The BEST WRITING on MATHEMATICS

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Introduction

Mircea Pitici

Welcome to reading the fifth anthology in our series of recent writings on mathematics. Almost all the pieces included here were first published during 2013 in periodicals, as book chapters, or online.

Much is written about mathematics these days, and much of it is good, amid a fair amount of uninspired writing. What does and what doesn't qualify as good is sometimes difficult to decide. I have learned that strictly normative criteria for what "good writing" on mathematics should be are of little utility, and in fact, might be counterproductive. Good writing comes in many styles. I know it when I see it, and I might see it differently from the way you see it. Making a selection called "the best" is inevitably subjective and inevitably doubles into an elimination procedure—a combination of successive positive and negative evaluations, always comparative, guided by the preferences, competence, and probity of the people involved, as well as by publishing and time constraints. I have described elsewhere how we reach the table of contents each year, but I find it useful to retell briefly the main steps of the process.

Driven by curiosity and by personal interest, I survey an enormous amount of academic and popular literature. I have done so for decades, even when access to bibliographic sources was much more difficult than it is today. I enjoy pondering what people say, making up my mind about what I read, and making up my mind independently of what I read. When pieces have a chance to be included in our anthology I take note and I put them aside, coming up with many titles each year (see the Notable Writings section at the end of the volume). From this broad collection I leave out pieces too long for the economy of our book or unaffordable because of copyright or reprinting issues. At this stage I usually look at a few dozen articles and I advance them to the publisher.

Princeton University Press asks independent reviewers to read and rate the articles, a procedure meant to guide us further. The reviewers always agree on some pieces but sharply disagree on others. When I consider the reports I also have in mind some goals I envisioned for the series well before I even found a publisher for this enterprise.

I want accessible but nontrivial content that presents for mathematicians and for the general public a wide assortment of informed and insightful perspectives on pure and applied mathematics, on the historical and philosophical issues related to mathematics, on topics related to the learning and the teaching of mathematics, on the practice and practicality of mathematics, on the social and institutional aspects in which mathematics grows and thrives, or on other themes related to mathematics. I aim to offer each volume as an instrument for gaining nondogmatic views of mathematics, impervious to indoctrination—as much as possible inclusive, panoramic, comprehensive, and reflecting a multitude of viewpoints that see through the apodictic nature of mathematics. No doubt some people dread this diversity of viewpoints when thinking about mathematics and adopt a defensive stance, retreating into overspecialization, frightened by opinions that upset their views on mathematics or by the specter of (what they consider to be) dilettantism. For what it is worth, *The Best Writing on Mathematics* series is meant as an antidote to the contagious power that emanates from such fears.

The final content of each book in the series is dependent on the literature available during the latest calendar year; therefore, each volume reflects to some extent the vagaries of fashion and, on the other hand, it is deprived of the topics temporarily out of favor with writers on mathematics. Thus each volume should properly be viewed in conjunction with the other volumes in the series.

Books are more important for what they make us think than for what we read in them. If we manage to destroy humanity but to leave all the books intact, there would be no consolation (except for the surviving worms and ants). Books are chances for the potential unknown of our imagination; their worth consists not only in their literal content but also in what they make us reflect—immediately or later, in accord with or in opposition to what we read. Books on mathematics, and mathematical books proper, have a special place in history. Over the past several centuries mathematical ideas, in conjunction with the enterprising élan of explorers, innovators, tinkerers, and the common folk, have served as catalysts for uncountable discoveries (e.g., geographic, technological, scientific, military, and domestic) and for the expansion of life possibilities, thus contributing to human phenotypic diversity.

At first thought opining on mathematics seems benign, having only an upside; yet it is neither trivial nor free of danger. Writing about mathematics is a form of interpreting mathematics; and interpreting mathematics, whether "elementary" or "advanced" mathematics, is not trifling, even for seasoned mathematicians. Interpreting mathematics leaves room for genuine differences of opinion; it allows you to stand your ground even if everybody else disagrees with you—and it's all about *mathematics,* a thinking domain on which people are supposed to agree, not to dissent. Private interpretation of mathematics is different from doing mathematics; it is a mark of personal worldview. In that sense it can indeed be dangerous and unsettling on occasion, but it is always rewarding as a full-mind activity free of the trappings that plague most institutionalized mathematics instruction (e.g., rote memorizing, repetitive learning, following strict rules, inside-the-box thinking, and dumb standardized assessment). Interpreting mathematics allows individuals to build niches in the social milieu by using unique thinking and acting features. For such a purpose mathematics is an inexhaustible resource, similar to but more encompassing than art. The freedom to interpret mathematics as I please compensates for the constraints inherent in the conventional content accepted by the prevailing communitarian view of mathematics. By interpreting, I recoup the range of imaginative possibilities that I gave up when submitting to the compelling rigor enforced by a chain of mathematical arguments. Interpreting mathematics supports my confidence to act not only on my supposed knowledge but also on my ignorance. It enables me to appropriate an epistemological payoff arrangement better than any other I can imagine, since my ignorance will always dwarf my hopelessly limited knowledge. Thus interpreting mathematics breaks paths toward opportunities but also opens doors to peril.

I learned firsthand, at great expense, that voicing opinions about mathematics is risky. I have seen it; I lived it. Long ago—a recent emigrant, poor and naive but buoyantly optimistic—I voiced some of my thoughts on the use and misuse of mathematics during classes at a leading business school. The displeased reaction to what I considered common-sense observations puzzled and befuddled me. To my dismay

I saw that complying, obeying, and blindly turning in assignments—*that* mattered, *that* was expected and appreciated, not hard thinking about the issues at hand. The more I spoke, the stranger the atmosphere became. For a while I lingered there surrounded by bristling silence, polite condescension, and impatient tolerance. The denouement came when a faculty member asked me why I went to that school if I thought I was so smart. The message, one of the many ensconced in that interpellation, was unmistakable: If you happen to have some ideas (and made the dumb error of coming here by borrowing a lot of money), better be careful how you handle them. The next morning I was out of there, at a staggering financial loss.

That episode had long-lasting consequences for me, highlighting the fickle rhetoric of the slogans encouraging free inquiry and open dialogue of ideas. When recurring effects and escalating misadventures added up over the ensuing years, I became a lot more cautious with what I said—yet matters kept slipping out of control, from terrible to worse, to desperate, until I chose not to say much anymore. If speaking up on mathematics proved to be such a disaster, fortunately I accomplished the next to "best" thing for me—the chance to give exposure to other people's diverse views on mathematics, in the volumes of *The Best Writing on Mathematics* series.

In conclusion to this preamble, I intend to include here pieces mostly *about* mathematics, not necessarily mathematical writing, although a bit of mathematical exposition makes its way into each volume. Expository mathematical writing is scrutinized, recognized, and rewarded by publication in the mathematics journals, consideration for professional awards, republication in extended versions as monographs, and circulation among the mathematical community. But writings *about* mathematics are spread wide and thin, in venues less frequented by mathematicians; my goal is to find what is notable in that literature and to make it easy for people to read it or at least to find it on their own, quickly and conveniently. In the books of this series you will meet a deliberate medley of styles, sources, and perspectives. With this choice for the content we succeed in placing in bookstores volumes that include authors who previously were known only inside the mathematical community and in circulating among mathematicians the names of interesting authors who care about mathematics from the outside.

Contents of the Volume

In the opening piece of the selection, Stephen Pollard reviews John Dewey's conception of human experience and notes that, in the philosopher's views, mathematics and its practice offer an integrative function that transcends the utilitarian goals commonly bestowed on mathematics by enhancing the minds and the lives of the people who study it.

Kenneth Cukier and Viktor Mayer-Schönberger discuss some of the consequences of the recent explosion in digital information, pointing out the qualitative shift it brings to data use and analysis, and concluding that these trends are radically changing our lives, work, and thinking about the world.

Tanya Khovanova poses and solves an ingenious puzzle invented decades ago by John Conway.

In the next piece John Conway observes that some true arithmetical statements are not provable and gives unsophisticated examples that support his assertion.

Brian Hayes traces the history of space-filling curves and offers surprising applications of the counterintuitive process that leads to their construction.

Lav Varshney and John Sun observe that nonlinear, logarithmic scaling is natural to our perception (but distorted into linear scaling by institutionalized education) and contend that, by reducing estimative errors, perceiving logarithmically confers evolutionary advantages.

Keith Devlin formulates negative and positive criteria for evaluating the instructive or noninstructive qualities of games that purport to help children learn mathematics and concludes that most games currently available do not deliver on the educational side.

Bahman Kalantari and Bruce Torrence describe certain features of the "graph" of a polynomial with a complex variable; then they use this intuitive geometrical guidance to draw the main lines in the chain of arguments that proves that any polynomial can be factored into a number of linear factors equal to its degree.

Nicole Lazar reviews the virtues and the pitfalls of using statistical methods to analyze and to create art.

Carlo Séquin catalogs varieties of geometrical shapes similar to a Klein bottle (a geometrical surface with a single face) and briefly indicates the mathematical elements that can help classify them rigorously.

Also starting with a Klein bottle surface, sarah-marie belcastro tackles issues related to the knitting of mathematical objects and illustrates her comments with photographs of needlework.

Marshall Gordon details the activities he developed for teaching eleventh graders the engineering applications of the mathematical study of quadratic quantities and relates the response these activities elicited from the students, stressing that his "multiple-centered" instruction reaches all students according to their levels of interest.

Penelope Dunham brings food into the classroom, not only to facilitate the intuitive understanding of the mathematical concepts encountered in a variety of undergraduate courses she devised but also to stimulate and reward the students.

Dov and Rina Zazkis find that the wonder part of mathematics resides in *doing* mathematics more than in contemplating mathematical results and present examples that link the nature of wonder in mathematics to encountering surprise, counterintuition, and the unexpected.

Francis Su's blog entry, based on a talk he gave as an awardee, is an impassioned plea for renouncing the infatuation with achievement and performance in education and focusing instead on the irreducible humanity that often goes missing in the interaction between teacher and student.

Uri Leron and Orit Zaslavsky reflect on the strengths, limitations, and educational value of *generic proofs*—that is, mathematical proofs based on an example that serves as a springboard for the main ideas supporting the generalization for all cases represented by the example.

John Conway and Joseph Shipman distinguish among different proofs of the same mathematical result based on proofs' dominant features, then exemplify what they call the *proof space* by comparing six ways of proving that the square root of the number 2 is not a proportion of integers.

Michael Barany shows that peculiar conservative inclinations led Augustin-Louis Cauchy to fuse algebraic and geometric thinking, thus inadvertently igniting the theoretical impulse that led to the rigor of modern mathematics.

Lawrence Brenton tells the long story of the speculation concerning the shape of the universe—and how its many avatars relate to geometrical ideas, old and new.

Mark Braverman explains the structural incompatibility between real numbers, which we represent intuitively as a continuous line, and the discrete nature of calculation achievable by mechanical computations. Then he examines the implications for the study of dynamical systems.

Adilson Motter and David Campbell start off by describing the deterministic mind-set characteristic of the exact sciences until about the middle of the twentieth century and continue by tracing the inroads made over the past half-century by the new chaos paradigm and its role in dynamical systems theory.

Roberto Behar, Pere Grima, and Lluís Marco-Almagro list and comment briefly on analogies they find useful for teaching statistics to students who take an introductory course.

David Gale and Lloyd Shapley's article is the only one in this volume that was first published long ago—although attention to it rekindled recently. Gale and Shapley found a well-determined and optimal solution to the problem of assigning people to institutions under conditions of uncertainty.

Jordan Ellenberg reports on an important theoretical breakthrough, Yitang Zhang's proof of the bounded gaps conjecture, and muses on the consequences it might have for the theory of numbers and for our knowledge of randomness.

More Writings on Mathematics

Writing on mathematics for a nonspecialist audience is now a full-time profession for some authors and a pastime for many others. The market is huge and growing. With the caveat that the following list covers only a small part of the publishing output in this area, here are some books that came to my attention over the past year.

I start by noting a volume meant as an overview of the state of mathematics in the United States and as a benchmark for future policy directions, *The Mathematical Sciences in 2025* written by a committee of the National Research Council (for accurate and complete references, please see the list of Books Mentioned at the end of the Introduction).

A book on some of the most important problems in mathematics and physics is Ian Stewart's *Visions of Infinity*. Other presentations of mathematical topics and/or how they spring up in life can be found in Paul Lockhart's *Measurement*, Göran Grimvall's *Quantify!*, Yutaka

Nishiyama's *The Mysterious Number 6174*, Günter Ziegler's *Do I Count?*, Daniel Tammet's *Thinking in Numbers*, David MacNeil's *Fundamentals of Modern Mathematics*, and the nicely illustrated *Mathematics* edited by Tom Jackson. A pocket-size compendium of mathematical notions and results, briefly stated, described, and illustrated, is *Math in Minutes* by Paul Glendinning. An anthology edited by Gina Kolata, *The New York Times Book of Mathematics*, gathers articles published by the newspaper over more than a century. An instructive etymological tool is *Origins of Mathematical Words* by Anthony Lo Bello. And a skeptical view of the powers of mathematical knowledge is laid out by Noson Yanofsky in *The Outer Limits of Reason*.

More technical books with chapters or parts accessible to nonspecialist readers are *The Tower of Hanoi* by Andreas Hinz and his colleagues, *Configurations from a Graphical Viewpoint* by Tomaž Pisanski and Brigitte Servatius, *Magnificent Mistakes in Mathematics* by Alfred Posamentier and Ingmar Lehmann, *The Joy of Factoring* by Samuel Wagstaff, and even Terence Tao's *Compactness and Contradiction*.

Informal introductions to statistics and probabilities are *Will You Be Alive 10 Years from Now?* by Paul Nahin, *The Basics of Data Literacy* by Michael Bowen and Anthony Bartley, *The Cartoon Introduction to Statistics* by Grady Klein and Alan Dabney, and *Dancing on the Tails of the Bell Curve* edited by Richard Altschuler.

Mathematicians have always been fond of telling their own life stories and the ideas that animated them—or others found that it is worth writing about mathematicians' lives and their ideas. The autobiographical literature is picking up speed, with the late Martin Gardner's *Undiluted Hocus-Pocus*, Benoit Mandelbrot's *The Fractalist*, Edward Frenkel's *Love and Math*, Larry Baggett's *In the Dark on the Sunny Side*, and *An Accidental Statistician* by George Box. Key autobiographical elements are also present in Reuben Hersh's anthology of articles *Experiencing Mathematics*. Phillip Schewe has written Freeman Dyson's story as *Maverick Genius*. A historical biography on Ada Lovelace is James Essinger's *A Female Genius*; others are Dirk van Dalen's *L. E. J. Brouwer* and *Vito Volterra* by Angelo Guerraggio and Giovanni Paoloni. A collection of historical studies is *Robert Recorde*, edited by Gareth Roberts and Fenny Smith. A children's book on Paul Erdős is *The Boy Who Loved Math* by Deborah Heiligman. Two books on the history of mathematics at Harvard, with emphasis on personalities, are *A History in Sum* by Steve Nadis and

Shing-Tung Yau, and *Carnap, Tarski, and Quine at Harvard* by Greg Frost-Arnold. And a history of computing through personalities is *Giants of Computing* by Gerard O'Regan.

Other recent works related to the history of mathematics are *If* A, *then* B by Michael Shenefelt and Heidi White, *Secret History* by Craig Bauer, *Classic Problems of Probability* by Prakash Gorroochurn, *Number Theory* by John Watkins and, with contemporary echoes, the collection *The Legacy of A. M. Turing* edited by Evandro Aggazi. In the philosophy of mathematics, I mention Mark Colyvan's *An Introduction to the Philosophy of Mathematics*, *The Applicability of Mathematics in Science* by Sorin Bangu, and *Plato's Problem* by Marco Panza and Andrea Sereni. A historical-philosophical study is Simon Duffy's *Deleuze and the History of Mathematics*. Highly interdisciplinary are Arturo Carsetti's *Epistemic Complexity and Knowledge Construction* and Pavel Pudlák's *Logical Foundations of Mathematics and Computational Complexity*. Another popular book on complexity, presenting the P versus NP problem, is *The Golden Ticket* by Lance Fortnow. Two books on the philosophy of mind that touch on mathematical reasoning are *Surfaces and Essences* by Douglas Hofstadter and Emmanuel Sander, and *Intuition Pumps and Other Tools for Thinking* by Daniel C. Dennett.

On mathematics education, too many books to mention here are published every year. For now I note first several addressed mainly to teachers and parents: a delightful little book by Stephen Brown titled *Insights into Mathematical Thought*; then *How Math Works* by Arnell Williams, *Success from the Start* by Rob Wieman and Fran Arbaugh, *Captivate, Activate, and Invigorate the Student Brain in Science and Math, Grades 6–12* by John Almarode and Ann Miller, *Math Power* by Patricia Kenschaft, and *One Equals Zero and Other Mathematical Surprises* by Nitsa Movshovitz-Hadar and John Webb. David Tall has published a wide-ranging study on *How Humans Learn to Think Mathematically*. Ed Dubinsky's APOS theory is presented in detail by a group of authors, including Dubinsky but also Ilana Arnon and others. An excellent anthology pieced from the 75 years of National Council of Teachers of Mathematics yearbooks was edited by Francis Fennell and William Speer, with the title *Defining Mathematics Education*. Other collective volumes are *Mathematics & Mathematics Education* edited by Michael Fried and Tommy Dreyfus, *Third International Handbook of Mathematics Education* edited by Ken Clements and his collaborators, *Reconceptualizing Early Mathematics Learning* edited by Lyn English and Joanne Mulligan, *Vital Directions for Mathematics Education Research* edited by Keith Leatham, and *Cognitive Activation in the Mathematics Classroom and Professional Competence of Teachers* edited by Mareike Kunter et al. A book of problems, mainly aimed at preparing students for mathematical competitions, is *Straight from the Book*, by Titu Andreescu and Gabriel Dospinescu.

Accessible books on mathematics and other disciplines or applied to various practical or entertainment endeavors are *Six Sources of Collapse* by Charles Hadlock, *Mathematical Morphology in Geomorphology and GISci* by Daya Sagar, *Mathematical Card Magic* by Colm Mulcahy, and *All the Right Angles* by Joel Levy. An algorithmic fusing of evolutionary, learning, and intelligence perspectives situated in the living environment of organisms is presented by Leslie Valiant in *Probably Approximately Correct*. A decoding and explaining of mathematical motifs in a popular TV series is given by Simon Singh in *The Simpsons and Their Mathematical Secrets*. Readers interested in connections between mathematics and music might find it useful to consult *The Geometry of Musical Rhythm* by Godfried Toussaint and *Mathematics and Music* by James Walker and Gary Don. Those passionate about using mathematics to make origami will find well-illustrated treasure troves of ideas in *Origami Tessellations* by Eric Gjerde and especially the massive *Origami Design Secrets* by the master of the trade, Robert Lang. A more general introduction to the mathematical motifs in the arts is *Manifold Mirrors* by Felipe Cucker. A collection of legal cases and situations that unveil the dangerous mishaps that occur in the (ab)use of mathematics in courtrooms is presented by Leila Schneps and Coralie Colmez in *Math on Trial*. Finally in this category, harnessing mathematics to the search for extraterrestrial life is the leitmotif in *Science, SETI, and Mathematics* by Carl DeVito.

Every year I include a few online resources, more or less randomly, as I notice them or other people point them out to me.

Philipp Legner is the creator of Mathigon (http://mathigon.org/), an attractive, colorful website where he uses new technologies to build realistic 3-D images of many geometric objects. Another page with good illustrations of solid geometry objects is Paper Models of Polyhedra (http://www.korthalsaltes.com/). A site hosting many images, applets, and videos illustrative of mathematical concepts, from elementary math to multivariable calculus, is Math Insight (http://mathinsight.org/about/mathinsight), maintained by a group of faculty from the University of Minnesota. Useful educational materials at the secondary school level can be found on the Basic Mathematics page (http://www.basic-mathematics.com/) and Interactive Math (http://www.mathsteacher.com.au/index.html); more advanced material is on the Art of Mathematics site (http://www.artofmathematics.org). AwesomeMath (https://www.awesomemath.org/) is a summer program at three universities geared toward gifted secondary students who aspire to receive training beyond school curricula and eventually qualify for advanced mathematical competitions. A web page constructed with information taken from Florian Cajori's book *A History of Mathematical Notations* is Earliest Uses of Various Mathematical Symbols (http://jeff560.tripod.com/mathsym.html). Nate Silver's online magazine FiveThirtyEight (http://fivethirtyeight.com/) posts critical articles on journalistic punditry invoking data and on other public uses of statistics. An online store specializing in mathematics manipulables is MathsGear (http://mathsgear.co.uk/).

Lastly, the recently restructured site of the Simmons Foundation, under the name *Quanta Magazine* (https://www.simonsfoundation.org/quanta/), hosts some of the best writing on mathematics; I did not include any of the *Quanta* pieces in this volume solely because they are freely and easily accessible online.

I encourage you to send comments, suggestions, and materials I might consider for future volumes to Mircea Pitici, P.O. Box 4671, Ithaca, NY 14852, or send electronic correspondence to mip7@cornell.edu; my Twitter handle is @MPitici.

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2014

Mathematics and the Good Life

Stephen Pollard

Introduction

A full account of mathematics will identify the distinctive contributions mathematics makes to our success as individuals and as a species. That, in turn, requires us to reflect both on what mathematicians do and on what it means for humanity to flourish. After all, we cannot say how mathematics contributes to our success unless we have a good sense of what mathematicians accomplish and a good idea of what it means for us to succeed. To give our discussion some focus, I offer the following proposition:

> Mathematics makes one substantial contribution to human prosperity: it enhances our instrumental control over physical and social forces.

I say "offer," not "endorse." In fact, I want to persuade you that this proposition is wrong because it presupposes either too narrow a conception of mathematical activity or too narrow a conception of human success. Mathematicians are not just "devices for turning coffee into theorems" (as Erdős may or may not have said). Furthermore, if they *were* such devices, humanity would be the worse for it, and this would be so even if the caffeine-fueled theorem mills were more efficient than real mathematicians at disgorging instrumentally useful product.[1]

The theorem-mill model leaves out at least two vital features of mathematical activity: mathematicians achieve deep insights and have intricately meaningful experiences. These features are vital because they are intrinsic goods for human beings: they are characteristically human ways to prosper. This is not news. It is an ancient idea: mathematics provides insights and experiences that, in themselves, ennoble our species. This idea is one of those enduring themes that make the history of philosophy a long, long conversation rather than a series of disconnected episodes. It can be enervating to feed on the sere remains of antique harvests. It can be energizing, though, to advance a conversation begun of old. We hope to do the latter, with John Dewey as our main interlocutor. We begin, however, with some inspiring words from Bertrand Russell that should make vivid the point of view I am promoting and give you some idea of how I mean to promote it.

A Glorious Torment

In the penultimate chapter of *Education and the Good Life,* Russell considers "the functions of universities in the life of mankind" [1926, p. 311]. He assumes that universities exist for two purposes: "on the one hand, to train men and women for certain professions; on the other hand, to pursue learning and research without regard to immediate utility" [Russell 1926, p. 306]. It is the latter theme, research for the sake of seeking and knowing, that causes Russell's language to soar.[2]

> I should not wish the poet, the painter, the composer or the mathematician to be preoccupied with some remote effect of his activities in the world of practice. He should be occupied, rather, in the pursuit of a vision, in capturing and giving permanence to something which he has first seen dimly for a moment, which he has loved with such ardour that the joys of this world have grown pale by comparison. All great art and all great science springs from the passionate desire to embody what was at first an unsubstantial phantom, a beckoning beauty luring men away from safety and ease to a glorious torment. The men in whom this passion exists must not be fettered by the shackles of a utilitarian philosophy, for to their ardour we owe all that makes man great. [Russell 1926, pp. 312–313]

There is a utilitarian reading even of this passage, a reading that Russell himself endorses. The utilitarian pursuit of knowledge, the quest for greater control over physical and social forces, is not "self-sustaining"; it needs to be "fructified by disinterested investigation" [Russell 1926, p. 312]. Our quest for control profits from seekers who take little or no interest in that quest. Our legitimate utilitarian interests can be served upon philosophical prose to draw it into the arena of thought. However, to make our lives whole and to increase our intelligent reflective control over our lives, we need to ponder and discuss human purposes and aspirations when we are not in the grip of aesthetic experiences. Discursive philosophy provides the intellectual base for *that* sort of intelligent reflection and inquiry. Dewey himself provides just such a base with his notion of a "balanced experience" whose cultivation is "the essence of morals" [Dewey 1916, p. 417].

To return for a moment to our main interest, we would like Dewey to teach us about mathematics. We might be expected, then, to focus on his remarks about mathematics. In fact, we largely ignore that material.[10] My slantwise strategy is to focus on a notion central to Dewey's aesthetics and philosophy of education: the aforesaid notion of experience.[11] This notion lies at the heart of Dewey's comprehensive vision of human prosperity and so guides us to a broader understanding of how mathematics helps us flourish as human beings. We now consider human experience in general and, more narrowly, the sort of experiences provided by mathematical inquiry.[12]

Mathematics as Experience

I would like to serve on a jury. I think it would be a good experience. Yes, that would be quite an experience. *An* experience with a definite beginning, a significant internal structure, and an end that is a completion, not just a termination. Experiences, in just this everyday sense, are the building blocks of a human life that is more than an animal existence, a life distinguished by the development and exercise of distinctively human capacities.

The basic conditions of an experience are not uniquely human. "No creature lives merely under its skin" [Dewey 1934a, p. 13]. We animals are all active in an environment that poses problems. How do I escape from the tiger? Where can I find water? Is the defendant guilty? We respond to those problems. If things work out well, we survive, we drink, we learn, we grow. There is a rhythm of problem, activity, resolution, growth in the lives of mollusks and mathematicians.

What is distinctively human is the conscious, reflective, intelligent integration of experiences into a life we can affirm as worthwhile, in which we exercise greater and greater control over a richer and richer array of experiences. If our effort at integration fails spectacularly enough, we will not even be sane [Dewey 1963, p. 44]. If we are unreflective, if we squander opportunities for growth, we will be not only stunted but unfree, "at the mercy of impulses into whose formation intelligent judgment has not entered" [Dewey 1963, p. 65].

So the quality of our experiences and the way those experiences build on one another matters desperately. Why is democracy better than tyranny? Why is kindness better than cruelty? There is, ultimately, a single reason: a democratic arrangement of society and a kindly attitude toward our fellows promote "a higher quality of experience on the part of a greater number" [Dewey 1963, p. 34].[13] Why should we care about the artworks of distant times and places? Because "the art characteristic of a civilization is the means for entering sympathetically into the deepest elements in the experience of remote and foreign civilizations" and, hence, their arts "effect a broadening and deepening of our own experience" [Dewey 1934a, p. 332]. What is education? "It is that reconstruction or reorganization of experience which adds to the meaning of experience, and which increases ability to direct the course of subsequent experience" [Dewey 1916, pp. 89–90].[14]

Experience, experience, experience. Why keep going on and on about experience when we are supposed to be talking about mathematics? Two reasons. First, if we take at all seriously a view that places experience at the heart of human prosperity, we have good reason to reflect on the quality of mathematical experiences and the capacity of those experiences to enrich subsequent experiences. Second, when we do so reflect, we better appreciate how mathematics contributes to human happiness in ways that are not narrowly instrumental. Here are five properties of mathematical experience we consider in turn.

1. Mathematics provides examples of "total integral experiences" [Dewey 1934a, p. 37; 1960, p. 153] of a particularly pure form.
2. Those experiences yield products that allow others to recreate the experiences.
3. Mathematics provides prime examples of experiences that "live fruitfully and creatively in subsequent experiences" [Dewey 1963, p. 28].
4. There is no necessary incompatibility between the ordered growth of mathematical experiences and other desirable forms of human growth.

5. Mathematical experiences not only allow for breadth of cultivation, but once such breadth is achieved, fit easily into the texture of a variegated life.

We start at the top of the list: the inner structure of a mathematical experience.

Experience

Chimpanzees negotiate environments that offer problems and opportunities. A chimp finds a nut that hides its tasty meat in a hard shell. The chimp finds a stone. The chimp thoughtfully manipulates the stone and the nut. Stone cracks shell. Here "the material experienced runs its course to fulfillment . . . a problem receives its solution . . . a situation . . . is so rounded out that its close is a consummation and not a cessation" [Dewey 1934a, p. 35; 1960, p. 151]. Whether the chimpanzee is a pioneer or just an apprentice, there is growth. "Life grows when a temporary falling out [damn this shell!] is a transition to a more extensive balance of the energies of the organism with those of the conditions under which it lives" [Dewey 1934a, p. 14]. As with the chimpanzee, so with the mathematician.

> . . . every experience is the result of interaction between a live creature and some aspect of the world in which it lives. . . . The creature operating may be a thinker in his study and the environment with which he interacts may consist of ideas instead of a stone. But interaction of the two constitutes the total experience that is had, and the close which completes it is the institution of a felt harmony. [Dewey 1934a, pp. 43–44; 1960, p. 160]

The drama of experience runs its characteristic course: problem, activity (physical or mental), success, growth. The problem may be a nut on the forest floor, an exercise in a book, or a "beckoning beauty" in some corner of your mind. Chimp or human, if you have any semblance of puzzle drive (to borrow Feynman's phrase), you are hooked. You *have* to figure it out. And when the nut is especially hard to crack, when it draws deeply on your resources, success is graced with an exultation that probably flows from somewhere deep in our primate nature. "Few joys," says Russell, "are so pure or so useful as this" [Russell 1926, p. 259]. As George Pólya observes, even minor triumphs have their savor.

> Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime. [Pólya 1957, p. v]

Again, a mathematical experience is no mean thing. Though Dewey does not command Russell's and Pólya's insider view of mathematics, he too appreciates this point.

> There are absorbing inquiries and speculations which a scientific man and philosopher will recall as "experiences" in the emphatic sense. In final import they are intellectual. But in their actual occurrence they were emotional as well; they were purposive and volitional . . . No thinker can ply his occupation save as he is lured and rewarded by total integral experiences that are intrinsically worthwhile. Without them he would never know what it is really to think and would be completely at a loss in distinguishing real thought from the spurious article. [Dewey 1934a, p. 37; 1960, p. 153]

To see the nut as a call to action, to work at the nut with mind, body, feeling, and purpose, to crack the nut and relish that crack in thought and feeling as a consummation of satisfying, goal-oriented, unconstrained action: that can be a peak primate experience. No wonder, then, that Dewey thinks "science should be taught so as to be an end in itself in the lives of students—something worthwhile on account of its own unique intrinsic contribution to the experience of life" [Dewey 1916, p. 282]. If we had to justify our mathematical preoccupations, we would make a good start by pointing to the character of creative mathematical experiences. But there is more.

Re-creation

Dewey observes that "Mathematics and formal logic . . . mark highly specialized branches of intellectual inquiry, whose working principles are very similar to those of works of fine art" [Dewey 1929b, p. 160]. One of the expectations artists and mathematicians share, one working principle, is that their creative experiences leave behind deposits.

Artists deposit novels, paintings, operas. Mathematicians deposit axioms, algorithms, proofs. These deposits are more than memorials of the creator's past activity (as Aristotle and Hegel emphasize). Neither artists nor mathematicians are jealous guardians of "some private, secret, and illicit mode of union with the eternal powers" [Dewey 1929a, p. 34]. Their creations both stimulate and guide *re-creation*. In both mathematics and art, "receptivity is not passivity" [Dewey 1934a, p. 52; 1960, pp. 169–170].

> . . . a beholder must *create* his own experience. And his creation must include relations comparable to those which the original producer underwent . . . there must be an ordering of the elements of the whole that is in form, though not in details, the same as the process of organization the creator of the work consciously experienced. Without an act of re-creation the object is not perceived as a work of art. [Dewey 1934a, p. 54; 1960, pp. 171–172]

And, we must add, without re-creation a proof is not experienced as a proof. Hermann Weyl, for one, insists emphatically on the distinction between a proof as an artifact, a lifeless "concatenation of grounds," and the insightful engagement with the artifact that yields the "experience of truth" [Weyl 1918, p. 11; 1994, p. 119]. Indeed, it is not enough "to get convicted, as it were, rather than convinced of a mathematical truth by a long chain of formal inferences and calculations leading us blindfolded from link to link" [Weyl 1985, p. 14]. There is a difference between confirming that a result follows and seeing why it follows.[15] A mathematical proof is not just an instrument of persuasion. It is the guise in which a mathematical experience haunts the world hoping to live anew in receptive minds. In Dewey's model at least, this new life requires a re-creation of creative experience. Re-creation may not measure up to creation but, then again, it ain't bad! I remember vividly the moment I was able to hold in my mind, all at once, a proof that every finitary closure space has a minimal closed basis. It would have been even more glorious if it had been *my* proof. But it was glorious nonetheless. There is a drive to *own* an insight, to grasp a solution in a grasping way. Puzzle drive, though, asks only that we come to understand, to survey the inner mechanism, to get at the *why*. It is wonderful just to *see* even when someone else shows us where to look and even when, to return to an earlier point, we have no idea how our insight might zest to life. If you are a locksmith, you will share your knowledge with your apprentice. A criminal safecracker *might* do the same (like Peter Wimsey's friend Bill Rumm). Then growth leads to growth through straightforward transmission. Growth leads to growth in another way too: through an evolutionary arms race between safecracker and safe-maker. So safecracking, even in its illicit form, can display both intrinsic value and contagious growth. But, as Dewey is quick to suggest, a good safecracker will probably not be good. What is likely to be missing is growth that builds a cohesive structure of balanced experiences.

> That a man may grow in efficiency as a burglar, as a gangster, or as a corrupt politician, cannot be doubted. But from the standpoint of growth as education and education as growth the question is whether growth in this direction promotes or retards growth in general. Does this form of growth create conditions for further growth, or does it set up conditions that shut off the person who has grown in this particular direction from the occasions, stimuli, and opportunities for continuing growth in new directions? What is the effect of growth in a special direction upon the attitudes and habits which alone open up avenues for development in other lines? [Dewey 1963, p. 36]

Dewey does not elaborate on the likely trajectory of burglars, gangsters, and corrupt politicians, but he seems confident that the utmost perfection of their art only retards capacities whose exercise makes human life worth living. Instead of following the career of our safecracker, we return to our main interest: the mathematician. And here we confront an uncomfortable fact: mathematics as a practice and profession rewards the obsessive pursuit of results. The more obsessive this pursuit, the more other departments of life suffer. It does not follow, of course, that mathematics makes people obsessive. It is at least as likely that obsessives are drawn to mathematics because it offers a form of sublimation with tangible personal and social benefits. Furthermore, every occupation can be overdone. The hours you spend comforting dying orphans are less admirable if your own children are fending for themselves. Your prowess as a brain surgeon comes at a steep price if your single-minded devotion leaves you unable to enjoy Mozart. Hammers are good tools because they have a productive use, not because it is impossible to misuse them. Similarly, a line of work offers personal benefits if it *can* contribute an important piece to a rich and balanced human life. If we require that it cannot do otherwise, then no line of work counts as personally worthwhile. Mathematics may have a reputation for harboring especially odd characters because, in this field, glaring social inadequacies are compatible with professional success. A captain of industry, no matter how vicious in other respects, necessarily has a highly developed aptitude for successful social interaction in a variety of public settings. A mathematician who can manage only the poorest showing in public can still land a spot at the Institute for Advanced Study. An optimistic conclusion is that mathematics does not stunt variegated growth more than other vocations; it is just more tolerant of inadequacies that are glaringly obvious. In any case, it is clearly possible for broadly cultivated individuals to enjoy profound mathematical experiences. (Hermann Weyl and Bertrand Russell leap to mind.) So one essential feature of a balanced life, breadth, is compatible with intense devotion to mathematics. But balance is more than breadth. Balance also requires the right relationship between the various pieces of an expansive life experience.

Integration

In *Still Life*, Noël Coward tracks an adulterous affair from innocent start to dismal finish. Each of the play's five scenes finds the lovers, Alec and Laura, in the refreshment room of the Milford Junction train station: a place for breaks, interruptions; somewhere to pause on the way somewhere else; a place where no one lives. We soon perceive that the protagonists' situation is tragic because the pieces do not fit. First of all, the pieces of their shared experience do not fit *with one another* to form something larger; there is no hope that the episodes of the affair will add up to an integral experience. The only available consummations are physical ones leading nowhere, offering no growth, opening no new prospects. Alec and Laura keep arriving nowhere going nowhere. There is also no hope that the pieces of the affair will fit with the other pieces of their lives. "All the circumstances of our lives," says Alec, "have got to go on unaltered." So their love has to be "enclosed," walled off from everything else, "clean and untouched" [Coward 1935, p. 30]. Everyday values apply with full force everywhere else, but, to admit them into this enclave would be like using a yardstick to take a person's weight.

"This is different—something lovely and strange and desperately difficult. We can't measure it along with the values of our ordinary lives" [Coward 1935, p. 25]. The effort to maintain their love as a pause, a suspension, a static nowhere proves unsustainable both practically and psychologically. They cannot build the enclave walls high enough: the world intrudes. And Laura finds the dis-integration in which they seek protection reproducing itself alarmingly inside her own personality. "I love them just the same, Fred I mean and the children, but it's as though it wasn't me at all—as though I were looking on at someone else" [Coward 1935, p. 25]. In the end, it just ends. I mention all this because of its relevance to the doctrine of mathematical platonism: the view that mathematical objects are causally inert, supernatural beings with no coordinates in space or time. Whatever one may think of this as metaphysics, one must acknowledge a certain moral insight: mathematics provides a static nowhere, a refreshment room of the mind where, as long as the cat is fed and the children tucked in, we can enjoy voluptuous experiences without harm to any living creature. Causal inertness is a figure for a kind of ethical inertness. The point is emphatically not that mathematical experiences lack ethical significance. The point is rather that mathematical experiences have so little capacity to interact toxically with other valuable experiences. What Alec and Laura find so desperately difficult, mathematicians manage without effort, not because they have special powers but because no special powers are required. The ethical inertness of mathematical experiences only heightens their ethical significance. They add their special grace without any threat to the other blessings of a richly balanced life.

To return, finally, to our original question: is heightened control over physical and social forces really the only substantial contribution of mathematics to human prosperity? If mathematicians were soulless logic machines or if human happiness consisted in satisfaction of animal needs, this might have been plausible. If we accept, on the other hand, that our prosperity depends vitally on the character and consequences of our experiences and that mathematics offers intrinsically valuable experiences that can grow contagiously without disturbing the balance of our lives, then the above thesis becomes wholly implausible. The instrumental contribution of mathematics to our success is immense, but that is not the only way mathematics helps us flourish. All living beings harness energies and consume earthly goods. Our unrivaled efficiency and rapacity mark an important quantitative distinction between ourselves and other species. More positively, we can be proud of our success in the battle against "chaos without." We should, however, be at least as proud of our victories over the "darkness within" in which mathematics plays no small role. Mathematics pervades scientific explanations and, as we have seen, is also an engine of understanding, a propagator of insightful experiences, in its own right.

This, too, is how we flourish, how we prosper, how we succeed as individuals and as a species.

When Is Good Mathematics Good?

Though my panegyric of mathematics is now complete, I would like to discuss one further issue: the question of whether nonmathematicians have any legitimate role in the evaluation of mathematics. It is perfectly clear that the answer to this question is "yes," and it will not take long to show why.

Let us assume that there is a social consensus about who the mathematicians are. Then it seems convenient to let *them* decide what counts as good mathematics and, indeed, what counts as mathematics at all. We would be using a social consensus about the use of one term ("mathematician") to regiment our use of two related terms ("good mathematics" and "mathematics").

Now, however, we notice that good mathematics can be good *for* various things—and it is not always an entirely mathematical question whether a bit of good mathematics (a bit of mathematics deemed good by the mathematicians) is good for some particular thing. Differential geometers need help from structural geologists to determine whether differential geometry has useful applications in structural geology. It is not an entirely mathematical question whether that bit of good mathematics is good for that purpose. In general, the question of the instrumental utility of mathematics is not an entirely mathematical question. So nonmathematicians have a legitimate role in the evaluation of mathematics—not, perhaps, *as* mathematics, but as a commodity that might benefit humanity in various ways.

But this is not the end of the story: we have seen that instrumental utility is not the only contribution mathematics makes to human happiness. Mathematics has additional ethical significance because it makes further contributions to human prosperity. Our assessment of those contributions is highly sensitive to our understanding of ultimate values, our account of what it means for human beings to prosper. It would be perverse in the extreme to leave it to the mathematicians to provide that account. Our best articulations of ultimate values are the fruits of an ages-old conversation between poets and playwrights and priests and all sorts of people—even philosophers. This paper is an example of how philosophers might legitimately appraise mathematics—not *as* mathematics—but as an ally in the battle against Old Night.

There is no guarantee that every bit of good mathematics will hold up well under this scrutiny. It is conceivable that some good mathematics will be found to add little fuel to the lamp of reason. Hermann Weyl reached just this conclusion about classical set theory using a standard of undoubted philosophical parentage. He thought it a hallmark of intellectual responsibility to insist on an especially strong grasp of *Sinn* (meaning) as a precondition for an especially high degree of *Evidenz* (evidentness, the experience that a proposition expresses an evident truth).[16] If, for whatever reason, we reject Weyl's harsh assessment of classical set theory, we might yet agree that constructivist or intuitionist alternatives add more or different fuel to the lamp of reason. That, too, would be a philosophically grounded evaluation of some good mathematics.

Suppose we are presented with such an evaluation. How do we decide whether that evaluation is plausible? Well, we would have to *look*. We would have to subject the evaluation to scrutiny informed by philosophy, mathematics, and any number of things. It would not be intellectually responsible to insist, as a matter of principle, that no extramathematical appraisal of mathematics merits our attention.[17] Furthermore, friends of mathematics should not welcome such a grant of immunity since it allows only a desiccated account of how mathematics contributes to the good life. That is no great favor. Though some bits may fare worse than others, mathematics on the whole does very well, thank you, when measured against a full-blooded conception of human prosperity.

Notes

1. *Cf.* Yehuda Rav's effort "to stir the slumber of those who still think of mathematicians as deduction machines rather than creators of beautiful theories and inventors of methods and concepts to solve humanly meaningful problems" [Rav 1999, p. 17].

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The Rise of Big Data: How It's Changing the Way We Think about the World

KENNETH CUKIER AND
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Everyone knows that the Internet has changed how businesses operate, governments function, and people live. But a new, less visible technological trend is just as transformative: "big data." Big data starts with the fact that there is a lot more information floating around these days than ever before, and it is being put to extraordinary new uses. Big data is distinct from the Internet, although the web makes it much easier to collect and share data. Big data is about more than just communication: the idea is that we can learn from a large body of information things that we could not comprehend when we used only smaller amounts.

In the third century BC, the Library of Alexandria was believed to house the sum of human knowledge. Today, there is enough information in the world to give every person alive 320 times as much of it as historians think was stored in Alexandria's entire collection—an estimated 1,200 exabytes' worth. If all this information were placed on CDs and they were stacked up, the CDs would form five separate piles that would all reach to the moon.

This explosion of data is relatively new. As recently as the year 2000, only one-quarter of all the world's stored information was digital. The rest was preserved on paper, film, and other analog media. But because the amount of digital data expands so quickly—doubling around every three years—that situation was swiftly inverted. Today, less than two percent of all stored information is nondigital.

Given this massive scale, it is tempting to understand big data solely in terms of size. But that would be misleading. Big data is also characterized by the ability to render into data many aspects of the world that have never been quantified before; call it "datafication." For example, location has been datafied, first with the invention of longitude and latitude, and more recently with GPS satellite systems. Words are treated as data when computers mine centuries' worth of books. Even friendships and "likes" are datafied, via Facebook.

This kind of data is being put to incredible new uses with the assistance of inexpensive computer memory, powerful processors, smart algorithms, clever software, and math that borrows from basic statistics. Instead of trying to "teach" a computer how to do things, such as drive a car or translate between languages, which artificial intelligence experts have tried unsuccessfully to do for decades, the new approach is to feed enough data into a computer that it can infer the probability that, say, a traffic light is green and not red or that, in a certain context, *lumière* is a more appropriate substitute for "light" than *léger*.

Using great volumes of information in this way requires three profound changes in how we approach data. The first is to collect and use a lot of data rather than settle for small amounts or samples, as statisticians have done for well over a century. The second is to shed our preference for highly curated and pristine data and instead accept messiness: in an increasing number of situations, a bit of inaccuracy can be tolerated because the benefits of using vastly more data of variable quality outweigh the costs of using smaller amounts of very exact data. Third, in many instances, we need to give up our quest to discover the cause of things, in return for accepting correlations. With big data, instead of trying to understand precisely why an engine breaks down or why a drug's side effect disappears, researchers can instead collect and analyze massive quantities of information about such events and everything that is associated with them, looking for patterns that might help predict future occurrences. Big data helps answer what, not why, and often that's good enough.

The Internet has reshaped how humanity communicates. Big data is different: it marks a transformation in how society processes information. In time, big data might change our way of thinking about the world. As we tap ever more data to understand events and make decisions, we are likely to discover that many aspects of life are probabilistic, rather than certain.

Approaching "n = *all*"

For most of history, people have worked with relatively small amounts of data because the tools for collecting, organizing, storing, and analyzing information were poor. People winnowed the information they relied on to the barest minimum so that they could examine it more easily. This was the genius of modern-day statistics, which first came to the fore in the late nineteenth century and enabled society to understand complex realities even when few data existed. Today, the technical environment has shifted 179 degrees. There still is, and always will be, a constraint on how much data we can manage, but it is far less limiting than it used to be and will become even less so as time goes on.

The way people handled the problem of capturing information in the past was through sampling. When collecting data was costly and processing it was difficult and time-consuming, the sample was a savior. Modern sampling is based on the idea that, within a certain margin of error, one can infer something about the total population from a small subset, as long the sample is chosen at random. Hence, exit polls on election night query a randomly selected group of several hundred people to predict the voting behavior of an entire state. For straightforward questions, this process works well. But it falls apart when we want to drill down into subgroups within the sample. What if a pollster wants to know which candidate single women under 30 are most likely to vote for? How about university-educated, single Asian American women under 30? Suddenly, the random sample is largely useless, since there may be only a couple of people with those characteristics in the sample, too few to make a meaningful assessment of how the entire subpopulation will vote. But if we collect all the data—"$n = \text{all}$," to use the terminology of statistics—the problem disappears.

This example raises another shortcoming of using some data rather than all of it. In the past, when people collected only a few data, they often had to decide at the outset what to collect and how it would be used. Today, when we gather all the data, we do not need to know beforehand what we plan to use it for. Of course, it might not always be possible to collect all the data, but it is getting much more feasible to capture vastly more of a phenomenon than simply a sample and to aim for all of it. Big data is a matter not just of creating somewhat larger samples but of harnessing as much of the existing data as possible about what is being studied. We still need statistics; we just no longer need to rely on small samples.

There is a trade-off to make, however. When we increase the scale by orders of magnitude, we might have to give up on clean, carefully curated data and tolerate some messiness. This idea runs counter to