The **BEST** **WRITING** on **MATHEMATICS**

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Mircea Pitici, Editor

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Introduction

Mircea Pitici

This volume is the tenth in our annual series of *The Best Writing on Mathematics*. For me, the editor, the series is a continuous enterprise of reading and curating the vast literature on mathematics (published in research journals, books, magazines, and online) fragmented yearly by our time-measuring conventions. I encourage the readers who want to enjoy fully the results of this endeavor and the reviewers who judge its merits to consult the series in its entirety, not only the current (or any other one) volume. In the introductions to the previous volumes in the series I detailed the procedure we follow to reach the selection; I will not repeat it here. The ten volumes available so far contain more than 200 pieces.

I selected the contents of this anthology from mostly 2018 materials. That, and the circumstance of appearing here together, might be the only characteristics common to the pieces in the book. The collection is once again purposefully eclectic, guiding the reader toward a non-dogmatic understanding of mathematics—panoramic, diverse, open to interpretive possibilities, and conducive to pertinent connections. Mathematics is unique in the asymmetry between its apparent singularity of method, circumscribed to the rigors of syllogistic reasoning, and the disconcerting multiplicity of its reverberations into other domains. Somehow disturbing is yet another asymmetry, between mathematicians' attempts to establish unambiguous, clear statements of facts, and the wide range of potentially harmful applications to which mathematical notions, methods, results, and conclusions are used, indiscriminately and independent of context. Discerning the proper use of mathematics in applications from the improper one is no trifling matter, as some of the contributors to this series of anthologies have pointed out over the years.

Before I present the articles you can find in the book, I remind you that this is not only an anthology of intriguing and stimulating readings but also a reference work meant to facilitate an easy orientation into the valuable literature on mathematics currently published in a broad array of sources. The list of books I give in this introduction and the additional lists grouped at the end of the book under the title "Notable Writings" are the main components of the bibliographic aspects of the series.

Overview of the Volume

To start the selection, Moon Duchin explains that the Markov chain Monte Carlo method, a geometric-statistical approach to the analysis of political districting, guards against the worst of many possible abuses currently taking place within elective political processes.

Theodore Hill describes the recent history of the fair division of a domain problem, places it in wider practical and *im*practical contexts, and traces the contributions of a few key mathematicians who studied it.

Paul Campbell examines some of the claims commonly made on behalf of learning mathematics and finds that many of them are wanting in the current constellation of teaching practices, curricula, and competing disciplines.

Roice Nelson introduces several puzzles whose ancestry goes back to the famous cube invented and commercialized by Ernő Rubik.

Kokichi Sugihara analyzes the geometry, the topology, and the construction of versatile three-dimensional objects that produce visual illusions when looked at from different viewpoints.

Kevin Hartnett traces the recent developments and the prospects of mathematical results that establish mirror symmetry between algebraic and simplectic geometry—an unexpected and only partly understood correspondence revealed by physicists.

James Propp presents a fresh approach to problems of discrete probability and illustrates it with examples of various difficulties.

Neil Sloane details some of the remarkable numerical sequences he included in the vast collection of integers he has organized and made available over the past several decades.

Alessandro Di Bucchianico, Laura Iapichino, Nelly Litvak, Frank van der Meulen, and Ron Wehrens point out specific theoretical advances in various branches of mathematics, which have contributed powerful applications to recent technologies and services.

Toby Cubitt, David Pérez-García, and Michael Wolf tell us how they explored the connections between certain open questions in quantum physics and classical results on undecidable statements in mathematics formulated by Kurt Gödel and Alan Turing.

Jeremy Avigad places in historical context and illustrates with recent examples the growing use of computation, not only in proving mathematical results but also in making hypotheses, verifying them, and searching for mathematical objects that satisfy them.

With compelling examples and well-chosen arguments, Reuben Hersh makes the case that mathematics is pluralistic on multiple levels: in content, in philosophical interpretation, and in practice.

Mary Leng subtly defends a position highly unpopular among mathematicians and in a small minority among the philosophers of mathematics, namely, the thesis that certain mathematical statements are questionable on the ground that they imply the existence of objects that might not exist at all—for instance abstract numbers.

Tiziana Bascelli and her collaborators (listed in alphabetical order), Piotr Błaszczyk, Vladimir Kanovei, Karin U. Katz, Mikhail G. Katz, Semen S. Kutateladze, Tahl Nowik, David M. Schaps, and David Sherry, discuss an episode of 17th-century nonstandard analysis to argue that clarifying both the historical ontology of mathematical notions and the prevalent procedures of past times is essential to the history of mathematics.

Noson Yanofsky invokes two paradoxes from the realm of numbers and a famous result from the mathematical theory of complexity to speculate about their potential to inform our understanding of daily life.

Andrew Gelman recommends several practices that will make the communication of statistical research, of the data, and of their consequences more honest (and therefore more informative) to colleagues and to the public.

Michael Barany narrates a brief history of the early Fields Medal and reflects on the changes that have taken place over the decades in the award's stated aims, as well as in the manner in which awardees are selected.

To conclude the selection for this volume, Melvyn Nathanson recalls some originalities of one of the most peculiar mathematicians, Paul Erdős.

More Writings on Mathematics

Besides the pieces included in the anthology, every year I suggest other readings, offering a quick overview of books that came to my attention recently, loosely grouped in several thematic categories. This list is lacunary; it consists only of books I consulted myself, either in the two excellent libraries accessible to me (at Syracuse University and Cornell University—thank you!) or sent to me by authors and publishers. Full references are included at the end of the introduction.

A direct marketplace competitor to this volume deserves a special mention: the outstanding anthology *The Prime Number Conspiracy*—in which the editor Thomas Lin included pieces previously published by the online magazine *Quanta*. An original celebration of fertile problems in mathematics, abundantly supplemented with theoretical introductions (sometimes quite technical for the general reader) is *100 Years of Math Milestones* by Stephan Ramon Garcia and Steven Miller.

Among books exploring the presence of mathematics in daily life, human activities, and the natural world, some titles are *The Beauty of Numbers in Nature* by Ian Stewart, *Humble Pi* by Matt Parker, *Weird Math* by David Darling and Agnijo Banerjee, *Outnumbered* by David Sumpter, and *The Logic of Miracles* by László Mérő. A somehow more technical expository book but widely accessible is *Exercises in (Mathematical) Style* by John McCleary.

Some books about data, statistics, and probability are *Using and Interpreting Statistics in the Social, Behavioral, and Health Sciences* by William Wagner III and Brian Joseph Gillespie, *Narrative by Numbers* by Sam Knowles, *The Essentials of Data Science* by Graham Williams, and *The Politics of Big Data* edited by Ann Rudinow Sætnan, Ingrid Schneider, and Nicola Green.

Two interesting books about mathematics in past cultures are *Scale and the Incas* by Andrew James Hamilton and *Early Rock Art of the American West* by Ekkehart Malotki and Ellen Dissanayake. Other books on the history of mathematics or on the role of mathematics in past societies are *A History of Abstract Algebra* by Jeremy Gray, *Reading Popular Newtonianism* by Laura Miller, *A People's History of Computing in the United States* by Joy Lisi Rankin, *Exact Thinking in Demented Times* by Karl Sigmund, and *Calculated Values* by William Deringer. Two recent histories of calendars, including the mathematics of calendars, are *Scandalous*

Hossenfelder, Sabine. *Lost in Math: How Beauty Leads Physics Astray*. New York: Basic Books, 2018.

Johnson, Eric. *Anxiety and the Equation: Understanding Boltzmann's Entropy*. Cambridge, MA: MIT Press, 2018.

Knowles, Sam. *Narrative by Numbers: How to Tell Powerful and Purposeful Stories with Data*. Abingdon, U.K.: Routledge, 2018.

Kolis, Mickey, and Cassandra Meinholz. *Brainball: Teaching Inquiry Math as a Team Sport.* Lanham, MD: Rowman & Littlefield, 2018.

Lapointe, Sandra. (Ed.) *Logic from Kant to Russell: Laying the Foundations for Analytic Philosophy*. New York: Routledge, 2019.

Lewis, Harry, and Rachel Zax. *Essential Discrete Mathematics for Computer Science*. Princeton, NJ: Princeton University Press, 2019.

Lin, Thomas. (Ed.) *Alice and Bob Meet the Wall of Fire: The Biggest Ideas in Science from* Quanta. Cambridge, MA: MIT Press, 2018.

Lin, Thomas. (Ed.) *The Prime Number Conspiracy: The Biggest Ideas in Math from* Quanta. Cambridge, MA: MIT Press, 2018.

Lindblad, Sverker, Daniel Pettersson, and Thomas S. Popkewitz. (Eds.) *Education by the Numbers and the Making of Society: The Expertise of International Assessments*. New York: Routledge, 2018.

Malotki, Ekkehart, and Ellen Dissanayake. *Early Rock Art of the American West: The Geometric Enigma*. Seattle: University of Washington Press, 2018.

Mancosu, Paolo. *Abstraction and Infinity*. Oxford, U.K.: Oxford University Press, 2016.

McCall, Grant S. *Strategies for Quantitative Research: Archaeology by Numbers*. Abingdon, U.K.: Routledge, 2018.

McCleary, John. *Exercises in (Mathematical) Style: Stories of Binomial Coefficients*. Washington, DC: Mathematical Association of America, 2018.

Mérő, László. *The Logic of Miracles: Making Sense of Rare, Really Rare, and Impossibly Rare Events.* New Haven, CT: Yale University Press, 2018.

Miller, Laura. *Reading Popular Newtonianism: Print, the* Principia*, and the Dissemination of Newtonian Science*. Charlottesville, VA: University of Virginia Press, 2018.

Moss, Sarah. *Probabilistic Knowledge*. Oxford, U.K.: Oxford University Press, 2018.

Neumayer, Eric, and Thomas Plümper. *Robustness Tests for Quantitative Research*. Cambridge, U.K.: Cambridge University Press, 2017.

Nothaft, C. Philipp E. *Scandalous Error: Calendar Reform and Calendrical Astronomy in Medieval Europe*. Oxford, U.K.: Oxford University Press, 2018.

Ornes, Stephen. *Math Art: Truth, Beauty, and Equations*. New York: Sterling Publishing, 2019.

Parker, Matt. *Humble Pi: A Comedy of Maths Errors*. London: Penguin Random House, 2019.

Peschard, Isabelle F., and Bas C. van Fraassen. (Eds.) *The Experimental Side of Modeling*. Minneapolis: University of Minnesota Press, 2018.

Piazza, Mario, and Gabriele Pulcini. (Eds.) *Truth, Existence, and Explanation*. Cham, Switzerland: Springer International Publishing, 2018.

Pleitz, Martin. *Logic, Language, and the Liar Paradox*. Münster, Germany: Mentis Verlag, 2018.

Quine, Willard Van Orman. *The Significance of the New Logic*. Cambridge, U.K.: Cambridge University Press, 2018.

Rankin, Joy Lisi. *A People's History of Computing in the United States*. Cambridge, MA: Harvard University Press, 2018.

Reingold, Edward M., and Nachum Dershowitz. *Calendrical Calculations: The Ultimate Edition*. Cambridge, U.K.: Cambridge University Press, 2018.

Reutlinger, Alexander, and Juha Saatsi. (Eds.) *Explanation beyond Causation: Philosophical Perspectives on Non-Causal Explanations.* Oxford, U.K.: Oxford University Press, 2018.

Sætnan, Ann Rudinow, Ingrid Schneider, and Nicola Green. (Eds.) *The Politics of Big Data: Big Data, Big Brother?* Abingdon, U.K.: Routledge, 2018.

Schwartz, Thomas. *Cycles and Social Choice: The True and Unabridged Story of a Most Protean Paradox.* Cambridge, U.K.: Cambridge University Press, 2018.

Sigmund, Karl. *Exact Thinking in Demented Times: The Vienna Circle and the Epic Quest for the Foundations of Science*. New York: Basic Books, 2017.

Sparrow, Giles. *What Shape Is Space? A Primer for the 21st Century*. London: Thames & Hudson, 2018.

Stewart, Ian. *The Beauty of Numbers in Nature: Mathematical Patterns and Principles from the Natural World*. Brighton, U.K.: Ivy Press, 2017 (first published in 2001).

Suh, Jennifer M., and Padmanabhan Seshaiyer. *Modeling Mathematical Ideas: Developing Strategic Competence in Elementary and Middle School*. Lanham, MD: Rowman & Littlefield, 2017.

Sumpter, David. *Outnumbered: From Facebook and Google to Fake News and Filter-Bubbles—The Algorithms That Control Our Lives*. London: Bloomsbury Sigma, 2018.

Tyson, Neil deGrasse, J. Richard Gott, and Michael A. Strauss. *Welcome to the Universe: The Problem Book*. Princeton, NJ: Princeton University Press, 2017.

Wagner, Roi. *Making and Breaking Mathematical Sense: Histories and Philosophies of Mathematical Practice*. Princeton, NJ: Princeton University Press, 2017.

Wagner, William E., III, and Brian Joseph Gillespie. *Using and Interpreting Statistics in the Social, Behavioral, and Health Sciences.* Los Angeles, CA: Sage, 2019.

Walsh, Andy. *Faith across the Multiverse: Parables from Modern Science*. Peabody, MA: Hendrickson Publishers, 2018.

Williams, Graham J. *The Essentials of Data Science: Knowledge Discovery Using R*. Boca Raton, FL: CRC Press, 2017.

Yackel, Carolyn, and sarah-marie belcastro. (Eds.) *Figuring Fibers.* Providence, RI: American Mathematical Society, 2018.

Yau, Shing-Tung, and Steve Nadis. *The Shape of a Life: One Mathematician's Search for the Universe's Hidden Geometry.* New Haven, CT: Yale University Press, 2019.

Geometry v. Gerrymandering

Moon Duchin

Gerrymandering is clawing across courtrooms and headlines nationwide. The U.S. Supreme Court recently heard cases on the constitutionality of voting districts that allegedly entrenched a strong advantage for Republicans in Wisconsin and Democrats in Maryland but dodged direct rulings in both. Another partisan gerrymandering case from North Carolina is winding its way up with a boost from an emphatic lower court opinion in August. But so far, it has been impossible to satisfy the justices with a legal framework for partisan gerrymandering. Part of the problem, as former Justice Anthony Kennedy noted in a 2004 case, is that courts high and low have yet to settle on a "workable standard" for identifying a partisan gerrymander in the first place. That is where a growing number of mathematicians around the country think we can help.

Two years ago, with a few friends, I founded a working group to study the applications of geometry and computing to redistricting in the United States. Since then, the Metric Geometry and Gerrymandering Group has expanded its scope and mission, becoming deeply engaged in research, outreach, training, and consulting. More than 1,200 people have attended our workshops around the country, and many of them have become intensely involved in redistricting projects. We think the time is right to make a computational intervention. The mathematics of gerrymandering is surprisingly rich—enough to launch its own subfield—and computing power is arguably just catching up with the scale and complexity of the redistricting problem. Despite our group's technical orientation, our central goal is to reinforce and protect civil rights, and we are working closely with lawyers, political scientists, geographers, and community groups to build tools and ideas in advance of the next U.S. Census and the round of redistricting to follow it.

In a country that vests power in elected representatives, there will always be skirmishes for control of the electoral process. And in a system such as that of our House of Representatives—where winner takes all within each geographical district—the delineation of voting districts is a natural battleground. American history is chock-full of egregious line-drawing schemes, from stuffing a district with an incumbent's loyalists to slicing a long-standing district three ways to suppress the political power of black voters. Many varieties of these so-called *packing and cracking* strategies continue today, and in the big data moment, they have grown enormously more sophisticated. Now more than ever, abusive redistricting is stubbornly difficult to even identify definitively. People think they know gerrymandering by two hallmarks—bizarre shapes and disproportionate electoral outcomes—yet neither one is reliable. So how do we determine when the scales are unfairly tipped?

The Eyeball Test

The 1812 episode that gave us the word "gerrymander" sprang from the intuition that oddly shaped districts betray an illegitimate agenda. It is named for Elbridge Gerry, who was governor of Massachusetts at the time. Gerry had quite a Founding Father pedigree—signer of the Declaration of Independence, major player at the U.S. Constitutional Convention, member of Congress, James Madison's vice president—so it is amusing to consider that his enduring fame comes from nefarious redistricting. "Gerry-mander," or Gerry's salamander, was the satirical name given to a curvy district in Boston's North Shore that was thought to favor the governor's Democratic-Republican party over the rival Federalists. A woodcut political cartoon ran in the *Salem Gazette* in 1813; in it, wings, claws, and fangs were suggestively added to the district's contours to heighten its appearance of reptilian contortions.

So the idea that erratic districts tip us off to wrongdoing goes a long way back, and the converse notion that close-knit districts promote democratic ideals is as old as the republic. In 1787, Madison wrote in *The Federalist Papers* that "the natural limit of a democracy is that distance from the central point which will just permit the most remote citizens to assemble as often as their public functions demand." In other words, districts should be transitable. In 1901, a federal apportionment act marked the first appearance in U.S. law of the vague desideratum that districts should be composed of "compact territory." The word "compact" then proliferated throughout the legal landscape of redistricting but almost always without a definition.

For instance, at a 2017 meeting of the National Conference of State Legislatures, I learned that after the last census, Utah's lawmakers took the commendable time and effort to set up a website, Redistrict Utah, to solicit proposed districting maps from everyday citizens. To be considered, maps were required to be "reasonably compact." I jumped at the opportunity to find out how exactly that quality was being tested and enforced, only to learn that it was handled by just tossing the funny-looking maps. If that sounds bad, Utah is far from alone. Thirty-seven states have some kind of shape regulation on the books, and in almost every case, the eyeball test is king.

The problem is that the outline of a district tells a partial and often misleading story. First, there can certainly be benign reasons for ugly shapes. Physical geography or reasonable attempts to follow county lines or unite communities of interest can influence a boundary, although just as often, legitimate priorities such as these are merely scapegoated in an attempt to defend the worst-offending districts. On the other hand, districts that are plump, squat, and symmetrical offer no meaningful seal of quality. Just this year, a congressional redistricting plan in Pennsylvania drafted by Republicans in the state legislature achieved strong compactness scores under all five formulas specified by Pennsylvania's supreme court. Yet mathematical analysis revealed that the plan would nonetheless lock in the same extreme partisan skew as the contorted plan, enacted in 2011, that it was meant to replace. So the justices opted for the extraordinary measure of adopting an independent outsider's plan.

Lopsided Outcomes

If shape is not a reliable indicator of gerrymandering, what about studying the extent to which elected representatives match the voting patterns of the electorate? Surely lopsided outcomes provide prima facie evidence of abuse. But not so fast. Take Republicans in my home state of Massachusetts. In the 13 federal elections for president and Senate since 2000, GOP candidates have averaged more than one third of the votes statewide. That is six times the level needed to win a seat in one

The issue is that the best counting techniques often rely on recursion—that is, solving a problem using a similar problem that is a step smaller—but two-dimensional spatial counting problems just do not recurse well without some extra structure. So complete enumerations must rely on brute force. Whereas a cleverly programmed laptop can classify partitions of small grids nearly instantly, we see huge jumps in complexity as the grid size grows, and the task quickly zooms out of reach. By the time you get to a grid of nine-by-nine, there are more than 700 trillion solutions for equinumerous rook partitions, and even a high-performance computer needs a week to count them all. This seems like a hopeless state of affairs. We are trying to assess one way of cutting up a state without any ability to enumerate—let alone meaningfully compare it against—the universe of alternatives. This situation sounds like groping around in a dark, infinite wilderness.

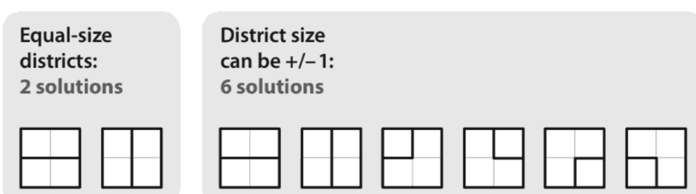
The good news is that there is an industry standard used across scientific domains for just such a colossal task: Markov chain Monte Carlo (MCMC). Markov chains are random walks in which where you go next is governed by probability, depending only on where you are now (at every position, you roll the dice to choose a neighboring space to move to). Monte Carlo methods are just estimation by random sampling. Put them together, and you get a powerful tool for searching vast spaces of possibilities. MCMC has been successfully used to decode prison messages, probe the properties and phase transitions of liquids, find provably accurate fast approximations for hard computational problems, and much more. A 2009 survey by the eminent statistician

How to Compare Countless Districting Plans

Markov chains are random walks around a graph or network in which the next destination is determined by a probability, like a roll of the dice, depending on the current position. Monte Carlo methods use random sampling to estimate a distribution of probabilities. Combined, Markov chain Monte Carlo (MCMC) is a powerful tool for searching and sampling from a vast space of scenarios, such as all the possible districting plans in a state. Attempts to use computational analysis to spot devious districting go back several decades, but efforts to apply MCMC to the problem are much more recent.

Dimensions; Districts	Equal-Size Districts	District Sizes Can Be Unequal (±1)
2×2 grid; 2 districts	2	6
3×3 grid; 3 districts	10	58
4×4 grid; 2 districts	70	206
4×4 grid; 4 districts	117	1,953
4×4 grid; 8 districts	36	34,524
5×5 grid; 5 districts	4,006	193,152
6×6 grid; 2 districts	80,518	?*
6×6 grid; 3 districts	264,500	?
6×6 grid; 4 districts	442,791	?
6×6 grid; 6 districts	451,206	?
6×6 grid; 9 districts	128,939	?
6×6 grid; 12 districts	80,092	?
6×6 grid; 18 districts	6,728	?
7×7 grid; 7 districts	158,753,814	?
8×8 grid; 8 districts	187,497,290,034	?
9×9 grid; 9 districts	706,152,947,468,301	?

*Mathematicians have not yet enumerated these solutions, which can require a week of computing or more. To find out more about the hunt for these numbers, visit www.mggg.org.



SIMPLE CASE. It is easy to enumerate all the ways to partition a small grid into equal-size districts. For a two-by-two grid with two districts of equal size, there are only two solutions. But if districts can vary in size, the number of solutions jumps to six.

Persi Diaconis estimated that MCMC drives 10 to 15% of the statistical work in science, engineering, and business, and the number has probably only gone up since then. Although computational analysis in redistricting goes back several decades, serious attempts to apply MCMC in that effort only started to appear publicly around 2014.

Imagine that officials in the state of Gridlandia hire you to decide if their legislature's districting plan is reasonable. If Gridlandia is a four-by-four grid of squares, and its state constitution calls for rook-contiguous districts, then you are in luck: There are exactly 117 ways to produce a compliant plan, and you can examine them all. You can set up a perfectly faithful model of this universe of districting plans by using 117 nodes to represent the valid plans and adding edges between the nodes to represent simple moves in which two squares in the grid swap their district assignments. The edges give you a way of conceptualizing how similar two plans are by simply counting the number of swaps needed to transform one to the other. (I call this structure a *metagraph* because it is a graph of ways to cut up another graph.) Now suppose that the state legislature is controlled by the Diamond party, and its rivals suspect that it has rigged the seats in its favor. To determine if that is true, one may turn to the election data. If the Diamond plan would have produced more seats for the party in the last election than, say, 114 out of 117 alternatives and if the same is true for several previous elections, the plan is clearly a statistical outlier. This is persuasive evidence of a partisan gerrymander—and you do not need MCMC for such an analysis.

The MCMC method kicks in when you have a full-sized problem in place of this small toy problem. As soon as you get past 100 or so nodes, there is a similar metagraph, but you cannot completely build it because of its forbidding complexity. That is no deal breaker, though. From any single plan, it is still easy to build out the local neighborhood by performing all possible moves. Now you can take a million, billion, or trillion steps and see what you find. There is mathematics in the background (ergodic theory, to be precise) guaranteeing that if you random-walk for long enough, the ensemble of maps you collect will have properties representative of the overall universe, typically long before you have visited even a modest fraction of nodes in your state space. This procedure lets you determine if the map you are evaluating is an extreme outlier according to various partisan metrics.

The cutting edge of scientific inquiry is to build more powerful algorithms and, at the same time, to devise new theorems that certify that we are sampling well enough to draw robust conclusions. There is an emerging scientific consensus around this method but there are also many directions of ongoing research.

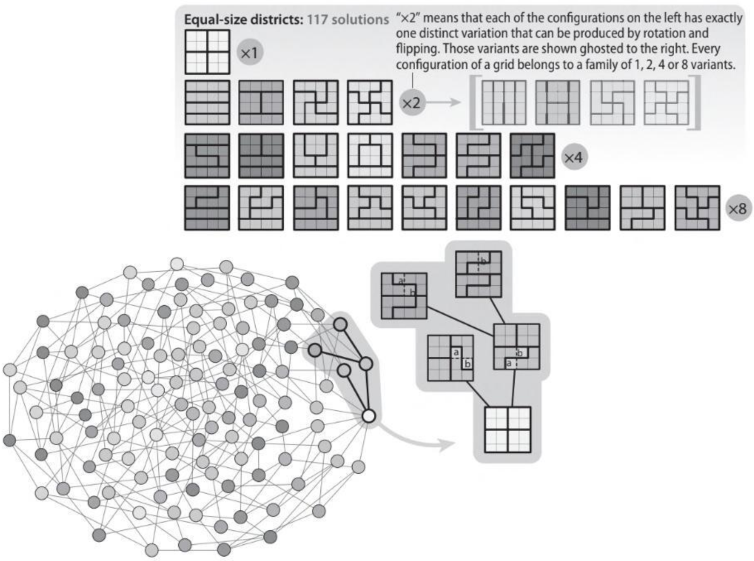
RIP Governor Gerry

So far, courts seem to be smiling on this approach. Two mathematicians—Duke University's Jonathan Mattingly and Carnegie Mellon University's Wes Pegden—have recently testified about MCMC approaches for the federal case in North Carolina and the state-level case in Pennsylvania, respectively.

Mattingly used MCMC to characterize the reasonable range one might observe for various metrics, such as seats won, across ensembles of districting plans. His random walk was weighted to favor plans that were deemed closer to ideal, along the lines of North Carolina state law. Using his ensembles, he argued that the enacted plan was an extreme partisan outlier. Pegden used a different kind of test, appealing to a rigorous theorem that quantifies how unlikely it is that a neutral plan would score much worse than other plans visited by a random walk. His method produces *p*-values, which constrain how improbable it is to find such anomalous bias by chance. Judges found both arguments credible and cited them favorably in their respective decisions.

For my part, Pennsylvania governor Tom Wolf brought me on earlier this year as a consulting expert for the state's scramble to draw new district lines following its supreme court's decision to strike down the 2011 Republican plan. My contribution was to use the MCMC framework to evaluate new plans as they were proposed, harnessing the power of statistical outliers while adding new ways to take into account more of the varied districting principles in play, from compactness, to county splits, to community structure. My analysis agreed with Pegden's in flagging the 2011 plan as an extreme partisan outlier—and I found the new plan floated by the legislature to be just as extreme, in a way that was not explained away by its improved appearances.

As the 2020 Census approaches, the nation is bracing for another wild round of redistricting, with the promise of litigation to follow. I hope the next steps will play out not just in the courtrooms but also in reform measures that require a big ensemble of maps made with open source tools to be examined before any plan is signed into law. In that way, the legislatures preserve their traditional prerogatives to commission and approve district boundaries, but they have to produce some guarantees that they are not putting too meaty a thumb on the scale.



Bigger Case. As the size of the grid grows, the number of possibilities for carving it up skyrockets. Dividing a four-by-four grid into four districts of equal size has 117 solutions. If the districts can vary in size by even one unit, there are 1,953 solutions. It does not take long before even the most powerful computers struggle to enumerate the possibilities for more complex grids. That situation presents a problem for anyone trying to detect manipulative maps by comparing the myriad ways to district a U.S. state. But MCMC can help.

We can efficiently explore valid districting plans by traveling randomly around a metagraph, defined by moves such as the unit swaps pictured. In the highlighted inset, each pattern has squares marked **a** and **b** whose district assignments are exchanged to arrive at the configuration of the pattern shown. The edges in the network represent these simple swap moves. The metagraph models the space of all valid districting plans and can be used to sample many billions of plans. Geometers are trying to understand the shape and structure of that universe. See also color images.

Computing will never make tough redistricting decisions for us and cannot produce an optimally fair plan. But it can certify that a plan behaves as though selected just from the stated rules. That alone can rein in the worst abuses and start to restore trust in the system.

general and aesthetically beautiful abstract concepts, soon to prove extremely powerful in a wide variety of mathematical and scientific fields.

The café tables had marble tops and could easily be written on in pencil and then later erased like a slate blackboard. Since the group often returned to ideas from previous meetings, they soon realized the need for a written record of their results and purchased a large notebook for documenting the problems and answers. The book, kept in a safe place by the café headwaiter and produced by him upon the group's next visit, was a collection of these mathematical questions, both solved and unsolved, that decades later became known in international mathematical circles as the *Scottish Book*.

The Ham Sandwich Problem

Problem No. 123 in the book, posted by Hugo Steinhaus, a senior member of the café mathematics group and a professor of mathematics at the University of Lemberg (now the University of Lviv), was stated as follows:

> Given are three sets A_1, A_2, A_3, located in the three-dimensional Euclidean space and with finite Lebesgue measure. Does there exist a plane cutting each of the three sets A_1, A_2, A_3, into two parts of equal measure?

To bring this question to life for his companions, Steinhaus illustrated it with one of his trademark vivid examples, one that reflected the venue of their meetings, and also perhaps their imminent preoccupation with daily essentials: Can every ordinary ham sandwich consisting of three ingredients, say bread, ham, and cheese, be cut by a planar slice of a knife so that each of the three is cut exactly in half?

A Simpler Problem

At the meeting where Steinhaus introduced this question, he reported that the analogous conclusion in two dimensions was true: Any two areas in a (flat) plane can always be simultaneously bisected by a single straight line, and he sketched out a solution on the marble tabletop. In the spirit of Steinhaus's food theme, let's consider the case where the two areas to be bisected are the crust and sausage on a pepperoni pizza. If the pizza happens to be a perfect circle, then every line passing through its center exactly bisects the crust.

To see that there is always a line that bisects both crust and sausage simultaneously, start with the potential cutting line in any fixed direction and rotate it about the center slowly, say, clockwise. If the proportion of sausage on the clockwise side of the arrow-cut happened to be 40% when the rotation began, then after the arrow-cut has rotated 180 degrees, the proportion on the clockwise side of the arrow-cut is now 60%. Because this proportion changed continuously from 40% to 60%, at some point it must have been exactly 50%, and at that point both crust and sausage have been exactly bisected (Figure 1).

On the other hand, if the pizza is not a perfect circle, as no real pizza is, then there may not be an exact center point such that every straight line through it exactly bisects the crust. But in this general noncircular case, again move the cutting line so that it always bisects the crust as it rotates, and note that even though the cutting line may not rotate around a single point as it did with a circular pizza, the same continuity

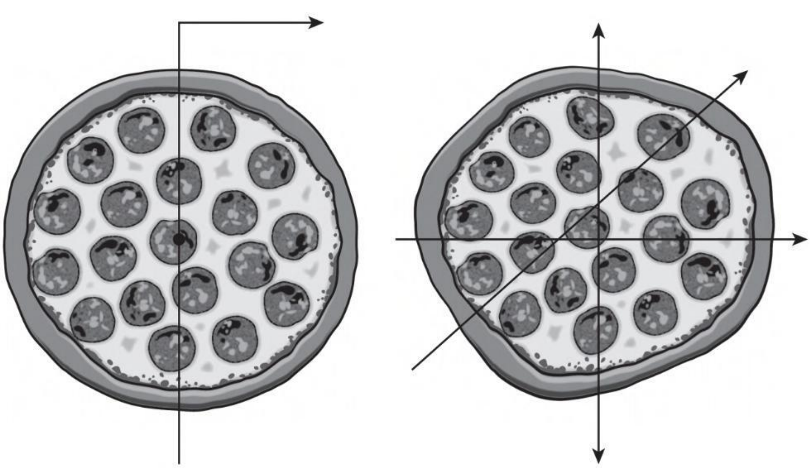


Figure 1. If a pizza is a perfect circle, then every line through the center bisects the crust. If the cut starts with 40% of the sausage clockwise from the arrow, after rotating 180 degrees, 60% of the sausage is clockwise from the arrow. So somewhere in between, the line hits 50% and the same cutting line bisects both crust and sausage. If the pizza is not a perfect circle, the crust-bisecting lines may not all pass through the same point, but the same argument applies.

argument applies. If the proportion clockwise of the north cut started at 40%, then when the cut arrow points south, that proportion will be 60%, which again completes the argument using the simple fact that to go continuously from 40 to 60, one must pass through 50. This simple but powerful observation, formally known as the intermediate value theorem, also explains why if the temperature outside your front door was 40 degrees Fahrenheit yesterday at noon and 60 degrees today at noon, then at some time in between, perhaps several times, the temperature must have been exactly 50 degrees.

Steinhaus's two-dimensional (pizza) version of the ham sandwich theorem may be used for gerrymandering. Instead of a pizza, imagine a country with two political parties whose voters are sprinkled through it in any arbitrary way. The pizza theorem implies that there is a straight line bisecting the country so that exactly half of each party is on each side of the line. Suppose, for example, that 60% of the voters in the United States are from party Purple and 40% are from party Yellow. Then there is a single straight line dividing the country into two regions, each of which has exactly 30% of the Purple on each side, and exactly 20% of the Yellow on each side, so the Purple have the strict majority on both sides. Repeating this procedure to each side yields four districts with exactly 15% Purple and exactly 10% Yellow in each. Again the majority party (in this case, Purple) has the majority in each district. Continuing this argument shows that whenever the number of desired districts is a power of two, there is always a straight-line partition of the country into that number of districts so that the majority party also has the majority of votes in every single district (Figure 2).

This repeated-bisection argument may fail, however, for odd numbers of desired districts. On the other hand, Sergei Bespamyatnikh, David Kirkpatrick, and Jack Snoeyink of the University of British Columbia found a generalization of the ham sandwich theorem that does the trick for any number of districts, power of two or not. They showed that for a given number of Yellow and Purple points in the plane (no three of which are on a line), there is always a subdivision of the plane into any given number of convex polygons (districts), each containing exactly the same numbers of Yellow points in each district, and the same number of Purple (Figure 3).

In his application of this theorem to gerrymandering, Soberón observed that for any desired number of districts, this theorem implies

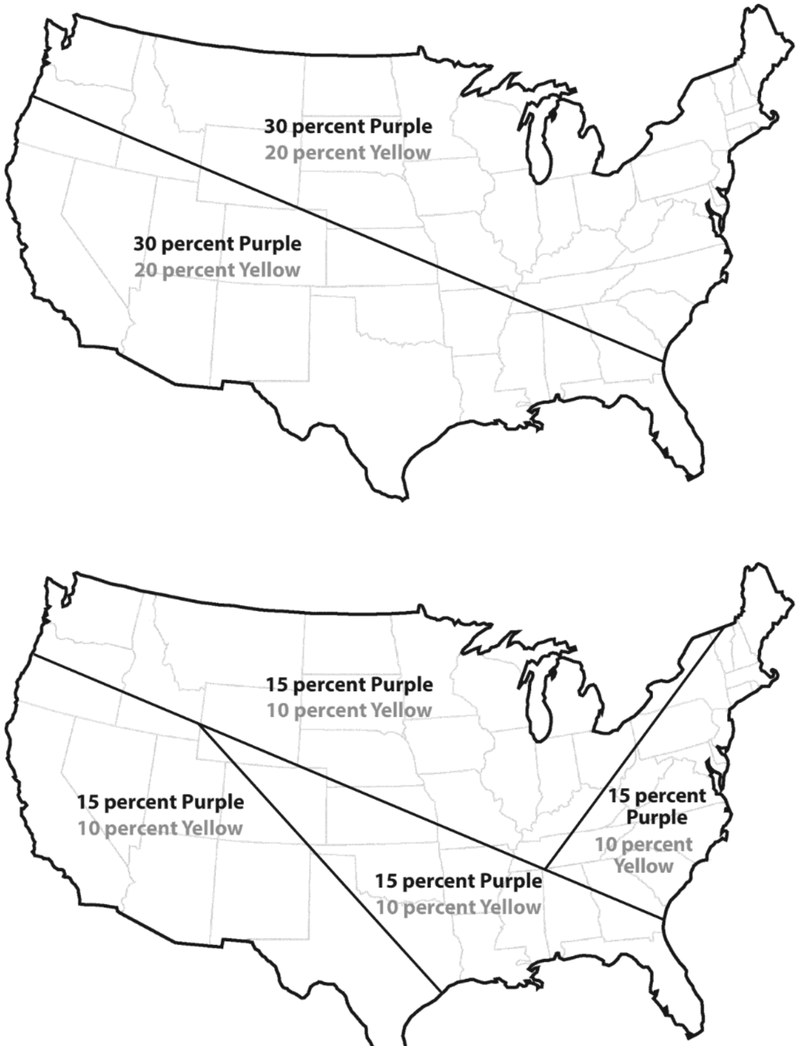


Figure 2. According to the two-dimensional (pizza) version of the ham sandwich theorem, there is a straight line across the United States so that exactly half of the Purple and half of the Yellow party voters are on either side (top). Bisecting each of those (bottom), the same argument shows that there are four regions with equal numbers of Purple and equal numbers of Yellow in each of them. Thus, the party with the overall majority also has the majority in each of the districts. See also color images.

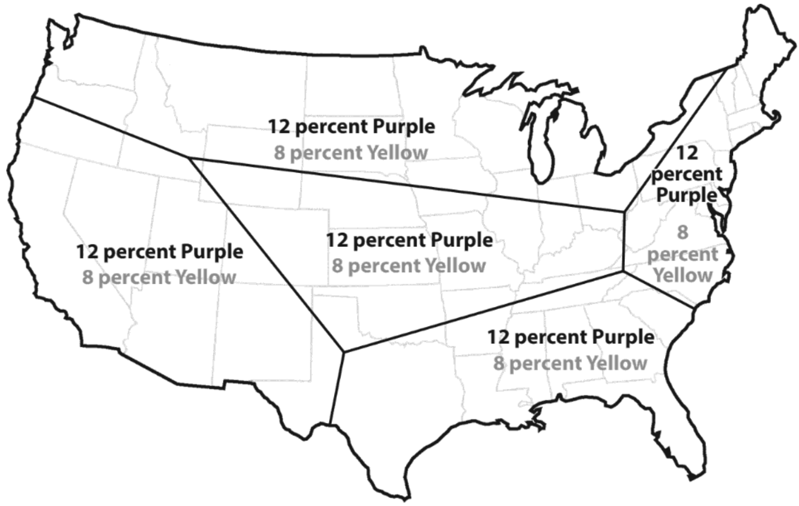


Figure 3. For odd numbers of desired districts, the repeated-bisection argument of the two-dimensional version of the ham sandwich theorem may fail. However, a generalization of the theorem works for any number of districts, by showing that for a given number of Purple or Yellow points in a plane (no three of which are on a line), there is always a subdivision of the plane into any given number of convex polygons, each of which contains exactly the same number of Yellow, and the same number of Purple, points. See also color images.

that there is always a subdivision into that number of polygonal districts so that each district has exactly the same number of Purple and exactly the same number of Yellow. Whichever party has the overall majority in the country also has the majority in every district. Thus, as he found, a direct application of the ham sandwich theory would not help fix the problem, but would actually make it worse, and the electorate should be wary if the person drawing congressional maps knows anything about that theory. No wonder the Supreme Court balked on all three of the most recent cases it has heard on partisan gerrymandering.

The Scottish Café

After giving his argument for the two-dimensional case of the ham sandwich theorem, Steinhaus then challenged his companions to prove the three-dimensional version. The same basic intermediate value theorem

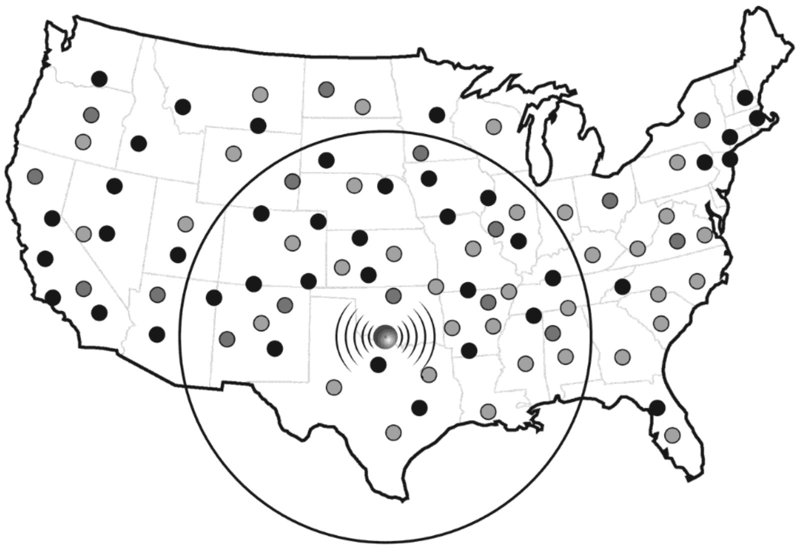


Figure 5. Mathematicians Arthur Stone and John Tukey of Princeton University extended the ham sandwich theorem to nonuniform distributions, higher dimensions, and a variety of other cutting surfaces and objects. One of their examples showed that a single circle simultaneously can bisect any three shapes in the plane. For instance, it is always possible to design the power and location of a telecommunications satellite so that its broadcasts reach exactly half the Yellow, half the Purple, and half the Teal (Independents). See also color images.

median planes, and median hyperplanes in higher dimensions. Using the Borsuk–Ulam theorem again, but this time applied to a different "midpoint median" function, it was straightforward to show that for any two arbitrary random distributions in the plane, or any three in space, there is always a line median or plane median, respectively, that has no more than half of each distribution on each side.

Some 20 years later, Columbia University economist Macartan Humphreys used this result to solve a problem in cooperative game theory. In a setting where several groups must agree on allocations of a fixed resource (say, how much of a given disaster fund should be allocated to medical, power, housing, and food), the objective is to find an allocation that no winning coalition could override in favor of another allocation. He showed that such equilibrium allocations exist precisely when they lie on "ham sandwich cuts."

Touching Planes

In explaining the beauties of the ham sandwich theorem to nonmathematician friends over beer and pizza, one of my companions noticed that often there is more than one bisecting line (or plane), and we saw that some bisecting lines might touch each of the objects, whereas others may not. I started looking at this observation more closely and discovered that in every case, I could always find a bisecting line or plane that touched all the objects. When I could not find a reference or proof of this concept, I posed the question to my Georgia Tech friend and colleague John Elton, who had helped me crack a handful of other mathematical problems: Is there always a bisecting plane (or hyperplane, in dimensions greater than 3) that also touches each of the objects?

Together, he and I were able to show that the answer is yes, which strengthens the conclusion of the classical ham sandwich theorem. For example, this improved version implies that at any instant in time in our solar system, there is always a single plane passing through three bodies—one planet, one moon, and one asteroid—that simultaneously bisects the planetary, the lunar, and the asteroidal masses in the solar system (Figure 6).

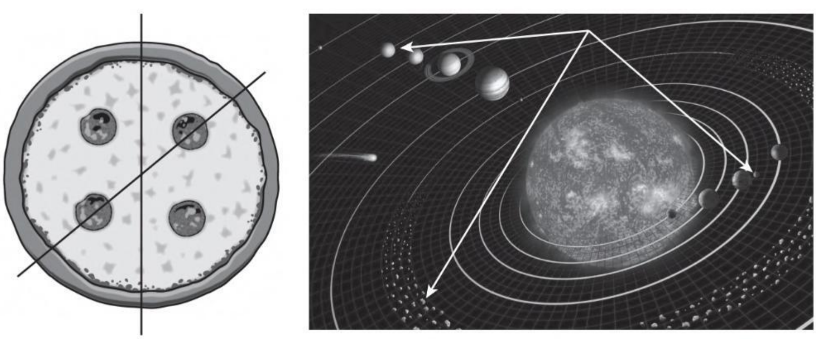


FIGURE 6. Some bisecting lines or planes may touch each of the objects, whereas others may not, as shown on the pizza above. Nevertheless, there is always a single bisecting line or plane (or hyperplane, in higher dimensions) that touches all of the objects. For example, at any instant in time in our solar system, there is always a single plane passing through three bodies—one planet, one moon, and one asteroid—that simultaneously bisects the planetary, the lunar, and the asteroidal masses in the solar system.

Diverse Divisions

The ideas underlying the ham sandwich theorem have also been used in diverse fields, including computer science, economics, political science, and game theory. When I asked my friend Francis Su, Harvey Mudd College mathematician and fair-division expert, about his own applications of the ham sandwich theorem, he explained how he and Forest Simmons of Portland Community College had used ham sandwich results to solve problems in *consensus halving*. In particular, they used it to show that given a territory and 2*n* explorers, two each of *n* different specialties (e.g., two zoologists, two botanists, and two archaeologists), there always exists a way to divide the territory into two regions and the people into two teams of *n* explorers (one of each type) such that each explorer is satisfied with their half of the territory.

As a more light-hearted application during a keynote lecture at Georgia Tech, Tel Aviv University mathematician Noga Alon described a discrete analog of the ham sandwich theorem for splitting a necklace containing various types of jewels, as might be done, he said, by mathematically oriented thieves who steal a necklace and wish to divide it fairly between them. Even though it had been offered as an amusement, his result had applications, including to very large scale integrated (VLSI) circuit designs where an integrated chip composed of two different types of nodes is manufactured in the shape of a closed circuit (much like a necklace), and may be restructured after fabrication by cutting and regrouping the pieces. Alon's theorem answers this question: How many cuts need to be made of the original circuit in order to bisect it into two parts, each containing exactly half of each type of node?

Revisiting the Café

Steinhaus published the proof of the ham sandwich theorem in the local Polish mathematical journal *Mathesis Polska* in 1938, the year of the infamously violent *Kristallnacht*. The Scottish Café mathematics gatherings continued for a few more years, despite the invasion of western Poland by the German army and the Soviet occupation of Lwów from the east, but the difficult times would soon disperse both scholars and their works. Ulam, a young man in his 20s and, like Steinhaus, also of

Jewish roots, had left with his brother on a ship for America just two weeks before the German invasion.

Banach, nearing 50 and already widely known for his discoveries in mathematics, was appointed dean of the University of Lwów's department of mathematics and physics by the Soviets after they occupied that city, under the condition that he promised to learn Ukrainian. When the Nazis in turn occupied Lwów, they closed the universities, and Banach was forced to work feeding lice at a typhus research center, which at least protected him from being sent into slave labor. (Banach, like many others, was made to wear cages of lice on his body, so that they could feed on his blood. The lice, which are carriers of typhus, were used in research efforts to create a vaccine against the disease.) Banach was able to help reestablish the university after Lwów was recaptured by the Soviets in 1944, but he died of lung cancer in 1945.

Although the correct statement of the crisp ham sandwich theorem had made it through the World War II mathematical grapevine perfectly, the proper credit for its discoverers was garbled en route, and Stone and Tukey mistakenly attributed the first proof to Ulam. Sixty years later, the record was set straight when a copy of Steinhaus's article in *Mathesis Polska* was finally tracked down, and we now know that Steinhaus posed the problem and published the first paper on it, but it was Banach who actually solved it first, using a theorem of Ulam's.

Today Banach is widely recognized as one of the most important and influential mathematicians of the twentieth century, and many fundamental theorems, as well as entire basic fields of mathematics, that are based on his work are now among the most extensively used tools in physics and mathematics.

Ulam went on to work as one of the key scientists on the Manhattan Project in Los Alamos, New Mexico, achieving fame in particular for the Teller–Ulam thermonuclear bomb design and for his invention of Monte Carlo simulation, a ubiquitous tool in economics, physics, mathematics, and many other areas of science, which is used to estimate intractable probabilities by averaging the results of huge numbers of computer simulations of an experiment.

After the war, Steinhaus would have been welcomed with a professorship at almost any university in the world, but he chose to stay in Poland to help rebuild Polish mathematics, especially at the university in Wrocław, which had been destroyed during the war. During those years in hiding, Steinhaus had also been breaking ground on the mathematics of fair division—the study of how to partition and allocate portions of a single heterogeneous commodity, such as a cake or piece of land, among several people with possibly different values. One of Steinhaus's key legacies was his insight to take the common vague concept of "fairness" and put it in a natural and concrete mathematical framework. From there, it could be analyzed logically, and it has now evolved into common and powerful tools. For example, both the website Spliddit, which provides free mathematical solutions to complicated everyday fair division problems from sharing rent to dividing estates, and the eBay auction system, which determines how much you pay—often below your maximum bid—are direct descendants of Steinhaus's insights on how to cut a cake fairly.

These ideas, born of a mathematician living and working clandestinely with little contact with the outside world for long periods of time and undoubtedly facing fair-allocation challenges almost daily, have inspired hundreds of research articles in fields from computer and political science to theoretical mathematics and physics, including many of my own. Steinhaus eventually became the first dean of the department of mathematics in the Technical University of Wrocław. Although I never met him in person, I had the good fortune to be invited to visit that university in December 2000, and it was my privilege to lodge in a special tower suite right above the mathematics department and to give a lecture in the Hugo Steinhaus Center.

Steinhaus had made the last entry in the original *Scottish Book* in 1941, just before he went into hiding with a Polish farm family, using the assumed name and papers of a deceased forest ranger. The *Scottish Book* itself also disappeared then, and when he came out of hiding and was able to rediscover the book, Steinhaus sent a typed version of it in Polish to Ulam at Los Alamos, who translated it into English. Mathematician R. Daniel Mauldin at the University of North Texas, a friend of Ulam, published a more complete version of the *Scottish Book*, including comments and notes by many of the problems' original authors. Their Problem 123, which evolved into the ham sandwich theorem, continues to fascinate and inspire researchers, and Google Scholar shows that eight decades later, several dozen new entries on the topic still appear every few months.

But what about that pesky gerrymandering problem? Negative results in science can also be very valuable; they can illuminate how a certain line of reasoning is doomed to failure and inspire searches in

Does Mathematics Teach How to Think?

Paul J. Campbell

What are the larger benefits of learning mathematics? We are not referring to what is variously termed number sense, numeracy, quantitative literacy, or quantitative reasoning.

In a tradition that goes back to Plato in his *Republic*, educators have maintained that mathematics beyond arithmetic is an essential component of an education: It "trains the mind," by teaching logical thinking and abstraction.

> *Mathematics* . . . *teaches* . . . *how to think* . . . . Reasoning is learned by practice and there is no better practice than mathematics. We have problems that can be solved by reasoning and we can see that our reasoning leads to correct answers. This is an advantage over any other subject . . . .
>
> [Dudley 2008, 2]

Hence, in an earlier era, students studied geometry from Euclid, memorizing and understanding proofs of theorems about idealized geometrical objects.

What habits of mind are ascribed to the learning of mathematics? Some are broadly based: identifying significant information; attention to appropriate detail and exactness; inculcating the discipline of committing to memory definitions, terminology, important facts, and frequently used techniques; following patterns and systematically applying rules; and constructing and evaluating logical arguments.

Others are more specific to mathematics: deduction and argument specifically from precise definitions and principles; discerning patterns from concrete examples; a spirit of generalizing; abstraction to remove less relevant detail; searching for justifying and/or falsifying arguments about conjectures; translation among domains (words, symbols, figures); reasoning with symbols; algorithmic approaches to calculation; investigation of exceptions, boundary situations, and limiting cases; and emphasis on numerical accuracy with suitable approximation.

We examine the claims about mathematics teaching how to think, considering the merits of some alternatives to mathematics, and then ask how mathematics *as taught* can realize those goals.

Alternatives to Mathematics

What about Latin? or Classical Greek? or Even Puzzles?

Claims of training the mind were traditionally made for learning Latin (and Greek) as part of a "classical" education. Some emphasis derived from the practicality of Latin as the medium of scholarly communication in the Middle Ages, and some from the desirability for clergy to read scripture in those languages. Today—in Germany, for example—Latin is still seen in applied terms, as a prerequisite for students aspiring to careers in law or medicine.

The study of any language involves categorization of parts of speech, declension of nouns and conjugation of verbs, memorization of vocabulary and terminology, learning numerous syntax rules and patterns (and their exceptions), and developing precision in expression—not to mention the language's oral component and translation between the oral and written components. Thus, language study involves and develops many of the broad-based habits of mind delineated above. Study of an accompanying culture has other humanistic values, which we do not consider here.

The claims for both mathematics and classical languages partake of the Theory of Formal Discipline (TFD), which asserts that certain fields of study develop general mental faculties, such as observation, attentiveness, discrimination, and reasoning [Aleven 2012], and these faculties have general applicability and transfer to other domains. TFD formalizes the hunches of Plato and similar notions of Locke.

Burger [2019] makes a case for puzzle solving to teach thinking. Part of the rationale is that because solving imaginative puzzles is scarcely a field of study and has no obvious applications, it is easier to focus on the thinking processes and their development.

Why Not Coding?

> Since the purpose of mathematics education is to improve the mind, *it does not matter much what mathematics is taught* . . . . [W]e must teach them something, so we teach them to follow rules to solve problems.
>
> [Dudley 2008, 2–3]

If the subject matter in which one develops and practices habits of mind does not matter, why not Latin? or chess? or coding, the writing of instructions for computers? "Coding for all!" meaning that all students should learn to program computers, has become a meme of contemporary U.S. culture. Why coding? Several rationales present themselves.

Coding as Social Welfare: Jobs in information technology (IT) pay well, hence coding could provide a socioeconomic "escalator" for upward mobility of students from disadvantaged backgrounds.

Coding as Career Insurance: In the wake of the Great Recession, STEM (science, technology, engineering, mathematics) jobs provide the living standard that Americans expect, and most such jobs are computer-related.

Coding as Educational Fad: Coding is the new literacy.

Coding as Competition: Since other countries have adopted "coding for all" (e.g., the United Kingdom in 2014), the United States needs to do so too, in order to be competitive in world markets of labor, technology, and commerce.

There are potential benefits to learning coding. Coding is an attractive opportunity that can enthuse some students. Coding for all fits well with the prevailing American business model for education as primarily job training: "Computer skills," like "math skills," are valued job skills.

Relevant to habits of mind, coding involves basic logic (e.g., Boolean logic and conditionals), demands attention to syntactical detail, and involves planning ahead—which, if not one of the habits of mind above, is a valuable trait in life.

Coding: Drawbacks and Practical Considerations

Do we need so many coders, when the demand for computer programmers is projected to decrease 8% by 2024 [Galvy 2016]? Would a career in coding be meaningful to many students? Can all students succeed at coding, or would it become just another resented obstacle (much as mathematics is now)? How low a standard would any measure of success have to meet? Would achievement gaps widen, with—just as with mathematics—some students considered "born to it" and others deemed hopeless? What other educational opportunities would instruction in coding replace? In most states, "computer science" can replace mathematics or science as a high school graduation requirement [Code.org 2016]; at some colleges, computer programming can satisfy a requirement to study a foreign language [Galvy 2016].

Because of fast obsolescence, learning the specifics of a particular programming language or computing platform today will not be good job training for tomorrow.

Anyway, there are nowhere near enough teachers for implementing coding for all (and there won't be enough as long as teachers' salaries remain far lower than those of coders and others in IT). Where could the needed money come from? (Hint: Not from a school's sports programs.)

Computer science is a liberal art [Jobs 1995] because to write a computer program involves managing complexity, much as the author of a book or the manager of a project must. You don't write a million-line computer program by writing the first line, then the second, and so on; and you don't do it all by yourself. What we emphasize in teaching computer science is not learning the ins and outs of the syntax and semantics of a particular programming language, but rather cultivating the art of solving a problem by breaking it down into manageable chunks that work together. But that's not simple coding.

Students find learning computer programming to be demanding (logic and syntax must be correct); frustrating (the logic can be wrong, and the computer demands perfect syntax); time-consuming (unlike a term paper that is written the night before it is due, a program is rarely "done" on the first try); but potentially fun (thanks to toy robots and easy-to-program graphics and animation).

Computer science is the science of information transfer. Its key question is, what can information-transfer machines do? Programming (coding) is making machines do those tasks. So, what students really need to learn is how to get machines to do what they want—even as the machines change with advances in technology.

What makes for a good programmer—and what should go into coding for all—may indeed be educationally valuable for all: logical