THE BIG BANG OF NUMBERS

How to Build the Universe Using Only Math

MANIL SURI

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THE BIG BANG OF NUMBERS

Introduction

THE POPE MADE ME WRITE THIS BOOK

ON MONDAY, SEPTEMBER 13, 2013, THE *NEW YORK Times* published my opinion piece "How to Fall in Love with Math." I awoke to find my email inbox overflowing with messages, not just from acquaintances but also from a bewildering number of strangers. Many more responses poured in online on the *Times* website—some enthusiastic about math, some scathing, all impassioned—clearly, I'd struck a nerve. By mid-afternoon, the number of posts had reached 360, and the paper closed the comments section. The article quickly climbed to the top of the *Times*'s most-emailed list and remained there for much of the next day.

The aim of my piece was to challenge the popular notion that mathematics is synonymous with *calculation*. Starting with arithmetic and proceeding through algebra and beyond, the message drummed into our heads as students is that we do math to "get the right answer." The drill of multiplication tables, the drudgery of long division, the quadratic formula and its memorization—these are the dreary memories many of us carry around from school as a result.

But what if we liberated ourselves from the stress of finding "the right answer"? What would math look like if delinked from this calculation-driven motivation? What, if anything, would remain of the subject?

The answer is *ideas*. That's what mathematics is truly about, the realm where it really comes alive. Ideas that engage and intrigue us as humans, that help us understand the universe. Ideas about the perfection of numbers, the nature of space and geometry, the spontaneous formation of patterns, the origins of randomness and infinity. The neat thing is that such ideas can be enjoyed without needing any special mathematical knowledge or being a computation whiz.

This is what I'd observed over the past decade and a half, during which, in addition to my day job as a mathematics professor, I'd been pursuing a dual career as a novelist. The juxtaposition put me in frequent contact with artists, writers, composers, journalists, and I was struck by the curiosity they expressed about math. Some had been good at it, but lost contact with the subject once they chose their career path; others had encountered difficulties learning it and viewed it as an unfulfilled intellectual challenge. Often, I was asked to give a talk not on my writing but about mathematics. "Something really exotic," a few would add, their eyes shiny with daring, as if venturing into an Indian restaurant and asking for the menu's hottest curry.

So I began talking about the mysteries of infinity (a topic that's spicy, but not overwhelmingly so), which eventually developed into an animated PowerPoint talk. I'd go to dinner at people's houses and once the plates had been cleared, ask if they'd like to see the show. (You know you've become a math evangelist when you carry such presentations on a flash drive in your wallet.) These activities got headier, more addictive. I started seeing myself as Florence Nightingale, administering math to the mathless; Johnny Appleseed, scattering math seeds like fairy dust everywhere I went. Some of my targets may have regarded me more like the Ancient Mariner and themselves as the cornered wedding guest. A few had to be rescued by their parents.

My novels (on India, not math) were doing well, so I was able to infiltrate even more venues where nonmathematicians congregated. My coolest coup was at the 2006 Berlin International Literature Festival, where a class of eleventh graders who thought they'd hear me speak about my second novel got my infinity talk instead. They seemed to like it—or at least sat through attentively, without fidgeting (the fact they were German may have had something to do with this).

By 2013, I'd begun to get a sense of the limits of such outreach efforts. So when my *New York Times* op-ed took off, I wondered if I'd finally hit it big. Nothing I'd ever written had ever gone "viral"—in fact, I wasn't even sure what numbers earned that characterization. By Thursday, my piece had climbed into the weekly top-ten list; by Friday, it had inched up a few slots more. Over the weekend, I watched obsessively as it crept into the top three, then nudged its way into second place. What I barely noticed was that the pope had chosen that very week to make some startlingly progressive statements about gays, abortion, and birth control. Just as I was about to claim my rightful pinnacle of victory, he appeared behind me from nowhere, bounding up the list in twos and threes. Quads flexing, cassock billowing, he made one final spectacular jump, to leapfrog clear over me and land in my number one spot.

Now, you may wonder if I developed a lingering grudge against the pope, if I've written this book to vindicate myself in an imagined *mano a mano* with him. Let me assure you that's not the case. I've completely forgiven him and will even be mailing him an autographed copy of the finished book at the Vatican to show no hard feelings remain.

However, his surprise appearance did have a crucial effect: it focused my attention on religion. In the popularity contest with mathematics, religion had handily won—as it almost surely would each time. What did it offer that math didn't? What lesson could one take away for math to draw people in, to compete in the attention economy we live in?

There's no shortage of answers to this question, but I was reminded of a quote I'd seen years earlier that had cut me to the quick. It was attributed to Rob Fixmer, a former editor of the *New York Times*, who was attempting to explain why math got so little media attention:

> Mathematics has no emotional impact. What physicists do challenges people's notions of origins and creations. Math doesn't challenge any fundamental beliefs or what it means to be human.

My immediate reaction was indignation—how could anyone malign something I loved this way? In time, I realized this might be an opinion many shared. Also, though the quote had compared math with physics, the same could be said while comparing math to religion. After all, both physics and religion seek to address the Big Questions—albeit from opposing perspectives: Where does everything come from? Why is the universe the way it is? How do we fit in? The two camps have been duking it out over the answers for centuries, begetting even more attention for themselves.

Math doesn't seem to have a dog to enter in this fight. The subject is abstract, agnostic—ready to describe and analyze phenomena, without having a position of its own to stake. That's the image perceived by most, anyway. Without such blockbuster spectacles as Genesis or the big bang, no wonder math has difficulty competing in the engagement sweepstakes.

But I'm here to tell you this picture of math is inaccurate. Math *does* have a compelling "origins" story, one that creates its basic building blocks out of nothing. With a little inventiveness, this narrative can be extended to show how with these building blocks—called *numbers*—the entire universe could plausibly be constructed. Big Questions do indeed get addressed along the way, with answers that come not from God or science but mathematics.

As I watched my article begin its descent on the *NYT* list, I realized that, as follow-up, I should write precisely the above kind of narrative. One that would flesh out my article's central assertion about math being more about ideas than calculation (which meant I'd need to severely limit formulas and equations!). Something that would not only convey the aesthetic pleasure of the subject but also reveal the deeper connections we—and our cosmos—have with it.

The thought experiment

The book you (and, God willing, the pope) are about to read emerged from that day's realization. Its premise is this. I'm going to put you, the reader, in the driver's seat and have you take on the task of creating the universe using only numbers, and the mathematics you formulate from them. We'll launch this adventure in the next chapter with the above-mentioned "origins" blast—math's very own creation spectacle!

Sitting at the controls of this thought experiment, you'll soon find yourself devising arithmetic, then geometry, then algebra, then physics—

all in response to the needs of your universe-in-progress. (This will incidentally answer the question "Why does algebra exist in the world?" asked by untold legions of unhappy schoolchildren.) The perspective you'll get is unusual, even radical: math as the life force of the universe, a top-down driving power that fashions everything that exists. This turns on its head the traditional way mathematics is understood. Rather than regarding it as something we devise to explain preexisting real-life phenomena (given to us by God or physics), we will view mathematics as the fundamental source of creation, with reality trying to follow its dictates as best it can.

Such a view is actually not new—it has precedents traceable all the way back to the ancient Greeks, particularly Plato. What differentiates us from Plato is that we don't assume all of mathematics already exists in some idealized form somewhere, waiting to be discovered, as he did. Rather, we *invent* math—from scratch, and through active, energetic exploration. Math that will *create* the universe, rather than *explain* something already in place.

There are several advantages to this reverse, hands-on approach. For one, it will enable you to get a firsthand taste of the playful nature of mathematics. This is something mathematicians often rhapsodize about but outsiders can find hard to access. Sitting in the driver's seat, you'll see how even simple arithmetic operations like addition and multiplication are, at heart, games. You'll be able to experiment with such games and ideas creatively, as if playing with an abstract set of toy building blocks or Lego bricks. Each time the inevitable question—"What good is this, anyway?"—comes up, the answer will be right there. After all, the components of the universe will, quite literally, be arising from your play!

Contrast this with the alternative, of starting with real-life phenomena and demonstrating how math can be used to approximately model them. Such efforts (as I've noticed in my own outreach) can come across as a "good for you" vitamin, with playfulness often smothered under the weight of technical elaboration. Using play and exploration to bring out the usefulness of math, as we do here, makes the connection feel more natural, effortless. The fact that we can embed math in a single continuous narrative helps show how its different areas are linked.

Another advantage of our approach is that it allows us a fresh look at the "unreasonable effectiveness of mathematics" in describing the universe (as Nobel laureate Eugene Wigner put it). This is a riddle that's central to the subject—how can something so abstract be so uncannily adept at explaining the reality we live in? Clearly, if we demonstrate that the mathematics in our thought experiment leads inevitably to the creation of *everything* in our universe (and only *our* universe, rather than some different one!) then we've come a long way toward changing "unreasonable" to "*very* reasonable" effectiveness.

Let me be up-front: our thought experiment won't quite achieve this kind of slam dunk. Trying to build everything just with math is, to put it mildly, a tad ambitious. However, proceeding step by step, we'll find out what other ingredients might be minimally needed, while getting to appreciate just how deeply numbers are hardwired into our experience. In particular, our flipped perspective will lead us to an insightful new interpretation of how Nature fits in. We'll view her as a building contractor, who has the task of turning our design for the universe into tangible reality. By casting her in this role, we'll be able to better understand why she doesn't follow our mathematical instructions to the letter, what role randomness plays.

Our explorations will also reveal that the universe we live in isn't the only one possible. That's because such basics commonly taken for granted—like size, distance, space—arise innately from mathematics. Consequently, you can make them strikingly different in any universe you create by defining them in alternative mathematical ways. For instance, we'll use the very physical art of crocheting to figure out the different fabrics possible for the universe's geometry. (It turns out that we don't quite know which is the correct one even for our own universe!) Expect to have your "fundamental beliefs" challenged as we explore such variations, thereby checking off one of the boxes in Fixmer's quote.

Another critical question raised in the quote was whether math has "emotional impact." It's true that mathematics, much more than art or music, is experienced more intellectually than viscerally. However, comprehension is often followed by a eureka moment, which is part of the emotional punch math packs. That's what our thought experiment is set up to deliver—whether through games that suddenly open up to reveal a deeper truth, or through "eye candy" fractals that transform into essential drivers of the universe when you engage with the math behind them. Look for such experiences particularly in the section on patterns, where we attempt to make the Mona Lisa prettier through a mathematical makeover.

We've adopted our top-down version of mathematics because it works so well in our exposition. But could this reflect reality? Could math truly be what guides our universe? We debate both sides of this ahead, but the section on physics contains evidence supporting our viewpoint. You'll see how mathematics actually *determines* physical principles like Newton's law of gravitation (and even a very basic idea used in general relativity), rather than just being a language used to explain them.

The penultimate section, on infinity, transports our thought experiment to a more expansive, philosophical realm. While previous sections bring out how mathematics informs such essential qualities as randomness, symmetry, and beauty, we now explore what it tells us about even deeper questions. Does knowledge have boundaries? Is omniscience possible? Can one list successive instants of time? The concept of infinity is essential for delving into such issues. Even though it's something that can only be envisaged, never physically attained, infinity is inextricably tied up with setting down the universe's blueprints; much of the reality around us involves the push and pull of the finite versus the infinite. Be on the lookout for a fictional interplanetary battle through which we'll steer our thought experiment to unlock infinity's secrets.

In the final section, we get down to the task of physically building our universe from the designs we've created. Iteration, where one state in time evolves to another, is the key mathematical process that facilitates construction. We explore how elementary rules of iteration can lead to complex, self-arising phenomena—fractal patterns, stripes on a zebra, "herd intelligence" in ants. Such *emergence* (as the spontaneous generation of complexity is called) could plausibly create life itself.

The issue of creation, of course, is also in the domain of religious philosophy, whose relationship to science we comment on at several points in the text. One of the most essential questions both ask is this: Why do we exist? Is it a result of randomness or intent?

This is the final enigma we address. Mathematics will reveal the answer at the end of our thought experiment.

Ground rules

The most stringent test of our ideas would be if we scrubbed our minds clean of all knowledge of both math and our universe, and started from a blank slate. We're going to do nothing of the sort. Which means that, consciously or unconsciously, such knowledge might seep in to inform everything we create. Will this damage the integrity of our experiment?

It won't. Many mathematical concepts can indeed be deduced quite naturally from the universe's components (e.g., a sphere from the universe's round objects, the Golden Ratio from sunflower seeds and pine cones). Although we'll often point out such links to familiarize you with a new concept, our charge will be to also find an *independent* path to develop the concept, one that ignores connections with the physical universe, and uses only the math we've created. In other words, our first guideline will be to always strive to maintain our top-down philosophy, by composing a narrative in which math drives the universe, rather than the universe providing us the math.

As far as prior knowledge of math goes, our second rule will be that we can't use a mathematical concept before we've developed it. For instance, we can't arrange numbers along a straight line if we haven't created lines yet. This will ensure we're actually constructing all the math we need as we go along.

Since we're following the math, we can expect (as mentioned earlier) for it to lead us to several alternative universes. We'll look into some of these variations, since they can provide interesting insights into our own reality. However, when a choice arises on how to proceed with our thought experiment, our guideline will be to pick the path most likely to lead us back to our own universe, since that's the one we want to build.

Finally, let's note that thought experiments have a particularly rich tradition in physics and philosophy, where they typically explore rather focused questions or concepts ("What does quantum superposition have to say about the health of Schrödinger's cat?"). We can afford to be more relaxed, since our thought experiment is just the vehicle—the true purpose of our journey is the understanding and appreciation of math. Consequently, we'll take some liberties—step outside our experiment at will, get Nature and the numbers to roam around as characters, have the pope materialize from time to time to offer commentary, and let a Greek chorus of physicists do the same.

How to read this book

As you've no doubt surmised, I'm aiming this book not only at novices but also at math enthusiasts. This includes inquiring high schoolers, teachers and parents, seasoned STEM readers, all the way up to hardened math professionals. If you've ever tried to get anyone excited about math, I promise you'll find new strategies within these pages.

However, a word of warning—some people's experiences with mathematics may have convinced them they can never find its ideas engaging. ("So—good for you, you're moved by the pure beauty of math. Please, take a cookie and congratulate yourself" is how one reader of my *NYT* article put it. Others were less generous.) We can all be resistant to the charms of different endeavors; my biggest blind spot—a problem, since I grew up in India—happens to be cricket (unlike everyone around me, I was always loath to watch it). So, try keeping an open mind as you embark on this book, with the perspective that you *can* enjoy math under the right circumstances. (I'll meanwhile think about giving another shot to cricket.)

Next, for novices, relax. Some of you might find it liberating to read about mathematical ideas for the first time without the nagging worry: Is this going to be on the test? Experience this math appreciation at your own pace—you can always come back to topics that you decide to just skim. Remember, even though the formulas I've used are both elementary and minimal in number, the ideas expressed can run rather deep.

Do read the footnotes, since they're interesting (that's why I added them!). Endnotes, which contain references, technical qualifications, and proofs, but also some engaging elaborations, can sometimes be more mathematically challenging to read. I'll alert you only to the most essential ones as we go along. That said, if you're a mathematician and feel something's missing, chances are it's addressed in one of the unflagged endnotes, so do check them.

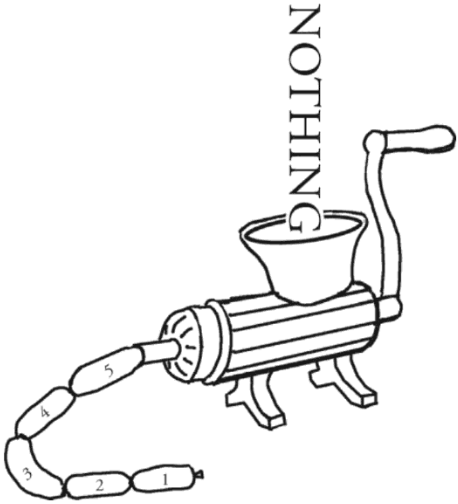
You'll notice I've organized this book in "days," as in Genesis. It's a metaphor I couldn't resist, once I realized our mathematical creation would have exactly seven stages. Although these stages don't correspond to what God creates in the Bible (light on Day 1, sky on Day 2, etc.), both progressions suggest how naturally each creation might lead to the next—in our case, arithmetic giving rise to geometry, geometry to algebra, and so on. Math becomes more of a story, not just like Genesis, but also like the narrative of how everything gets built up from the big bang in physics. And who doesn't love a story, as they say.

This brings me to my final note. Both religion and science might have, for each reader, their own place. Despite any apparent irreverence, my aim isn't to supplant or disrespect either. Rather, it's to take you on a thought experiment that introduces another worldview, one fashioned around mathematics. It's an offering that even the pope, my most treasured potential reader, shouldn't have a problem with.*

* This is a good spot to make clear that Pope Francis (who's been the pope while I've been writing this book) has not participated in this work in any way beyond his chance entanglement with me on the *NYT* "most emailed" list. The comments, actions, and thoughts I attribute to him ahead are fictional, though details about his educational background are correct.

Day 1

ARITHMETIC



Creating numbers out of nothing

1.

SETTING UP THE BIG BANG

THERE'S A HICCUP WITH THE VERY FIRST STEP OF creating our building blocks, the numbers. Why waste time on "creating" something so obvious? Can't we just assume the numbers exist, and take it from there? After all, even the God of Genesis comes equipped with numbers—witness the way the six days of creation are counted off, with a rest on Day 7. And surely physics also assumes numbers' existence—otherwise one couldn't have the precise values of physical constants needed for the universe to work. Are mathematicians so arrogant that they're going to nitpick with both science and God? Surely this is an exercise as unnecessary as—to put it in my terms—watching cricket?

Fine. I was hoping to give you some insight into how mathematicians operate, but we can skip this step. Provided you just answer one question. What is a number?

That's easy. "One, two, three . . . ," you start, but these are just examples. "A value used to count" or "A symbol used to quantify" are only descriptions—not particularly satisfying ones at that, since they use other undefined terms like "count" and "quantify." Notice how hard it is to actually pin down the definition of a number. It's my second-favorite question to quell an uppity math class. (My favorite is "Define math.")

The problem is we conflate the identity of numbers too much with their function, associating them only with counting and measurement. Perhaps this is a legacy from our early ancestors, who, after all, were motivated by purely practical needs when they embarked on their arduous quest for numbers. Asking them to characterize their invention might have elicited different replies through the ages: notches on a wolf bone, knots on a cord, hieroglyphics and pictographs, all the way to the Indo-Arabic symbols used now. They'd have dismissed us as precious (or worse, demented) had we persisted in asking them to define the essence of a number.

Plato would be an exception. He believed numbers to be abstract entities, living in their own idealized universe—a universe containing galaxies of perfect concepts that exist independently of the material realm. The most humans could hope for was to get occasional glimpses of this cosmos, coming away with discoveries they could use in their own world.

Like Plato, we need to understand the nitty-gritty of numbers. We need to understand the material we're building our universe from.* Unlike him, we won't take the easy way out by assuming numbers are givens, in existence since time immemorial. Rather, we'll *construct* them ourselves, much like an artisanal builder who personally bakes every brick for a house. That's the scout's motto for mathematicians: be prepared to build everything from scratch. As promised, there *will* be a blast!

So imagine you're at a starting point that predates the universe—*all* universes, even Plato's. God has not yet commenced building and neither has physics—in fact, both are waiting somewhat testily for you to give them the numbers first. Given their penchant for sudden eruptions, you'd better get to work.

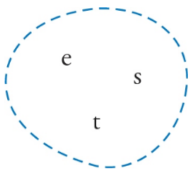
Except you literally have *nothing* to work with. No energy, no matter, no symbols, no objects to count. You can't even imagine yourself suspended alone in empty space, because space hasn't been invented yet. Even the *concept* of space, along with that of length or area or dimension, has yet to be formulated. Time is hazy as well—perhaps it's waiting to be initiated. You've never felt so isolated, never seen a "nothing" so absolute, so stark. A "nothing" you realize is your only resource—the commodity you'll have to construct your numbers from.

Now, King Lear may have warned that nothing can beget only nothing, but both religion and physics have been enchanted by the opposite, utopian, notion. *Creatio ex nihilo* is the ultimate magic trick, whereby oceans, skies, planets, the universe, life itself, are all created out of nothing (*nihilo*). Either with the help of a divine entity, or unaided, through a spacetime "singularity" (as physicists might put it). Every religion has its own variant—if the universe isn't being blown out by Brahma in a single breath, then it's rising from the infinite sea of chaos posited by the ancient Egyptians.

The reason I call *creatio ex nihilo* a magic trick is that the "nothing" you start with always needs an asterisk (a singularity for the big bang, a Supreme Creator for Genesis). Some of these asterisks can be quite pronounced—making the *nihilo* part essentially a con. What's remarkable about the version you're about to use for the numbers is that the asterisk needed is so trifling, so close to nothing itself. It's called the *empty set*. Let's start by understanding what this is.

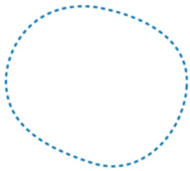
The empty set

By *set* we'll simply mean a collection of objects. For instance, we could pictorially denote the set of letters in the word "set" as shown. The nature of the boundary or arrangement of the objects inside doesn't matter, just the fact that they belong.



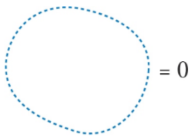
You can think of all sorts of sets: the set of digits greater than 2 but less than 8, the set of cats in your house, the set of trees in your yard.

What happens, though, if you don't have any cats or trees? That's when you get the *empty* set—conveniently visualized as a boundary with nothing enclosed.



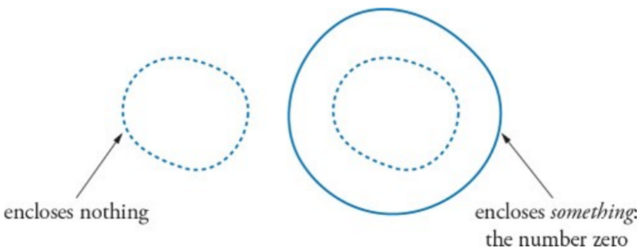
As other examples, consider the set of negative numbers greater than 2, the set of six-headed cats, or (gratuitous swipe) the set of truthful politicians.

Now here's where *creatio ex nihilo* kicks in: numbers can be generated from the empty set. To do so in a stand-alone, mathematically useful way, we need to transcend our cultural training. Think of numbers as sets—in fact, begin by defining the "number" zero as the empty set.

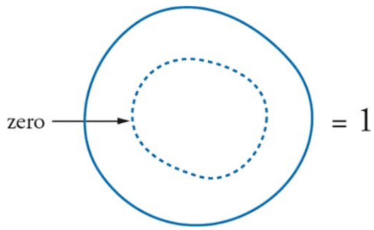


This might feel strange, but it fills a gap—it pins down zero precisely, invests it with an independent mathematical identity. In fact, it's essentially how the Hindus invented zero—they gazed into the void, a mainstay of Hindu and Buddhist philosophy, and realized it could be represented as a number.

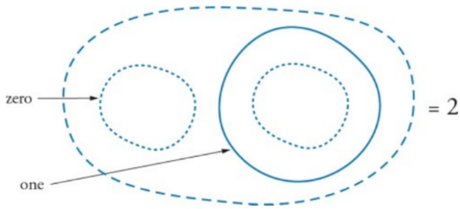
Zero happened to be the final digit the Hindus needed, since they already had the others from their counting experience. For us, the process is reversed—we need to start at zero and build from the bottom up. So how do we create the number *one*?



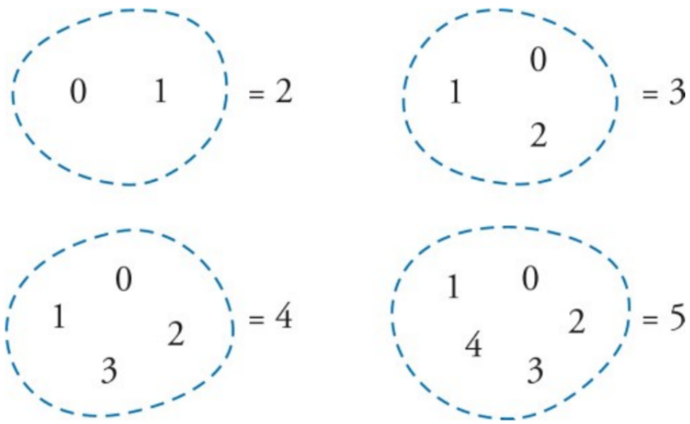
To do this, imagine a new solid boundary around the dotted boundary that represents the empty set. As before, the dotted boundary encloses nothing. But notice that the solid boundary *does* enclose something: the "number" zero we've just created. This solid boundary therefore represents a new set—one that is no longer empty. Define the number *one* to be this new set, that is, one is the set which contains the previous number zero.



Continue this way. Having figured out how to get from zero to one, we use the same procedure to get from one to its successor. All we have to do is visualize a new set, containing both zero and one, and call this new number *two*.



From two to *three*, three to *four*—we can simply make new sets each time, containing all the numbers already created. It's like igniting the fuse to a chain reaction—one that gives successors over and over again (try creating the next few in the process illustrated below).



You might object that these creations have little resemblance to the numbers we use in real life for counting. However, notice that "two" is a set containing exactly *two* elements—the numbers zero and one. "Three" is a set containing exactly *three* elements—zero, one, two. A little like a die's face value being indicated by the number of dots on it.

What's the point of all this? For one, it gives you a glimpse into how mathematicians think, building up the subject brick by brick with such elemental definitions (just as physicists might with subatomic particles). More importantly, these "defining sets" give you something to latch on to when thinking of numbers. They're a demonstration that numbers exist as their own independent entities, just as Plato asserted.

So let's birth them, make them the first inhabitants of the universe you create. Let's have that kickoff explosion I proposed (remember the chain-reaction fuse that was lit?). Imagine yourself adrift in nothingness, corralling it somehow into an empty set, astounded at the newborn Zero you've just created. Tinkering around some more, delivering One and Two and Three, still uncertain about this process you've stumbled upon. The Big Bang of Numbers catches you by surprise—a spontaneous blast, like something a youngster experimenting with a chemistry set might ignite. The numbers burst forth, each one generating the next. Minuscule stars, fiery from your act of creation—they streak through your lonely universe, filling it with light and companionship.

* Plato's building blocks were not numbers but geometric solids, identified with the basic elements of the universe (the tetrahedron was fire; the cube was earth; the octahedron, air; the icosahedron, water; the dodecahedron, the universe as a whole). He expounded on this in *Timaeus*, where he also aimed to explain the formation of the universe, as we do here. However, he did this based on philosophical and theological principles, rather than mathematical ones, such as we're using. Interestingly, Plato's choice of the cube as the earth element has been shown to have some truth to it. The most natural way rocks fracture when broken up is indeed, on average, into six-sided cube-like chunks.

2.

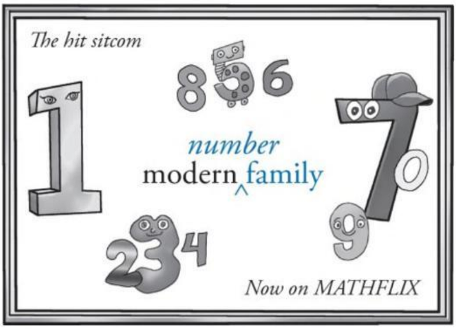
GAMES NUMBERS PLAY

YOU'RE DOING RATHER WELL FOR SOMEONE WHO'S just delivered a litter that's countless. Even God, if given credit for every being that's ever existed, may have had only a finite number of offspring. To top it off, your feat of conception was clearly immaculate. As creator, you should be pretty pleased with yourself.

But there's a problem. Your numbers are too abstract, and born into a bit of a nebulous jumble. Wouldn't it be nice if they had personalities to make them more real—even to set them apart, like ordinary children? Hold this thought as we hear from a student of mine who might be of help.

Numbers are people too, my friend

Lili has something unusual to share in a math seminar I'm teaching. Numbers always appear pigmented in their own unique colors for her: one is orange, two purple, three pink, and so on. The technical term for her condition is synesthesia, she informs us, and it can take many forms. Instead of seeing colors when viewing letters or numbers, some might associate fragrances with sounds, others might even be able to "taste" words. For her, the association is so strong that coloring books traumatized her as a child—each time a region labeled one wasn't orange or two wasn't purple, she'd cringe at the unnaturalness of it all. I feel instantly guilty about my color choices chalking numbers in class. Apologies, Lili, for making you squirm—much as teachers enjoy occasionally torturing their students, this wasn't on purpose.



Lili tells us of an extra dimension to her synesthesia: numbers also come with their own genders and personalities. For instance, One is a hard-working single mom, Two is her shy son who might be gay, Three is a sweet and nurturing elder sister, while Seven is a cool dude—the kind with a backward baseball cap, sunglasses, and a skateboard. It can get as hectic as a television sitcom, Lili says.*

Clearly, numbers are alive for Lili in a way they're not for most of us. Could you take a cue from her, to better see numbers as entities in themselves? Maybe even ascribe personas, consistent with the roles they'll play in the mathematical universe to come?

Start with Zero, who we'll make, quite arbitrarily, a "she." (See the endnote on genders and pronouns I've used.) Not only is Zero the originator of an entire tribe, but she's a thinker, a *yogini*, forever pondering the deepest mysteries of numberdom. How does she differ from *nothing*, for instance? What exactly is this entity called a set? Are you truly her creator, and if so, what dispensation allowed you to harness emptiness? *Creatio ex nihilo*, as I cautioned, always comes with an asterisk, and your firstborn is relentless in trying to zero in (so to speak) on the trick involved. Her motives, though, are pristine—she's driven by introspection, not mischief.†

In contrast with Zero's meditative nature, One—or Uno, as he prefers to be called—is practical and down-to-earth, a real doer. Were numbers human, he'd be the kid you'd shoot hoops or (horrors) play cricket with. He takes charge of the rest of the brood, tinkers with both his own and others' defining sets. Ever on the lookout for problems to solve using calculation, One—sorry, Uno—is always eager to forge ahead with math.

Two has little of his older brother's brashness. He's placid, stable, good at forging compromises, a cherisher of noble qualities like symmetry and evenness. He does chafe a tiny bit, however, at always being in the shadow of Uno's accomplishments.

Such anthropomorphizing is fun, and we'll keep it handy to help us understand some key ideas ahead. Viewing numbers as denizens of your universe will breathe new life into addition, multiplication, and other such operations. That said, you might already see that assigning *every* number its own personality isn't going to work. The tribe is too vast—even Lili's synesthesia stops at ten.

Fortunately, we don't need personalities to set these denizens apart, since each already comes with a prominent identifying feature—its number (of course!). In fact, this feature is important in terms of not only mathematics but also language. Let's take a short detour to see how it gives rise to some basic semantics.

Numbers and comparison

"More," "less," "big," "small"—such terms have no meaning in your universe yet, since size and counting haven't been invented. Numbers so far are just defining sets, floating around like bits of mathematical protoplasm. It's only after you attach appropriate context to them that you'll be able to use them to make comparisons, and ascribe meaning to such terms.

How to attach such context? Recall the numbers emerged in a definite *order* from their Big Bang. So, call a number "less than" or "greater than" another depending on who came first in this progression. Three is less than Five, whom it preceded, but greater than One, whom it succeeded; Six is greater than Four but less than Nine, and so on. Once you get adept at this, you can even start developing a subjective sense of calling a number "big" or "small," based on how many other numbers lie between it and Zero. (Note, though, that numbers you call "big" might become "small" when you compare them with even bigger numbers, so these adjectives are relative.)

Long and short, near and far, rich and poor—numbers will help you define all these concepts, as you proceed with your universe. All derive from the sequence in which numbers succeeded each other during birth.

The ability to compare, however, is a double-edged sword. Once you possess it, the idea of inequality is born. This is a notion that will mesmerize humans (once they appear) in droves. The desire to best will become one of the driving obsessions of civilization, unleashing monstrous inequities in your world. Inequities that could arguably be traced back to and blamed on numbers.

So much so that perhaps the story of Adam and Eve should be rewritten. Perhaps it was the forbidden tree of numeracy, not knowledge, that got them banished from Eden. Perhaps the serpent's offering was a zero, not an apple.

Addition and multiplication

But we're getting way ahead in your creation process. So let's return to the *naturals*—the name ‡ you give the numbers 0, 1, 2, . . . Now that you've produced them, what next? You're probably eager to deploy them in some spectacular feat of creation that will astound both physicists and the pope.

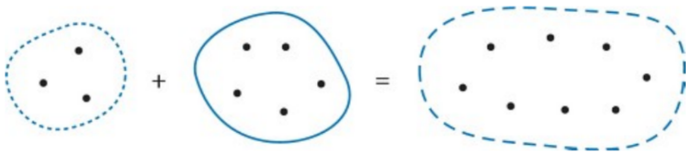
Unfortunately, that's not how math's going to work. At least not in this early stage, when your numbers can't be applied to anything—for the simple reason that nothing else exists! Instead, let's take a page from the typical view mathematicians have of their subject: that it's really a form of play. One invents a bunch of rules as one might for tic-tac-toe or sudoku or Scrabble, and sees where they lead, what deductions can be made from them. The English mathematician G. H. Hardy said it most famously: not only did he call mathematics a game but he also declared being enamored most of all with the game's uselessness!

Except it doesn't necessarily stay that way. Over time, even the most whimsical game might develop practical uses. Hardy himself has been spinning around at quite a clip in his grave for decades, since the parts of his own research he treasured most for their uselessness turned out to be the most flagrantly applicable (e.g., in cryptography).

You're going to have to hope for something similar. For now, give up the vision of erecting the universe's grand monuments. Instead, invent some games to play with the naturals and cross your fingers something comes of it.

This is where giving the numbers life pays its dividend. Imagine them dart and race around as they might for Lili, my synesthetic student. In your unformed universe, though, there's literally nowhere to go, nothing to see. The numbers quickly get bored with such limited opportunities for play. Some of the more rambunctious ones start roughhousing among themselves.

You're forced to intervene to restore order. As you're about to pull apart Three and Five, you pause. Why do they remind you of Eight, grappling with each other like that? You realize it's because if you count up Three's defining set, and keep going with Five's, you end up with exactly the same number of elements as in Eight's defining set. Congratulations! You've just discovered the "game" of *addition*.

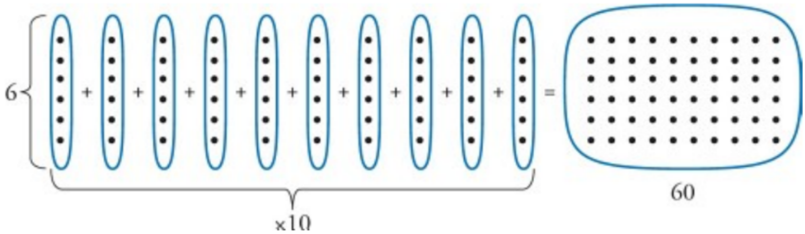


You start entertaining yourself (and your numbers!) with this game. Input two natural numbers (Zero, One, Two, etc.) as players and you get another natural number as the outcome. You discover some interesting facts. First, whether you take the "sum" Three + Five or Five + Three, the answer, Eight, remains the same. (Picture what the above sketch would look like for Five + Three instead of Three + Five.) More strikingly, you can play with *any* pair of numbers—the game always culminates successfully, in the sense that there's always a number waiting in your set of naturals to assume the role of the sum.

You soon cook up another game. This one's played by adding together copies of the *same* number. You know One plus One gives Two, Four + Four + Four gives Twelve, Five + Five + Five + Five gives Twenty. It becomes tedious to list long sums involving the same number, so why not abbreviate such strings? Six + Six + Six . . . , repeated ten times, gives Sixty, which you shorten to "Six times Ten equals Sixty." And then you simply start writing this as

Six × Ten = Sixty.

You call this *multiplication*—a game that again depends on pairs of numbers as input.



In time, you discover tricks for calculating "products," and invent multiplication tables. You're especially entertained by watching small numbers multiply each other to vault toward bigger targets. The games bring you closer to your numbers, help you understand them better. Even if you don't see any immediate use for such play.

Representing numbers

It turns out the multiplication game *does* end up having a practical use: it yields a convenient way to represent numbers. So far, you've just been calling them by their names, "Zero," "Five," "Twenty," and so on. You've also come up with symbols to represent the first few of them—the usual digits 0 to 9, say. What you'd like is to design a separate such symbol for each of your building blocks, perhaps □ for Ten, Δ for Eleven, ☺ for Twelve, and so on. But this seems impossible, since the numbers are endless. Not only would you run out of ideas, but nobody could remember such a limitless lineup of symbols.

Multiplication comes to the rescue. You notice that Twenty-one equals Two times Ten plus One, while Thirty-six equals Three times Ten plus Six.

Writing these observations out, you see a natural method of representation emerge:

$$\text{Twenty-one} = 2 \times \text{Ten} + 1 = 21,$$

$$\text{Thirty-six} = 3 \times \text{Ten} + 6 = 36.$$

In other words, the multiplication by Ten and the subsequent addition of units can be treated as implicit steps left unstated. With this convention, you need only the ten symbols for Zero through Nine to be able to represent *any* number. For instance, Fifty-four, which is 5 × Ten + 4, can be represented as 54, and Seventy-three, which is 7 × Ten + 3, as 73. Interpret your own age this way, and it reveals how many complete decades—plus extra years over—you've lived.

If you suspect the choice of using multiples of Ten as your base is arbitrary, you're correct. For a while, you try experimenting with using other numbers as base as well.§ But for some reason, Ten seems just right to use as base—not too big and not too small.¶ Of course, humans started using this base because they had ten fingers. But in this thought experiment, where humans don't exist as yet, we'll refrain from justifying our choice by appealing to this fact.#

What about the symbol for Ten itself? Using the convention you've developed, you write

$$\text{Ten} = 1 \times \text{Ten} + 0,$$

so Ten = 10. Also,

$$\text{Hundred} = 10 \times \text{Ten} + 0,$$

so you represent it as 100.

The perfection of the naturals

Let's take a moment to savor the aesthetics of what you've created. This, after all, is one of our goals—to get a feel for the so-called elegance of math. Hard as it might be to believe, there's beauty to be gleaned even from something you've plugged through thousands of times before—operations as humdrum as addition, as mundane as multiplication.

Consider how your naturals are both complete and self-contained. No matter how you add or multiply them, the outcome is always another natural just like them. Players are never in danger of crossing a boundary and stepping off the universe by virtue of their sum or product getting too big (something that would happen had there been a largest natural). There are also no unfilled holes that could pose a hazard during play (as might have happened had a number like 10 or 20 been somehow skipped in the creation process). Every possible number that could ever be an outcome is included.

I want to impress on you how special such a situation is in our world. For instance, chemical elements are the building blocks of matter, but not every pair of chemicals can be made to combine (e.g., sodium reacts with chlorine, but it won't with potassium). Or consider Scrabble. Try placing a random string of letters on the board, and you're likely to encounter a "hole"—a word not in the dictionary, and hence inadmissible.

Your naturals, on the other hand, appear perfect. They seem to have no omissions or limits. The games you've invented proceed without a hitch on their seamless playing field. The system runs as smoothly as well-oiled machinery. It therefore comes as a shock when you see this perfection blown away.

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* Interestingly, she's not alone in experiencing numbers with personalities—there's even been a play called *Numesthesia* written about her kind of synesthesia.

† Zero's questions are all valid issues that come up for anyone laying the logical foundations of mathematics. In most cases, one has to address them through technical assumptions or "axioms"—for example, the axiom "The empty set exists." We'll take a closer look at axioms when we explore geometry on Day 2. For now, let us mention that unexpected problems can arise if one defines sets "naively" without axioms, as we have done (see endnote).

‡ The name "naturals" can be traced back to the French mathematician Nicolas Chuquet, who referred to the "natural progression" of numbers 1, 2, 3, . . . in a 1484 treatise. Some omit 0 from the list, but as we've seen, everything follows from 0, so it's *natural* to include it. (Those who omit 0 from the naturals sometimes characterize the numbers 0, 1, 2, . . . as constituting a new set called the "wholes.")

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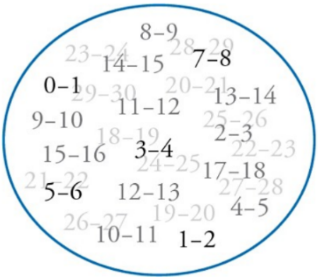
around with the defining sets for Zero and One, but there are no obvious modifications that would yield a set for Minus One.

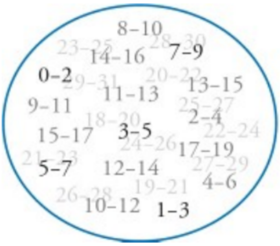
Meanwhile, you realize this same Minus One might also work as the answer to another stuck subtraction: 1 – 2. In fact, you could use it to liberate all such locked pairs involving two successive naturals: 2 – 3, 3 – 4, and so on. You start gathering together such pairs, a painstaking task. There's a mess of unconsummated subtractions from which you have to tweeze out just the ones you want. Once you finish, you herd them all into a set you label −1.



= −1

You're contemplating your handiwork when you have a sudden thought. Could this be the defining set you were after? The more you think of it, the more you're convinced it's true. It may look nothing like previous specimens, but you've just formulated the defining set for Minus One. It consists of every subtraction where the second natural number is one more than the first.

This recipe is easy to extend. You corral together all the differences 0 – 2, 1 – 3, 2 – 4, . . . , where the second number is *two* more than the first. This set, you name the negative number −2. Proceeding this way, you similarly create −3, −4, −5, and so on.



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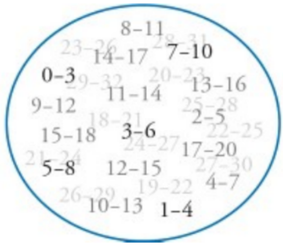

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