The BEST WRITING on MATHEMATICS

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Mircea Pitici, Editor

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Introduction

Mircea Pitici

The eighth volume of *The Best Writing on Mathematics* brings you a new collection of diverse, surprising, and well-written pieces, all published originally during 2016 in academic journals, scientific magazines, or mass media. In addition to the selection, at the end of the book I offer a copious reference section of notable writings and sources for those of you who want to find out more about mathematics on your own; that supplement is important for the goals of the series, serving the research needs of interested readers.

I hope that this series illustrates the versatility of mathematics and that of its interpretations; I also hope that the series helps readers gain a rich panoramic view of mathematics, as opposed to the impoverished parochial view promoted at all levels by our education system. The more facets of mathematics we discover, the more aware we are that mathematics has become a behemoth of human thought, with tentacles reaching into many of the ingenious innovations that fill our personal and collective lives with technological wonders and with deadly perils.

In a memorable line from the movie *Stand and Deliver* (1988), the mathematics teacher Jaime Escalante tells his students in a run-down Los Angeles community that "math is the great equalizer"—meaning, perhaps, that learning mathematics opens up life possibilities for achievement to everyone, regardless of their ethnicity, social condition, and family status. My own avatars teach me that, like everything else people say about mathematics, Escalante's proclamation is both true *and* untrue, depending on the perspective one takes and on the life caprices one encounters. Against a long and parsimonious tradition that associates mathematics with recipes of ready-made clichés, I aim to show with this series that mathematics is more interesting than the most interesting writing about it—and, more, that this statement remains valid even if we replace the attribute "interesting" with one of its antonyms, or even with most other attributes. Such a sweeping statement will sound disconcerting to the unaware mind, but it is intriguing to the inquisitive mind. Mathematics is a domain of clarity *and* obscurity, of enchantment *and* boredom, of unperturbed neatness *and* of puzzling paradox, of apodictic truth *and* of arguable interpretation. The pieces collected in this volume once again demonstrate the dynamic coexistence of opposite characteristics of mathematics—and show that mathematics is anything but the dull subject serviced by an increasingly powerful but stultifying educational bureaucracy unable to grasp, appreciate, promote, and teach the creative and imaginative sides of mathematics.

Overview of the Volume

In the same vein as the previous books in the series, this volume contains both expository and interpretive pieces on mathematics and aspects of life in the mathematical community, historical and contemporary.

To open the selection, Philip Davis sees mathematicians as producers and shows, with many examples from the past and from the present, that treating mathematical results as "products" is neither far-fetched nor outrageous; it is just an observation supported by abundant evidence but still denied by many mathematicians.

Evelyn Lamb explains why it is useful to know that certain big numbers are primes—and why people are finding primes among a variety of numbers of certain algebraic expression.

Kevin Hartnett describes the work of geometers and physicists who attempt to discover similarities among various random processes.

Siobhan Roberts glosses on the idiosyncratic mathematical achievements of a peculiar centenarian, the recently celebrated Richard Guy.

Lloyd Trefethen shows that the precision of mathematical statements obscures the multitude of contexts in which we can interpret such results.

Gerald Alexanderson reviews biographical contributions inspired by Srinivasa Ramanujan's life and work, and he tells us the intriguing circumstances under which he acquired a bronze bust of the famous Indian mathematician.

Larry Riddle brings abstract algebra to the study of systems of functions to create beautiful fractal images.

Marc Frantz contributes with elements of projective geometry to the wider context (within optics and perception) of the moon tilt illusion.

Mohammadhossein Kasraei, Yahya Nourian, and Mohammadjavad Mahdavinejad study how the Persian architectural element Girih was used in the construction of three Iranian domes; they also analyze the relationship between dome curvature and the polygonal division of the dome's base circle.

Jo Boaler and Lang Chen summarize studies from several disciplines to conclude that children's degree of dexterity with "finger math" is important to their mathematical development.

Sinéad Breen and Ann O'Shea rethink the design of undergraduate mathematics education, proposing that the central role held by the pairing of content and techniques should be replaced by "threshold concepts," which they define and characterize in their piece.

John Mason pleads for a mathematics education attentive to the circumstantial elements that occasion learning—as opposed to the dogmatism of normative theories so popular with researchers.

Viktor Blåsjö exemplifies with a geometric-algebraic construction taken from Leibniz's work the changing meaning of mathematics and mathematics notation over the past few centuries.

Carlo Séquin and Raymond Shiau examine a famous painting by Fra' Luca Pacioli to determine whether the plane rendering of a spatial geometric object is genuine, and they bring the topic to the present by offering a computerized version of that representation.

Jeremy Gray asks what would have passed as most valuable mathematical research, most worthy of award-winning consideration, a century and a half ago and examines in that context the work of several mathematicians prominent at the time.

Noson Yanofsky illustrates with an abundance of examples different types of mathematical and scientific limitations, from logical and physical to mental and practical.

Jean-Pierre Marquis defines abstraction and "levels" of abstraction in mathematics, distinguishing between the axiomatic method and the abstract method. Then he infers the philosophical consequences of using the latter.

Robert Bain considers pro and con arguments for the proposition that human reasoning, beliefs, and decision making actively adjust based on evidence and probability expectations.

Speaking of expectations, in the last piece of the anthology, Graham Southorn describes quantitative methods used in forecasting and explains why they never achieve certainty in our ever more complex world.

More Writings on Mathematics

Among recent books on mathematics deserving special mention are the following: the *Sourcebook in the Mathematics of Medieval Europe and North Africa* edited by Victor Katz; the path-opening *Visualizing Mathematics with 3D Printing* by Henry Segerman; the book-and-catalogue *Mathematics* edited by David Rooney for the Science Museum of London; and two interdisciplinary books reaching both close to and far away from mathematics, *The Oxford Handbook of Generality in Mathematics and the Sciences* edited by Karine Chemla, Renaud Chorlay, and David Rabouin, and the massive *Handbook of Geomathematics* edited by Willi Freeden, Zuhair Nashed, and Thomas Sonar.

Expository books on mathematics are *Elements of Mathematics* by John Stillwell, *The Circle* by Alfred Posamentier and Robert Geretschläger, *Algebra* by Peter Higgins, *Fractals* by Kenneth Falconer, *Combinatorics* by Robin Wilson, *Measurement* by David Hand, *Some Applications of Geometric Thinking* by Bowen Kerins et al., *Thinking Geometrically* by Thomas Sibley, *Geometry in Problems* by Alexander Shen, *Problem-Solving Strategies in Mathematics* by Alfred Posamentier and Stephen Krulik, *An Interactive Introduction to Knot Theory* by Inga Johnson and Allison Henrich, *Circularity* by Ron Aharoni, *Can You Solve My Problems?* by Alex Bellos, and *Summing It Up* by Avner Ash and Robert Gross.

Mathematics in life (including gambling and games) is described and interpreted in such books as *The Calculus of Happiness* by Oscar Fernandez, *Fluke* by Joseph Mazur, *Man vs. Mathematics* by Timothy Revell and Joe Lyward, *In Praise of Simple Physics* by Paul Nahin, *The Mathematics that Power Our World* by Joseph Khoury and Gilles Lamothe, *Living by Numbers* by Steven Connor, *The Perfect Bet* by Adam Kucharski, *The Joy of SET* by Liz McMahon and her coauthors, *That's Maths* by Peter Lynch, and *Math Squared* by Rachel Thomas and Maryanne Freiberger; Daniel Levitin takes a broad perspective in *A Field Guide to Lies*.

In the history of mathematics and biography, recently I noticed *A Brief History of Mathematical Thought* by Luke Heaton, *Infinite Series in a History of Analysis* by Hans-Heinrich Körle, *Turing* by Jack Copeland,

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Mathematical Products

Philip J. Davis

A prominent mathematician recently sent me an article he had written and asked me for my reaction. After studying it, I said that he was proposing a mathematical "product" and that as such it stood in the scientific marketplace in competition with nearby products. He bridled and was incensed by my use of the word "product" to describe his work. Our correspondence terminated. What follows is an elaboration of what I mean by mathematical products and how I situate them within the mathematical enterprise.

Civilization has always had a mathematical underlay, often informal, and not always overt. I would say that mathematics often lies deep in formulaic material, procedures, conceptualizations, attitudes, and now in chips and accompanying hardware. In recent years, the mathematization of our lives has grown by leaps and bounds. A useful point of view is to think of this growth in terms of products. Mathematical products serve a purpose; they can be targeted to define, facilitate, enhance, supply, explain, interpret, invade, complicate, confuse, and create new requirements or environments for life.

What? Mathematical "products"? Products in an intellectual area that is reputed to contain the finest result of pure reason and logic: a body of material that in its early years was in the classical quadrivium along with astronomy and music? How gross of me to bring in the language of materialistic commerce and in this way sully or besmirch the reputation of what are clean, crisp idealistic constructions! Products are the routine output of factories, not of skilled craft workers whose sharp minds frequently reside far above the usual rewards of life. The notion that mathematics has products or that its content is merchandise might tarnish both its image and the self-image of the creators of this noble material.

(plugs). On MATLAB's website, you can find a list of MATLAB's available products, listed openly and labeled clearly as "products." Investment and insurance schemes are called "products."

A product can be sold, e.g., a handheld computer or the *Handbook of Mathematical Functions*. A product can be licensed for usage, or it can be made available as a freebie. In the case of taxes (*qua* mathematical product), it is "promoted" by laws and threats of punishment. Rubik's Cube, a mathematical product, caught the imagination and challenged the wits of millions of people and has earned fortunes. Sudoku, a mathematical puzzle, is sold in numerous formats. If a product is income producing, its sellers can be taxed. A product can be copyrighted or patented; the owners of such can be contested or sued for infringement.

1.4 Competitive Aspects of Mathematical Products

A mathematical product is often subject to competition from nearby products. Think of the innumerable ways of solving a set of linear equations. Textbooks, a source of considerable income, compete in a mathematical marketplace that involves educationists, testing theorists and outfits, unions, publishers, parents' groups, and local state and national governments.

1.5 Social Aspects of Mathematical Products

If a mathematical product finds widespread usage, it may have social, economic, ethical, legal, or political implications or consequences. The repugnant Nuremberg Racial Laws in Germany in 1935, with their numerical criteria, caused incredible suffering. DNA sequencing and its interpretations is a relatively new branch of applied mathematics, resulting in a host of new products. In a number of states, the level of mathematical tests for the lower school grades has been questioned. The social consequences of mathematical products, benign or otherwise, may not emerge for many years.

1.6 Legal Aspects of Mathematical Products

There are innumerable examples of this. The U.S. Constitution is full of number processes. Consider

> Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons. (Later Amended!)

Some mathematical products have been subject to judicial review. As an example, the mathematical scheme for the 2010 Census was vetted and restricted by the U.S. Supreme Court.

An example of a statutory product is the method of least proportions used to allocate representatives in Congress. It was approved by the Supreme Court in *Department of Commerce v. Montana*, 503 U.S. 442 (1992). A multiple regression model used in an employment discrimination class action is another such example; it was approved by the Supreme Court in *Bazemore v. Friday*, 478 U.S. 385 (1986).

1.7 Logical or Philosophical Aspects of Mathematical Products

A mathematical product, considered as such, is neither true nor false. Of course, it may embody certain principles of deductive logic, but these do not automatically make the employment of the product plausible or advisable. A product can be made plausible, moot, or useless on the basis of certain internal or external considerations. An interesting historical example of this is the dethroning of Euclidean geometry as the unique geometry by the discovery of non-Euclidean geometries.

A product may raise or imply philosophical questions, such as the distinction between the subjective and the objective or between the qualitative and the quantitative, between the deterministic and the probabilistic, the tangible and the intangible, the hidden and the overt.

Numerical indexes of this thing and that thing abound. Cases of subjectivity occur when a product asks a person or a group of people to pass judgment on some issue: "On a scale of zero to ten, how much do you like tofu?" The well-known *Index of Economic Freedom* embodies a number of items, expressed numerically:

> We measure ten components of economic freedom, assigning a grade in each using a scale from 0 to 100, where 100 represents

the maximum freedom. The ten component scores are then averaged to give an overall economic freedom score for each country. The ten components of economic freedom are: Business Freedom | Trade Freedom | Fiscal Freedom | Government Size | Monetary Freedom | Investment Freedom | Financial Freedom | Property Rights | Freedom from Corruption | Labor Freedom

1.8 Moral Aspects of Mathematical Products

Society asks many questions. Does the manner of taking the U.S. Census account properly for the homeless? Are tests in algebra slanted toward certain subcultures? Does the tremendous role that mathematics plays in war raise questions or angst in the minds of those who are responsible for its application? Are results of IQ testing being misused?

2 Judgments of Mathematical Products

As mentioned, mathematical products serve a purpose; they can be targeted to define, facilitate, enhance, or invade any of the requirements or aspects of life. Ultimately, a mathematical product can be judged in the same way that any product can be judged: by the response of its targeted users or purchasers. In the case of a mathematical product, what criteria are in play? The cheapest? The most convenient? The most useful? The most comprehensive? The most accurate? The most original? The most seminal? The most reassuring? The safest or least vulnerable? The most esthetic? The most moral? Is the product unique? Are there pressures from investors or the various foundations that support their production?

Is "survival of the fittest" a good description of the judgment process? Probably not. There are fashions in the product world attracting both excited consumers and producers. Time, chance, and what the larger world requires, appreciates, or suffers from mathematizations are always in play to determine survival.

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The Largest Known Prime Number

Evelyn Lamb

Earlier this week, BBC News reported an important mathematical finding, sharing the news under the headline "Largest Known Prime Number Discovered in Missouri." That phrasing makes it sound a bit like this new prime number—it's $2^{74,207,281} - 1$, by the way—was found in the middle of some road, underneath a street lamp. That's actually not a bad way to think about it.

We know about this enormous prime number thanks to the Great Internet Mersenne Prime Search. The Mersenne hunt, known as GIMPS, is a large distributed computing project in which volunteers run software to search for prime numbers. Perhaps the best-known analogue is SETI@home, which searches for signs of extraterrestrial life. GIMPS has had a bit more tangible success than SETI, with 15 primes discovered so far. The shiny new prime, charmingly nicknamed M74207281, is the fourth found by University of Central Missouri mathematician Curtis Cooper using GIMPS software. This one is 22,338,618 digits long.

A prime number is a whole number whose only factors are 1 and itself. The numbers 2, 3, 5, and 7 are prime, but 4 is not because it can be factored as 2×2. (For reasons of convenience, we don't consider 1 to be a prime.) The M in GIMPS and in M74207281 stands for Marin Mersenne, a 17th-century French friar who studied the numbers that bear his name. Mersenne numbers are 1 less than a power of 2. Mersenne primes, logically enough, are Mersenne numbers that are also prime. The number 3 is a Mersenne prime because it's one less than 2^2, which is 4. The next few Mersenne primes are 7, 31, and 127.

M74207281 is the 49th known Mersenne prime. The next largest known prime, $2^{57,885,161} - 1$, is also a Mersenne prime. So is the one after that. And the next one. And the next one. All in all, the 11 largest known primes are Mersenne.

Why do we know about so many large Mersenne primes and so few large non-Mersenne ones? It's not because large Mersenne primes are particularly common, and it's not a spectacular coincidence. That brings us back to the road and the street lamp. There are several different versions of the story. A man, perhaps he's drunk, is on his hands and knees underneath a streetlight. A kind passerby, perhaps a police officer, stops to ask what he's doing. "I'm looking for my keys," the man replies. "Did you lose them over here?" the officer asks. "No, I lost them down the street," the man says, "but the light is better here."

We keep finding large Mersenne primes because the light is better there.

First, we know that only a few Mersenne numbers are even candidates for being Mersenne primes. The exponent *n* in $2^n - 1$ needs to be prime, so we don't need to bother to check $2^6 - 1$, for example. There are a few other technical conditions that make certain prime exponents more enticing to try. Finally, there's a particular test of primeness—the Lucas–Lehmer test—that can only be used on Mersenne numbers.

To understand why the test even exists, let's take a detour to explore why we bother finding primes in the first place. There are infinitely many of them, so it's not like we're going to eventually find the biggest one. But aside from being interesting in a "math for math's sake" kind of way, finding primes is good business. RSA encryption, one of the standard ways to scramble data online, requires the user (perhaps your bank or Amazon) to come up with two big primes and multiply them together. Assuming that the encryption is implemented correctly, the difficulty of factoring the resulting product is the only thing between hackers and your credit card number.

This new Mersenne prime is not going to be used for encryption any time soon. Currently, we only need primes that are a few hundred digits long to keep our secrets safe, so the millions of digits in M74207281 are overkill, for now.

You can't just look up a 300-digit prime in a table. (There are about 10^{297} of them. Even if we wanted to, we physically could not write them all down.) To find large primes to use in RSA encryption, we need to test randomly generated numbers for primality. One way to do this is trial division: Divide the number by smaller numbers and see if you ever get a whole number back. For large primes, this takes way too

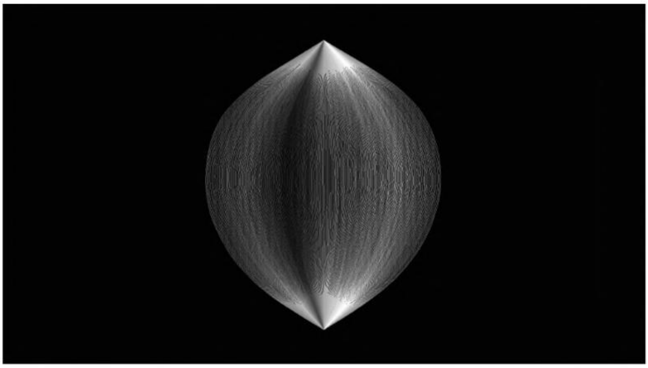

FIGURE 1. Randomness increases in a structure known as an "SLE curve." Photo by Jason Miller.

"You take the most natural objects—trees, paths, surfaces—and you show they're all related to each other," Sheffield said. "And once you have these relationships, you can prove all sorts of new theorems you couldn't prove before."

In the coming months, Sheffield and Miller will publish the final part of a three-paper series that for the first time provides a comprehensive view of random two-dimensional surfaces—an achievement not unlike the Euclidean mapping of the plane.

"Scott and Jason have been able to implement natural ideas and not be rolled over by technical details," said Wendelin Werner, a professor at ETH Zurich and winner of the Fields Medal in 2006 for his work in probability theory and statistical physics. "They have been basically able to push for results that looked out of reach using other approaches."

A Random Walk on a Quantum String

In standard Euclidean geometry, objects of interest include lines, rays, and smooth curves like circles and parabolas. The coordinate values of the points in these shapes follow clear, ordered patterns that can be described by functions. If you know the value of two points on a line, for instance, you know the values of all other points on the line. The same is true for the values of the points on each of the rays in Figure 2, which begin at a point and radiate outward.

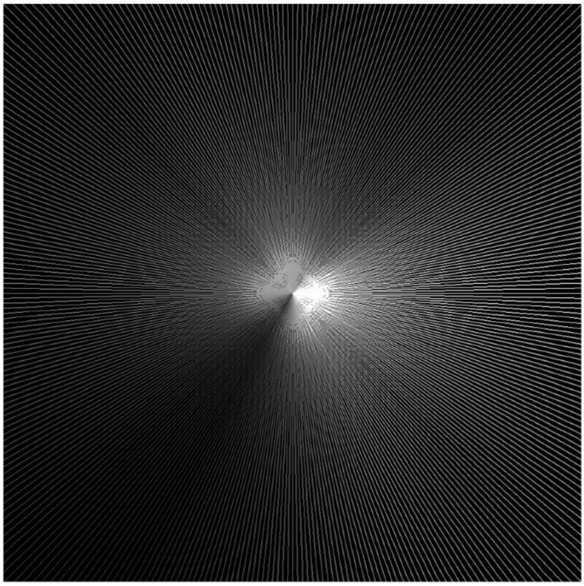

FIGURE 2. Rays constructed by a function that introduces no randomness. Photo by Scott Sheffield.

One way to begin to picture what random two-dimensional geometries look like is to think about airplanes. When an airplane flies a long-distance route, like the route from Tokyo to New York, the pilot flies in a straight line from one city to the other. Yet if you plot the route on a map, the line appears to be curved. The curve is a consequence of mapping a straight line on a sphere (Earth) onto a flat piece of paper.

If Earth were not round, but were instead a more complicated shape, possibly curved in wild and random ways, then an airplane's trajectory (as shown on a flat two-dimensional map) would appear even more irregular, like the rays in Figure 3.

Each ray represents the trajectory an airplane would take if it started from the origin and tried to fly as straight as possible over a randomly fluctuating geometric surface. The amount of randomness that characterizes the surface is dialed up in Figures 4 and 5—as the randomness increases, the straight rays wobble and distort, turn into increasingly jagged bolts of lightning, and become nearly incoherent.

Yet incoherent is not the same as incomprehensible. In a random geometry, if you know the location of some points, you can (at best)

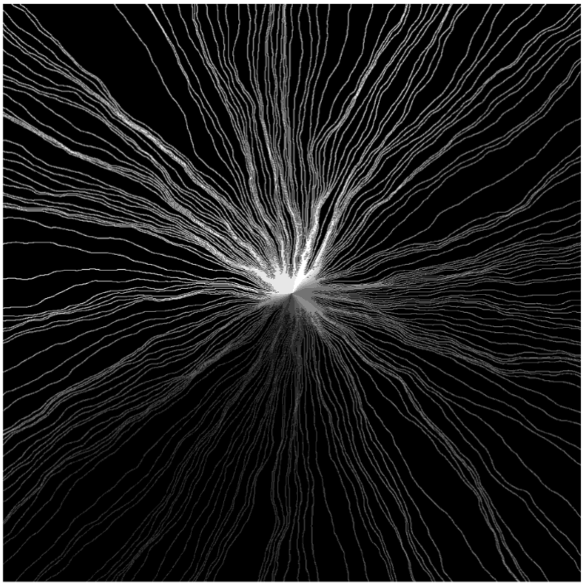

FIGURE 3. Rays with randomness set to a value of kappa = 4/101. Photo by Scott Sheffield. See also color image.

assign probabilities to the location of subsequent points. And just as a loaded set of dice is still random, but random in a different way than a fair set of dice, it's possible to have different probability measures for generating the coordinate values of points on random surfaces.

What mathematicians have found—and hope to continue to find—is that certain probability measures on random geometries are special and tend to arise in many different contexts. It is as though nature has an inclination to generate its random surfaces using a very particular kind of die (one with an uncountably infinite number of sides). Mathematicians like Sheffield and Miller work to understand the properties of these dice (and the "typical" properties of the shapes they produce) just as precisely as mathematicians understand the ordinary sphere.

The first kind of random shape to be understood in this way was the random walk. Conceptually, a one-dimensional random walk is the kind of path you'd get if you repeatedly flipped a coin and walked one way for heads and the other way for tails. In the real world, this type of movement first came to attention in 1827 when the English botanist

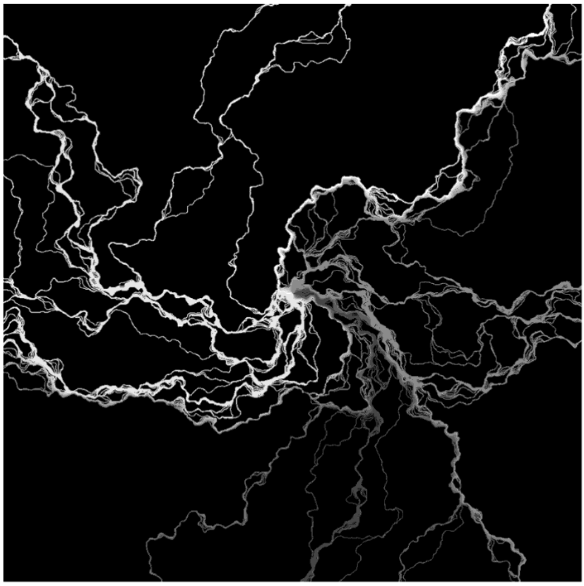

Figure 4. Rays with randomness set to a value of kappa = 4/5. Photo by Scott Sheffield. See also color image.

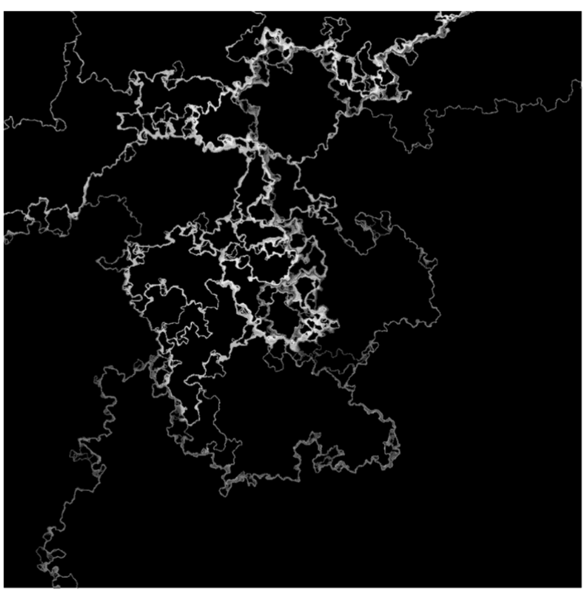

Figure 5. Rays with randomness set to a value of kappa = 2. Photo by Scott Sheffield.

Robert Brown observed the random movements of pollen grains suspended in water. The seemingly random motion was caused by individual water molecules bumping into each pollen grain. Later, in the 1920s, Norbert Wiener of MIT gave a precise mathematical description of this process, which is called Brownian motion.

Brownian motion is the "scaling limit" of random walks—if you consider a random walk where each step size is very small, and the amount of time between steps is also very small, these random paths look more and more like Brownian motion. It's the shape that almost all random walks converge to over time.

Two-dimensional random spaces, in contrast, first preoccupied physicists as they tried to understand the structure of the universe.

In string theory, one considers tiny strings that wiggle and evolve in time. Just as the time trajectory of a point can be plotted as a one-dimensional curve, the time trajectory of a string can be understood as a two-dimensional curve. This curve, called a *worldsheet*, encodes the history of the one-dimensional string as it wriggles through time.

"To make sense of quantum physics for strings," said Sheffield, "you want to have something like Brownian motion for surfaces."

For years, physicists have had something like that, at least in part. In the 1980s, physicist Alexander Polyakov, who's now at Princeton University, came up with a way of describing these surfaces that came to be called Liouville quantum gravity (LQG). It provided an incomplete but still useful view of random two-dimensional surfaces. In particular, it gave physicists a way of defining a surface's angles so that they could calculate the surface area.

In parallel, another model, called the Brownian map, provided a different way to study random two-dimensional surfaces. Where LQG facilitates calculations about area, the Brownian map has a structure that allows researchers to calculate distances between points. Together, the Brownian map and LQG gave physicists and mathematicians two complementary perspectives on what they hoped were fundamentally the same object. But they couldn't prove that LQG and the Brownian map were in fact compatible with each other.

"It was this weird situation where there were two models for what you'd call the most canonical random surface, two competing random surface models, that came with different information associated with them," said Sheffield.

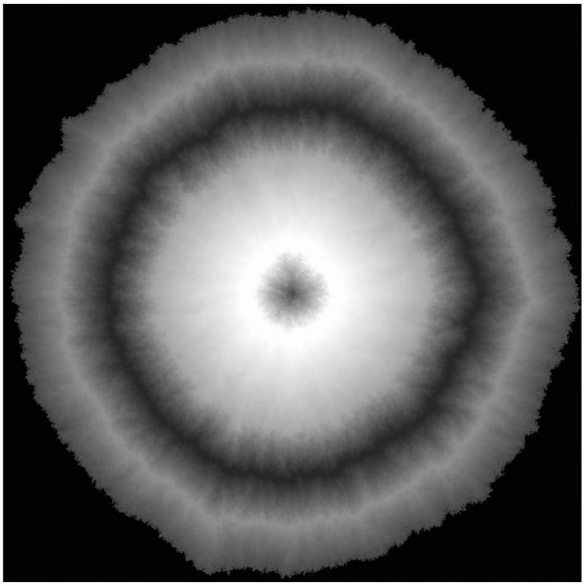

FIGURE 6. Eden growth with gamma equal to 0.25. Photo by Jason Miller.

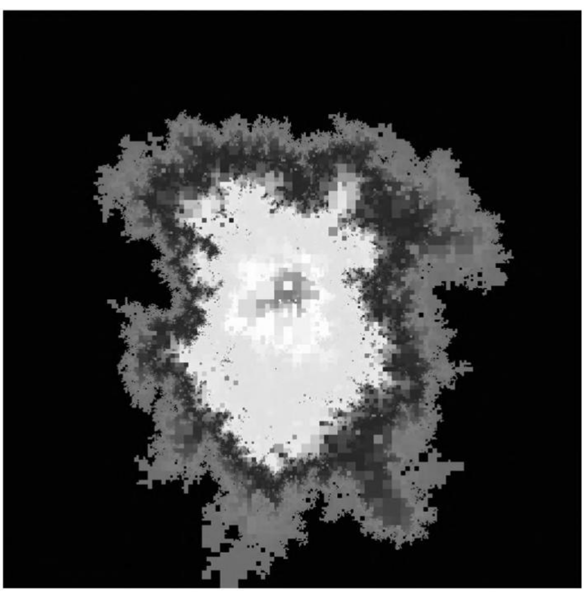

FIGURE 7. Eden growth with gamma equal to 1.25. Photo by Jason Miller.

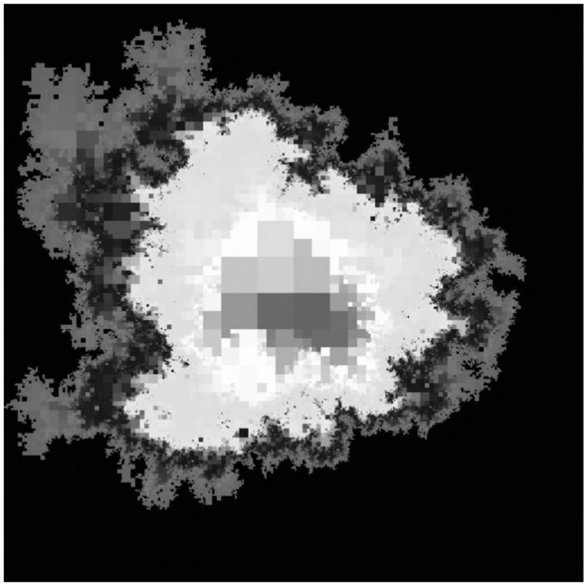

FIGURE 8. Eden growth with gamma equal to $\sqrt{8/3}$. Photo by Jason Miller. See also color image.

pressure fluctuations in a hurricane. Yet Sheffield and Miller realized that they needed to figure out how to model Eden growth on very random LQG surfaces in order to establish a distance structure equivalent to the one on the (very random) Brownian map.

"Figuring out how to mathematically make [random growth] rigorous is a huge stumbling block," said Sheffield, noting that Martin Hairer of the University of Warwick won the Fields Medal in 2014 for work that overcame just these kinds of obstacles. "You always need some kind of amazing clever trick to do it."

Random Exploration

Sheffield and Miller's clever trick is based on a special type of random one-dimensional curve that is similar to the random walk except that it never crosses itself. Physicists had encountered these kinds of curves for a long time in situations where, for instance, they were studying the boundary between clusters of particles with positive and negative spin (the boundary line between the clusters of particles is a one-dimensional

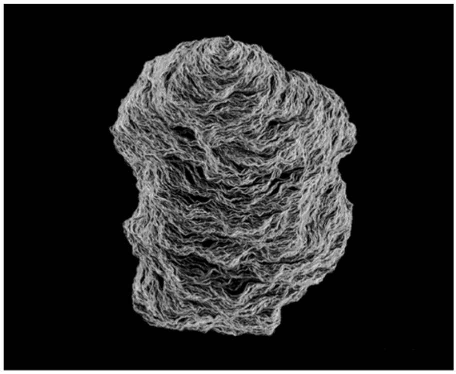

Figure 9. An example of an SLE curve. Photo by Jason Miller.

path that never crosses itself and takes shape randomly). They knew these kinds of random, noncrossing paths occurred in nature, just as Robert Brown had observed that random crossing paths occurred in nature, but they didn't know how to think about them in any kind of precise way. In 1999, Oded Schramm, who at the time was at Microsoft Research in Redmond, Washington, introduced the SLE curve (for Schramm–Loewner evolution) as the canonical noncrossing random curve (Figure 9).

Schramm's work on SLE curves was a landmark in the study of random objects. It's widely acknowledged that Schramm, who died in a hiking accident in 2008, would have won the Fields Medal had he been a few weeks younger at the time he'd published his results. (The Fields Medal can be given only to mathematicians who are not yet 40.) As it was, two people who worked with him built on his work and went on to win the prize: Wendelin Werner in 2006 and Stanislav Smirnov in 2010. More fundamentally, the discovery of SLE curves made it possible to prove many other things about random objects.

"As a result of Schramm's work, there were a lot of things in physics they'd known to be true in their physics way that suddenly entered the realm of things we could prove mathematically," said Sheffield, who was a friend and collaborator of Schramm's.

For Miller and Sheffield, SLE curves turned out to be valuable in an unexpected way. In order to measure distance on LQG surfaces, and thus show that LQG surfaces and the Brownian map were the same,

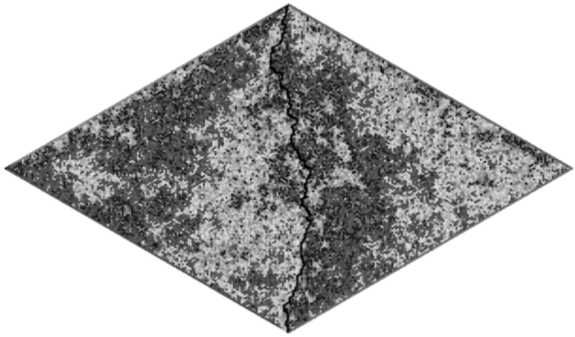

FIGURE 10. An SLE curve with kappa equal to 0.5. Photo by Jason Miller. See also color image.

they needed to find some way to model random growth on a random surface. SLE proved to be the way.

"The 'aha' moment was [when we realized] you can construct [random growth] using SLEs and that there is a connection between SLEs and LQG," said Miller.

SLE curves come with a constant, kappa, which plays a similar role to the one gamma plays for LQG surfaces. Where gamma describes the roughness of an LQG surface, kappa describes the "windiness" of SLE curves. When kappa is low, the curves look like straight lines. As kappa increases, more randomness is introduced into the function that constructs the curves and the curves turn more unruly, while obeying the rule that they can bounce off of, but never cross, themselves. Figure 10 is an SLE curve with kappa equal to 0.5, and Figure 11 is an SLE curve with kappa equal to 3.

Sheffield and Miller noticed that when they dialed the value of kappa to 6 and gamma up to the square root of eight-thirds, an SLE curve drawn on the random surface followed a kind of exploration process. Thanks to works by Schramm and by Smirnov, Sheffield and Miller knew that when kappa equals 6, SLE curves follow the trajectory of a kind of "blind explorer" who marks her path by constructing a trail as she goes. She moves as randomly as possible, except that whenever she bumps into a piece of the path she has already followed, she turns away from that piece to avoid crossing her own path or getting stuck in a dead end.

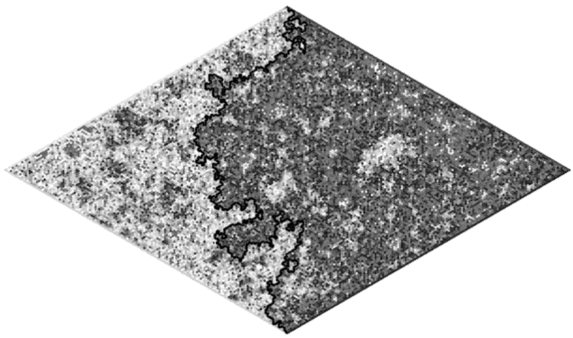

Figure 11. An SLE curve with kappa equal to 3. Photo by Jason Miller. See also color image.

"[The explorer] finds that each time her path hits itself, it cuts off a little piece of land that is completely surrounded by the path and can never be visited again," said Sheffield.

Sheffield and Miller then considered a bacterial growth model, the Eden model, that had a similar effect as it advanced across a random surface: It grew in a way that "pinched off" a plot of terrain that, afterward, it never visited again. The plots of terrain cut off by the growing bacteria colony looked exactly the same as the plots of terrain cut off by the blind explorer. Moreover, the information possessed by a blind explorer at any time about the outer unexplored region of the random surface was exactly the same as the information possessed by a bacterial colony. The only difference between the two was that while the bacterial colony grew from all points on its outer boundary at once, the blind explorer's SLE path could grow only from the tip.

In a paper posted online in 2013, Sheffield and Miller imagined what would happen if, every few minutes, the blind explorer were magically transported to a random new location on the boundary of the territory she had already visited. By moving all around the boundary, she would be effectively growing her path from all boundary points at once, much like the bacterial colony. Thus they were able to take something they could understand—how an SLE curve proceeds on a random surface—and show that with some special configuring, the curve's evolution exactly described a process they hadn't been able to understand, random growth. "There's something special about the