstein's model is a hot area in pure mathematics, driving the invention of new techniques to deal with the highly nonlinear PDEs underlying his theory. In other words, pure mathematics made Einstein's physics possible, which in turn opened up new fields for pure mathematics.

A spectacular recent example of the interconnections between pure and applied mathematics involves prime numbers. No one (especially G. H. Hardy, as I mentioned) suspected that prime numbers could ever be useful in the real world, yet they are now the foundation of the encryption techniques that allow online financial transactions. This application is a small part of an industry of theoretical work on new algorithms for discrete problems—in particular, their classification by the order of magnitude of their speed—which is the bread and butter of computer science.

I want to describe another example of the intertwining of pure and applied mathematics in which I was personally involved. Computer vision research concerns writing computer code that will interpret camera and video input as effectively as humans can with their brains, by identifying all the objects and actions present. When this problem was first raised in the 1960s, many people believed that it was a straightforward engineering problem and would be solved in a few years. But fifty years later, computers still cannot recognize specific individuals from their faces very well or name the objects and read all the signs in a street scene. We are getting closer: Computers are pretty good at talked about a variety of such connections *among* these fields. But even within pure mathematics, amazing connections between remote areas are uncovered all the time. In the last decade, for example, ideas from number theory have led to progress in the understanding of the topology of high-dimensional spheres.

It will be difficult to fully repair the professional split between pure and applied mathematics and between mathematics and physics. One reason for this difficulty is that each academic field has grown so much, so that professionals have limited time to read work outside their specialties. It is not easy to master more than a fraction of the work in any single field, let alone in more than one. What we need, therefore, is to work harder at explaining our work to each other. This book, though it is addressed mostly to lay people, is a step in the right direction.

As I see it, the major obstacle is that there are two strongly conflicting traditions of writing and lecturing about mathematics. In pure mathematics (but not exclusively), the twentieth century saw the development of an ideal exposition as one that started at the most abstract level and then gradually narrowed the focus. This style was especially promoted by the French writing collaborative "Bourbaki." In the long tradition of French encyclopedists, the mathematicians forming the Bourbaki group sought to present the entire abstract structure of all mathematical concepts in one set of volumes, the *Éléments de Mathématique*. In that treatise the real numbers, which most of us regard as a starting point, only appeared midway into the series as a special "locally compact topological field." In somewhat less relentless forms, their orientation has affected a large proportion of all writing and lecturing in mathematics.

An opposing idea, promoted especially in the Russian school, is that a few well-chosen examples can illuminate an entire field. For example, one can learn stochastic processes by starting with a simple random walk, moving on to Brownian motion, its continuous version, and then to more abstract and general processes. I remember a wonderful talk on hyperbolic geometry by the mathematician Bill Thurston, where he began by scrawling with yellow chalk on the board: He explained that it was a simple drawing of a fire. His point was that in hyperbolic space, you have to get much, much closer to the fire to warm up than you do in Euclidean space. Along with such homey illustrations, there is also the precept "lie a little." If we insist on detailing all the technical qualifications of a theorem, we lose our readers or our audience very fast. If we learn to say things simply and build up slowly from the concrete to the abstract, we may be able to build many bridges among our various specialties. For me, this style will always be *The Best Writing on Mathematics,* and this book is full of excellent examples of it.

Introduction

Mircea Pitici

A little more than eight years ago I planned a series of "best writing" on mathematics with the sense that a sizable and important literature does not receive the notice, the consideration, and the exposure it deserves. Several years of thinking on such a project (for a while I did not find a publisher interested in my proposal) only strengthened my belief that the best of the nontechnical writings on mathematics have the potential to enhance the public reception of mathematics and to enrich the interdisciplinary and intradisciplinary dialogues so vital to the emergence of new ideas.

The prevailing view holds that the human activity we conventionally call "mathematics" is mostly beyond fruitful debate or personal interpretation because of the uncontested (and presumably uncontestable) matters of fact pertaining to its nature. According to this view, mathematics speaks for itself, through its cryptic symbols and the efficacy of its applications.

At close inspection, the picture is more complicated. Mathematics has been the subject of numerous disputes, controversies, and crises—and has weathered them remarkably well, growing from the resolution of the conundrums that tested its strength. By doing so, mathematics has become a highly complex intellectual endeavor, thriving at the ever-shifting intersection of multiple polarities that can be used to describe its characteristics. Consequently, most people who are engaged with mathematics (and many people disengaged from it) do it on a more personal level than they are ready to admit. Writing is an effective way of informing others on such individualized positioning vis-à-vis mathematics. A growing number of authors—professionals and amateurs—are taking on such a task. Every week new books on mathematics are published, in a dazzling blossoming of the genre hard to imagine even a decade ago (I mention a great number of these titles later in this introduction). This recent flourishing confirms that, just as mathematics offers unlimited possibilities for asking new questions, formulating new problems, opening new theoretical vistas, and rethinking old concepts, narrating our individualized perspectives on it is equally potent in expressivity and in impact.

By editing this annual series, I stand for the wide dissemination of insightful writings that touch on any aspect related to mathematics. I aim to diminish the gap between mathematics professionals and the general public and to give exposure to a substantial literature that is not currently used systematically in scholarly settings. Along the way, I hope to weaken or even to undermine some of the barriers that stand between mathematics and its pedagogy, history, and philosophy, thus alleviating the strains of hyperspecialization and offering opportunities for connection and collaboration among people involved with different aspects of mathematics. If, by presenting in each volume a snapshot of contemporary thinking on mathematics, we succeed in building a useful historical reference, in offering an informed source of further inquiry, and in encouraging even more exceptional writing on mathematics, so much the better.

Overview of the Volume

In the first article of our selection, Mario Livio ponders the old question of what makes mathematics effective in describing many features of the physical universe and proposes that its power lies in the peculiar blending between the human ingenuity in inventing flexible and adaptable mathematical tools and the uncanny regularities of the universe.

Timothy Gowers brings the perspective of a leading research mathematician to another old question, that concerning features of discovery and elements of invention in mathematics; he discusses some of the psychological aspects of this debate and illustrates it with a wealth of examples.

In a succession of short pieces, Peter Rowlett and his colleagues at the British Society for the History of Mathematics present the unexpected applications, ricocheting over centuries, of notions and results long believed to have no use beyond theoretical mathematics.

Brian Hayes puzzles over the proportion between the volume of a sphere and that of the cube circumscribed to it, in various concludes that, despite various degrees of success, she remains an agnostic in this matter.

Bonnie Gold argues that everyone teaching mathematics does it according to certain philosophical assumptions about the nature of mathematics—whether the assumptions are explicit or remain implicit.

Susanna S. Epp examines several uses of the concept of "variable" in mathematics and opines that, from an educational standpoint, the best is to treat variables as placeholders for numerals.

David Mumford and Sol Garfunkel plead for a broad reform of the U.S. system of mathematics education, more attuned to the practical uses of mathematics for the citizenry and less concerned with the high-stakes focus on testing currently undertaken in the United States.

Jeremy Gray surveys recent trends in the study of the history of mathematics as compared to research on the history of science and examines the possibility that the two might be somehow integrated in the future.

Charlotte Simmons writes about Augustus De Morgan as a mentor of other mathematicians, an aspect less known than the research contributions of the great logician.

Giuseppe Bruno, Andrea Genovese, and Gennaro Improta review several formulations of various routing problems, with wide applications to matters of mathematical optimization.

Special curves were at the forefront of mathematical research about three centuries ago, and one of them, the cycloid, attracted the attention (and the rivalry) of the most famous mathematicians of the time—as Gerald L. Alexanderson shows in his piece on the Bernoulli family.

Fernando Gouvêa examines Georg Cantor's correspondence, to trace the original meaning of Cantor's famous remark "I see it, but I don't believe it!" and to refute the ulterior, psychological interpretations that other people have given to this quip.

Ian Hacking explains that the enduring fascination and the powerful influence of mathematics on so many Western philosophers lie in the experiences engendered on them by learning and doing mathematics.

Richard Elwes delves into the subtleties of mathematical infinity and ventures some speculations on the future clarification of the problems it poses.

Finally, Mark Colyvan illustrates the basic mathematics involved in the games of choice we encounter in life, whenever we face processes that require successive alternative decisions.

The number of interdisciplinary and applicative books that build connections between mathematics and other domains continues to grow fast. On mathematics and music, we recently have *A Geometry of Music* by Dmitri Tymoczko and *The Science of String Instruments,* edited by Thomas D. Rossing. Some remarkable books on mathematics and architecture are now available, including *The Function of Form* by Farshid Moussavi (marvelously illustrated); *Advances in Architectural Geometry 2010,* edited by Cristiano Ceccato and his collaborators; *The New Mathematics of Architecture* by Jane and Mark Burry; the 30th anniversary reissue of *The Dynamics of Architectural Form* by Rudolf Arnheim; and *Matter in the Floating World,* a book of interviews by Blaine Brownell.

Among the books on mathematics and other sciences are Martin B. Reed's *Core Maths for the Biosciences*; *BioMath in the Schools,* edited by Margaret B. Cozzens and Fred S. Roberts; *Chaos: The Science of Predictable Random Motion* by Richard Kautz (a historical overview); *Some Mathematical Models from Population Genetics* by Alison Etheridge; and *Mathematics Meets Physics*, a collection of historical pieces (in English and German) edited by Karl-Heinz Schlote and Martina Schneider.

Everyone expects some books on mathematics and social sciences; indeed, this time we have *Mathematics of Social Choice* by Christoph Börgers, *Bond Math* by Donald J. Smith, *An Elementary Introduction to Mathematical Finance* by Sheldon M. Ross, and *E. E. Slutsky as Economist and Mathematician* by Vincent Barnett. A highly original view on mathematics, philosophy, and financial markets is *The Blank Swan* by Elie Ayache. And an important collection of papers concerning statistical judgment in the real world is David A. Freedman's *Statistical Models and Causal Inference*.

More surprising reaches of mathematics can be found in *Magical Mathematics* by Persi Diaconis and Ron Graham, *Math for the Professional Kitchen* (with many worksheets for your convenience) by Laura Dreesen, Michael Nothnagel, and Susan Wysocki, *The Hidden Mathematics of Sport* by Rob Eastaway and John Haigh, *Face Geometry and Appearance Modeling* by Zicheng Liu and Zhengyou Zhang, *How to Fold It* by Joseph O'Rourke, and *Mathematics for the Environment* by Martin E. Walter. Sudoku comes of (mathematical) age in *Taking Sudoku Seriously* by Jason Rosenhouse and Laura Taalman. More technical books, but still interdisciplinary and accessible, are *Viewpoints: Mathematical Perspective and Fractal Geometry in Art* by Marc Frantz and Annalisa Crannell and *Infinity: New Research Frontiers,* edited by Michael Heller and W. Hugh Woodin.

Many new books have been published recently in mathematics education, too many to mention them all. Several titles that caught my attention are Tony Brown's *Mathematics Education and Subjectivity*, Hung-Hsi Wu's unlikely voluminous *Understanding Numbers in Elementary School Mathematics*, Judith E. Jacobs' *A Winning Formula for Mathematics Instruction*, as well as *Upper Elementary Math Lessons* by Anna O. Graeber and her collaborators, *The Shape of Algebra in the Mirrors of Mathematics* by Gabriel Katz and Vladimir Nodelman, and *Geometry: A Guide for Teachers* by Judith and Paul Sally. Keith Devlin offers an original view of the connections between computer games and mathematics learning in *Mathematics Education for a New Era.* Among the many volumes at the National Council of Teachers of Mathematics, notable is the 73rd NCTM Yearbook, *Motivation and Disposition*, edited by Daniel J. Brahier and William R. Speer; *Motivation Matters and Interest Counts* by James Middleton and Amanda Jansen; and *Disrupting Tradition* by William Tate and colleagues. NCTM also publishes many books to support the professional development of mathematics teachers. A good volume for preschool teachers is *Math from Three to Seven* by Alexander Zvonkin. With an international perspective are *Russian Mathematics Education,* edited by Alexander Karp and Bruce R. Vogeli; *International Perspectives on Gender and Mathematics Education,* edited by Helen J. Forgasz and her colleagues; and *Teacher Education Matters* by William H. Schmidt and his colleagues. *Mathematics Teaching and Learning Strategies in PISA*, published by the Organisation for Economic Co-operation and Development, contains a wealth of statistics on global mathematics education.

An excellent volume at the intersection of brain research, psychology, and education, with several contributions focused on learning mathematics, is *The Adolescent Brain,* edited by Valerie F. Reyna and her collaborators.

Besides the historical biographies mentioned above, several other contributions to the history of mathematics are worth enumerating. Among thematic histories are Ranjan Roy's *Sources in the Development of Mathematics*, a massive and exhaustive account of the growth of the theory of series and products; *Early Days in Complex Dynamics* by Daniel S. Alexander and collaborators; *The Origin of the Logic of Symbolic Mathematics* by Burt C. Hopkins; *Lobachevski Illuminated* by Seth Braver; *Mathematics in Victorian Britain,* edited by Raymond Flood and collaborators; *Journey through Mathematics* by Enrique A. González-Velasco; and

Histories of Computing by Michael Sean Mahoney. Two remarkable books that weave the history of mathematics and European arts are *Between Raphael and Galileo* by Alexander Marr and *The Passionate Triangle* by Rebecca Zorach.

Several historical editions are newly available, for instance, Lobachevsky's *Pangeometry,* translated and edited by Athanase Papadopoulos; *80 Years of Zentralblatt MATH,* edited by Olaf Teschke and collaborators; and Albert Lautman's *Mathematics, Ideas, and the Physical Real*. Other historical works are *The Theory That Would Not Die* by Sharon Bertsch McGrayne, *From Cardano's Great Art to Lagrange's Reflections* by Jacqueline Stedall, *Chasing Shadows* by Clemency Montelle, and *World in the Balance* by Robert P. Crease.

In philosophy of mathematics, a few books concern personalities: *After Gödel* by Richard Tieszen; *Kurt Gödel and the Foundations of Mathematics,* edited by Matthias Baaz et al.; *Spinoza's Geometry of Power* by Valtteri Viljanen; *Bolzano's Theoretical Philosophy* by Sandra Lapointe; and *New Essays on Peirce's Mathematical Philosophy,* edited by Matthew E. Moore. Other recent volumes on the philosophy of mathematics and its history are Paolo Mancosu's *The Adventure of Reason*, Paul M. Livingston's *The Politics of Logic*, Gordon Belot's *Geometric Possibility*, and *Fundamental Uncertainty,* edited by Silva Marzetti Dall'Aste Brandolini and Roberto Scazzieri.

Mathematics meets literature in William Goldbloom Bloch's *The Unimaginable Mathematics of Borges' Library of Babel* and, in a different way, in *All Cry Chaos* by Leonard Rosen (where the murder of a mathematician is pursued by a detective called Henri Poincaré). Hans Magnus Enzensberger, the German writer who authored the very successful book *The Number Devil*, has recently published the tiny booklet *Fatal Numbers*.

For other titles the reader is invited to check the introduction to the previous volumes of *The Best Writing on Mathematics*.

As usual, at the end of the introduction I mention several interesting websites. A remarkable bibliographic source is the online list of references on Benford's Law organized by Arno Berger, Theodore Hill, and Erika Rogers (http://www.benfordonline.net/). Other good topic-oriented websites are the MacTutor History of Mathematics archive from the University of St. Andrews in Scotland (http://www-history.mcs.st-and.ac.uk/), Mathematicians of the African Diaspora (MAD) (http://www.math.buffalo.edu/mad/), the Famous Curves index (http://www

-history.mcs.st-and.ac.uk/Curves/Curves.html), the National Curve Bank (http://curvebank.calstatela.edu/index/index.htm), and Free Mathematics Books (http://www.e-booksdirectory.com/mathematics.php). An intriguing site dedicated to the work of Alexandre Grothendieck, one of the most intriguing mathematicians alive, is the Grothendieck Circle (http://www.grothendieckcircle.org/). An excellent website for mathematical applications in science and engineering is Equalis (http://www.equalis.com/). Among websites with potential for finding materials for mathematical activities are the one on origami belonging to Robert Lang, a contributor to this volume (http://www.langorigami.com/index.php4); many other Internet sources for the light side of mathematics can be found conveniently on the personal page maintained by Greg Frederickson of Purdue University (http://www.cs.purdue.edu/homes/gnf/hotlist.html).

I hope you, the reader, find the same value and excitement in reading the texts in this volume as I found while searching, reading, and selecting them. For comments on this book and to suggest materials for consideration in preparing future volumes, I encourage you to send correspondence to me: Mircea Pitici, P.O. Box 4671, Ithaca, NY 14852.

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truths? Many great mathematicians—including David Hilbert, Georg Cantor, and quite a few of the group known as Nicolas Bourbaki—have shared Einstein's view, associated with a school of thought called Formalism. But other illustrious thinkers—among them Godfrey Harold Hardy, Roger Penrose, and Kurt Gödel—have held the opposite view, Platonism.

This debate about the nature of mathematics rages on today and seems to elude an answer. I believe that by asking simply whether mathematics is invented or discovered, we ignore the possibility of a more intricate answer: both invention and discovery play a crucial role. I posit that together they account for why math works so well. Although eliminating the dichotomy between invention and discovery does not fully explain the unreasonable effectiveness of mathematics, the problem is so profound that even a partial step toward solving it is progress.

Invention and Discovery

Mathematics is unreasonably effective in two distinct ways, one I think of as active and the other as passive. Sometimes scientists create methods specifically for quantifying real-world phenomena. For example, Isaac Newton formulated calculus largely for the purpose of capturing motion and change, breaking them up into infinitesimally small frame-by-frame sequences. Of course, such active inventions are effective; the tools are, after all, made to order. What is surprising, however, is their stupendous accuracy in some cases. Take, for instance, quantum electrodynamics, the mathematical theory developed to describe how light and matter interact. When scientists use it to calculate the magnetic moment of the electron, the theoretical value agrees with the most recent experimental value—measured at 1.00115965218073 in the appropriate units in 2008—to within a few parts per trillion!

Even more astonishing, perhaps, mathematicians sometimes develop entire fields of study with no application in mind, and yet decades, even centuries, later physicists discover that these very branches make sense of their observations. Examples of this kind of passive effectiveness abound. French mathematician Évariste Galois, for example, developed group theory in the early 1800s for the sole purpose of determining the solvability of polynomial equations. Very broadly, groups are algebraic structures made up of sets of objects (say, the integers) united under the edges of individual objects and at distinguishing between straight and curved lines and between different shapes, such as circles and ellipses—abilities that probably led to the development of arithmetic and geometry. So, too, the repeated human experience of cause and effect at least partially contributed to the creation of logic and, with it, the notion that certain statements imply the validity of others.

Selection and Evolution

Michael Atiyah, one of the greatest mathematicians of the 20th century, has presented an elegant thought experiment that reveals just how perception colors which mathematical concepts we embrace—even ones as seemingly fundamental as numbers. German mathematician Leopold Kronecker famously declared, "God created the natural numbers, all else is the work of man." But imagine if the intelligence in our world resided not with humankind but rather with a singular, isolated jellyfish, floating deep in the Pacific Ocean. Everything in its experience would be continuous, from the flow of the surrounding water to its fluctuating temperature and pressure. In such an environment, lacking individual objects or indeed anything discrete, would the concept of number arise? If there were nothing to count, would numbers exist?

Like the jellyfish, we adopt mathematical tools that apply to our world—a fact that has undoubtedly contributed to the perceived effectiveness of mathematics. Scientists do not choose analytical methods arbitrarily but rather on the basis of how well they predict the results of their experiments. When a tennis ball machine shoots out balls, you can use the natural numbers 1, 2, 3, and so on, to describe the flux of balls. When firefighters use a hose, however, they must invoke other concepts, such as volume or weight, to render a meaningful description of the stream. So, too, when distinct subatomic particles collide in a particle accelerator, physicists turn to measures such as energy and momentum and not to the end number of particles, which would reveal only partial information about how the original particles collided because additional particles can be created in the process.

Over time, only the best models survive. Failed models—such as French philosopher René Descartes's attempt to describe the motion of the planets by vortices of cosmic matter—die in their infancy. In contrast, successful models evolve as new information becomes available.

For instance, very accurate measurements of the precession of the planet Mercury necessitated an overhaul of Newton's theory of gravity in the form of Einstein's general relativity. All successful mathematical concepts have a long shelf life: The formula for the surface area of a sphere remains as correct today as it was when Archimedes proved it around 250 BC. As a result, scientists of any era can search through a vast arsenal of formalisms to find the most appropriate methods.

Not only do scientists cherry-pick solutions, they also tend to select problems that are amenable to mathematical treatment. There exists, however, a whole host of phenomena for which no accurate mathematical predictions are possible, sometimes not even in principle. In economics, for example, many variables—the detailed psychology of the masses, to name one—do not easily lend themselves to quantitative analysis. The predictive value of any theory relies on the constancy of the underlying relations among variables. Our analyses also fail to fully capture systems that develop chaos, in which the tiniest change in the initial conditions may produce entirely different end results, prohibiting any long-term predictions. Mathematicians have developed statistics and probability to deal with such shortcomings, but mathematics itself is limited, as Austrian logician Gödel famously proved.

Symmetry of Nature

This careful selection of problems and solutions only partially accounts for the success of mathematics in describing the laws of nature. Such laws must exist in the first place! Luckily for mathematicians and physicists alike, universal laws appear to govern our cosmos: An atom 12 billion light-years away behaves just like an atom on Earth; light in the distant past and light today share the same traits; and the same gravitational forces that shaped the universe's initial structures hold sway over present-day galaxies. Mathematicians and physicists have invented the concept of symmetry to describe this kind of immunity to change.

The laws of physics seem to display symmetry with respect to space and time: They do not depend on where, from which angle, or when we examine them. They are also identical to all observers, irrespective of whether these observers are at rest, moving at constant speeds, or accelerating. Consequently, the same laws explain our results, whether the experiments occur in China, Alabama, or the Andromeda galaxy—and whether we conduct our experiment today or someone else does a billion years from now. If the universe did not possess these symmetries, any attempt to decipher nature's grand design—any mathematical model built on our observations—would be doomed because we would have to continuously repeat experiments at every point in space and time.

Even more subtle symmetries, called gauge symmetries, prevail within the laws that describe the subatomic world. For instance, because of the fuzziness of the quantum realm, a given particle can be a negatively charged electron or an electrically neutral neutrino, or a mixture of both—until we measure the electric charge that distinguishes between the two. As it turns out, the laws of nature take the same form when we interchange electrons for neutrinos or any mix of the two. The same holds true for interchanges of other fundamental particles. Without such gauge symmetries, it would have been difficult to provide a theory of the fundamental workings of the cosmos. We would be similarly stuck without locality—the fact that objects in our universe are influenced directly only by their immediate surroundings rather than by distant phenomena. Thanks to locality, we can attempt to assemble a mathematical model of the universe much as we might put together a jigsaw puzzle, starting with a description of the most basic forces among elementary particles and then building on additional pieces of knowledge.

Our current best mathematical attempt at unifying all interactions calls for yet another symmetry, known as supersymmetry. In a universe based on supersymmetry, every known particle must have an as-yet undiscovered partner. If such partners are discovered (for instance, once the Large Hadron Collider at CERN near Geneva reaches its full energy), it will be yet another triumph for the effectiveness of mathematics.

I started with two basic, interrelated questions: Is mathematics invented or discovered? And what gives mathematics its explanatory and predictive powers? I believe that we know the answer to the first question: Mathematics is an intricate fusion of inventions and discoveries. Concepts are generally invented, and even though all the correct relations among them existed before their discovery, humans still chose which ones to study. The second question turns out to be even more complex. There is no doubt that the selection of topics we address mathematically has played an important role in math's perceived effectiveness. But mathematics would not work at all were there no universal features to be discovered. You may now ask: Why are there universal laws of nature at all? Or equivalently: Why is our universe governed by certain symmetries and by locality? I truly do not know the answers, except to note that perhaps in a universe without these properties, complexity and life would have never emerged, and we would not be here to ask the question.

More to Explore

The Unreasonable Effectiveness of Mathematics in the Natural Sciences. Eugene Wigner in *Communications in Pure and Applied Mathematics*, Vol. 13, No. 1, pages 1–14; February 1960.

Pi in the Sky: Counting, Thinking, and Being. John D. Barrow. Back Bay Books, 1992.

Creation v. Discovery. Michael Atiyah in *Times Higher Education Supplement*, September 29, 1995.

Is God a Mathematician? Mario Livio. Simon & Schuster, 2009.

Scientific American Online, review of *Is God a Mathematician?* 2009.

favour of Platonism only needs *some* of mathematics to be discovered: If it turns out that there are two broad kinds of mathematics, then perhaps one can understand the distinction and formulate more precisely what mathematical discovery (as opposed to the mere producing of mathematics) is.

As the etymology of the word “discover” suggests, we normally talk of discovery when we find something that was, unbeknownst to us, already there. For example, Columbus is said to have discovered America (even if one can question that statement for other reasons), and Tutankhamun’s tomb was discovered by Howard Carter in 1922. We say this even when we cannot directly observe what has been discovered: For instance, J. J. Thompson is famous as the discoverer of the electron. Of greater relevance to mathematics is the discovery of facts: We discover *that* something is the case. For example, it would make perfectly good sense to say that Bernstein and Woodward discovered (or contributed to the discovery) that Nixon was linked to the Watergate burglary.

In all these cases, we have some phenomenon, or fact, that is brought to our attention by the discovery. So one might ask whether this transition from unknown to known could serve as a definition of discovery. But a few examples show that there is a little more to it than that. For instance, an amusing fact, known to people who like doing cryptic crosswords, is that the words “carthorse” and “orchestra” are anagrams. I presume that somebody somewhere was the first person to notice this fact, but I am inclined to call it an observation (hence my use of the word “notice”) rather than a discovery. Why is this? Perhaps it is because the words “carthorse” and “orchestra” were there under our noses all the time and what has been spotted is a simple relationship between them. But why could we not say that the relationship is discovered even if the words were familiar? Another possible explanation is that once the relationship is pointed out, one can easily verify that it holds: You don’t have to travel to America or Egypt, or do a delicate scientific experiment, or get access to secret documents.

As far as evidence for Platonism is concerned, the distinction between discovery and observation is not especially important: If you notice something, then that something must have been there for you to notice, just as if you discover it, then it must have been there for you to discover. So let us think of observation as a mild kind of discovery rather than as a fundamentally different phenomenon.

realm and brought the rules into existence. A more appealing picture would be that they selected the rules of cricket from a vast "rule space" that consists of all possible sets of rules (most of which give rise to terrible games). A drawback with this second picture is that it fills up the abstract realm with a great deal of junk, but perhaps it really is like that. For example, it is supposed to contain all the real numbers, all but countably many of which are undefinable.

Another argument against the idea that one brings an abstract concept into existence when one invents it is that the concepts that we invent are not fundamental enough: They tend to be methods for dealing with other objects, either abstract or concrete, that are much simpler. For example, the rules of cricket describe constraints on a set of procedures that are carried out by 22 players, a ball, and two wickets. From an ontological point of view, the players, ball, and wickets seem more secure than the constraints on how they behave.

Earlier, I commented that we do not normally talk of inventing a single work of art. However, we do not discover it either. A commonly used word for what we do is "create." And most people, if asked, would say that this kind of creation has more in common with invention than with discovery, just as observation has more in common with discovery than with invention.

Why is this? Well, in both cases what is brought into existence has many arbitrary features: If we could turn the clock back to just before cricket was invented and run the world all over again, it is likely that we would see the invention of a similar game, but unlikely that its rules would be identical to those of the actual game of cricket. (One might object that if the laws of physics are deterministic, then the world would develop precisely as it did the first time. In that case, one could make a few small random changes before the rerun.) Similarly, if somebody had accidentally destroyed *Les Desmoiselles d'Avignon* just after Picasso started work on it, forcing him to start again, it is likely that he would have produced a similar but perceptibly different painting. By contrast, if Columbus had not existed, then somebody else would have discovered *America* and not just some huge landmass of a broadly similar kind on the other side of the Atlantic. And the fact that "carthorse" and "orchestra" are anagrams is independent of who was the first to observe it.

With these thoughts in mind, let us turn to mathematics. Again, it will help to look at some examples of what people typically say about various famous parts of the subject. Let me list some discoveries, some observations, and some inventions. (I cannot think of circumstances where I would definitely want to say that a piece of mathematics was created.) Later I will try to justify why each item is described in the way it is.

A few well-known discoveries are the formula for the quadratic equation, the absence of a similar formula for the quintic, the monster group, and the fact that there are infinitely many primes. A few observations are that the number of primes less than 100 is 25, that the last digits of the powers of 3 form the sequence 3, 9, 7, 1, 3, 9, 7,1, . . . , and that the number 10,001 factors as 73 times 137. An intermediate case is the fact that if you define an infinite sequence $z_0, z_1, z_2, \ldots$ of complex numbers by setting $z_0 = 0$ and $z_n = z_{n-1}^2 + C$ for every $n > 0$, then the set of all complex numbers C for which the sequence does not tend to infinity, now called the Mandelbrot set, has a remarkably complicated structure. (I regard this as intermediate because, although Mandelbrot and others stumbled on it almost by accident, it has turned out to be an object of fundamental importance in the theory of dynamical systems.)

On the other side, it is often said that Newton and Leibniz independently invented calculus. (I planned to include this example, and was heartened when, quite by coincidence, on the day that I am writing this paragraph, there was a plug for a radio program about their priority dispute, and the word "invented" was indeed used.) One also sometimes talks of mathematical theories (as opposed to theorems) being invented: It does not sound ridiculous to say that Grothendieck invented the theory of schemes, though one might equally well say "introduced" or "developed." Similarly, any of these three words would be appropriate for describing what Cohen did to the method of forcing, which he used to prove the independence of the continuum hypothesis. From our point of view, what is interesting is that the words "invent," "introduce," and "develop" all carry with them the suggestion that some general technique is brought into being.

A mathematical object about which there might be some dispute is the number i, or more generally the complex number system. Were complex numbers discovered or invented? Or rather, would mathematicians normally refer to the arrival of complex numbers into mathematics using a discovery-type word or an invention-type word? If you type the phrases "complex numbers were invented" and "complex numbers were discovered" into Google, you get approximately the same number of hits (between 4,500 and 5,000 in both cases), so there appears to be no clear answer. But this too is a useful piece of data. A similar example is non-Euclidean geometry, though here "discovery of non-Euclidean geometry" outnumbers "invention of non-Euclidean geometry" by a ratio of about 3 to 1.

Another case that is not clear-cut is that of *proofs*: Are they discovered or invented? Sometimes a proof seems so natural—mathematicians often talk of "the right proof" of a statement, meaning not that it is the only correct proof but that it is the one proof that truly explains why the statement is true—that the word "discover" is the obvious word to use. But sometimes it feels more appropriate to say something like, "Conjecture 2.5 was first proved in 1990, but in 2002 Smith came up with an ingenious and surprisingly short argument that actually establishes a slightly more general result." One could say "discovered" instead of "came up with" in that sentence, but the latter captures better the idea that Smith's argument was just one of many that there might have been and that Smith did not simply stumble on it by accident.

Let us take stock at this point, and see whether we can explain what it is about a piece of mathematics that causes us to put it into one of the three categories: discovered, invented, or not clearly either.

The nonmathematical examples suggest that discoveries and observations are usually of objects or facts over which the discoverer has no control, whereas inventions and creations are of objects or procedures with many features that could be chosen by the inventor or creator. We also drew some more refined, but less important, distinctions within each class. A discovery tends to be more notable than an observation and less easy to verify afterward. And inventions tend to be more general than creations.

Do these distinctions continue to hold in much the same form when we come to talk about mathematics? I claimed earlier that the formula for the quadratic was discovered, and when I try out the phrase "the invention of the formula for the quadratic," I find that I do not like it, for exactly the reason that the solutions of $ax^2 + bx + c$ *are* the numbers $(-b \pm \sqrt{b^2 - 4ac})/2a$. Whoever first derived that formula did not have any choice about what the formula would eventually be. It is, of course, possible to notate the formula differently, but that is another matter. I do not want to get bogged down in a discussion of what it means for two formulas to be "essentially the same," so let me simply say that the formula itself was a discovery but that different people have *come up with* different ways of expressing it. However, this kind of concern will reappear when we look at other examples.

The insolubility of the quintic is another straightforward example. *It is* insoluble by radicals, and nothing Abel did could have changed that. So his famous theorem was a discovery. However, aspects of his *proof* would be regarded as invention—there have subsequently been different looking proofs. This notion is particularly clear with the closely related work of Galois, who is credited with the invention of group theory. (The phrase "invention of group theory" has 40,300 entries in Google, compared with 10 for "discovery of group theory.")

The monster group is a more interesting case. It first entered the mathematical scene when Fischer and Griess predicted its existence in 1973. But what does that mean? If they could refer to the monster group at all, then does that not imply that it existed? The answer is simple: They predicted that a group with certain remarkable *properties* (one of which is its huge size—hence the name) existed and was unique. So to say "I believe that the monster group exists" was shorthand for "I believe that there exists a group with these amazing properties," and the name "monster group" was referring to a hypothetical entity.

The existence and uniqueness of the monster group were indeed proved, though not until 1982 and 1990, respectively, and it is not quite clear whether we should regard this mathematical advance as a discovery or an invention. If we ignore the story and condense 17 years to an instant, then it is tempting to say that the monster group was there all along until it was discovered by group theorists. Perhaps one could even add a little detail: Back in 1973, people started to have reason to suppose that it existed, and they finally bumped into it in 1982.

But how did this "bumping" take place? Griess did not prove in some indirect way that the monster group had to exist (though such proofs are possible in mathematics). Rather, he *constructed* the group. Here, I am using the word that all mathematicians would use. To construct it, he constructed an auxiliary object, a complicated algebraic structure now known as the Griess algebra, and showed that the symmetries of this algebra formed a group with the desired properties. However, this method is not the only way of obtaining the monster group: There are other constructions that give rise to groups that have the same properties, I do not have a complete answer to this question, but I suspect that the reason it is a somewhat difficult example is similar to the reason that the monster group is difficult, which is that one can “construct” the complex numbers in more than one way. One approach is to use something like the way they were constructed historically (my knowledge of the history is patchy, so I shall not say *how* close the resemblance is). One simply introduces a new symbol, i, and declares that it behaves much like a real number, obeying all the usual algebraic rules, and has the additional property that $i^2 = -1$. From this setup, one can deduce that

$$(a+bi)(c+di) = ac + bci + adi + bdi^2 = (ad - bd) + (ad + bc)i$$

and many other facts that can be used to build up the theory of complex numbers. A second approach, which was introduced much later to demonstrate that the complex number system was consistent if the real number system was, is to define a complex number to be an ordered pair (a, b) of real numbers, and to stipulate that addition and multiplication of these ordered pairs are given by the following rules:

$$\begin{aligned}(a,b)+(c,d) &= (a+c, b+d)\\(a,b)+(c,d) &= (ac-bd, ad+bc)\end{aligned}$$

This second method is often used in university courses that build up the number systems rigorously. One proves that these ordered pairs form a field under the two given operations, and finally one says, “From now on I shall write $a + bi$ instead of (a, b).”

Another reason for our ambivalence about the complex numbers is that they feel less real than real numbers. (Of course, the names given to these numbers reflect this notion rather unsubtly.) We can directly relate the real numbers to quantities such as time, mass, length, temperature, and so on (though for this usage, we never need the infinite precision of the real number system), so it feels as though they have an independent existence that we observe. But we do not run into the complex numbers in that way. Rather, we play what feels like a sort of game—imagine what would happen if -1 *did* have a square root.

But why in that case do we not feel happy just to say that the complex numbers were invented? The reason is that the game is much more interesting than we had any right to expect, and it has had a huge influence even on those parts of mathematics that are about real numbers or me point out just one source of choice and arbitrariness: Often a proof requires one to show that a certain mathematical object or structure exists (either as the main statement or as some intermediate lemma), and often the object or structure in question is far from unique.

Before drawing any conclusions from these examples, I would like to discuss briefly another aspect of the question. I have been looking at it mainly from a linguistic point of view, but, as I mentioned right at the beginning, it also has a strong psychological component: When one is doing mathematical research, it sometimes feels more like discovery and sometimes more like invention. What is the difference between the two experiences?

Since I am more familiar with myself than with anybody else, let me draw on my own experience. In the mid-1990s, I started on a research project that has occupied me in one way or another ever since. I was thinking about a theorem that I felt ought to have a simpler proof than the two that were then known. Eventually, I found one (here I am using the word that comes naturally); unfortunately it was not simpler, but it gave important new information. The process of finding this proof felt much more like discovery than invention because by the time I reached the end, the structure of the argument included many elements that I had not even begun to envisage when I started working on it. Moreover, it became clear that there was a large body of closely related facts that added up to a coherent and yet-to-be-discovered theory. (At this stage, they were not proved facts, and not always even precisely stated facts. It was just clear that "something was going on" that needed to be investigated.) I and several others have been working to develop this theory, and theorems have been proved that would not even have been stated as conjectures 15 years ago.

Why did this work feel like discovery rather than invention? Once again, it is connected with control: I was not selecting the facts I happened to like from a vast range of possibilities. Rather, certain statements stood out as obviously natural and important. Now that the theory is more developed, it is less clear which facts are central and which more peripheral, and for that reason, the enterprise feels as though it has an invention component as well.

A few years earlier, I had a different experience: I found a counterexample to an old conjecture in the theory of Banach spaces. To do this, I constructed a complicated Banach space. This construction felt partly like an invention—I did have arbitrary choices, and many other counterexamples have subsequently been found—and partly like a discovery—much of what I did was in response to the requirements of the problem and felt like the natural thing to do, and a similar example was discovered independently by someone else (and even the later examples use similar techniques). So this is another complicated situation to analyze, but the reason it is complicated is simply that the question of how much control I had is a complicated one.

What conclusion should we draw from all these examples and from how we naturally seem to regard them? First, it is clear that the question with which we began is rather artificial. For a start, the idea that either all of mathematics is discovered or all of mathematics is invented is ridiculous. But even if we look at the origins of individual pieces of mathematics, we are not forced to use the word "discover" or "invent," and we often don't.

Nevertheless, there does seem to be a spectrum of possibilities, with some parts of mathematics feeling more like discoveries and others more like inventions. It is not always easy to say which are which, but there does seem to be one feature that correlates strongly with whether we prefer to use a discovery-type word or an invention-type word. That feature is the control that we have over what is produced. This feature, as I have argued, even helps to explain why the doubtful cases are doubtful.

If this difference is correct (perhaps after some refinement), what philosophical consequences can we draw from it? I suggested at the beginning that the answer to the question did not have any bearing on questions such as "Do numbers exist?" or "Are mathematical statements true because the objects they mention really do relate to each other in the ways described?" My reason for that suggestion is that pieces of mathematics have objective features that explain how much control we have over them. For instance, as I mentioned earlier, the proof of an existential statement may well be far from unique, for the simple reason that there may be many objects with the required properties. But this statement is a straightforward mathematical phenomenon. One could accept my analysis and believe that the objects in question "really exist," or one could view the statements that they exist as moves in games played with marks on paper, or one could regard the objects as convenient fictions. The fact that some parts of mathematics are unexpected