The **BEST WRITING** on **MATHEMATICS**

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Mircea Pitici, Editor

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Introduction

Mircea Pitici

This is the ninth volume in our series of remarkable writings on mathematics. The pieces you will read here were initially published during 2017 in various venues, including academic and other professional journals, book chapters, online publications, or newspapers. Except for a few technical mathematical notions required to understand select pieces, the book is accessible to the public that does not specialize in mathematics; yet the book will also interest mathematicians and scientists. Aiming to a wide audience has been, and remains, one of our goals when we prepare every volume.

The origins of *The Best Writing on Mathematics* series go back about fifteen years, to a time when my frustration with the clichés about mathematics I was reading and hearing made me curious to know opinions about mathematics not only from mathematicians but also from outsiders. I quickly discovered that a considerable literature on mathematics authored by mathematicians and by nonmathematicians exists and thrives. Despite its richness in ideas, it is mostly ignored in academic institutions, as if it did not exist and it had no instructional value. For several years, I mulled over the idea of editing such a series, and I attempted to start it; yet life difficulties and the disinterest toward my proposal from the publishers I approached stopped the project in its tracks.

Since 2010, the volumes in this series have contained more than two hundred pieces by authors with diverse backgrounds. These articles range in style from tightly argued theoretical positions on issues related to mathematics to bold speculations on the limits of the applicability of mathematics. Overall, the series is meant to convey to its readers the extraordinary ramifications of the influence of mathematics on contemporary mind, life, and society—and to stimulate connections we usually overlook when we talk about mathematics. The ninth volume is no exception from the general profile of the series.

Overview of the Volume

In the first article in the book, Francis Su gives us an impassioned credo, which he delivered as president of the Mathematical Association of America. Su finds common threads between mathematics on one side and play, beauty, truth, justice, and love on the other side—and from these associations concludes that mathematics contributes to human flourishing.

Margaret Wertheim proposes that certain sentient organisms other than humans, and even nonliving artifacts, perform mathematics whenever they enact rigorous geometric or algebraic patterns—that is, they do so without conscious intelligence.

Robert Thomas considers prevalent views on the "beauty" of mathematics and argues that the quality of making us interested and curious to do mathematics is at least as valuable aesthetically as the quality of pleasing us.

Marijn Heule and Oliver Kullmann write that automated computer proofs are useful and meaningful even if we cannot understand them—and describe how such methods work, detailing one example of proof done by what they call brute reason.

Peter Denning explains how the growth of computational power enabled scientists to change their disciplines from within by adding simulation and information process analysis to the established practices of experimenting and theorizing.

Robbert Dijkgraaf points out that the dynamic of the long interaction between mathematics and physics is reversing; where traditionally mathematics influenced physics, lately physics branches such as string and quantum theories occasion breakthroughs in mathematics, possibly leading toward a new type of mathematics, which he tentatively calls "quantum mathematics."

Erik Demaine and his coauthors describe a planar tangle toy, examine some of the topological configurations available through manipulating it, answer some of the mathematical questions it poses, and formulate a few open problems related to it.

James Grime shows us how to build subtly mischievous dice for playing slightly unfair games!

Arthur Benjamin, Joseph Kisenwether, and Ben Weiss consider another game: bingo. They observe and prove that, contrary to unexamined expectations, bingo winning patterns are asymmetrical, with completed horizontal rows occurring more frequently than completed vertical columns.

Peter Winkler summarizes a plethora of conflicting mathematical, psychological, semantic, and psychological arguments advanced over the past two decades in connection with a question of probability. He purposefully settles nothing, convinced that the controversy surrounding it will continue forever.

José Ferreirós delves into Eugene Wigner's intellectual biography and finds that Wigner's conception of mathematics and some of his consequential epistemological claims were influenced to a considerable degree by his professional friends and associates.

Chris Arney introduces us to the fundamentals of mathematical modeling, including to the necessary theoretical components of modeling, the current and potential scope of applications, and the essential bibliography specialized in mathematical modeling.

Nancy Emerson Kress offers precise advice to school students (and implicitly to teachers, but useful to everyone) on how to approach and to solve problems in mathematics.

On a topic that I have wanted to include in *The Best Writing on Mathematics* anthologies for a long time, Benjamin Braun and his coauthors define active classroom instruction, describe its benefits and variants, and prepare instructors inclined to adopt active teaching methods for some of the challenges and opportunities they are likely to experience.

In a brief note relating more detailed research, Daniel Mansfield and Norman Wildberger explain that in ancient Babylon the mathematicians (or perhaps the surveyors) of the time practiced a different trigonometry than we do today—simpler but precise, founded on the sexagesimal system of numeration.

Isabel Serrano, Lucy Odom, and Bogdan Suceavă examine the structure of mathematics in one of the most important encyclopedic works of premodern times, the *Etymologies* authored by Isidore of Seville in the seventh century.

Michael Barany traces the current awareness of the importance of mathematics in society to developments that started shortly before, but especially during, the Second World War; Barany tells the story of the people who accomplished the major shift in research funding from a long tradition of private sponsorship to one supported in considerable part by the state.

Finally, Caroline Yoon argues that writing and doing mathematics have more in common than we usually admit—and encourages mathematicians (as well as budding mathematicians) to transfer their mathematical skills into writing competencies. Her advice coincides with one of the main goals of *The Best Writing on Mathematics* series.

More Writings on Mathematics

Every year we offer further reading suggestions on mathematics, chosen from recent publications. Toward the end of the book you can find a long list of pieces I considered for selection in this volume. Here, I list some of the books that have come to my attention lately.

I make special mention of the graphically exceptional collection *The Arts of Ornamental Geometry*, edited by Gülru Necipoğlu and the collection of interviews with Russian mathematicians edited by Andrei Sobolevski under the title *Mathematical Walks*.

Among books on mathematics in daily life, including puzzles and games, you can pick from *Gladiators, Pirates and Games of Trust* by Haim Shapira, *Chancing It* by Robert Matthews, *We Are Data* by John Cheney-Lippold, *A Survival Guide to the Misinformation Age* by David Helfand, *The Math Behind . . .* by Colin Beveridge, and *The Power of Networks* by Christopher Brinton and Mung Chiang.

You will find plenty more expository and exciting mathematics at an accessible level in *Foolproof* by Brian Hayes, *Mathematics Rebooted* by Lara Alcock, *Arithmetic* by Paul Lockhart, and even in (the now second volume of) *The Mathematics of Various Entertaining Subjects* edited by Jennifer Beineke and Jason Rosenhouse. More challenging for the mathematically uninitiated, yet appealing to the interested learner with a solid background in school mathematics are *Prime Numbers and the Riemann Hypothesis* by Barry Mazur and William Stein, *Introduction to Experimental Mathematics* by Søren Eilers and Rune Johansen, *Modern Cryptography and Elliptic Curves* by Thomas Shemanske, *A Conversational*

Beineke, Jennifer, and Jason Rosenhouse. (Eds.) *The Mathematics of Various Entertaining Subjects*, Vol. 2. Princeton, NJ: Princeton University Press, 2017.

Beveridge, Colin. *The Math Behind. . .: Discover the Mathematics of Everyday Events.* Richmond Hill, Canada: Firefly Books.

Blatt, Ben. *Nabokov's Favorite Word Is Mauve: What the Numbers Reveal about the Classics, Bestsellers, and Our Own Writing*. New York: Simon and Schuster, 2018.

Boccuni, Francesca, and Andrea Sereni. (Eds.) *Objectivity, Realism, and Proof.* Cham, Switzerland: Springer International Publishing, 2017.

Bouk, Dan. *How Our Days Became Numbered: Risk and the Rise of the Statistical Individual*. Chicago: University of Chicago Press, 2018.

Brinton, Christopher G., and Mung Chiang. *The Power of Networks: Six Principles That Connect Our Lives*. Princeton, NJ: Princeton University Press, 2017.

Carlson, David L. *Quantitative Methods in Archaeology Using R*. Cambridge, U.K.: Cambridge University Press, 2017.

Cellucci, Carlo. *Rethinking Knowledge: The Heuristic View*. Cham, Switzerland: Springer Science+Business Media, 2017.

Charbonneau, Paul. *Natural Complexity: A Modeling Handbook*. Princeton, NJ: Princeton University Press, 2017.

Cheney-Lippold, John. *We Are Data: Algorithms and the Making of Our Digital Selves*. New York: New York University Press, 2017.

Cooke, Roger. *It's about Time: Elementary Mathematical Aspects of Relativity*. Providence, RI: Mathematical Association of America, 2017.

Cooper, S. Barry, and Andrew Hodges. (Eds.) *The Once and Future Turing: Computing the World*. Cambridge, U.K.: Cambridge University Press, 2016.

Damodaran, Aswath. *Narrative and Numbers: The Value of Stories in Business*. New York: Columbia Business School Publishing, 2017.

Dawson, Jeremy. *Analysing Quantitative Survey Data for Business and Management Students*. Thousand Oaks, CA: Sage, 2017.

Diaconis, Persi, and Brian Skyrms. *Ten Great Ideas about Chance*. Princeton, NJ: Princeton University Press, 2017.

Eilers, Søren, and Rune Johansen. *Introduction to Experimental Mathematics*. Cambridge, U.K.: Cambridge University Press, 2017.

Enrich, David. *The Spider Network: The Wild Story of a Math Genius, a Gang of Backstabbing Bankers, and One of the Greatest Scams in Financial History*. New York: HarperCollins, 2017.

Everett, Caleb. *Numbers and the Making of Us: Counting and the Course of Human Cultures.* Cambridge, MA: Harvard University Press, 2017.

Fagone, Jason. *The Woman Who Smashed Codes: A True Story of Love, Spies, and the Unlikely Heroine Who Outwitted America's Enemies* [Elizebeth Smith Friedman]. New York: HarperCollins, 2017.

Farrell, Simon, and Stephan Lewandowsky. *Computational Modeling of Cognition and Behavior*. Cambridge, U.K.: Cambridge University Press, 2018.

Gödel, Kurt. *Logic Lectures: Gödel's Basic Logic Course at Notre Dame*. Belgrade, Serbia: Logical Society of Belgrade, 2017.

Graney, Christopher M. *Mathematical disquisitions: The Booklet of Theses Immortalized by Galileo*. Notre Dame, IN: University of Notre Dame Press, 2017.

Grötschel, Iris, and Martin Grötschel. *Mathematical Berlin: Science, Sights, and Stories*. Berlin, Germany: Berlin Story Verlag, 2016.

Hayes, Brian. *Foolproof, and Other Mathematical Meditations*. Cambridge, MA: MIT Press, 2017.

Helfand, David J. *A Survival Guide to the Misinformation Age: Scientific Habits of Mind.* New York: Columbia University Press, 2017.

Hill, Ted. *Pushing Limits: From West Point to Berkeley and Beyond.* Providence, RI: American Mathematical Society, and Washington, DC: Mathematical Association of America, 2017.

Holden, Joshua. *The Mathematics of Secrets: Cryptography from Caesar Cyphers to Digital Encryption.* Princeton, NJ: Princeton University Press, 2017.

Horsten, Leon, and Philip Welch. *Gödel's Disjunction: The Scope and Limits of Mathematical Knowledge.* New York: Oxford University Press, 2016.

Humphreys, Macartan. *Political Games: Mathematical Insights on Fighting, Voting, Lying and Other Affairs of State.* New York: W. W. Norton, 2016.

Huttegger, Simon M. *The Probabilistic Foundations of Rational Learning.* Cambridge, U.K.: Cambridge University Press, 2017.

Katzner, Donald W. *Models, Mathematics, and Methodology in Economic Explanation.* Cambridge, U.K.: Cambridge University Press, 2018.

Keller, Dana K. *The Tao of Statistics: A Path to Understanding (with No Math).* Los Angeles: Sage, 2017.

Landry, Elaine. *Categories for the Working Philosopher.* Oxford, U.K.: Oxford University Press, 2017.

Larvor, Brendan. (Ed.) *Mathematical Cultures: The London Meetings 2012–2014.* Cham, Switzerland: Springer International Publishing, 2016.

Lenhard, Johannes, and Martin Carrier. (Eds.) *Mathematics as a Tool: Tracing New Roles of Mathematics in the Sciences.* Cham, Switzerland: Springer International Publishing, 2017.

Lim, Tina Su Lyn, and Donald B. Wagner. *The Continuation of Ancient Mathematics: Wang Xiaotong's* Jigu suanjing*, Algebra and Geometry in 7th-Century China.* Copenhagen, Denmark: Nordic Institute for Asian Studies, 2017.

Lima, Manuel. *The Book of Circles: Visualizing Spheres of Knowledge.* Princeton, NJ: Princeton Architectural Press, 2017.

Lockhart, Paul. *Arithmetic.* Cambridge, MA: Harvard University Press, 2017.

Matthews, Robert. *Chancing It: The Laws of Chance and How They Can Work for You.* London: Profile Books, 2017.

Mazur, Barry, and William Stein. *Prime Numbers and the Riemann Hypothesis.* Cambridge, U.K.: Cambridge University Press, 2016.

Merchant, Carolyn. *Autonomous Nature: Problems of Prediction and Control from Ancient Times to the Scientific Revolution.* Abingdon, U.K.: Routledge, 2016.

Morgan, Daniel Patrick. *Astral Sciences in Early Imperial China: Observation, Sagehood and the Individual.* Cambridge, U.K.: Cambridge University Press, 2017.

Necipoğlu, Gülru. (Ed.) *The Arts of Ornamental Geometry: A Persian Compendium on Similar and Complementary Interlocking Figures.* Leiden, Netherlands: Brill, 2017.

Peterson, Martin. *The Ethics of Technology: A Geometric Analysis of Five Moral Principles.* New York: Oxford University Press, 2017.

Pollack, Paul. *A Conversational Introduction to Algebraic Theory: Arithmetic beyond* $\mathbb{Z}$. Providence, RI: American Mathematical Society, 2017.

Rosenbaum, Paul R. *Observation and Experiment: An Introduction to Causal Inference.* Cambridge, MA: Harvard University Press, 2017.

Roski, Stefan. *Bolzano's Conception of Grounding.* Frankfurt am Main, Germany: Vittorio Klostermann, 2017.

Rovelli, Carlo. *Reality Is Not What It Seems: The Journey to Quantum Gravity.* New York: Random House, 2017.

Serres, Michel. *The Birth of Physics*. London, UK: Rowman & Littlefield, 2018 [French orig. 1977].

Serres, Michel. *Geometry: The Third Book of Foundations*. London: Bloomsbury Academic, 2017.

Shapira, Haim. *Gladiators, Pirates and Games of Trust: How Game Theory, Strategy and Probability Rule Our Lives*. London: Watkins, 2017.

Shemanske, Thomas R. *Modern Cryptography and Elliptic Curves: A Beginner's Guide*. Providence, RI: American Mathematical Society, 2017.

Sobolevski, Andrei. (Ed.) *Mathematical Walks: A Collection of Interviews*. Moscow: Skolkovo Institute of Science and Technology, 2017.

Stanton, Jeffrey M. *Reasoning with Data: An Introduction to Traditional and Bayesian Statistics Using R*. New York: Guilford Press, 2017.

Stillwell, John. *Reverse Mathematics: Proofs from the Inside Out*. Princeton, NJ: Princeton University Press, 2017.

Sullivan, Josh, and Angela Zutavern. *The Mathematical Corporation: Where Machine Intelligence and Human Ingenuity Achieve the Impossible*. New York: Perseus Books, 2017.

Taschner, Rudolf. *Game Changers: Stories of the Revolutionary Minds behind Game Theory*. Amherst, NY: Prometheus Books, 2017.

Venkatasubramanian, Venkat. *How Much Inequality Is Fair? Mathematical Principles of a Moral, Optimal, and Stable Capitalist Society*. New York: Columbia University Press, 2017.

Watson, Peter. *Convergence: The Idea at the Heart of Science*. New York: Simon and Schuster, 2016.

Watts, Edward J. *Hypatia: The Life and Legend of an Ancient Philosopher*. Oxford, U.K.: Oxford University Press, 2017.

Weissman, Martin H. *An Illustrated Theory of Numbers*. Providence, RI: American Mathematical Society, 2017.

West, Geoffrey. *Scale: The Universal Laws of Growth, Innovation, Sustainability, and the Pace of Life in Organisms, Cities, Economies, and Companies*. New York: Penguin Press, 2017.

White, John K. *Do the Math! On Growth, Greed, and Strategic Thinking*. Los Angeles, CA: Sage, 2013.

Wilmott, Paul, and David Orrell. *The Money Formula: Dodgy Finance, Pseudo Science, and How Mathematicians Took Over the Markets*. Chichester, U.K.: John Wiley & Sons, 2017.

Wilson, Matthew W. *New Lines: Critical GIS and the Trouble of the Map*. Minneapolis: University of Minnesota Press, 2017.

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For Simone, to *read* someone means to interpret or make a judgment about them. She's saying, "Every being cries out silently to be judged differently." I sometimes wonder if Simone was crying out about herself. For she, too, loved and participated in mathematics, but she was always comparing herself to her brother. She wrote [**6**, p. 64],

> At fourteen I fell into one of those fits of bottomless despair that come with adolescence, and I seriously thought of dying because of the mediocrity of my natural faculties . . . the exceptional gifts of my brother, who had a childhood and youth comparable to those of Pascal, brought my own inferiority home to me. I did not mind having no visible successes, but what did grieve me was the idea of being excluded from that transcendent kingdom to which only the truly great have access and wherein truth abides. I preferred to die rather than live without that truth.

We know Simone loved mathematics because she used mathematical examples in her philosophical writing. And you'll find her in photos of Bourbaki with her brother.

I often wonder what her relationship to mathematics would be like if she weren't always in André's shadow.

Every being cries out silently to be read differently.

As president of the Mathematical Association of America (MAA), you might think that my relationship to mathematics has always been solid. I don't like the word "success," but people look at me and think I'm successful, as if the true measure of mathematical achievement is the grants I've received or the numerous papers I've published.

Like Christopher, I've had a proclivity for mathematics since youth. But I grew up in a small rural town in South Texas, with limited opportunities. Most of my high school peers didn't even attend college. I did because my dad was a college professor, but my parents didn't know about the many mathematical opportunities I now know exist.

My love for math deepened at the University of Texas, and I managed to get admitted to Harvard for my Ph.D. But I felt out of place there, since I did not come from an Ivy League school, and unlike my peers, I did not have a full slate of graduate courses when I entered. I felt like Simone Weil, standing next to future André Weils, thinking I would never be able to flourish in mathematics if I were not like them.

I was told by one professor that I didn't belong in graduate school. That forced me to consider, among other things, why I wanted to do mathematics. And in fact, that is essentially the one big question that I'd like for you to consider today.

Why do mathematics?

This is a simple question, but worth considerable reflection. Because how you answer will strongly determine *who* you think should be doing mathematics, and *how* you will teach it.

Why is Christopher sitting in a prison cell studying calculus, even though he won't be using it as a free man for another twenty-five years? Why was Simone so captivated by transcendent mathematical truths? Why should anyone persist in doing math or seeing herself as a mathematical person when others are telling her in subtle and not so subtle ways that she doesn't belong?

And in this present moment, the world is also asking what its relationship with mathematics should be. Amid the great societal shifts wrought by the digital revolution and a shift to an information economy, we are witnessing the rapid transformation of the ways we work and live. And yet we hear voices in the public sphere, saying "high school students don't need geometry" or "let's leave advanced math for the mathematicians." And some mathematicians won't admit it, but they signal exactly the same thing by refusing to teach lower-level math courses or viewing the math major as a means to weed out those they don't think are fit for graduate school.

Our profession is threatened by voices like these from within, and without, who are undermining how society views mathematics and mathematicians. And the view of our profession is dismal. The 2012 report from the President's Council of Advisors on Science and Technology pegs introductory math courses as the major obstacle keeping students from pursuing STEM majors. We are not educating our students as well as we should, and like most injustices, this hurts those who are most vulnerable.

I want us as a mathematical community to move forward in a different way. It may require us to change our view of who should be doing mathematics and how we should teach it. But this way will be no less rigorous and no less demanding of our students. And yet it will draw more people into mathematics because they will see how mathematics connects to their deepest human desires.

So if you asked me, "Why do mathematics?" I would say, "Mathematics helps people flourish."

Mathematics is for human flourishing.

The well-lived life is a life of human flourishing. The ancient Greeks had a word for human flourishing, *eudaimonia*, which they viewed as the good composed of all goods. There is a similar word in Hebrew: *shalom*, which is used as a greeting. Shalom is sometimes translated "peace," but the word has a far richer context. To say "shalom" to someone is to wish that they will flourish and live well. And Arabic has a related word: *salaam*.

A basic question, taken up by Aristotle, is this: How do you achieve human flourishing? What is the well-lived life? Aristotle would say that flourishing comes through the exercise of virtue. The Greek concept of virtue is excellence of character that leads to excellence of conduct. So it includes more than just moral virtue; for instance, courage and wisdom are also virtues.

What I hope to convince you of today is that the practice of mathematics cultivates virtues that help people flourish. These virtues serve you well no matter what profession you choose. And the movement toward virtue happens through basic human desires.

I want to talk about five desires we all have. The first of these is play.

1. Play

It is a happy talent to know how to play.
—Ralph Waldo Emerson [2, p. 138]

Think of how babies play. Play is hard to define, but we can think of a few qualities that characterize it. For instance, play should be *fun* and *voluntary*, or it wouldn't be play. There is usually some *structure*—even babies know that "peekaboo" follows a certain pattern—but there is lots of *freedom* within that structure. That freedom leads to investigation of some sort, like "where will you appear if we play peekaboo one more time?" There is usually *no great stake* in the outcome. And the investigation can often lead to some sort of *surprise*, like appearing in a different place in peekaboo. Of course, animals play too, but what characterizes human play is the enlarged role of mind and the *imagination*.

Think about Rubik's Cube or the game Set. There's interplay between structure and freedom and no great stake in the outcome, but there's investigation that can lead to the delight of solving the cube or finding sets of matching cards.

Mathematics makes the mind its playground. We play with patterns, and within the structure of certain axioms, we exercise freedom in exploring their consequences, joyful at any truths we find. We even have a whole area of mathematics known as "recreational mathematics"! Do you know another discipline that has a "recreational" subfield? Is there a "recreational physics" or "recreational philosophy"?

And mathematical play builds virtues that enable us to flourish in every area of our lives. For instance, math play builds **hopefulness**—when you sit with a puzzle long enough, you are exercising hope that you will eventually solve it. Math play builds **community**—when you share in the delight of working on a problem with another human being. And math play builds **perseverance**—just as weekly soccer practices build up the muscles that make us stronger for the next game, weekly math investigations make us more fit for the next problem, whatever that is, even if we don't solve the current problem. It's why the MAA supports programs like the American Mathematics Competitions and the Putnam Competition. We help kids flourish through building hopefulness, perseverance, and community. This year, you may have heard that the U.S. team, which MAA trained, won the International Math Olympiad for the second time in a row. What you might not have heard is that Po-Shen Loh, who coached our team, invited teams from other countries to train with them to prepare. You see, our priority was community over competition. This action was so impressive to the Singaporean prime minister that he publicly thanked President Obama for this remarkable collaboration. This was true play: teams in friendly competition.

Play is part of human flourishing. You cannot flourish without play.

And if mathematics is for human flourishing, we should "play up" the role of play in how we teach and who we teach. Everyone can play. Everyone enjoys play. Everyone can have a meaningful experience in mathematical play.

And teaching play is hard work! It's actually harder than lecturing because you have to be ready for almost anything to happen in the classroom, but it's also more fun. Play is part of what makes inquiry-based learning and other forms of active learning so effective. There's overwhelming evidence that students learn better with active learning. This year, in the Conference Board of the Mathematical Sciences, I signed a statement with presidents of other math organizations endorsing active learning, available on the CBMS website. And if you want to see the evidence for active learning, we've included some background information in this statement.

So, teach play.

Another basic human desire is beauty.

2. Beauty

It is impossible to be a mathematician
without being a poet in soul.
*—Sónya Kovalévsky [**3**, p. 316]*

Who among us does not enjoy beautiful things? A beautiful sunset. A sublime sonata. A profound poem. An elegant proof.

Mathematicians and scientists are awed by the simplicity, regularity, and order of the laws of the universe. These are called "beautiful." They feel transcendent. Why should mathematics be as powerful as it is? This is what Nobel prize–winning physicist Eugene Wigner called "the unreasonable effectiveness of mathematics" to explain the natural sciences. And Einstein asked, "How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?"

And mathematicians are not satisfied with just any proof of a theorem. We often look for the best proofs, the simplest or most pleasing. Mathematicians have a special word for this—we say a proof is "elegant." Paul Erdős often spoke of "The Book" that God keeps, in which all the most elegant proofs of theorems are recorded.

Pursuing mathematics in this way cultivates the virtues of **transcendence** and **joy**. By joy, I refer to the wonder or awe or delight in the beauty of the created order. By transcendence, I mean the ability to embrace the mystery of it all. There's a transcendent joy in experiencing the beauty of mathematics.

And mathematics builds the virtue of **circumspection**. We know the limits of our arguments, and we don't overgeneralize. I like what my friend Rachel Schwell said:

"I think math helps me make fewer sweeping generalizations about people. For example, I wouldn't assume a person is, say, uneducated, just because she is, say, poor, just as I can't assume a number is say, positive, because it is an integer. I can't even assume it is positive if I know it's nonnegative, even if, probability-wise, it probably is positive! So I don't leap to automatic associations as much."

Can we help our students see that the virtue of circumspection is important in life?

A fourth basic human desire is justice.

4. Justice

Justice. To be ever ready to admit that another person is something quite different from what we read when he is there (or when we think about him). Or, rather, to read in him that he is certainly something different, perhaps something completely different from what we read in him. Every being cries out silently to be read differently.
*—Simone Weil [**7**, p. 188]*

Akemi was a student of mine who did research with me as an undergraduate. Her innovative paper linking game theory and phylogenetics was published in a highly regarded mathematical biology journal. She went to a top research university for her Ph.D. So I was surprised when I learned that Akemi quit after one year.

She told me that she had many negative experiences. Her advisor was never willing to meet with her, and she had faced uncomfortable experiences as a woman. She told me one example:

> At the beginning of the course, I consistently got 10/10 on my homework assignments which were all graded by the TA. One day, Jeff [a mutual friend] told me that he was hanging out with our TA and someone asked the TA how the analysis class was doing. He went on and on about some "guy" named Akemi and how perfect "his" homeworks were and how clearly they were written,

> etc. Jeff told him I was a girl and the TA was shocked. (Jeff told me this story because he thought it was funny that someone both didn't know my sex from my name and reacted so dramatically to finding out.) After that, I never got remotely close to 10/10 on my assignments and my exams were equally harsh—most of the reasons for docked points were vague with comments like "give more detail." I didn't feel like my understanding of the material diminished that quickly or dramatically.

I hope you agree something is not right with this picture. If a certain kind of anger wells up in you, you are experiencing a telltale sign of flourishing: the desire for justice. Justice means setting things right. And justice is a powerful motivator to action.

Justice is required for human flourishing. We flourish—we experience shalom—when we treat others justly and when we are treated justly.

Simone Weil realized that correcting injustice must involve changing how we view others: "to read in him that he is certainly something different, perhaps something completely different from what we read in him. Every being cries out silently to be read differently."

Now before we are too quick to censure Akemi's TA, we have to realize that the problem of reading others differently begins with ourselves. The TA may not have even realized he was doing this. This is the problem of *implicit bias*: unconscious stereotypes that subtly affect our decisions. One of the best experiences I had in MAA leadership was attending a workshop on implicit bias, in which I realized in a powerful way how I am biased even though I try not to be. We all do it without realizing it. Numerous experiments confirm results of the following kind: When given two nearly identical resumes except that one has a positively stereotyped name and one has a negatively stereotyped name (e.g., woman or minority), judges rate the positively stereotyped resume higher. This happens even if judges come from the negatively stereotyped group.

This is why good practices are important. The MAA now has a document for selection committees called "Avoiding Implicit Bias" that lists a number of practices that have been shown in research to mitigate the effects of implicit bias, such as taking time to make decisions or generating a large candidate pool. These are good practices even if you don't believe that bias exists. That document is now distributed with every MAA committee assignment.

You see, we have to recognize that even if people are just, even if they desire to be just, a society may not be just if its structures and practices are not also just. And the only way a whole society can flourish is if the society is a just society. It is often said that the mark of a just society is how it treats its most vulnerable members.

So I ask, with great humility, are we a just community?

If you believe that mathematics is for human flourishing, and we teach mathematics to help people flourish, you will see, if you look around the room, that we aren't helping all our students flourish. The demographics of the mathematical community do not look like the demographics of America. We have left whole segments out of the benefits of the flourishing available in our profession.

So we have to talk about race, and that's hard. It can bring up complicated emotions, even more so with all that has taken place in our nation in the past year. In our community, we have to become more comfortable talking about race, listening to each other's experiences, and being willing to recognize it's there. If you want to treat others with dignity and they are hurting, you don't ignore their pain. You ask, "What are you going through?"

It's not enough to say, "I don't think about race" because in a community, how one member is doing affects the whole community. And for those of us not in the dominant racial group, we don't have the luxury of saying, "I don't think about race" because racial issues affect us on a daily basis. So let me encourage all of us to try having these conversations, to be quick to listen, slow to speak, and quick to forgive each other when we say something stupid. That'll happen if you start to have conversations, and we just have to have grace for each other if we make mistakes—it's better than not talking.

So if we're going to have conversation, I'll start. I grew up in Texas in a white and Latino part of the state, and I realized early on that my family had different customs from my friends—my clothes were different, the food in my lunchbox was different—and these things were causing me not to fit in. I wanted to be white. Not Latino, because white people got more respect, and as an Asian, I was getting picked on all the time. I had no role models for being Asian-American. So I tried hard to act white, even if I couldn't look white.

On the other hand, in Chinese communities, I also don't fit in. I don't speak Chinese. I don't act Chinese. At Chinese restaurants, I'm viewed as white. Did you know that at authentic Chinese restaurants, there is often a special menu, a secret menu, that they only give to Chinese people? It has all the good stuff. I don't get that menu unless I ask for it. In fact, they discourage me, saying, "You won't like the stuff on that menu."

As mathematicians, who gets to see our secret menu? Whom do we shepherd toward taking more math courses? Whom do we discourage from looking at that menu?

Don't let me sound as if I'm complaining about my race. There are ways in which I benefited from being Asian. People expected me to do better at math and science, and I'm sure that's part of why I did. Because I now know there is a recognized literature on "expectancy effects," that teacher expectations do affect student performance.

The first time I didn't feel like a minority was when I moved to California. There are so many Asian-Americans there. In Texas, I would commonly get the question, "Your English is so good! Where are you from?"

"Texas."

"No, where are you *really* from?" That never happens in California, and there's a feeling of freedom I have in not having to counter these verbal stings.

These days, I'm used to being at math conferences and seeing a sea of white faces. So even I was a little bit surprised that when I was elected MAA president, a prominent blogger on race issues for Asian-Americans wrote a blog post about it. His name is Angry Asian Man. He looked at the photos of past MAA presidents on our website, and given how many Asians he expected to be in math, he noted that they were all white except for me and wrote a sarcastic post entitled

"Finally, an Asian guy who's good at math."

I am the first president of color of either the American Mathematical Society (AMS) or MAA. Minorities, including Asians, are easy to overlook when you think about who would make a good leader. This situation may not be intentional, but when you are asked to think about who is fit for this or that role, you often think of people just like the people who have been in office. So it is easy for implicit bias to creep in.

I raise this discussion out of deep affection for the mathematical community. I want us to flourish, and there are ways in which we can do better.

In 2015, I had the great pleasure of running MSRI-UP, a summer research program for students from underrepresented backgrounds: first-generation college students, Latino and African-American kids. I asked them to help me prepare this talk, to tell me about obstacles they've faced doing mathematics.

One of them, who did wonderful work that summer, told me about her experience in an analysis course after she got back. She said, "Even though the class was really hard, it was more difficult to receive the humiliations of the professor. He made us feel that we were not good enough to study math and he even told us to change to another 'easier' profession." As a result of this and other experiences, she switched her major to engineering.

Let me be clear: There is no good reason to tell a student she doesn't belong in math. That's the student's decision, not yours. You see the snapshot of her progress, but you don't see her trajectory. You can't know how she will grow and flourish in the future. But you can help her get there.

Of course, you should give forthright counsel to students about skills they might need to develop further if they want to go on in mathematics, but if you see mathematics as a means to help them flourish, why wouldn't you encourage them to take more math?

Another student that summer, Oscar, told me about his experience as a math major. Unlike his peers and because of his background, he did not enter college with any advanced placement credit. He says,

> I noticed how different my trajectory was, however, while I was in my Complex Analysis course. A student was presenting a solution on the board which required a bit of a complicated derivation halfway through. They skipped over a number of steps, citing "I don't think I need to go through the algebra. . . . we all tested out of Calculus here anyway!" with my professor nodding in agreement and some students laughing. I quietly commented that Calculus

admit when we make mistakes and to learn from them. We have hopefulness that our labor is never in vain and the transcendent belief that our work will bear fruit in the flourishing of our students.

Because what I am asking you to do is something you already know, at the heart of the teacher–student relationship, pulls us toward virtue.

I'm asking you to love.

5. Love

If I speak in the tongues of mortals and of angels, but have not love, I am a noisy gong or a clanging cymbal.
*—Paul the Apostle [**4**, p. 2017]*

Love is the greatest human desire. And to love and be loved is a supreme mark of human flourishing. For it serves the other desires—play, truth, beauty, and justice—and it is served by them.

Every being cries out silently to be read differently. Every being cries out silently to be loved. Christopher, in prison, wasn't looking only for mathematical advice. He was looking for connection, someone to reach out to him in his mathematical space and say, "I see you, and I share the same transcendent passion for math that you do, and you belong here."

When I was in the depths of despair in graduate school, struggling over many nonacademic things with a professor who had said I don't belong, already interviewing for jobs because I was sure I was going to quit, one professor reached out to me, became my advocate. And he said, "I would rather see you work with me than quit." So now I stand before you to ask you to find a struggling student, love that student, be his advocate!

I'll close with this reflection by Simone Weil. After wrestling with her own insecurity in mathematics, she saw that there was a path to virtue through her struggle and that her struggle could help others. She wrote [**6**, pp. 115–116],

> The love of our neighbour in all its fullness simply means being able to say to him: "What are you going through?" It is a recognition that the sufferer exists, not only as a unit in a collection, or a

specimen from the social category labelled "unfortunate", but as a man, exactly like us, who was one day stamped with a special mark by affliction. For this reason it is enough, but it is indispensable, to know how to look at him in a certain way.

This way of looking is first of all attentive. The soul empties itself of all its own contents in order to receive into itself the being it is looking at, just as he is, in all his truth.

Only he who is capable of attention can do this.

So it comes about that, paradoxical as it may seem, a Latin prose or a geometry problem, even though they are done wrong, may be of great service one day, provided we devote the right kind of effort to them. Should the occasion arise, they can one day make us better able to give someone in affliction exactly the help required to save him, at the supreme moment of his need.

For an adolescent, capable of grasping this truth and generous enough to desire this fruit above all others, studies could have their fullest spiritual effect, quite apart from any particular religious belief.

Academic work is one of those fields which contain a pearl so precious that it is worthwhile to sell all our possessions, keeping nothing for ourselves, in order to be able to acquire it.

Simone Weil had found a path through struggle to virtue. She understood that mathematics is for human flourishing. And the mathematical experience cannot be separated from love:

The love between friends who play with a mathematical problem.

The love between teacher and student growing together toward virtue.

The love of a community like the Mathematical Association of America working with each other toward a common goal: through the knowledge and virtues wrought by mathematics, to help everyone flourish.

Thank you for the opportunity to serve you these last two years. Shalom and salaam, my friends. Grace and peace to you. May you and all your students flourish.

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References

1. D. Brewster, *Memoirs of the Life, Writings, and Discoveries of Sir Isaac Newton*, Vol. II. Thomas Constable, Edinburgh, 1855.
2. R. W. Emerson, *Emerson in His Journals*. Ed. J. Porte. Harvard Univ. Press, Cambridge, MA, 1982.
3. S. Kovalévsky, *Sónya Kovalévsky: Her Recollections of Childhood*. Trans. I. F. Hapgood. Century, New York, 1895.
4. *The New Oxford Annotated Bible with Apocrypha (NRSV)*. Oxford Univ. Press, New York, 2010.
5. *Vulgate New Testament*. Samuel Bagster and Sons, London, 1872.
6. S. Weil, *Waiting for God*. Trans. E. Craufurd. G. P. Putnam's Sons, New York, 1951.
7. ——, *Gravity and Grace*. Trans. A. Wills. G. P. Putnam's Sons, New York, 1952.

How To Play Mathematics

MARGARET WERTHEIM

What does it mean to know mathematics? Since math is something we teach using textbooks that demand years of training to decipher, you might think the *sine qua non* is intelligence—usually "higher" levels of whatever we imagine that to be. At the very least, you might assume that knowing mathematics requires an ability to work with symbols and signs. But here's a conundrum suggesting that this line of reasoning might not be wholly adequate. Living in tropical coral reefs are species of sea slugs known as nudibranchs, adorned with flanges embodying hyperbolic geometry, an alternative to the Euclidean geometry that we learn about in school, and a form that, over hundreds of years, many great mathematical minds tried to prove impossible.

Sea slugs have at least the rudiments of brains; they generally possess a few thousand neurons, whose large size has made these animals a model organism for scientists studying basic neuronal functioning. This tiny number isn't nearly enough to enable the slug to formulate any representation of abstract signs, let alone an ability to mentally manipulate them, and yet, somehow, a nudibranch materializes in the fibers of its very being a form that genius-level human mathematicians didn't discover until the nineteenth century; and when they did, it nearly drove them mad. In this instance, complex brains were an impediment to understanding.

Nature's love affair with hyperbolic geometry dates to at least the Silurian age, more than 400 million years ago, when sea floors of the early Earth were covered in vast coral reefs. Many species of corals, then and now, also have hyperbolic structures, which we immediately recognize by the frills and crenellations of their forms. Although corals are animals, they have only simple nervous systems and can't be said to have a brain. A head of coral is actually a colonial organism made up of thousands of individual polyps growing together; collectively, they grow a vascular system, a respiratory system, and a crude gastrointestinal system through which all the individuals of the colony eat and breathe and share nutrients. Nothing like a brain exists, and yet the colony can organize itself into a mathematical surface disallowed by Euclid's axiom about parallel lines. Strike two against "higher intelligence."

Ask any high schooler what the angles of a triangle add up to, and she'll say, "180 degrees." That isn't true on a hyperbolic surface. Ask our student what's the circumference of a circle, and she'll say, "2π times the radius." That's also not true on a hyperbolic surface. Most of the geometric rules we're taught in school don't apply to hyperbolic surfaces, which is why mathematicians such as Carl Friedrich Gauss were so disturbed when finally forced to confront the logical validity of these forms, and hence their mathematical existence. So worried was Gauss by what he was discovering about hyperbolic geometry that he didn't publish his research on the subject: "I fear the howl of the Boetians if I make my work known," he confided to a friend in 1829. To their universal horror, other mathematicians soon converged on the same conclusion, and the genie of non-Euclidean geometry was let loose.

But can we say that sea slugs and corals *know* hyperbolic geometry? I want to argue here that in some sense they do. Absent the apparatus of rationalization and without the capacity to form mental representations, I'd like to postulate that these humble organisms are skilled geometers whose example has powerful resonances for what it means for us *humans* to know math—and also profound implications for teaching this legendarily abstruse field.

I'm not the first person to have considered the mathematical capacities of nonsentient things. Toward the end of Richard Feynman's life, the Nobel Prize–winning physicist is said to have become fascinated by the question of whether atoms are "thinking." Feynman was drawn to this deliberation by considering what electrons do as they orbit the nucleus of an atom. In the earliest days of atomic science, atoms were conceived as little solar systems, with the electrons orbiting in simple paths around their nuclei, much as a planet revolves around its sun. Yet in the 1920s, it became evident that something much more mathematically complex was going on; in fact, as an electron buzzes around its nucleus, games and animation; we'd be able to watch whole movies akin to the marvelous holographic projection of Princess Leia in the original *Star Wars* film.

Calculating transforms for complex objects requires vast computational powers and skills as yet unachieved by human CGH practitioners. Nonetheless, simple chemicals interacting with light on a piece of film manage to enact Fourier transforms of complicated scenes. Acting together, wave fronts of light and atoms execute a beautiful piece of mathematical encoding, and when the light plays back through the film they *do* the de-encoding. As such, where a photograph is a representation, a hologram is a performance.

Fourier came to his equation in the early 1800s, not to describe images (the origin of holograms dates to the 1940s), but to describe heat flow, and it turns out that his mathematics also leads to enormously powerful applications in the audio domain. Why does a piece by Mozart sound so different when played on a flute or a violin? One way of explaining it is that, although both instruments are playing the same sequence of notes, the Fourier transform of the sound produced by each one is different. The transform reveals the sonic DNA of the instrument's sound, giving us a precise description of its harmonic components (formally, it describes the set of pure sine waves that make up the sound). With software, audio engineers can analyze the transform of a musical recording and tell you what kind of instrument was playing; moreover, they can tweak the transform to bring out qualities they like and filter out ones they don't. By fiddling with the math, one can sculpt the sound to suit particular tastes.

Calculating Fourier transforms of sounds is a lot easier than calculating the transforms of visual scenes, and software engineers have created programs to simulate musical instruments (e.g., Apple's GarageBand), effectively giving users a sim-orchestra on their laptops for the price of an app. Advances in Fourier-based sound simulation have revolutionized the economics of the music business, including movie scoring. Now you don't need an actual orchestra to produce stirring strings to accompany a heroine's triumph; you can conjure them from the virtual depths, generated through mathematics.

Whereas music synthesis demonstrates how we can use mathematics to create something powerful out of a vacuum, here I'm more interested in what happens in actual concert halls. Each great hall has its own unique "sound"; each room acts as a filter for the music, tweaking and sculpting its Fourier transform. Contemporary acoustic engineers use Fourier techniques when designing new concert halls, manipulating the architecture of the space, for example, adding baffles in specific places, all aided by software that simulates how sounds will react within the space. If the engineers do their job well, there will be no "dead spots," and the hall will sing with warmth and resonance. Here we have a mathematical performance between the sound waves, the architecture, and the surfaces of the walls.

Some music schools now have electronic "practice rooms," where, through software, you can dial up a Fourier-based simulation of a cathedral or a tin shed and hear what your playing would sound like in different spaces. However, music connoisseurs will tell you that no simulation is a substitute for physical reality, which is why revered concert halls, such as Vienna's Musikverein, or New York's Carnegie Hall, won't be replaced by software any time soon. It's interesting that most of the best-rated halls were built before 1901, a fact that the acoustic legend Leo Beranek has attributed to their *lack* of fancy architecture (their resolutely shoe-box shape) and their lightly upholstered seats. From the perspective I'm adopting, even the *chairs* can be said to be participating in the mathematical performance enacted in a concert hall. Score another home run for nonsentience.

Since at least the time of Pythagoras and Plato, there's been a great deal of discussion in Western philosophy about how we can understand the fact that many physical systems have mathematical representations: the segmented arrangements in sunflowers, pinecones, and pineapples (Fibonacci numbers); the curve of nautilus shells, elephant tusks, and rams' horns (logarithmic spirals); music (harmonic ratios and Fourier transforms); atoms, stars, and galaxies, which all now have powerful mathematical descriptors; even the cosmos as a whole, now represented by the equations of general relativity. The physicist Eugene Wigner has termed this startling fact "the unreasonable effectiveness of mathematics." Why does the real world actualize math at all? And so much of it? Even arcane parts of mathematics, such as abstract algebras and obscure bits of topology, often turn out to be manifest somewhere in nature. Most physicists still explain this by some form of philosophical Platonism, which in its oldest form says that the universe is molded by mathematical relationships that precede the material world. To