The BEST WRITING on MATHEMATICS

2016

Mircea Pitici, Editor

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PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540
In the United Kingdom: Princeton University Press,
6 Oxford Street, Woodstock, Oxfordshire OX20 1TW

press.princeton.edu

ISBN (pbk.) 978-0-691-17529-4

This book has been composed in Perpetua

Printed on acid-free paper. ∞

Printed in the United States of America

1 3 5 7 9 10 8 6 4 2

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Introduction

Mircea Pitici

For the seventh consecutive year, we offer an anthology of recent writings on mathematics, easily accessible to general readers but intriguing enough to interest professional mathematicians curious about the reach of mathematics in the broader concert of ideas, disciplines, society, and history. With one exception (which escaped my radar two years ago), these pieces were first published during the calendar year 2015 in various venues, including professional journals, mass media, and online. In the introductions to the previous volumes and in several interviews, I detailed the motivations and the procedures that underlie the making of the books in this series; before I turn to the content of the current volume, I remind you of some of those tenets, then I point out a few constant and a few changing features that will help you, the reader, consider this enterprise in long-term perspective. I see each year's edition as the product of a convention imposed by our inevitable subservience to calendar strictures; each volume should be considered together with the other volumes, an integral part of the series. It seems to me that the reviewers of all the volumes published so far ignored this aspect. To rectify somehow this prevailing perception of discontinuity, in the near future we will make available a combined index of the pieces published so far.

The main message of the series is that there is a lot more to mathematics than formulas and learning by rote—a lot more than the stringency of proof and the rigor usually associated with mathematics (and held so dear by mathematicians). Mathematics has interpretive sides with endless possibilities, many made manifest by writing in natural language. By now this kind of literature has produced veteran practitioners who are easily recognized by the public but also talented newcomers. It is a new genre that has spun a vast literature but remains completely ignored in educational settings. I hope that *The Best Writing on Mathematics* series helps educators and perhaps policy makers see that it is worth broadening students' understanding of mathematics. Mathematics is an extraordinarily diverse phenomenon molded by the human mind but anchored in a realm both abstract and concrete at the same time. That is not easy to comprehend and to master if we restrict the learning of mathematics to the use of conventional mathematical symbols and notations. I am happy to see that in the books of this series we can bring together constituencies that are sometimes uncomfortable with each other, such as mathematicians and mathematics educators, "purists" and mavericks, entrepreneurs and artists. This series can contribute to a change in the perception of mathematics, toward a broad view that includes the anchoring of mathematics in social, cultural, historical, and intellectual phenomena that hold huge stakes in the working of contemporary humanity. For what it is worth, with its good qualities and its faults, this series undoubtedly reflects my vision, even though other people also add opinions along the way. I propose the initial selection of pieces. I choose most of the pieces myself, but occasionally I receive suggestions from other people (including their own productions). I always welcome suggestions, and I try to remain blind to my correspondents' eventual self-interest and self-promotion. I start from the premise that people are well-intended. I consider everything that comes to my attention, regardless of the manner in which it reaches me. The nontechnical literature on mathematics is immense and it is impossible for me to survey all of it; if people come forward to point out something new to me, I am grateful. A few of the pieces brought to my attention by their authors rightly made it into volumes; most of them did not.

Seizing an opportunity mediated by Steven Strogatz and accepted by Vickie Kearn, I did the first volume in this series (2010) quickly, in just two months, pulling all-nighters in the library at Cornell University; it had a slightly different structure than subsequent volumes. Starting with the second volume (2011) we settled on a template and on certain routines, schedules, and processes that work smoothly—with occasional bumps in the road, to keep us on our toes! This is a fast-paced series; there is not much room for delays and second-guessing.

For the final selection of texts, we take guidance from several professional mathematicians who grade each of about sixty to seventy pieces I initially propose for consideration; we also have to consider constraints related to length, copyright, and diversity of topics and authors. Thus, the content of the books is dependent on the changing literature available but the format is fairly stable. Yet changes in format do occur, gradually. For instance, starting with the second volume, I added the long list of also-rans, supplemented later by a list of special journal issues and now by two new lists, one of remarkable book reviews and another of interviews. The profile of the volumes in *The Best Writing on Mathematics* has shifted from anthology-only to anthology *and* reference work for additional sources about mathematics. By now, in each book we offer not only a collection of writings but also a fairly detailed list of supplementary resources, worth the attention of researchers who might study on their own the complex phenomenon of mathematics and its reverberations in contemporary life. The section containing notable writings, which I present toward the end of each volume, takes me a lot more time to prepare than the rest of the book. Many of the sources I list there are difficult to find for most readers of our series; I hope the list will stimulate you to seek at least a few other readings, in addition to those included here. In this volume another new feature is the printing of figures in full color within the pieces where they belong (not in a separate insert of the book).

I receive a lot of feedback on this series, most of it informal, most of it good, pleasing, cheery, even exalted. The more formal feedback comes in book reviews. It is fascinating and reassuring to compare the reviews. Almost every aspect criticized by some reviewer is praised by other reviewers. Where some people find faults, others see virtue. The point I am making is that regardless of what *would* be desirable to include in the volumes of this series, we are circumscribed by the material extant out there. I cannot include a certain type of article if I encounter nothing of that sort or if we cannot overcome republishing hurdles.

Inevitably, the final selection of pieces for each volume is slightly tilted each year toward certain themes. This happens despite my contrary tendency, to reach a selection as varied as possible, unbiased, illustrating as many different perspectives on mathematics as the recent literature allows. The attentive reader will notice that in this edition a group of contributions refer to the dynamic tension between the object and the practice of "pure" versus "applied" mathematics. This topic is intertwined with the millennia-old history of mathematics and has been addressed by many mathematicians, nonmathematicians, and philosophers.

Overview of the Volume

I now offer a brief outline of each contribution to the volume, with the caveat that the sophisticated arguments made by our authors and the assumptions that support those arguments are worth reading in detail, comparing, corroborating, and contrasting.

Hyman Bass reveals the learning experiences that shaped his views of mathematics and places them in a broader discussion of the complex rapport between mathematics and its pedagogy.

Daniel Silver presents H. G. Hardy's blazing ideas from that most famous exposition of a working "pure" mathematician, *A Mathematician's Apology*—and gives us a potpourri of reactions that followed its publication.

In a piece that can be seen as a genuine continuation on the same theme, Hannah Elizabeth Christenson and Stephan Ramon Garcia detail the fortuitous confluence of mathematics, cricket, and genetics—all of which conspired to make Hardy, at least for once, an *applied* mathematician.

Derek Abbott contends that the applicability of mathematics is considerably overrated, takes the decidedly anti-Platonist position that mathematics is entirely a human construct, and proposes that accepting these unpopular tenets will "accelerate progress."

Aided by suggestive and clear illustrations, Burkard Polster manages to use no formulas in a plain-language proof of a geometric result concerning circles arranged in a rectangle—and leaves us with a few challenges on similar topics.

Joshua Bowman considers the problem of periodic paths of billiard balls in pools of triangular shape; he gives some known results and mentions a few open problems.

In his second piece included in this volume, Burkard Polster teaches us how to invent our own variants of the card game *Spot It!*

Jennifer Quinn reports on a contest in mathematical virtuosity, with participants so ingenious that the victor cannot be settled on the spot;

all this comes from one of the newly popular, enormous, always sold-out arenas of mathematical-gladiatorial disputes!

In a compelling piece that combines elements of group, string, and number theories, Erica Klarreich weaves together considerations made by several mathematicians and physicists struck by the connections they find among fields of research apparently far apart.

Davide Castelvecchi describes the avatars of the proposed proof for the *abc* conjecture in numbers theory and relates them to the peculiarities of the proof's author, Shinichi Mochizuki.

Kevin Hartnett tells the decisive story of a recent breakthrough, Ciprian Manolescu's proof that some spaces of dimension higher than three cannot be subdivided.

Steven Strogatz presents a brilliantly insightful proof of the Pythagorean theorem from the young Albert Einstein.

In a semiautobiographical recollection, Brian Greene outlines the evolving speculative assumptions of the physicists who developed the mathematical underpinnings of string theory.

Tanya Khovanova, Eric Nie, and Alok Puranik describe step by step a generalization about hexagonal grids of a fractal iteration first proposed by Stanislaw Ulam.

Marc Frantz shows us beautiful fractal images and explains how he constructed them.

Joseph Dauben and Marjorie Senechal took strolls through the Metropolitan Museum in New York and discovered a treasure trove of mathematical content—some in plain view, some deftly hidden in details.

Alan Schoenfeld places the contentious Common Core Standards for School Mathematics in perspective, relating them to previous programmatic documents in U.S. education and discussing their rapport with a few important issues in mathematics instruction.

Katharine Beals and Barry Garelick take a decidedly critical stand toward the Common Core, arguing that the standards run against developmental constraints and may impede the learning of mathematics, not promote it.

David Acheson, Peter Turner, Gilbert Strang, and Rachel Levy present four visions of teaching applied mathematics, each of them emphasizing different elements they see as important in such work.

David Richeson details the very early history of one of the most widely recognized mathematical facts, the constancy of the proportion between the circumference and the diameter.

Isabel Serrano and Bogdan Suceavă reconsider the contribution of the medieval thinker Nicholas Oresme to the history of the idea of curvature and conclude that Oresme deserves more credit than he is usually given in accounts of the history of geometrical concepts.

In a similar reconsideration but with an opposite twist than Serrano and Suceavă, Viktor Blåsjö contends that Gottfried Wilhelm Leibniz receives *too much* credit for "proving" the fundamental theorem of calculus.

Amy Shell-Gellasch examines some connections between geometric curves and mechanical drawing devices, with a focus on the drawing toy Spirograph.

John Stillwell ponders what qualities might account for the "depth" of mathematical results and compares from this viewpoint a number of well-known theorems.

Drawing on their work experience, Marco Puts, Piet Daas, and Ton de Waal propose solutions for the instances in which errors find their way into large sets of data.

Brian Hayes summarizes the problematic current state of achievement in the quest to construct probabilistic (as opposed to deterministic) programming languages.

In a piece that combines psychological insights and mathematical rigor, Jorge Almeida analyzes several common misperceptions of the people who attempt to rationalize the most likely outcome of the lottery process.

Andrew Gelman explains why some research claims based on null hypothesis significance tests are spurious, especially in psychological studies.

Howard Wainer and Richard Feinberg point out that the practice of administering long tests under the uncritical pretense that they result in accurate assessments leads to colossal wastes of time, when aggregated to a social scale.

In the last piece of the volume, Ian Stewart attempts to circumscribe popular mathematics as a genre, affirms its value for the public, and advises those who want to contribute to it.

More Writings on Mathematics

In every edition of this series, I list other writings on mathematics published mainly during the previous year, to stimulate readers in their search for broad views of mathematics. The writings mentioned are either books and online resources (listed in the introduction) or articles (listed in the section of notable writings). Most of these items are not highly technical, but occasionally mathematical virtuosity or even theory does enter the picture; a clear distinction between technical and nontechnical writings on mathematics is not possible. The pretense that these bibliographies are comprehensive would be quixotic in the present state of publishing on mathematics. I list only items I have seen—courtesy of authors and publishers who sent me books (thank you!) or through the valuable services of the Cornell University Library and, in a few instances, of the Z. Smith Reynolds Library at Wake Forest University, where my wife worked for four years. For additional books I refer the reader to the monthly new books section published by *The Notices of the American Mathematical Society* and to the excellent website for book reviews hosted by the Mathematical Association of America.

Among the books that came to my attention over the past year, three collective volumes stand out: *The Princeton Companion to Applied Mathematics* edited by Nicholas Higham, *The First Sourcebook on Asian Research in Mathematics Education* edited by Bharath Sriraman, Jinfa Cai, and Kyeong-Hwa Lee, and the anthology *An Historical Introduction to the Philosophy of Mathematics* edited by Russell Marcus and Mark McEvoy.

Some books in which the authors focus on mathematics as it commingles with our lives, in its habitual, entertaining, puzzling, and gaming aspects, are *The Proof and the Pudding* by Jim Henle, *A Numerate Life* by John Allen Paulos, *Truth or Truthiness* by this year's contributor to our volume Howard Wainer, *The Magic Garden of George B and Other Logical Puzzles* by the almost centenarian Raymond Smullyan, *Problems for Metagrobologists* by David Singmaster, as well as the collective volumes *The Mathematics of Various Entertaining Subjects* edited by Jennifer Beineke and Jason Rosenhouse, *Numbers and Nerves* edited by Scott and Paul Slovic, and even *Digital Games and Mathematics Learning* edited by Tom Lowrie and Robyn Jorgensen (the last one with an emphasis on educational issues). More expository books are *Math Geek* by Raphael Rosen, *A Mathematical Space Odyssey* by Claudi Alsina and Roger Nelsen, *Beautiful, Simple, Exact, Crazy* by Apoorva Khare and Anna Lachowska, *Single Digits* by Marc Chamberland, *Math Girls Talk about Integers* and *Math Girls Talk about Trigonometry* by Hiroshi Yuki, and *Sangaku Proofs* by Marshall Unger.

An area enjoying phenomenal growth is the history of mathematics, in which I include the history of mathematical ideas and the histories of mathematical people. Here are some recent titles: *I, Mathematician,* a remarkable collection of pieces by working mathematicians, edited by Peter Casazza, Steven G. Krantz, and Randi Ruden; *The War of Guns and Mathematics* edited by David Aubin and Catherine Goldstein; *Music and the Stars* by Mary Kelly and Charles Doherty; *The Real and the Complex* by Jeremy Gray; and *Taming the Unknown* by Victor Katz and Karen Hunger Parshall. Biographical works on remarkable mathematicians are *Leonhard Euler* by Ronald Calinger, *The Scholar and the State: In Search of Van der Waerden* by Alexander Soifer, *Genius at Play: The Curious Mind of John Horton Conway* by Siobhan Roberts, *Fall of Man in Wilmslow* (a novel that starts with the death of Alan Turing) by David Lagercrantz, and *The Astronomer and the Witch: Johannes Kepler's Fight for His Mother* by Ulinka Rublack. Two collective volumes in the same biographical category are *Oxford Figures* edited by John Fauvel, Raymond Flood, and Robin Wilson; and *Lipman Bers* edited by Linda Keen, Irwin Kra, and Rubí Rodríguez. *Birth of a Theorem* by Cédric Villani is autobiographical. Collected works and new editions of old books include *The G. H. Hardy Reader* edited by Donald J. Albers, Gerald Alexanderson, and William Dunham; *Birds and Frogs* by Freeman Dyson; *A Guide to Cauchy's Calculus* by Dennis Cates; and *Tartaglia's Science of Weights and Mechanics in the Sixteenth Century* by Raffaele Pisano and Danilo Capecchi. Historical in perspective with a strong sociological component (and on an actual topic) is *Inventing the Mathematician* by Sara Hottinger.

Recent books on philosophical aspects of mathematics are *Mathematics, Substance and Surmise* edited by the son-and-father pair Ernest and Philip Davis; *G.W. Leibniz, Interrelations between Mathematics and Philosophy* edited by Norma Goethe, Philip Beeley, and David Rabouin; *Mathematical Knowledge and the Interplay of Practices* by José Ferreirós; and *The Not-Two* by Lorenzo Chiesa. Wide-ranging and difficult to categorize is *Algorithms to Live By*, by Brian Christian and Tom Griffiths.

A great number of books on mathematics education are published every year; it is not feasible for me to mention all that literature. Here are a few recent titles that came to my attention: *Confessions of a 21st Century Math Teacher* and *Math Education in the U.S.* by our contributor Barry Garelick, *What's Math Got to Do with It?* by Jo Boaler, *Mathematical Mindsets* by Jo Boaler and Carol Dweck, *More Lessons Learned from Research* edited by Edward Silver and Patricia Ann Kenney, *Assessment to Enhance Teaching and Learning* edited by Christine Suurtamm, *How to Make Data Work* by Jenny Grant Rankin, and the refreshingly iconoclastic *Burn Math Class* by Jason Wilkes.

In connections between mathematics and other domains, including directly applied mathematics and highly interdisciplinary works, recent titles are *Great Calculations* by Colin Pask, *Creating Symmetry* by Frank Farris, *Why Information Grows* by César Hidalgo, *Data Mining for the Social Sciences* by Paul Attewell and David Monaghan, *Knowledge Is Beautiful* by David McCandless, *Exploring Big Historical Data* by Shawn Graham, Ian Milligan, and Scott Weingart, *Topologies of Power* by John Allen, *Understanding and Using Statistics for Criminology and Criminal Justice* by Jonathon Cooper, Peter Collins, and Anthony Walsh, *Multilevel Modeling in Plain Language* by Karen Robson and David Pevalin, a new edition of *The Essentials of Statistics* by Joseph Healey, and *Mathematics for Natural Scientists* by Lev Kantorovich.

Other friendly books on topics in statistics are *The Seven Pillars of Statistical Wisdom* by Stephen Stigler, *The 5 Things You Need to Know about Statistics* by William Dressler, *Bayes' Rule* by James Stone, *Statistics without Mathematics* by David Bartholomew, and *Beginning Statistics* by Liam Foster, Ian Diamond and Julie Jeffries.

Highly visual is the new volume in *The Best American Infographics* series edited by Gareth Cook.

A work of fiction is *Zombies and Calculus* by Colin Adams.

I hope that you, the reader, will enjoy reading this anthology at least as much as I did while working on it. I encourage you to send comments, suggestions, and materials I might consider for (or mention in) future volumes to Mircea Pitici, P.O. Box 4671, Ithaca, NY 14852 or electronic correspondence to mip7@cornell.edu.

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2016

Mathematics and Teaching

Hyman Bass

Induction

I was not born to be a mathematician. Like many, I was drawn to mathematics by great teaching. Not that I was encouraged or mentored by supportive and caring teachers—such was not the case. It was instead that I had as teachers some remarkable mathematicians who made the highest expression of mathematical thinking visible and available to be appreciated. This was like listening to fine music with all of its beauty, charm, and sometimes magical surprise. Though not a musician, I felt that this practice of mathematical thinking was something I could pursue with great pleasure and capably so, even if not as a virtuoso. And I had the good fortune to be in a time and place where such pursuits were comfortably encouraged.

The watershed event for me was my freshman (honors) calculus course at Princeton. The course was directed by Emil Artin, with his graduate students John Tate and Serge Lang among its teaching assistants. It was essentially a Landau-style course in real analysis (i.e., one taught rigorously from first principles). Several notable mathematics research careers were launched by that course. Amid this cohort of brilliant students, I hardly entertained ideas of an illustrious mathematical future, but I reveled in this ambience of *beautiful thinking*, and I could think of nothing more satisfying than to remain a part of that world. It was only some fifteen years later that I came to realize that this had not been a more-or-less standard freshman calculus course.

Certain mathematical dispositions that were sown in that course remain with me to this day and influence both my research and my teaching. First is the paramount importance of proofs as the defining source of mathematical truth. A theorem is a distilled product of a proof, but the proof is a mine from which much more may often be profitably extracted. Proof analysis may show that the argument in fact proves much more than the theorem statement captures. Certain hypotheses may not have been used, or only weakly used, and so a stronger conclusion might be drawn from the same argument. Two proofs may be observed to be structurally similar, and so the two theorems can be seen to be special cases of a more unifying claim. The most agreeable proofs explain rather than just establish truth. And the logical narrative clearly distinguishes the illuminating turn from technical routine.

Artin himself once reflected on teaching in a review published in 1953:

> We all believe that mathematics is an art. The author of a book, the lecturer in a classroom tries to convey the structural beauty of mathematics to his readers, to his listeners. In this attempt, he must always fail. Mathematics is logical to be sure, each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that, its perception should be instantaneous. We have all experienced on some rare occasion the feeling of elation in realising that we have enabled our listeners to see at a moment's glance the whole architecture and all its ramifications.

Two things of a more social character about mathematics also impressed me. First, the standards for competent and valid mathematical work appeared to be clear, objective, and (so I thought at the time) culturally neutral. The norms of mathematical rigor were for the most part universal and shared across the international mathematics community. Of course, mathematical correctness is not the whole story; there is also the question of the interest and significance of a piece of mathematical work. On this score, matters of taste and aesthetics come into play, but there is still, compared with other fields, a remarkable degree of consensus among mathematicians about such judgments. One circumstance that readily confers validation is a rigorous solution to a problem with high pedigree, meaning that it was posed long ago and has so far defied the efforts of several recognized mathematicians.

A social expression of this culture of mathematical norms stood out to me. Success in mathematics was independent of outward trappings, physical or social, of the individual. This was in striking contrast with almost every other domain of human endeavor. People, for reasons of physical appearance, affect, or personality, were often less favored in nonmathematical contexts. But, provided that they met mathematical norms, such individuals would be embraced by the mathematics community. At least so it seemed to me, and this was a feature of the mathematical world that greatly appealed to me. I have since learned that, unfortunately, many mathematicians individually compromised this cultural neutrality, allowing prejudice to discourage the entry of women and other culturally defined groups into the field.

One effect of this intellectual indifference to social norms in mathematics is that a number of accomplished mathematicians do not present the socially favored images of appearance and/or personality, and so the field is sometimes caricatured as being one of brilliant but socially maladroit and quirky individuals. On the contrary, it is a field with the full range of personality types, and it is distinctive instead for its lack of the kind of exclusion based on personality or physical appearance that infects most other domains of human performance. Perhaps the intellectual elitism and sense of aristocracy common to many mathematicians act as a counterpoint to this social egalitarianism.

The second social aspect of mathematics that stood out to me concerned communication about mathematics to nonexperts. Throughout my student years, undergraduate and graduate, I was amazed and excited by the new horizons being opened up. I enthusiastically tried to communicate some of this excitement to my nonmathematical friends, from whom I had eagerly learned so much about their own studies. These efforts were increasingly frustrated despite my efforts to put matters in analytically elementary terms. I think perhaps that I had already become too much of a mathematical formalist and considered the formal rendering of the ideas an important part of the message. This rendering was often inaccessible to my friends despite my enthusiasm. As my research career entered more abstract theoretical domains, I gradually, and with disappointment, retreated somewhat from efforts to talk to others about what I did as a mathematician. Samuel Eilenberg, my mentor when I first joined the Columbia University faculty, once said something to the effect that

> Mathematics is a performance art, but one whose only audience is fellow performers.

I remain to this day deeply interested in this communication problem, and I have admired and profited from the growing number of authors who have found the language, representations, styles, and narratives with which to communicate the nature of mathematics, its ideas, and its practices. Also, my current studies of mathematics teaching have reopened this question, but now in a somewhat different context. For twelve years of schooling, mathematics has a captive audience of young minds with a natural mathematical curiosity too often squandered. And these children's future teachers are students in the mathematics courses we teach.

Mathematical Truth and Proof

Each discipline has its notions of truth, norms for the nature and forms of allowable evidence, and warrants for claims. Mathematics has one of the oldest, most highly evolved, and well-articulated systems for certifying knowledge—deductive proof—dating from ancient Greece and eventually fully formalized in the twentieth century. There may be philosophical arguments about allowable rules of inference and about how generous an axiom base to admit, and there may be practical as well as philosophical issues about the production and verification of highly complex and lengthy, perhaps partly machine-executed, "proofs." But the underlying conceptual construct of (formal) proof is not seriously thrown into question by such productions, only whether some social or artifactual construction can be considered to legitimately support or constitute a proof.

Mathematical work generally progresses through a trajectory that I would describe as

Exploration → discovery → conjecture → proof → certification

In my work with Deborah Ball (2000, 2003), we have described the first three phases as involving *reasoning of inquiry* and the last two as involving *reasoning of justification*. The former is common to all fields of science. The latter has only a faint presence in mathematics education, even though it is the distinguishing characteristic of mathematics as a discipline.

Deductive proof accounts for a fundamental contrast between mathematics and the scientific disciplines: they honor very different epistemological gods. Mathematical knowledge tends much more to be cumulative. New mathematics builds on, but does not discard, what came before. The mathematical literature is extraordinarily stable and reliable. In science, by contrast, new observations or discoveries can invalidate previous models, which then lose their scientific currency. The contrast is sharpest in theoretical physics, which historically has been the science most closely allied with the development of mathematics. I. M. Singer is said to have once compared the theoretical physics literature to a blackboard that must be periodically erased. Some theoretical physicists—Richard Feynman, for instance—enjoyed chiding the mathematicians' fastidiousness about rigorous proofs. For the physicist, if a mathematical argument is not rigorously sound but nonetheless leads to predictions that are in excellent conformity with experimental observation, then the physicist considers the claim validated by nature, if not by mathematical logic; nature is the appropriate authority. The physicist P. W. Anderson once remarked, "We are talking here about theoretical physics, and therefore of course mathematical rigor is irrelevant."

On the other hand, some mathematicians have shown a corresponding disdain for this free-wheeling approach of the theoretical physicists. The mathematician E. J. McShane once likened the reasoning in a "physical argument" to that of "the man who could trace his ancestry to William the Conqueror, with only two gaps."

Even some mathematicians eschew heavy emphasis on rigorous proof in favor of more intuitive and heuristic thinking and of the role of mathematics to help explain the world in useful or illuminating ways. In general, they do not necessarily scorn rigorous proof, only consign it to a faintly heeded intellectual superego. But I would venture nonetheless, that, for most research mathematicians, the notion of proof and its quest are at least tacitly central to their thinking and their practice as mathematicians. And this would apply even to mathematicians who, like Bill Thurston, view what mathematicians do as not so much the production of proofs, but as "advancing mathematical understanding" (Thurston 1994). It would be hard to find anyone with the kind of mathematical understanding and function of which Thurston speaks who has not already assimilated the nature and significance of mathematical proof.

At the same time, the writing of rigorous mathematical proofs is not the work that mathematicians actually do, for the most part, or what they most cherish and celebrate. Such tributes are conferred instead on

A typical form of compression arises from the unification of diverse phenomena as special cases of a single construct (for example, groups, topological spaces, Hilbert spaces, measure spaces, and categories and functors) about which enough of substance can be said in general to constitute a useful unifying theory. This leads to another salient feature of mathematics: *abstraction*. Most mathematics has its roots in science and so ultimately in the "real world." But mathematics, even that contrived to solve real-world problems, *naturally* generates its own problems, and so the process continues with these, leading to successive stages of unification through abstraction until the mathematics may be several degrees removed from its empirical origins. It is a happy miracle that this process of pursuing *natural mathematical questions* repeatedly reconnects with empirical reality in unexpected and unplanned ways. This is the "unreasonable effectiveness" of which Wigner wrote (1960) and of which Varadarajan (2015) writes eloquently in this collection of essays (Casazza et al. 2015).

Abstraction is often thought to separate mathematics from science. But even among mathematicians, there are different professed dispositions toward abstraction. Some mathematicians protest that they like to keep things "concrete," but a bit of reflection on what they consider to be concrete shows it to be far from such for an earlier generation. Indeed, I noticed while I was a graduate student that the extent to which a mathematical idea was considered abstract seemed more a measure of the mathematician's age than of the cognitive nature of the idea. As new ideas become assimilated into courses of instruction, they become the daily bread and butter of initiates, all the while remaining novel and exotic to many of their elders.

Whereas compression is an essential instrument for the ecological survival of mathematics, its very virtue presents a serious obstacle for the teaching of mathematics. The knowledgeable and skillful mathematician has assimilated and internalized years of successive compression, streamlining of ideas, and habits of mind. But a young learner of mathematics still lives and thinks in a "mathematically decompressed" world, one that has become hard for the mathematically trained person to imagine, much less remember. This phenomenon presents a special challenge to teachers of mathematics to children. And, interestingly, it requires a special kind of knowledge of mathematics itself that is neither easy for nor common among otherwise mathematically knowledgeable adults (Ball et al. 2005, 2008).

Teaching Mathematics

Among the questions to which our editors invited us to direct attention was, "How are we research mathematicians viewed by others?" For one community, (school) mathematics teachers and education researchers, I have some firsthand knowledge of this after more than a decade of interdisciplinary work in mathematics education, and what I have learned I find both interesting and important for mathematicians to understand. I began this essay with an account of how great university teaching drew me into mathematics. Now, in my close study of elementary mathematics teaching, I have a changed vision of what teaching entails.

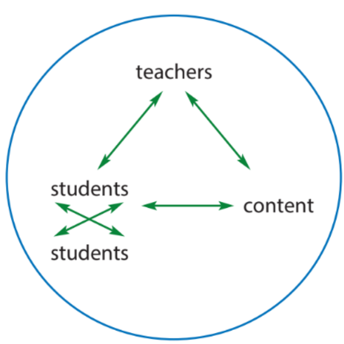
Of course many mathematicians, like mathematics educators, are seriously interested in the mathematics education of students. But there are significant differences in how the two communities, broadly speaking, see this enterprise. The mathematicians' interests, naturally enough, are directed primarily at the graduate and undergraduate levels and perhaps somewhat at the secondary level. In contrast, the interests of the educators are predominantly aimed at the primary through secondary levels. But there is a broadening overlap in the ranges of interest of the two communities, and some fundamental aspects of what constitutes quality teaching are arguably independent of educational levels. Nonetheless, in areas of common focus there are often profound differences of perspective and understanding across the two communities about what constitutes quality mathematics instruction. I expand on this later.

There is also a difference in educational priorities between the mathematics and mathematics education communities, a difference whose importance cannot be overstated. Mathematicians are naturally interested in "pipeline" issues, the rejuvenation of the professional community, and so the induction and nurturing of talented and highly motivated students into high-level mathematical study. Mathematics educators, on the other hand, are professionally committed to the improvement of public education at scale, with the aim of *high levels of learning for all students* and with a heavy emphasis on the word "all." Although these two agendas are not intrinsically in conflict or competition, they are often seen to be so, and this can have resource and policy implications. Public education in the United States, despite long and costly interventions, continues to perform poorly in international comparisons in terms of meeting workplace demands and even in providing basic literacies. Moreover, there is a persistent achievement gap reflected in underperformance of certain subpopulations that the educational system historically has not served well. These are the "big frontline problems" of mathematics education, and they are just as compelling and urgent to mathematics educators as the big research problems are to mathematicians.

Mathematicians have an excellent tradition of nurturing students of talent. What they are less good at is identifying potential talent. The usual indices, high test scores and precocious accomplishment, are easy enough to apply, but these indices typically overlook students of mathematical promise whom the system has not encouraged or given either the expectation of or opportunity for high performance. As a result, mathematical enrichment programs, if not sensitively designed, can sometimes perpetuate the very inequities that mathematics educators are trying to mitigate.

My main focus here, however, is on teaching. Much of what I have learned I owe to work with my colleague and collaborator, Deborah Ball. To facilitate what I want to say, it is helpful to use a schematic developed by Deborah and her colleagues David K. Cohen and Stephen W. Raudenbush (2003) to describe the nature of instruction.

The "Instructional Triangle"



The concept proposed by this image is that instruction is about the interactions of the teacher with students (and of students with each other) around content. The double arrows emphasize that these interactions are dynamic; in particular, changing any one element of the picture significantly alters the whole picture.

Now what I suggest is that most mathematicians' conception of instruction lives primarily at the content corner of the triangle, with its school incarnation expressed in terms of curriculum (including both standards documents and curriculum materials). In this view, with high-quality curriculum in place, the mathematically competent teacher has only to implement that curriculum with intellectual fidelity, and then attentive and motivated students will learn. Of course, mathematicians have little direct influence on schoolchildren, but they often have explicit, discipline-inspired ideas about what mathematics children should learn and how. In the case of teachers, mathematicians do bear some direct responsibility for their mathematical proficiency, since many teachers learn much of the mathematics they know in university mathematics courses. Many studies have pointed to weak teacher content knowledge as a major source of underperformance in public education. Of the many remedies proposed, mathematicians have generally favored more and higher level mathematics courses as a requirement for certification. Unfortunately, none of the interventions based on these ideas so far undertaken have yet produced the desired gains in student achievement for a broad range of students. Note that all of this discussion and debate resides on the content–teacher side of the triangle, with little explicit attention yet given to the role of the students in instruction. Effective teachers teach both math and children.

An educational counterpart to the above "one-sided" (teacher–content) stance is an intellectually adventurous, but somewhat romantic, view of school teaching, this time on the teacher–student side of the triangle. This approach proposes offering tangible and somehow "real-world" related mathematical activities with which to engage learners but leaving the development of mathematical ideas to largely unrestrained student imagination and invention. In such cases, the discipline of the mathematical ideas may be softened to the point of dissipation.

The most refined understanding of mathematics teaching, of which Deborah Ball's work is exemplary, insists that teaching must coordinate attention to the integrity of the mathematics and appropriate learning goals with attention to student thinking, which needs to be honored and made an integral part of the instruction. In this view, the effective teacher has not only a deep understanding of and fascination with the mathematics being taught but also a dual knowledge of and fascination with student thinking. The underlying premise is that children have significant mathematical ideas, albeit imperfectly expressed, and that a part of the teacher's work is to recognize the presence of those ideas (for which a sophisticated knowledge of mathematics is needed!), to give them appropriate validation, and to help students shape them into a more developed articulation and understanding. It is the coordination of these dual spheres of attention, both to the mathematics and to students' thinking, that makes effective teaching the intricate and skilled work that it is.

This is a kind of professional practice that discursive rhetoric cannot adequately capture. It is best conveyed through an examination of teaching practice itself. So let me offer a vignette. Consider the teaching of mathematical proving. This teaching is typically first done in high school geometry, but there are good reasons to argue that it should be done developmentally, starting in the early grades. What might this mean or look like? After all, young children have no concept of anything approaching mathematical proof, and no one seriously argues that this should be formally taught to them. First of all, what is the intellectual imperative for proving that can be made meaningful to young children? Our view is that it is the persistent question, "Why are things true?" Deductive proof is the refined method devised by mathematicians to answer that very question. Learning the methods of deductive proving takes time, but the question "Why are things true?" is itself immediately compelling. The underlying pedagogy is that students progressively learn the methods of constructing compelling mathematical arguments in the course of repeatedly trying to convince others of things they have good reasons to believe to be true. In other words, students can be helped to construct the infrastructure of proving in the very course of proving things (to the best of their growing abilities).

To illustrate this idea, I turn to an episode from a third-grade class (eight-year-olds). The children have been exploring even and odd numbers. Although they do not yet have formal definitions, they rely on intuitive ideas of even and odd based on notions of fair-sharing, and they can identify whether particular (small) numbers are even or odd. They begin to notice addition patterns, like: even + even = even, even + odd = odd, and odd + odd = even. The teacher asks them whether they can "prove" that "Betsy's conjecture" (odd + odd = even) is *always* validate and give space to these ideas, but also to help students reshape them in mathematically productive ways. The teacher needs to know how to instruct with questions more than with answers and to give students appropriate time, space, and resources to engage these questions. The actions to accomplish this, though purposeful, structured, and carefully calibrated to the students, are also subtle, deliberately focusing on student performance. And so, to the naive observer, it often can appear that "the teacher is not doing much." The central problem of teacher education is to provide teachers with the knowledge and skills of such effective practice.

The idea of developmental learning, such as we saw with the third graders, was also a feature of my calculus course back at Princeton in 1951. I first learned in that course what (a strange thing) a real number was (mathematically),[1] even though I had been working comfortably with real numbers throughout the upper high school grades. With regard to proving, while we learned, through witnessing stunning examples and through practice problems how to construct reasonably rigorous proofs, it was still always the case that the claims in question seemed at least reasonable and sufficiently meaningful that general mathematical intuition could guide us. But I was at first stymied by a problem on one of our take-home exams. A function $f(x)$ was defined by: $f(x) = 0$ for x irrational, and $f(\mathrm{x}) = 1/q$ if x is rational, equal to p/q in reduced form. The question was, "Where is f continuous?" At first sight, this question seemed outrageous. How could one possibly answer it? It was impossible to sketch the graph. Intuition was useless. After some reflection, I resigned myself to the fact that all we had to work with were definitions of f and of continuity. It was a great revelation to me that these definitions and a modest amount of numerical intuition sufficed to answer the question. This was a great lesson about the nature and the power of deductive reasoning, to which my enculturation, though at a different level, was not so different from that of the third graders in our example.

The construction and timing of such a problem was itself a piece of instruction. I wish now that we had video records of Emil Artin's calculus teaching that I could show to an accomplished teacher, like Deborah, to analyze the pedagogical moves, including interaction with students, of which his practice was composed. Much as I profited from that instruction, I was not prepared at the time to see the craft of its construction.

Note

1. An equivalence class of Cauchy sequences of rational numbers.

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In Defense of Pure Mathematics

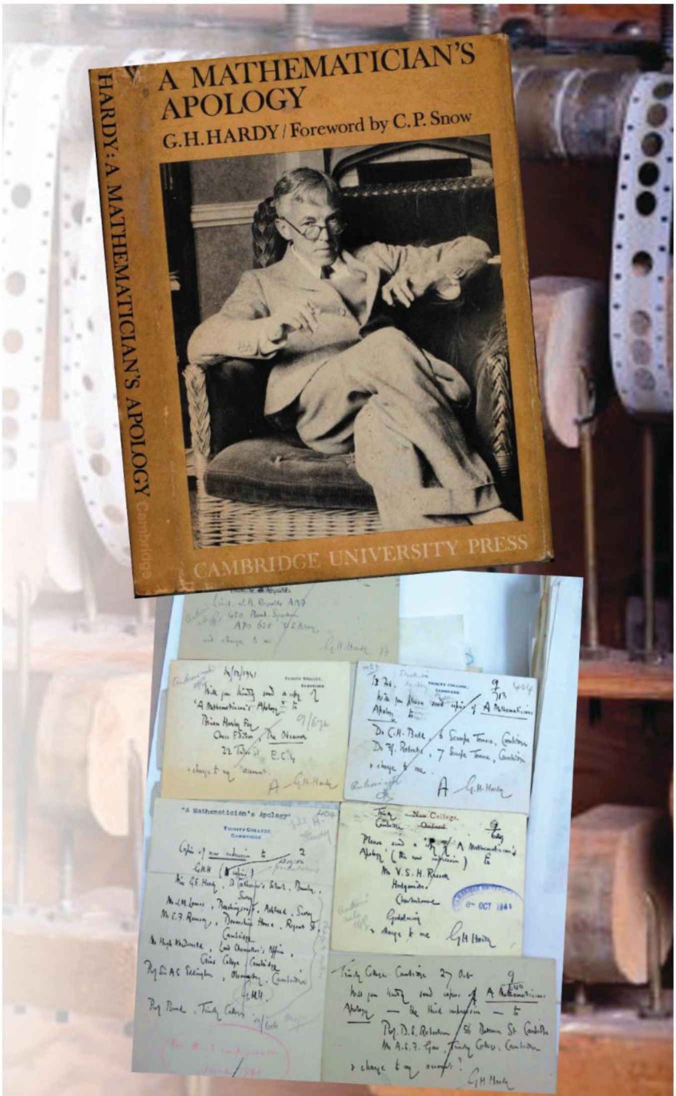
Daniel S. Silver

Godfrey Harold Hardy was one of the greatest number theorists of the twentieth century. Mathematics dominated his life, and only the game of cricket could compete for his attention. When advancing age diminished his creative power, and a heart attack at sixty-two robbed his physical strength, Hardy composed *A Mathematician's Apology*. It was an *apologia* as Aristotle or Plato would have understood it, a self-defense of his life's work.

"A mathematician," Hardy contended, "like a painter or poet, is a maker of patterns. . . . The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way." It was a personal and profound view of mathematics for the layman, unlike anything that had appeared before. The book, which this year reaches the seventy-fifth anniversary of its original publication, is a fine and most accessible description of the world of pure mathematics.

Ever since its first appearance, *A Mathematician's Apology* has been a lightning rod, attracting angry bolts for its dismissal of applied mathematics as being dull and trivial. The shaft that lit up the beginning of a review in the journal *Nature* by Nobel laureate and chemist Frederick Soddy was particularly piercing: "This is a slight book. From such cloistral clowning the world sickens."

Hardy's opinions about the worth of unfettered thought were strong, but stated with "careful wit and controlled passion," to borrow words of the acclaimed author Graham Greene. They continue to find sympathetic readers in many creative fields. They were prescient at the time and remain highly relevant today.



The second edition of *A Mathematician's Apology* featured Hardy's now-iconic photograph on the dust jacket (Jacket image courtesy of Cambridge University Press). For the first edition, Hardy sent postcards requesting that presentation copies be sent to colleagues, including C. D. Broad and J. E. Littlewood, the physicist Sir Arthur Eddington, chemist and novelist C. P. Snow, cricketer John Lomas (to whom he dedicated the book), and his sister Gertie. He also requested copies be sent to colleagues in the United States. Postcards photograph courtesy of Cambridge University Library.

The Art of Argument

Hardy was born on February 7, 1877, in Cranleigh, Surrey. His parents valued education, but neither had been able to afford university.

Hardy grew up to be a scholar, a sportsman, an atheist, and a pacifist, but, above all, he was an individualist. In an obituary of him, the mathematician E. C. Titchmarsh recalled: "If he dined at high table in tennis clothes it was because he liked to do so, not because he had forgotten what he was wearing."

Hardy began a famous collaboration with analyst J. E. Littlewood in 1911. Two years later, they would pore over strange, handwritten mathematical manuscripts that had been sent unsolicited by a young Indian civil servant, Srinivasa Ramanujan. Together they would decide that it was the work of a true genius. After considerable effort, Hardy succeeded in bringing Ramanujan to Cambridge, where Hardy was a professor. The "one romantic incident in my life" is how Hardy described his discovery and collaboration with his young protégé, who tragically died of illness seven years later.

In 1919, Hardy moved to Oxford University, where his eccentricities thrived. In his rooms he kept a picture of Vladimir Lenin. He shunned mechanical devices such as telephones, would not look into a mirror, rarely allowed his photograph to be taken, and was very shy about meeting people. Nevertheless, he was a superb conversationalist, able to carry on talk about many subjects (including, of course, cricket). Titchmarsh recalled: "Conversation was one of the games which he loved to play, and it was not always easy to make out what his real opinions were."

Mathematician George Pólya had a similar recollection: "Hardy liked to shock people mildly by stating unconventional views, and he liked to defend such views just for the sake of a good argument, because he liked arguing."

It is clear that Hardy enjoyed teasing his audience, something one should keep in mind when reading *A Mathematician's Apology*.

Work for Second-Rate Minds

Many reviews of *A Mathematician's Apology* appeared during the first few years of its life. Most were favorable. The author of one such review, published in *The Spectator* in 1940, was Graham Greene. Hardy he and Wilhelm Weinberg independently proved is well known today as the Hardy–Weinberg principle.

But for Hardy, who lived through two world wars, number theory provided a retreat that was, thankfully, useless to military planners. Borrowing from his own article, "Mathematics in Wartime," published in the journal *Eureka* in 1940, Hardy wrote in *Apology* the same year that "When the world is mad, a mathematician may find in mathematics an incomparable anodyne."

According to Hardy, the mathematician's world is directly linked to reality. Theorems are non-negotiable. In contrast, he says, the scientist's reality is merely a model. "A chair may be a collection of whirling electrons, or an idea in the mind of God," he declared. "Each of these accounts of it may have merits, but neither conforms at all closely to the suggestions of common sense." The pure mathematician need not be tethered to physical facts. In Hardy's words, "'Imaginary' universes are so much more beautiful than this stupidly constructed 'real' one."

Frederick Soddy, who had helped the world understand radioactivity, was disgusted by such sentiments. In his review in *Nature*, he said that if Hardy were taken seriously, then the "real mathematician" would be a "religious maniac."

Hardy was aware of Soddy's review. He might have been amused by it. A letter to him from R. J. L. Kingsford at Cambridge University Press, dated January 1941, concluded, "I quite agree that Soddy's amazing review in *Nature* is a most valuable advertisement. I enclose a copy of the review, herewith."

Another condemnation of *A Mathematician's Apology* came from E. T. Bell, a mathematician and science fiction writer who is best remembered for his 1937 book *Men of Mathematics.* In his review, published in *The Scientific Monthly* in 1942, Bell recommended Hardy's book to "solemn young men who believe they have a call to preach the higher arithmetic to mathematical infidels." He concluded, "Congenital believers will embrace [Hardy's book] with joy, possibly as a compensation for the loss of their religious beliefs of their childhood."

It Won't Make a Nickel for Anyone

Hardy had intended to publish *A Mathematician's Apology* with Cambridge University Press at his own expense. However, Press Secretary S. C. Roberts recognized the value of the ninety-page essay and endorsed it to the Syndics, the governing body of the press. The book was reprinted twice, the second time in 1948, the year following Hardy's death. In June 1952, Hardy's sister wrote to the press:

> As *A Mathematician's Apology* is now impossible to get, both first hand and second hand, I expect that you will in time be reprinting it; I think that it would be a good idea to have a photograph of my brother in it granted that it did not make it too expensive. I have the negative of which the enclosed photograph is an enlargement; it is an amateur snap and extremely characteristic.

The photograph that she had sent would eventually appear on the dust jacket of the second edition, and has become a well-known image of her brother.

Reissuing *A Mathematician's Apology* would be difficult. Inflation in Britain had made it impossible to reprint so small a book at a reasonable sales price. Some sort of material would be needed to extend it, but none of Hardy's academic lectures seemed appropriate.

Hardy (far right) and his protégé Srinivasa Ramanujan (center) are shown with colleagues at Cambridge University. After sending Hardy many theorems in letters, Ramanujan worked closely with Hardy for five years on various aspects of number theory, including highly composite numbers, which are positive integers with more divisors than any smaller positive integer. Ramanujan received a doctorate from Cambridge for this work in 1916. Reproduced by permission of Cambridge University Press.

In 1959, eleven years later, chemist and writer C. P. Snow suggested that he might write an introduction. It seemed a superb idea. Snow had advised Cambridge University Press during the war years. He was a close friend of Hardy and had offered him advice about the book. However, Snow would not commit to a deadline. An internal memorandum from the current Secretary, R. W. David, in September 1966 complained that "we have been chasing Snow for copy at roughly yearly intervals."

After many incidents, the second edition of *A Mathematician's Apology* was finally in bookstores by the end of 1967. Snow's contribution added literary charm. It began:

> It was a perfectly ordinary night at Christ's high table, except that Hardy was dining as a guest. . . . This was 1931, and the phrase was not yet in English use, but in later days they would have said that in some indefinable way he had star quality.

As Cambridge University Press anticipated, the new edition of *A Mathematician's Apology* was received well in the United States. Byron Dobell, an author and editor in New York who helped many young writers, including Tom Wolfe and Mario Puzo, seasoned his praise with a sprinkle of caution:

> It is the kind of book you wish was being read by all your friends at the very moment when you are reading it yourself. It is one of those secret, perfect works that makes most writing seem like a mixture of lead and mush. It's the under-the-counter book we're touting this month. It has nothing to do with anything but the joy of life and mind. The price is $2.95 and, with a title like that, it won't make a nickel for anyone.

Physical Connections

The second edition of *A Mathematician's Apology* appeared as mathematics was becoming increasingly abstract. Many mathematicians rejoiced at this change of direction in their field. Others lamented. If the trend continued, some believed, mathematics would become irrelevant.

One mathematician who celebrated was University of Chicago professor Marshall Stone. His article "The Revolution in Mathematics," which first appeared in the journal *Liberal Education* in 1961 and was reprinted the same year in *American Mathematical Monthly*, saw abstraction bridging areas of mathematics that had previously been isolated islands of thought. The identification of mathematics and logic, he argued, was greatly responsible:

> Mathematics is now seen to have no necessary connections with the physical world beyond the vague and mystifying one implicit in the statement that thinking takes place in the brain. The discovery that this is so may be said without exaggeration to mark one of the most significant intellectual advances in the history of mankind.

Stone noted a paradox: Increasing abstraction was spawning new applications. He listed the mathematical theory of genetics and game theory, as well as the mathematical theory of communications with contributions to linguistics.

A very different opinion was expressed the following year in "Applied mathematics: What is needed in research and education," published in *SIAM Review*. It was the transcript of a symposium chaired by mathematician H. J. Greenberg. Its panel consisted of mathematicians George Carrier, Richard Courant, and Paul Rosenbloom, and physicist C. N. Yang. Stone's article, with its embrace of abstraction, was discussed with alarm. The panel members urged a more traditional vision of mathematics, one that draws its inspiration from science. Courant's warning sounded like a review of *A Mathematician's Apology*:

> We must not accept the old blasphemous nonsense that the ultimate justification of mathematical science is "the glory of the human mind." Mathematics must not be allowed to split and diverge towards a "pure" and an "applied" variety.

Despite Courant's warning, a line between pure and applied mathematics exists at most universities today. Too often it is a battle line, witnessing skirmishes over scant resources and bruised egos. It is a line that has perhaps been blurred a bit by pure mathematicians' widespread use of computers and technology's urgent need for sophisticated algorithms. Mathematicians who share Hardy's sentiments might feel reluctant to express them in the face of soaring costs of higher education. Students with mounting debts have become increasingly impatient with teachers who digress from material directly needed for their exams. Administrators drool over research grants in medicine and cybersecurity while finding less filling the meager grants awarded in pure mathematics.

The line between pure and applied mathematics might be blurred, but it will not soon be erased. As long as it exists, G. H. Hardy's *A Mathematician's Apology* will be read and—usually—enjoyed. No finer summary can be offered than that written by J. F. Randolph in his 1942 review:

> This book is not only about mathematics, it is about ideals, art, beauty, importance, significance, seriousness, generality, depth, young men, old men, and G. H. Hardy. It is a book to be read, thought about, talked about, criticized, and read again.

G. H. Hardy: Mathematical Biologist

Hannah Elizabeth Christenson
and Stephan Ramon Garcia

Godfrey Harold Hardy (1877–1947), the magnificent analyst who "discovered" the enigmatic Ramanujan and penned *A Mathematician's Apology*, is most widely known outside of mathematics for his work in genetics. Hardy's fame stems from a condescending letter to the editor in *Science* concerning the stability of genotype distributions from one generation to the next. His result is known as the Hardy–Weinberg law, and every biology student learns it today.

How did Hardy, described by his colleague, C. P. Snow, as "the purest of the pure" [8], become one of the founders of modern genetics? What would Hardy say if he knew that he had earned scientific immortality for something so mathematically simple?

In a lecture delivered by R. C. Punnett (of Punnett square fame), the statistician Udny Yule raised a question about the behavior of the ratio of dominant to recessive traits over time. This led Punnett to question why a population does not increasingly tend toward the dominant trait. He was confused and brought the question to his colleague, G. H. Hardy, with whom he frequently played cricket (for the complete story, see references [2 and 3]).

Under certain natural assumptions, Hardy demonstrated that there is an equilibrium at which the ratio of different genotypes remains constant over time (this result was independently obtained by the German physician Wilhelm Weinberg). There is no deep mathematics involved; the derivation of the Hardy–Weinberg law involves only "mathematics of the multiplication-table type" [6]. Hardy's brief letter dismisses Yule's criticism of Mendelian genetics:

> I am reluctant to intrude in a discussion concerning matters of which I have no expert knowledge, and I should have expected