The BEST WRITING on MATHEMATICS

2015

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Mircea Pitici, Editor

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Contents

Introduction

Mircea Pitici

This is the sixth anthology in our series of recent writings on mathematics selected from professional journals, general interest publications, and Internet sources. All pieces were first published in 2014, roughly in the form we reproduce (with one exception). Most of the volume is accessible to readers who do not have advanced training in mathematics but are curious to read well-informed commentaries about it.

What do I want by sending this book into the world? What kind of experience I want the readers to have? On previous occasions I answered these questions in detail. To summarize anew my intended goal and my vision underlying this series, I use an extension of Lev Vygotsky's concept of "zone of proximate development." Vygotsky thought that a child learns optimally in the twilight zone where knowing and not knowing meet—where she builds on already acquired knowledge and skills, through social interaction with adults who impart new knowledge and assist in honing new skills. Adapting this idea, I can say that I aspire to make the volumes in this series ripe for an optimal impact in the imaginary zone of proximal reception of their prospective audience. This means that the topics of some contributions included in these books might be familiar to some readers but novel and instructive for others. Every reader will find intriguing pieces here.

Besides offering a curated collection of articles, each book in this series doubles into a reference work of sorts, for the recent nontechnical writings on mathematics—with the caveat that I decline any claim to being comprehensive in this attempt. The list of book titles I give at the end of the introduction and the list of notable writings at the end of the volume contain a few entries published prior to the 2014 calendar year, in an acknowledgment that in previous volumes I overlooked materials worth mentioning. The same is surely the case for this year. The fast pace of the series, the immense quantity of literature I survey, and the convention subtly ensconced in calling a "year" the interval from January 1 to December 31 not only make such lapses inevitable but to a high degree determine the content of the book(s). Were we to look at the same literature from July 1 of one year until June 30 of the next year, the books in this series would look very different from what you can read between these covers. That is why each volume should be seen in conjunction with the others, part of a serialized enterprise meant to facilitate the access to and exchange of ideas concerning diverse aspects of the mathematical experience.

In this volume a greater number of contributions than in the previous volumes concern mathematical games and puzzles. For many centuries and in many cultures, recreational mathematics used to be seen as a benign amusement of no immediate utility. That enduring but now old-fashioned perception has gradually changed over the past century because of at least two broad phenomena. First, the history of the most salient branches of contemporary mathematics (algebra, modern algebra, geometry, probability, number theory, graph theory, knot theory, topology, combinatorics, and even calculus) has been either rooted into or decisively influenced by "recreational" problems. Second, talented writers and popularizers of recreational mathematics (the most famous of whom was Martin Gardner) found a large audience in the public, enjoyed appreciation from select but remarkable mathematicians, and built a devoted following of like-minded authors who carry on working in the same vein, encouraged by the lasting impact of their predecessors. Recreational mathematics has a rich and sophisticated history studied in the past by a few authors who contributed brief works (notably David Singmaster); recently the scholarship is growing rapidly, as illustrated by a special issue of the journal *Historia Mathematica* dedicated entirely to recreational mathematics. Nowadays good recreational mathematics is placed midway between the intelligent but mathematically untrained public and the mathematics professionals, by virtue of linking easy-to-understand problems to serious mathematics. In other words the problems posed in high-quality recreational mathematics are comprehensible to the layperson, while pursuing and understanding the ideas developed in the solutions occasioned by the problems might require an independent learning effort the reader is free to undertake or not. Thus the context of good recreational mathematics straddles the popular and the pedagogical, having the dual value of intellectual the one administered by courts under the slogan "in the best interest of the child.") Since then I became a lot more cautions with my suggestions concerning mathematics; I learn from my errors, the hard way.

Yet I still venture a bit in talking about mathematics, here and there—now prudent, aware that attempting to crack the thought establishments is fraught with dangers. As an example, I can say that the process of editing each volume in this series is a lesson in working with uncertainty while at the same time interpreting mathematics. The contrast between my limited knowledge and the limitless possibilities available to all the people who gloss on mathematics offers me palpable practice for a general mnemonic that serves well in other endeavors. I generalized it into a theoretical and practical principle, which I call "the paradox of reward." The paradox of reward says that in a competitive, fair, unpredictable, and infinitely complex environment, the most valuable knowledge is to know how to be rewarded for ignorance; in other words, more reward is available for taking advantage of ignorance (if one finds such a path to reward) than it is for taking advantage of knowledge. Of course I am *not* saying that ignorance is preferable to knowledge; it is *not*. I am saying that in certain environments harvesting rewards off ignorance is (by far) more valuable than seeking rewards for knowledge. This might seem to have little to do with mathematics; yet to my mind it is nothing else *but* interpreting mathematics, in a world so complex that ignorance is unavoidable but ignoring its benefits is avoidable. This subject is a lot vaster than I sketched in these few sentences, and it has consequences not only for learning and teaching mathematics but also for incorporating the private interpretation of mathematics in strategic thinking. Yet I am mindful of the dangers of venturing too far in speaking unconventionally about mathematics and interpreting mathematics, so I leave it for another occasion.

Overview of the Volume

I feel rewarded to collect in this book thoughts and perspectives on mathematics I could never think up myself.

Michael Barany and Donald MacKenzie locate the center of the mathematical activity done in institutional settings (and occasionally in private homes) at the blackboard; they note that blackboards are key objects that influence the organizing of the research and teaching spaces, while chalk-writing on blackboards influences the logistics and the overall manner of mathematical communication.

Pradeep Mutalik finds that the repeated experience of feeling right when we suddenly comprehend the solution to a problem or a puzzle has had a positive evolutionary role in defining us as humans, both cognitively and emotionally.

Colm Mulcahy and Dana Richards write an informed centennial appreciation of the life and work of Martin Gardner, that remarkable polymath who inspired many mathematicians and laypersons to take up mathematical games and similar challenges.

Arthur Benjamin and Ethan Brown teach us how to construct an unlimited number of customized magic squares by improvising on a few ingenious templates.

Toby Walsh starts with the popular Candy Crush game as a guidepost for his discussion of the factors that determine the difficulty of solving computational problems in mathematics.

Marianne Freiberger takes us to the billiard room; she explains how the trajectory of a ball rolling on the pool table leads to mathematically complex problems related to chaos theory, the conductivity of metals, and other . . . infinite surprises.

Erik R. Tou models juggling numerically, to show that it is mathematically similar to the morphing game of transforming one word into a very different one by incremental steps that admit changes of only one letter at a time.

Scott Aaronson dissects the intricacies of the notion of randomness and connects it to the study of paradoxes, complexity, and quantum mechanics.

Dana Mackenzie describes how biologists, physicists, and mathematicians interact(ed) to overcome theoretical obstacles encountered in the birth and growth of synthetic biology.

In a similar vein, Natalie Wolchover describes the interdisciplinary efforts undertaken by researchers interested in the Tracy-Widom distribution associated with phase transitions in interactive systems of various types.

Eli Maor and Eugen Jost present (and illustrate beautifully) the basic geometric properties of the logarithmic spiral, cycloid, epicycloids, and hypocycloids—some of the best-known curves studied, over the centuries, in connection to natural phenomena and physical motions.

Burkard Polster analyzes the mathematical properties of several noncircular shapes of constant width and shows us how they have been applied to various gadgets and playful devices.

The quickest way to summarize the brief article by Annalisa Crannell, Marc Frantz, and Fumiko Futamura is to say that they look at Dürer's perspective drawing from several different perspectives!

Vi Hart and Henry Segerman ask whether there are groups of symmetries that can be visualized using real-life objects but have never been represented as such—and propose a novel modeling of the quaternion group.

John Conway and Alex Ryba use wordplay and a humorous ingenuity to discuss the merits of several different proofs they give to an old geometry problems that looks deceptively easy (until you try to solve it).

Gila Hanna and John Mason discuss the many facets, relative merits, and theoretical pedigrees of various terms used by mathematicians or mathematics educators to qualify the worthiness of proofs—with the main reference to a similar attempt by Timothy Gowers.

Jim Fey, Sol Garfunkel, and their coauthors formulate five tenets they consider important to be taken as guiding principles for the mathematics education reform at high school level (in the United States).

Guili Zhang and Miguel A. Padilla compare multiple aspects of mathematics instruction in China and the United States, based on previous theoretical and empirical studies.

Against commonly held wisdom, Benoît Rittaud and Albrecht Heeffer argue that the pigeonhole principle, usually attributed to Dirichlet, was stated in writing at least two centuries earlier in *Selectae Propositiones*, a book by Jean Leurechon.

Lisa Rougetet traces the earliest written descriptions of the popular game of Nim to a treatise written at the beginning of the sixteenth century by Luca Pacioli and follows the subsequent European developments of the game since then.

Jan von Plato considers the context of mathematical ideas and the personalities that shaped German mathematician Gerhard Gentzen's ordinal proof theory—and how this work relates (or does not!) with a theorem by Reuben Goodstein.

James Franklin illustrates with several well-chosen examples the local-global synergy in mathematics, one of the many conceptual polarities that characterize mathematical thinking.

Carlo Cellucci reviews many opinions on what constitutes mathematical beauty and its role in mathematics; he concludes that aesthetic factors play an indirect epistemological role in discovery via their selective role in choosing what hypotheses to consider.

Mark Balaguer argues that philosophers of mathematics are mainly concerned with the meaning of mathematical discourse and that the semantic theories they adhere to can lead to claims hardly acceptable for the mathematicians.

Steven Strogatz discerns three broad types of rapport with mathematics in the general public and tells us how he honed his talent for writing about mathematics by paying attention to the writing qualities of masters in similar trades.

Domenico Napoletani, Marco Panza, and Daniele C. Struppa examine the methodological underpinnings and the philosophical implications of using ever-more powerful computing techniques in the modeling of complex phenomena.

Andrew Gelman and Eric Loken caution that evidential claims of statistical significance in research journals are often spurious because of multiple factors related to the gathering of data and its interpretation; they give several suggestive examples.

Jeffrey S. Rosenthal tells the true story of how his statistical expertise led to the discovery and prosecution of fraudulent lottery winnings in Ontario, Canada.

David Hand explains a bias of expectations that precludes us from perceiving the increased likelihood of coincidences following the rapid combinatorial growth of possibilities that comes with the increase of the number of simple events.

More Writings on Mathematics

Every year I started this section by naming one book outstanding among all others. This time I cannot decide on only one; I give two, both excellent and badly needed reference books: Lizhen Ji's *Great Mathematics Books of the Twentieth Century* and *Encyclopedia of Mathematics Education*, edited by Stephen Lerman.

Now, as usual, I roughly group the other titles by theme (full references are at the end of the introduction); some of the books listed here are not easy to categorize, but I made ad hoc choices for the sake of expediency.

A handful of books blend mathematical ideas with describing the world from nonmathematical viewpoints, sometimes with a strong historical perspective—a growing trend I signaled previously in these pages and picking up steam lately. Thus are Jeff Suzuki's *Constitutional Calculus*, Anders Engberg-Pedersen's *Empire of Chance*, Keith Tribe's *The Economy of the World*, along with *The Norm Chronicles* by Michael Blastland and David Spiegelhalter, and *How Reason Almost Lost Its Mind* by Paul Erickson and colleagues. Less shy with giving mathematics a prominent role in daily life are *Grapes of Math* by Alex Bellos and *Mathematics and the Real World* by Zvi Artstein.

In the history of mathematics I note Joseph Mazur's *Enlightening Symbols*, Alexander Amir's *Infinitesimal*, Amir Aczel's *Finding Zero*, Jiri Hudecek's *Reviving Ancient Chinese Mathematics*, Vedveer Arya's *Indian Contributions to Mathematics and Astronomy*,David Reamer's *Count Like an Egyptian*, and, slightly more technical but with well-chosen historical vignettes, *From Mathematics to Generic Programming* by Alexander Stepanov and Daniel Rose. Books focused on personalities are *John Napier* by John Havil, *How Euler Did It Even More* by Edward Sandifer, *Beyond Banneker* by Erika Walker, and *Peter Lax, Mathematician* by Reuben Hersh. A broad swipe of history is covered in *Classical Mathematics from Al-Khwarīzmī to Descartes* by Roshdi Rashed. Theme-based histories are *Whatever Happened to the Metric System?* by John Marciano, *The New Math* by Christopher Phillips, and *The Palgrave Centenary Companion to "Principia Mathematica"* edited by Nicholas Griffin and Bernard Linsky. Focused on the formerly communist countries are *Mathematics across the Iron Curtain* by Christopher Hollings and *Pearls from a Lost City* by Roman Duda. Other (auto)biographical or celebratory volumes are *Gordon Welchman, Bletchley Park's Architect of Ultra Intelligence* by Joel Greenberg, *Wearing Gauss's Jersey* by Dean Hathout, *My Life and Functions* by Walter Hayman, as well as the collective volumes *Arnold* edited by Boris Khesin and Serge Tabachnikov, *Alexandre Grothendieck* edited by Leila Schneps, and *Four Lives [of Raymond Smullyan]* edited by Jason Rosenhouse.

Some books on the interactions between mathematics and other disciplines: *The Oxford Handbook of Computational and Mathematical Psychology* edited by Jerome Busemeyer and his collaborators; *Biographical Encyclopedia of Astronomers* with Thomas Hockey as editor-in-chief; *Scientific Visualization* edited by Charles Hansen et al.; *Bond Math* by Donald Smith; *Measuring and Reasoning [in Life Sciences]* by Fred Bookstein; *Mathematics*

Finally, mathematicians and others who need to write mathematical script can pick up *Practical LATEX*, the latest book by George Grätzer.

A few months ago I started to use Twitter (@mpitici). This led me to find many online resources on mathematics. A supplementary section of this introduction containing dozens of Internet links is available online at http://press.princeton.edu/titles/10558.html.

I encourage you to send comments, suggestions, and materials I might consider for future volumes to Mircea Pitici, P.O. Box 4671, Ithaca, NY 14852; or electronic correspondence to mip7@cornell.edu.

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A Dusty Discipline

Michael J. Barany and
Donald MacKenzie

How does one see a mathematical idea? Can it be heard, touched, or smelled?* If you spend enough time around mathematicians in the heat of research, you tend to believe more and more that when it comes to mathematics, the materials matter. To many, mathematical ideas look and sound and feel and smell a lot like a stick of chalk slapping and then gliding along a blackboard, kicking up plumes of dust as it traces formulas, diagrams, and other mathematical tokens.

Chalk and blackboards first made their mark in higher education at elite military schools, such as the École Polytechnique in France and West Point in the United States, at the start of the nineteenth century. Decades of war and geopolitical turmoil, combined with sweeping changes to the scale and social organization of governments, put a new premium on training large corps of elite civil and military engineers. Mathematics was their essential tool, and would also become a gateway subject for efficiently sorting the best and brightest. Blackboards offered instructors a way of working quickly and visibly in front of the large groups of students who would now need to know mathematics to a greater degree than ever before. They also furnished settings of discipline, both literal and figurative, allowing those instructors to examine and correct the work of many students at once or in succession as they solved problems at the board.

In the two intervening centuries, the importance of chalk and blackboards for advanced mathematics grew and grew. Blackboards reigned

* This essay is the authors' adaptation of "Chalk: Materials and Concepts in Mathematics Research," in Catelijne Coopmans, Michael Lynch, Janet Vertesi, and Steve Woolgar (eds.), Representation in Scientific Practice Revisited (Cambridge: MIT Press, 2014), pp. 107–29.

as the dominant medium of teaching and lecturing for most of the twentieth century, and continue to be an iconic presence in countless settings where mathematics is learned, challenged, and developed anew. As, indeed, it routinely is. While schoolbook mathematics seems as though it has been settled since time immemorial (it has not been, but that is another story), the mathematics taking place in universities and research institutes is changing at a faster rate than ever before. New theorems and results emerge across the world at such a dizzying pace that even the brightest mathematicians sometimes struggle to keep up with breakthroughs in their own and nearby fields of study. Long gone are the days when a single mathematician could even pretend to have a command of the latest ideas of the entire discipline.

The problem of keeping up might seem to lead one toward high technology, but to a surprising extent it leads back to the blackboard. When Barany followed the day-to-day activities of a group of university mathematicians, he found the blackboard was most prominent in their weekly seminar, when they gather after lunch to hear a local or invited colleague's hour-long presentation on the fruits and conundrums of recent and ongoing work. But blackboards are also present in offices, and even the departmental tea room. Their most frequent use came when mathematicians return to the seminar or other rooms to teach mathematics to students, just as their predecessors did 200 years ago. Wherever they are, blackboards serve as stages for learning, sharing, and discussing mathematics.

Blackboards are still more pervasive when one searches for them in unexpected places. The archetypal blackboard is a large rectangular slab of dark gray slate mounted on a wall, but over their history most blackboards have been made of other materials, some of which are not even black. (This includes the dark green boards in the seminar room of Barany's subjects.) Characteristics of blackboard writing can be found in the pen-and-paper notes researchers scribble for themselves or jot for colleagues. Gestures and ways of referring to ideas in front of a board translate readily to other locations. The blackboard is one part writing surface and two parts state of mind. To understand the blackboard, we realized, is to understand far more about mathematics than just seminars, lectures, and the occasional chalk marking in an office.

Even as blank slates, blackboards are laden with meaning. Because they are large and mostly immobile, they greatly affect how other features of offices or seminar rooms can be arranged. Entering a seminar room, one knows where to sit and look even when the speaker has not yet arrived, and the same principle holds for the different kinds of situations present in offices. We noted that when one arrangement of desks and chairs did not quite work it was the desks and chairs, rather than the blackboard, that were rearranged. Staring blankly at its potential users, a blackboard promises a space for writing and discussion. Depending on the context, too much writing on the board prompted users to use an eraser long before anyone intended to use the newly cleared space. Having a blank space available at just the right moment was important enough that mathematicians anticipated the need far in advance, trading present inconvenience for future chalk-based possibilities.

When blackboards are in use, more features come into play. They are big and available: large expanses of board are visible and markable at each point in a presentation, and even the comparatively small boards in researchers' offices are valued for their relative girth. Blackboards are visually shared: users see blackboard marks in largely the same way at the same time. They are slow and loud: the deliberate tapping and sliding of blackboard writing slows users down and makes it difficult to write and talk at the same time, thereby shaping the kinds of descriptions possible at the board. As anyone who has fussed with a video projector or struggled with a dry-erase marker that was a bit too dry appreciates, blackboards are robust and reliable, with very simple means of adding or removing images.

As surfaces, blackboards do more than host writing. They provide the backdrop for the waves, pinches, and swipes with which mathematicians use their hands to illustrate mathematical objects and principles. They also fix ideas to locations, so that instead of having to redescribe a detailed idea from earlier in the talk a lecturer can simply gesture at the location of the chalk writing that corresponded to the prior exposition. We were surprised to find that such gestures are used and seem to work whether or not the chalk writing had been erased in the interim—although sometimes the speaker had to pause after finding that the relevant expression was no longer where it was expected to be.

In addition to these narrative uses in a lecture, locations on the blackboard can have a specifically mathematical significance. Mathematical arguments often involve substituting symbolic expressions for one another, and on the board this can be done by smudging out the old expression and writing the new one in the now-cloudy space where the old one had been. This ability to create continuity between old and new symbols is so important in many cases that speakers frequently will struggle to squeeze the new terms in the too-small space left by the old ones rather than rewrite the whole formula, even when the latter approach would have been substantially easier to read. Boards are also large enough to let the speaker create exaggerated spaces between different parts of a formula, permitting the speaker to stress their conceptual distinctness or to leave room for substitutions and transformations.

And what of the chalk marks themselves? One rarely thinks of what *cannot* be written with chalk, a tool that promises the ability to add and remove marks from a board almost at will. The chalk's shape, its lack of a sharp point, and the angle and force with which it must be applied to make an impression all conspire to make certain kinds of writing impossible or impractical. Small characters and minute details prove difficult, and it is hard to differentiate scripts or weights in chalk text. Board users thus resort to large (sometimes abbreviated) marks, borrow typewriter conventions such as underlining or overlining, or employ board-specific notations such as "blackboard bold" characters to denote certain classes of mathematical objects.

Not every trouble has a work-around. Similar to a ballpoint pen or pencil on paper, chalk must be dragged along the board's surface to leave a trace. Entrenched mathematical conventions from the era of fountain pens, such as "dotting" a letter to indicate a function's derivative, stymie even experienced lecturers by forcing them to choose between a recognizable dotting gesture and the comparatively cumbersome strokes necessary to leave a visible dot on the board.

These practical considerations have profound effects on how mathematics is done and understood. For one, blackboard writing does not move very well. This means that whenever one makes an argument at a blackboard one must reproduce each step of the argument at the board from scratch. Nothing is pre-written, and (in the ideal of mathematical argument) nothing is pre-given as true. Proofs and constructions proceed step by step, and can be challenged by the audience at each point. It is not possible in a rigorous mathematical presentation, unlike in other disciplines, to drop a mountain of data in front of an interlocutor and then move straight to one's interpretations and conclusions.

In a blackboard lecture, those taking notes write along with the speaker. In a classroom setting, lecturers can expect that most of their board writing will be transcribed with little further annotation. Fewer audience members take notes during a seminar, but the expectation of transcription persists. In particular, what is written on the chalkboard is, in a good lecture, largely self-contained. The division between the speaker's writing and speech parallels a similar division in any mathematical argument. Such arguments combine commentary and explanation (like the presenter's speech) with a rigorous formal exposition that, mathematically, is supposed to stand on its own (like the presenter's writing), even if it might be difficult to understand without the commentary.

Going along step by step with an argument produced at a blackboard gives mathematicians the chance to break an argument that can be extremely difficult to comprehend globally into smaller steps that are possible to understand for many in the audience. This local-but-not-global way of viewing colleagues' work is indispensable in a discipline as vast and quickly changing as mathematics. For while the basic steps of mathematical arguments are often shared among specialists in related areas, the nuances and particularities of a single mathematician's work can be opaque even to recent collaborators. A proof may be true or valid universally, but mathematicians must make sense of it in their own particular ways. So blackboards offer a means of communication in both the obvious sense—as things on which to write—and a more subtle sense in terms of a step-by-step method of exposition.

The objects of that exposition also have certain features enforced by blackboard writing. Any writing on the board can be corrected, annotated, or erased at the board user's will. It is common to see lecturers amend statements as new information becomes relevant, often after a query from the audience. The blackboard lets speakers make those amendments without a messy trail of scribbles or crossings-out, preserving the visual integrity of the record that remains on the board. In this way, speaker and audience alike can believe that the ultimate mathematical objects and statements under consideration maintain a certain conceptual integrity despite all the messy writing and rewriting needed to understand and convey them. This view is a key part of mathematical Platonism, which contends that mathematical objects and truths exist independent of human activity, and represents a central position in the philosophy of mathematics, albeit with many variations.

To judge whether this seemingly grandiose claim is tenable, we need to isolate what characteristics of humans are *qualitatively* different from other intelligent animals and especially from our close ape relatives.

Does the difference lie in what we term our complex social human emotions—love, empathy, shame, jealousy, political intrigue, and the like? Not at all, as any pet lover knows—pets regularly exhibit such emotions, and political intrigue is well known in apes. We share many behaviors with animals, and although we execute them with greater complexity and sophistication as a result of our greater intelligence, they do not define us.

Is it tool use or problem solving that makes us different? No. The use of simple tools and the ability to solve problems to obtain food or other extrinsic rewards is well known in animals.

What is different about human beings is *our underlying emotional attitude to problem solving*. We seek out puzzles and learning for fun. This makes us learning machines in the area of our choice, whether it be tracking prey or navigating difficult terrain. *Aha!* experiences help us master an area of learning unique to our species: spontaneous syntactic language. We enjoy art, music, and humor: cognitive experiences that seem to be without any short-term practical purpose. And we can form models of the world and understand it. "The most incomprehensible thing about the universe it that it is comprehensible," Albert Einstein famously declared. As we shall see, it is the cognitive-emotional links in our brains, of which the *Aha!* experience is the most dramatic manifestation, that makes all this possible.

Our brains have cognitive modules for language, face recognition, social interaction, numerical manipulations, motor planning, and so on. But as we just saw, even disparate cognitive processes have the same emotional concomitants when a solution is found. The modules all use the same reward mechanism.

What exactly is this unifying *Aha!* experience? At its strongest, it is a flash of insight that instantly shifts our worldview. It is accompanied by intense pleasure and the confident realization that the answer is right: No external validation is needed. There is a sense of rightness, of things falling into place, like a puzzle piece that can fit only one way. There is a strong memory of the insight, and the feeling is somewhat addictive: You want to come back for more.

Another important characteristic is that this feeling is an intrinsic, impersonal reward—it is not related to the utility of the result.

This is perhaps most extremely illustrated in a statement made by the Cambridge mathematician G. H. Hardy to a friend, the philosopher Bertrand Russell: "If I could prove by logic that you would die in five minutes, I should be sorry you were going to die, but my sorrow would be very much mitigated by pleasure in the proof!"

Math enthusiasts know that puzzle solving is intrinsically fun, but seeking out puzzles is not a universal activity by any means. What relevance does the *Aha!* experience have to the vast number of human beings who don't care for puzzles, mathematical or otherwise? Here's the kicker: The same emotional reaction of joy and certainty is experienced when the brain solves a puzzle that is subconscious—when a person is not even aware that he or she has solved a puzzle!

Such puzzles are constantly being solved by the cognitive, visual, and auditory systems of all humans in day-to-day activities. The cognitive puzzles we need to solve all the time require abstraction, pattern recognition, generalization, the solving of equations, and rule-based induction—things that mathematicians do consciously. And when these puzzles are solved, our brains reward themselves by a similar positive emotional reaction.

As Gestalt psychology has shown, some functions of the brain are global: common across modules. The brain has general algorithms that can recognize good solutions to any kind of problem. Let's look at some examples to try to understand what these are.

Figure 1 shows a stereogram puzzle of the type popularized by the *Magic Eye* book series. When you relax your eyes, allowing the two guide circles at the top to come together, and staying focused on the pattern, some hidden three-dimensional objects emerge. Finding this image elicits the same emotional elements as the *Aha!* experience—positive reinforcement with no doubts at all.

In fact, every act of recognition—whether visual, auditory, or conceptual—is an *Aha!* experience. Cognitively, it is triggered by a change in an initially disordered internal representation to one that makes sense. Order is created out of disorder; the new representation is more compact and coherent. It is much easier to have a bunch of splotches coherently organized into the shape of a recognized object than to account for them individually.

Thus, what brings on the *Aha!* experience is something that can be termed a *decrease in cognitive entropy*. Our brains appear to have a built-in

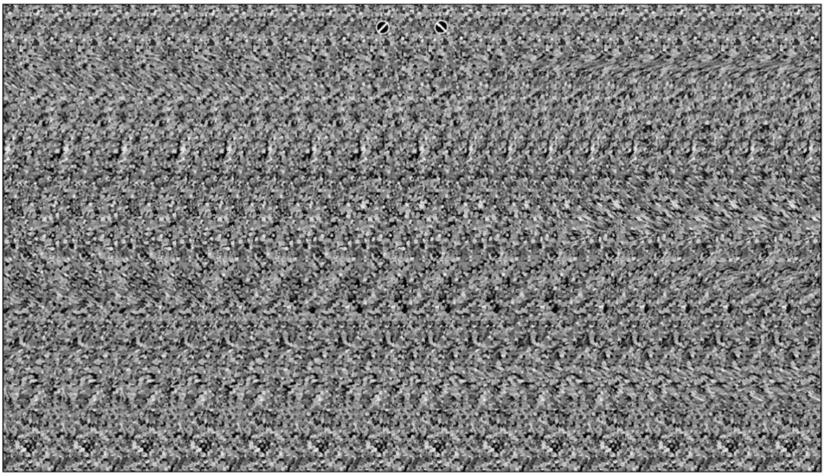


Figure 1. What mathematical objects do you see in this picture? (The answer is at the end of the article.) See also color image.

algorithm that triggers the familiar emotional *Aha!* reaction whenever a simple coherent explanation fits disorderly input. The famous principle of parsimony in problem solving—Occam's razor—is apparently built in to our brains.

This powerful principle also helps us learn language. When a child learns to speak, the number of words he or she knows grows slowly at first, and then at around 18 months, suddenly takes off at an exponential rate. The reason seems to be that every child inductively discovers the rule that every object has a name. From then on, the child hounds its parents into feeding it names . . . and the rest is history.

The experience of discovering the name rule occurs too early for most of us to remember, but Helen Keller had it at the age of seven and here's how she described it: "I knew then that 'w-a-t-e-r' meant the wonderful cool something that was flowing over my hand. That living word awakened my soul, gave it light, hope, joy, set it free!"

The certainty and joy she describes clearly identify this as a true *Aha!* experience. *This certitude and pleasure is extremely important to learning language because the child cannot turn to anyone else for validation of its conclusions: It still has to learn language!* Cognitively, the unification of independent representations caused by this induced rule represents a large decrease in cognitive entropy quite similar to the visual case.

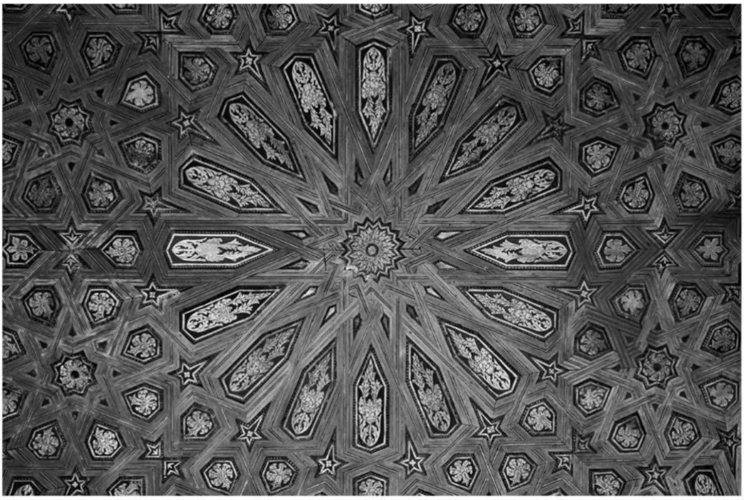


FIGURE 2. Beautiful woodwork on the ceiling of the Alhambra in Granada, Spain. See also color image.

Mini *Aha!* experiences continue to guide language learning and, in fact, all independent learning throughout childhood.

This emotional reaction that favors low cognitive entropy in the solution of unconscious problems gives a natural explanation for those uniquely human aesthetic pursuits: art and music. We find regular visual patterns like the one in Figure 2 pleasing. We love symmetry. Our visual system makes recognized patterns pop out. Symmetry and observed patterns reduce the representational requirement of a visual object, triggering pleasurable reactions.

Music is pleasurable for the same reason. Musical scales consist of notes in simple integer ratios: 1:2, 1:3, 5:4, and so on. The pleasure associated with such ratios is based on the fact that sound-makers in the environment essential to our survival, such as predators, prey, and vibrating inanimate objects, give out resonant frequencies in integer ratios.

To parcel out environmental sounds accurately, the brain has to be able to identify integer ratios in the mishmash of frequencies that we hear. So in effect, our auditory system tries to solve Diophantine equations. When it does so, *Aha!* There is a reduction of cognitive entropy and we feel pleasure. Also, musical rhythm is a compact organization of time intervals, creating, essentially, symmetric patterns in time. Of course, there is a lot more to aesthetics than these basic elements, but the underlying intrinsic pleasure of low cognitive entropy motivates us to follow these pursuits.

The same drive to detect existing patterns in aesthetics extends to finding hitherto unknown patterns in humor and creativity. As Arthur Koestler outlined in his brilliant book *The Act of Creation*, humor and creativity are linked because they both arise from finding new patterns of reasoning that are intrinsically appealing: those that decrease cognitive entropy. Once we find such new patterns, we can celebrate those that are valid and weed out those that don't quite work in the real world and are therefore funny.

Koestler tells the joke about the man who came home to find his wife in bed with a priest and, instead of reacting angrily, went out onto the balcony and pretended to bless an imaginary congregation. His explanation to the priest was "You are doing my job, so let me do yours." This creative pattern of thinking—reciprocity—is valid in many situations, but not in this one. So we find it funny: Humor is the brain's way of saying, "Nice try, but you are reasoning on thin ice here."

Neuroimaging studies confirm that both cognition and emotion are involved in the *Aha!* effect. There is increased brain activity in the more recently evolved brain structures of the cerebral cortex—specifically, the anterior superior temporal gyrus and the right hemisphere—during the *Aha!* effect. But there is also increased activation of the right hippocampus, which is involved in memory, and of more primitive brain structures that are powerfully involved in emotion, motivation, and even addiction, such as the amygdala.

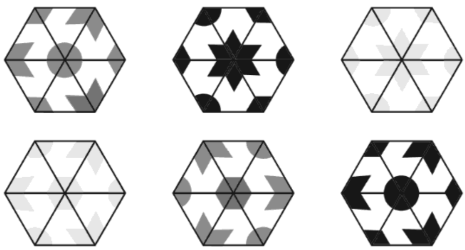
It is a signal achievement of human brain evolution that it has managed to link the results of our most sophisticated cognitive processes with our most primitive pleasure centers. It makes evolutionary sense: If you were to make an animal with no imposing physical traits that had to live off its wits, you would provide it an internal reward when it solved a problem. And that's exactly what evolution has done.

All primitive human societies have experts that excel in particular fields of knowledge: language, reckoning, navigating by the stars, tracking, and so on. Unlike, say, insect societies, this expertise is not innate but self-cultivated. *Aha!* experiences in childhood in a particular field can accentuate variations in intrinsic ability, leading the child to seek problems in, and master, a particular field. The almost addictive way to appreciate his groundbreaking columns may be simply to reread them—or to discover them for the first time, as the case may be. Perhaps our celebration here of his work and the seeds it planted will spur a new generation to understand just why recreational mathematics still matters in 2015.

From Logic to Hexaflexagons

For all his fame in mathematical circles, Gardner was not a mathematician in any traditional sense. At the University of Chicago in the mid-1930s, he majored in philosophy and excelled at logic but otherwise ignored mathematics (although he did audit a course called "Elementary Mathematical Analysis"). He was, however, well versed in mathematical puzzles. His father, a geologist, introduced him to the great turn-of-the-century puzzle innovators Sam Loyd and Henry Ernest Dudeney. From the age of 15, he published articles regularly in magic journals, in which he often explored the overlap between magic and topology, the branch of mathematics that analyzes the properties that remain unchanged when shapes are stretched, twisted, or deformed in some other way without tearing. For example, a coffee mug with a handle and a doughnut (or bagel) are topologically the same because both are smooth surfaces with one hole.

In 1948 Gardner moved to New York City, where he became friends with Jekuthiel Ginsburg, a mathematics professor at Yeshiva University and editor of *Scripta Mathematica*, a quarterly journal that sought to



Six different pictures can be made to appear after a single decorated strip of paper is folded into a flat hexagonal structure called a hexahexaflexagon and then twisted and reflattened multiple times, as Gardner demonstrated in *Scientific American* in December 1956. (For a cutout you can use to make your own hexaflexagon, go to http://www.scientificamerican.com/editorial/martin-gardner-centennial/)

extend the reach of mathematics to the general reader. Gardner wrote a series of articles on mathematical magic for the journal and, in due course, seemed to fall under the influence of Ginsburg's argument that "a person does not have to be a painter to enjoy art, and he doesn't have to be a musician to enjoy good music. We want to prove that he doesn't have to be a professional mathematician to enjoy mathematical forms and shapes, and even some abstract ideas."

In 1952 Gardner published his first article in *Scientific American* about machines that could solve basic logic problems. Editor Dennis Flanagan and publisher Gerard Piel, who had taken charge of the magazine several years earlier, were eager to publish more math-related material and became even more interested after their colleague James Newman authored a surprise best seller, *The World of Mathematics*, in 1956. That same year Gardner sent them an article about hexaflexagons—folding paper structures with properties that both magicians and topologists had started to explore. The article was readily accepted, and even before it hit newsstands in December, he had been asked write a monthly column in the same vein.

Gardner's early entries were fairly elementary, but the mathematics became deeper as his understanding—and that of his readers—grew. In a sense, Gardner operated his own sort of social media network but at the speed of the U.S. mail. He shared information among people who would otherwise have worked in isolation, encouraging more research and more findings. Since his university days, he had maintained extensive and meticulously organized files. His network helped him to extend those files and to garner a wide circle of friends, eager to contribute ideas. Virtually anyone who wrote to him got a detailed reply, almost as though they had queried a search engine. Among his correspondents and associates were mathematicians John Horton Conway and Persi Diaconis, artists M. C. Escher and Salvador Dalí, magician and skeptic James Randi, and writer Isaac Asimov.

Gardner's diverse alliances reflected his own eclectic interests—among them literature, conjuring, rationality, physics, science fiction, philosophy, and theology. He was a polymath in an age of specialists. In every essay, it seems, he found a connection between his main subject and the humanities. Such references helped many readers to relate to ideas they might have otherwise ignored. For instance, in an essay on "Nothing," Gardner went far beyond the mathematical concepts of zero and the empty set—a set with no members—and explored the concept of nothing in history, literature, and philosophy. Other readers flocked to Gardner's column because he was such a skillful storyteller. He rarely prepared an essay on a single result, waiting instead until he had enough material to weave a rich tale of related insights and future paths of inquiry. He would often spend 20 days on research and writing and felt that if he struggled to learn something, he was in a better position than an expert to explain it to the public.

Gardner translated mathematics so well that his columns often prompted readers to pursue topics further. Take housewife Marjorie Rice, who, armed with a high school diploma, used what she learned from a Gardner column to discover several new types of tessellating pentagons, five-sided shapes that fit together like tiles with no gaps. She wrote to Gardner, who shared the result with mathematician Doris Schattschneider to verify it. Gardner's columns seeded scores of new findings—far too many to list. In 1993, though, Gardner himself identified the five columns that generated the most reader response: ones on Solomon W. Golomb's polyominoes, Conway's Game of Life, the nonperiodic tilings of the plane discovered by Roger Penrose of the University of Oxford, RSA cryptography, and Newcomb's paradox [*see box entitled "An Unsolved Problem"*].

Polyominoes and Life

Perhaps some of these subjects proved so popular because they were easy to play with at home, using common items such as chessboards, matchsticks, cards, or paper scraps. This was certainly the case when, in May 1957, Gardner described the work by Golomb, who had recently explored the properties of polyominoes, figures made by joining multiple squares side by side; a domino is a polyomino with two squares, a tromino has three, a tetromino has four, and so forth. They turn up in all kinds of tilings, logic problems, and popular games, including modern-day video games such as Tetris. Puzzlers were already familiar with these shapes, but as Gardner reported, Golomb took the topic further, proving theorems about what arrangements were possible.

Certain polyominoes also appear as patterns in the Game of Life, invented by Conway and featured in *Scientific American* in October 1970.

Puzzle Sampler

Test Yourself

Recreational math puzzles fall into many broad categories and solving them draws on a variety of talents, as the examples here, some of which are classics, show. (For the answers, go to http://www.scientificamerican.com/editorial/martin-gardner-centennial/)

Some puzzles call for little more than basic reasoning. For instance, consider this brain teaser: There are three on/off switches on the ground floor of a building. Only one operates a single lightbulb on the third floor. The other two switches are not connected to anything. Put the switches in any on/off order you like. Then go to the third floor to see the bulb. Without leaving the third floor, can you figure out which switch is genuine? You get only one try.

Cryptarithms serve up harder tests of a puzzler's abilities. In these problems, each letter corresponds to a single digit. For instance, can you figure out which digit each letter represents to make the sum at the right work?

```
  SEVEN
  SEVEN
  SEVEN
  SEVEN
  SEVEN
  SEVEN
+ SEVEN
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 FORTY9
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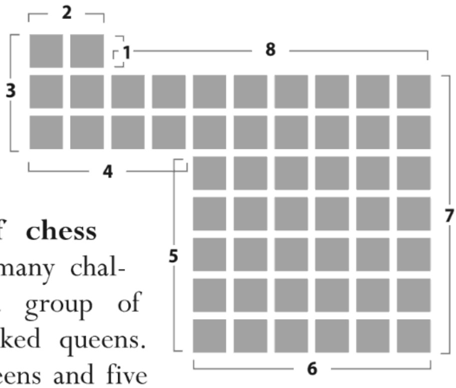
A knack for visualization is helpful for solving geometric stumpers. Can you picture a solid pyramid consisting of a square base and four equilateral triangles, alongside a solid tetrahedron with four faces identical to those of the pyramid's triangles? Now glue one triangle face of the pyramid to a triangle on the tetrahedron. How many faces does the resultant polyhedron have? It's not seven!

Puzzlers, like mathematicians, must sometimes solve challenges that reflect general problems or require the construction of logical proofs. Think about the class of polygons known as serial isogons. All adjacent sides meet at 90 degrees, and the sides are of increasing length: 1, 2, 3, 4, and so on. The simplest isogon, with sides 1–8, is shown at the right. This is the only serial isogon known to tile the plane. But there are more isogons. Can you prove that the number of their sides must always be a multiple of 8?



The properties of chess pieces play a part in many challenges, including in a group of problems about unattacked queens. Imagine three white queens and five black queens on a 5 × 5 chessboard. Can you arrange them so that no queen of one color can attack a queen of the other color? There is only one solution, excluding reflections and rotations.

The game involves "cells," entries in a square array marked as "alive" or "dead," that live (and can thus proliferate) or die according to certain rules—for instance, cells with two or three neighbors survive, whereas those with no, one, or four or more neighbors die. "Games" start off with some initial configuration, and then these groupings evolve according to the rules. Life was part of a fledgling field that used "cellular automata" (rule-driven cells) to simulate complex systems, often in intricate detail. Conway's insight was that a trivial two-state automaton, which he designed by hand, contained the ineffable potential to model complex and evolutionary behavior.

After Gardner's column appeared, the Game of Life quickly attracted a cultlike following. "All over the world mathematicians with computers were writing Life programs," Gardner recalled. His dedicated readership soon produced many surprising findings. Mathematicians had long known that a short list of axioms can lead to profound truths, but the Life community in the early 1970s experienced it firsthand. Some 40 years later Life continues to spark discoveries: a new self-constructing pattern known as Gemini—which copies itself and destroys its parent pattern while innovatively moving in an oblique

The community exploring the properties of Penrose tilings has made a number of advances since, including finding that the patterns display a property called self-similarity, also enjoyed by fractals, structures that repeat at different scales. (Fractals, too, gained widespread popularity in large part because of Gardner's December 1976 column about them.) And Penrose tiles have also led to the discovery of quasicrystals, which have an orderly but aperiodic structure. Nobody was more delighted about the connection than Gardner, who commented, "They are wonderful examples of how a mathematical discovery, made with no inkling of its applications, can turn out to have long been familiar to Mother Nature!"

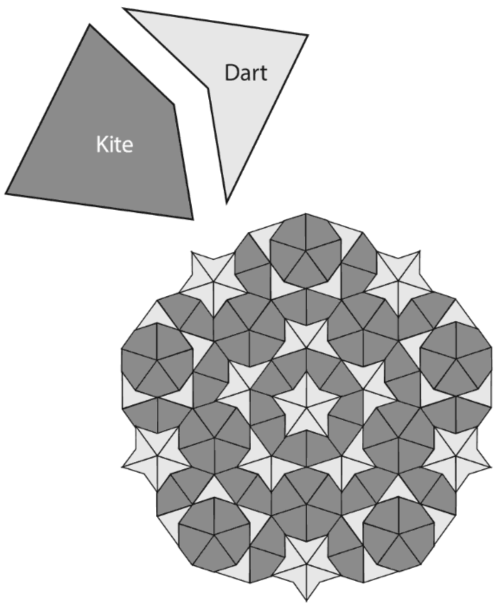
In August 1977 Gardner anticipated another modern-day development: the use of electronic mail for personal communication "in a few decades." This prediction opened a column that introduced the world to RSA cryptography, a public-key cryptosystem based on trapdoor functions—ones that are easy to compute in one direction but not in the opposite direction. Such systems were not new in the mid-1970s, but computer scientists Ron Rivest, Adi Shamir, and Leonard Adleman (after whom RSA is named) introduced a different kind of trapdoor using large prime numbers (those divisible only by one and themselves). The security of RSA encryption stemmed from the apparent difficulty of factoring the product of two sufficiently large primes.

Before publishing their result in an academic journal, Rivest, Shamir, and Adleman wrote to Gardner, hoping to reach a large audience quickly. Gardner grasped the significance of their innovation and uncharacteristically rushed a report into print. In the column, he posed a challenge, asking readers to attempt to decode a message that would require them to factor a 129-digit integer, an impossible task at that time. Gardner wisely prefaced the challenge with an Edgar Allan Poe quotation: "Yet it may be roundly asserted that human ingenuity cannot concoct a cipher which human ingenuity cannot resolve." And indeed, only 17 years later, a large team of collaborators, relying on more than 600 volunteers and 1,600 computers, cracked the code, revealing that the secret message read: "The magic words are squeamish ossifrage." RSA challenges continued for many years, ending only in 2007.

After Gardner

Gardner's love of play went hand in hand with his impish sense of fun. A 1975 April Fools' Day column featured "six sensational discoveries that somehow or another have escaped public attention." All were plausible—and false. For instance, he claimed that Leonardo da Vinci invented the flush toilet. Allusions to "Ms. Birdbrain" and the psychic-powered "Ripoff rotor" were meant to alert readers to the gag nature of the column, but hundreds failed to get the joke and sent Gardner animated letters.

In 1980 Gardner decided to retire his column to concentrate on other writing projects. *Scientific American* quickly introduced a successor: Douglas Hofstadter. He wrote 25 columns, entitled Metamagical



Penrose tiles are remarkable for producing "aperiodic" patterns: given an infinite supply, they will fill the floor without gaps such that the initial configuration never repeats exactly. Gardner wrote about Penrose tiles called kites and darts in January 1977. To ensure aperiodicity, the tiles must be laid according to certain rules. The starting grouping above is named "the infinite star pattern."

Themas—an anagram of Mathematical Games—many of which discussed artificial intelligence, his own specialty. A. K. Dewdney followed, penning seven years of Computer Recreations. Ian Stewart's Mathematical Recreations column ran for the next decade. Later Dennis Shasha wrote a long series of Puzzling Adventures, based on computing and algorithmic principles, subtly disguised. "Martin Gardner was an impossible act to follow," Stewart once commented. "What we did try to do was replicate the spirit of the column: to present significant mathematical ideas in a playful mood."

For the past two decades the spirit of the column has lived on at invitation-only, biennial Gathering 4 Gardner conferences, where mathematicians, magicians, and puzzlers assemble to share what they wish they could still share via Mathematical Games. Gardner himself attended the first two. In recent years participants have ranged from old friends, such as Golomb, Conway, Elwyn Berlekamp, Richard Guy, and Ronald Graham, to rising stars, such as computer scientist Erik Demaine and video maven Vi Hart, and some very young blood in the form of talented teenagers Neil Bickford, Julian Hunts, and Ethan Brown. Following Gardner's death in 2010, spin-off Celebration of Mind parties, which anyone can attend (or host), have been held all over the world every October in his honor.

Although Gardner is gone, there are good reasons to take inspiration from his work and to champion recreational mathematics today. Noodling over puzzles and related activities often leads to important discoveries, as shown, if only briefly, in this article. Almost every essay Gardner wrote gave rise to communities of enthusiasts and specialists. A great number of his columns could now be expanded into books—entire shelves of books even. In addition, thinking about a problem from a mathematical perspective can be enormously valuable for clarity and rigor. Gardner never thought of recreational mathematics as a set of mere puzzles. The puzzles were a gateway to a richer world of mathematical marvels.

In his final, retrospective *Scientific American* article in 1998, Gardner reflected that the "line between entertaining math and serious math is a blurry one. . . . For 40 years I have done my best to convince educators that recreational math should be incorporated into the standard curriculum. It should be regularly introduced as a way to interest young students in the wonders of mathematics. So far, though, movement in this direction has been glacial."

Today the Internet hosts scores of math-related apps, tutorials, and blogs—including many different Game of Life apps of varying quality—and social media can connect like-minded aficionados faster than Gardner ever could. But maybe that speed has a downside: Web-based experiences are perfect for quick "Interesting!" responses, but it takes careful reflection to reach revelatory "Aha!" moments. We believe that part of the success of Gardner's column was that he and his audience took the trouble to exchange detailed ideas and craft thoughtful answers. Only time will tell if a new community of puzzlers—in a less patient era—will pick up Gardner's mantle and propel future generations to fresh insights and discoveries.

Challenging Magic Squares for Magicians

Arthur T. Benjamin and Ethan J. Brown

Magic squares are fascinating, and not just for mathematicians. Most people are intrigued to see numbers arranged in a box where every row, column, and diagonal have the same magical sum. Four-by-four magic squares, like the one shown in Figure 1, are especially intriguing because so many sets of four entries have the same magic total. Notice how every row, column, diagonal, broken diagonal, 2×2 box, and more, add up to 34.

Magic Squares with a Given Total

Many magicians, including the authors of this paper, create magic squares as parts of their shows. Typically, an audience member is asked for a number (say between 30 and 100) and the magician quickly creates a magic square and shows off the many ways that their total is obtained. As described in many magic books (such as [**5**]), the quickest and easiest way to create a magic square with total T is to modify the square in Figure 1 to create the square in Figure 2.

8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

Figure 1. A magic square with magic total 34.

(a)

A	B	C	D
$C-x$	$D+x$	$A-x$	$B+x$
$D+x$	$C+x$	$B-x$	$A-x$
B	$A-2x$	$D+2x$	C

(b)

6	**20**	**9**	**9**
8	10	5	21
10	10	19	5
20	4	11	9

FIGURE 5. Construction 2 for a double birthday magic square.

the numbers are repeated. Not only do the bottom corner squares (intentionally) duplicate two of the numbers in the top row, but some of the numbers in the second and third rows are duplicates as well.

Magic Squares with Three Random Digits

For the ultimate challenge, the authors thought it would be especially impressive to allow members of the audience to choose the total and three numbers to be placed in any three of the squares. (Naturally, asking the audience for four numbers and the total could lead to an impossible problem.) The solution, as performed by the second author, takes advantage of previous Constructions 1 and 2.

Ask members of the audience to choose any three squares, then ask for any three numbers to go inside those squares (say between 1 and 20). Finally, ask another audience member to provide the total (say between 30 and 80). Note that the first three squares can be chosen in $\binom{16}{3} = 560$ ways. In $4^4 = 256$ of those situations, the three chosen locations will correspond to three different letters from Figure 4(a), say using letters A, B, and C.

For example, if the chosen numbers are 3, 11, and 13 in the squares prescribed in Figure 6(a), then Construction 1 provides a solution by letting $A = 3$, $B = 14$, and $C = 15$. If the prescribed total is $T = 41$, then that would force $D = T - (A + B + C) = 9$, and the square could be completed as in Figure 6(b).

If the three locations do not correspond to different letters in Figure 4(a), then they often correspond to different letters when Figure 4(a) is turned counterclockwise 90 degrees, as displayed in Figure 7(a). For

(a)

3			
			11
	13		

(b)

3	14	15	9
18	6	6	**11**
7	**13**	16	5
13	8	4	16

FIGURE 6. Constructing a magic square with total 41 with three prescribed squares.

example, in Figure 7(b), the prescribed numbers 6, 5, and 18 all use B numbers from Figure 4(a), but they correspond to $A = 6$, $B = 8$, and $D = 23$ in Figure 7(a). So with a prescribed total of 49, the magician need only "tilt his or her head" and apply the same process of Construction 1 to reach the completed square in Figure 7(b). We call this process Construction 1R.

Conveniently, in the $4^2 = 16$ situations when all three prescribed squares use the same letter in Figure 4(a), those squares will correspond to different letters in the square of Figure 7(a). (Indeed, that would be true even if there were four prescribed squares with the same letter.) Moreover, $4 \cdot 6 \cdot 6 = 144$ situations are of the form YYZ (with $Z \neq Y$) in Figure 4(a), that correspond to three different letters in Figure 7(a). To see this, there are four choices for Y, then $\binom{4}{2} = 6$ ways to choose which two squares in Figure 5(a) to use with that letter. Those will correspond to two different letters in Figure 7(a), say A and B. Then there are $8 - 2 = 6$ choices for the remaining square corresponding to a C or

(a)

D	$B - 3$	$A + 2$	$C + 1$
C	$A + 3$	$B + 2$	$D - 5$
B	$D - 3$	$C - 2$	$A + 5$
A	$C + 3$	$D - 2$	$B - 1$

(b)

23	**5**	8	13
12	9	10	**18**
8	20	10	11
6	15	21	7

FIGURE 7. Constructing a magic square with total 49 with three prescribed squares using Construction 1R.

D in Figure 7(a) that is not one of the remaining *Y*'s of Figure 4(a). Thus using Construction 1 or 1R, we can handle $256 + 16 + 144 = 416$ of the 560 ways of choosing three locations for the prescribed numbers. Note that if two locations were prescribed, instead of three, then Construction 1 or 1R handles all $\binom{16}{2} = 120$ possibilities.

Combinatorial aside. The letters in Figure 4(a) and its rotation are orthogonal Latin squares, in that their combined letters give all 16 possible ordered pairs, as shown in Figure 8. This allows us to count the number of ways to choose three squares that will correspond to different letters in both matrices. The *first* square can be chosen 16 ways, then the *second* square can be chosen $3^2 = 9$ ways (three choices for the first coordinate, then three choices for the second coordinate), then the *third* square can be chosen $2^2 = 4$ ways. Since order does not matter, there are $(16 \cdot 9 \cdot 4)/3! = 96$ ways to select three squares that will satisfy both conditions. Hence, by the principle of inclusion-exclusion, the number that satisfies at least one condition is $256 + 256 - 96 = 416$, as previously noted.

We are left with $560 - 416 = 144$ situations like the one in Figure 9. This can also be counted directly. To create a *YYZ* in Figures 4(a) and 7(a), there are four choices for *Y* to be used in Figure 4(a), and then $\binom{4}{2} = 6$ ways to pick their locations. These will necessarily correspond to different letters in Figure 7(a), say *A* and *B*, so there are $8 - 2 = 6$ ways to pick the other *A* or *B* square in Figure 7(a). The example in Figure 9 can be extended to a magic square neither by Construction 1 (since it's of type *AAB*) nor Construction 1R (since it's of type *BBD*). When this happens, we resort to "plan *x*."

AD	*BB*	*CA*	*DC*
CC	*DA*	*AB*	*BD*
DB	*CD*	*BC*	*AA*
BA	*AC*	*DD*	*CB*

Figure 8. The letters of Figures 4(a) and 7(a) are orthogonal Latin squares.

8	**17**		
		5	

FIGURE 9. A square that foils Constructions 1 and 1R.

When Constructions 1 and 1R fail us, we go back to the birthday magic square of Figure 5(a), which we repeat in Figure 10(a). This has the same letter pattern as Construction 1, but it also has a variable parameter x that can be exploited. For example, in the example of Figure 9, we can assign $A = 8$, $B = 17$, and $x = 3$. We still have two degrees of freedom, so after the total is assigned, say $T = 52$, we can arbitrarily choose two numbers that add to $T - (A + B) = 27$, say $C = 21$ and $D = 6$, and the result is the magic square of Figure 10(b). We call this process Construction 2.

This process successfully handles $4 \cdot 4 \cdot 6 = 96$ of the 144 situations. For these YYZ situations, there are four choices for Y, then $\binom{4}{2} - 2 = 4$ choices for which pair we can use. (For example, with $Y = A$, we disallow picking A with $A - 2x$ or picking $A - x$ twice.) Then the third square can be chosen six ways to form YYZ in the rotated square. This can result in the occasional unaesthetic appearance of negative numbers. (For instance, if in the last example, the number 17 was replaced by 1, then the third number on the main diagonal would be $B - x = -2$.)

(a)

A	B	C	D
$C - x$	$D + x$	$A - x$	$B + x$
$D + x$	$C + x$	$B - x$	$A - x$
B	$A - 2x$	$D + 2x$	C

(b)

8	**17**	21	6
18	9	**5**	20
9	24	14	5
17	2	12	21

FIGURE 10. "Completing the square" with Construction 2.

Of the remaining 48 situations, $4 \cdot 1 \cdot 6 = 24$ of them use two prescribed squares in Construction 2 with a difference of $2x$:

- A and $A - 2x$,
- $B + x$ and $B - x$,
- $C - x$ and $C + x$,
- D and $D + 2x$.

In each case, six choices of the third square will not be satisfied by Construction 1R. For example, suppose that the boxes $C - x$, $C + x$, and $B + x$ are prescribed, as in Figure 11(a).

In order to apply Construction 2, the squares containing $C - x$ and $C + x$ must differ by an even number. (While we are willing to put up with negative numbers, we do not tolerate half-integers!) The simplest remedy is to restrict the parity of one of the entries. For example, if your volunteer chooses $C - x$ to be 3 and $B + x$ to be 14, then indicates square $C + x$, the magician can say, "To make this really interesting, give me any odd number between 1 and 20." (A more complex remedy without the parity restriction will be given in the final section.) Say the volunteer chooses $C + x = 15$, then $C = 9$, $x = 6$, and $B = 8$. If the prescribed total is 72 and you freely choose $A = 30$ and $D = 72 - (30 + 8 + 9) = 25$, then the square can be completed as in Figure 11(b).

The only remaining situation to consider, which occurs in just $4 \cdot 1 \cdot 6 = 24$ of the 560 possible cases, is when two of the prescribed squares are required to be equal by Construction 2. In other words, among the three prescribed squares, two have label $A - x$ or B or C or $D + x$, where the labels are given in Figure 11(a). Conveniently, in all 24 of these situations, when the magician mentally rotates the matrix 180 degrees, we obtain the previously considered case where the two

(a)

A	B	C	D
$\mathbf{C - x}$	$D + x$	$A - x$	$\mathbf{B + x}$
$D + x$	$\mathbf{C + x}$	$B - x$	$A - x$
B	$A - 2x$	$D + 2x$	C

(b)

30	8	9	25
3	31	24	**14**
31	**15**	2	24
8	18	37	9

FIGURE 11. An "even" more challenging situation.