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THE GLORIOUS GOLDEN RATIO

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Prometheus Books
59 John Glenn Drive
Amherst, New York 14228-2119

Published 2012 by Prometheus Books

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Amherst, New York 14228–2119
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WWW.PROMETHEUSBOOKS.COM

16 15 14 13 12 5 4 3 2 1

Library of Congress Cataloging-in-Publication Data

Posamentier, Alfred S.
The glorious golden ratio / by Alfred S. Posamentier and Ingmar Lehmann.
p. cm.
ISBN 978–1–61614–423–4 (cloth : alk. paper)
ISBN 978–1–61614–424–1 (ebook)
1. Golden Section. 2. Lehmann, Ingmar II. Title.

QA466.P67 2011
516.2'04--dc22

2011004920

Printed in the United States of America on acid-free paper

Contents

Acknowledgments

The authors wish to extend sincere thanks for proofreading and useful suggestions to Dr. Michael Engber, professor emeritus of mathematics at the City College of the City University of New York; Dr. Manfred Kronfeller, professor of mathematics at Vienna University of Technology, Austria; Dr. Bernd Thaller, professor of mathematics at Karl Franzens University–Graz, Austria; and Dr. Sigrid Thaller, professor of mathematics at Karl Franzens University–Graz, Austria. We are very grateful to Heino Hellwig of the Humboldt University–Berlin for contributing the chapter covering biology and for suggestions offered beyond that chapter. With gratitude we thank Dr. Ana Lucia Braz Dias, professor of mathematics at Central Michigan University, for contributing the chapter on fractals. And to Peter Poole our thanks for some wise suggestions throughout the book. We wish to thank Linda Greenspan Regan for her editorial assistance, Peggy Deemer for her extraordinary expertise in technical editing, and Jade Zora Ballard for putting the book into final shape.

Introduction

Few mathematical concepts, if any, have an impact on as many aspects of our visual and intellectual lives as the golden ratio. In the simplest form, the golden ratio refers to the division of a given line segment into a unique ratio that gives us an aesthetically pleasing proportion. This proportion is formed in the following way: The longer segment (*L*) is to the shorter segment (*S*) as the entire original segment (*L*+*S*) is to the longer segment. Symbolically, this is written as $\frac{L}{S}=\frac{L+S}{L}$.

Let us consider a rectangle whose length is *L* and whose width is *S*, and whose dimensions are in the golden ratio. We call this a golden rectangle, which derives its name from the apparent beauty of its shape: a view supported through numerous psychological studies in a variety of cultures. The shape of the golden rectangle can be found in many architectural masterpieces as well as in famous classical works of art.

When the golden ratio is viewed in terms of its numerical value, it seems to infiltrate just about every aspect of mathematics. We have selected those manifestations of the golden ratio that allow the reader to appreciate the beauty and power of mathematics. In some cases, our endeavors will open new vistas for the reader; in other cases, they will enrich the reader's understanding and appreciation for areas of mathematics that may not have been considered from this unusual vantage point. For example, the golden ratio is a value, frequently referred to by the Greek letter ϕ (phi), which has the unique characteristic in that it differs from its reciprocal by 1, that is, $\phi-\frac{1}{\phi}=1$. This unusual characteristic leads to a plethora of fascinating properties and genuinely connects ϕ to such familiar topics as the Fibonacci numbers and the Pythagorean theorem.

In the field of geometry, the applications of the golden ratio are practically boundless, as are their beauty. To fully appreciate their visual aspects, we will take you through a journey of geometric experiences that will include some rather unusual ways of constructing the golden ratio, as well as exploring the many surprising geometric figures into which the golden ratio is embedded. All this requires of the reader is to be merely fortified with nothing more than some elementary high school geometry.

Join us now as we embark on our journey through the many wonderful appearances of the golden ratio, beginning with a history of these sightings dating from before 2560 BCE all the way to the present day. We hope that throughout this mathematical excursion, you will get to appreciate the quotation by the famous German mathematician and scientist Johannes Kepler (1571–1630), who said, “Geometry harbors two great treasures: One is the Pythagorean theorem, and the other is the golden ratio. The first we can compare with a heap of gold, and the second we simply call a priceless jewel.”[1] This “priceless jewel” will enrich, entertain, and fascinate us, and perhaps open new doors to unanticipated vistas.

Chapter 1

Defining and Constructing the Golden Ratio

As with any new concept, we must first begin by defining the key elements. To define the golden ratio, we first must understand that the ratio of two numbers, or magnitudes, is merely the relationship obtained by dividing these two quantities. When we have a ratio of 1:3, or $\frac{1}{3}$, we can conclude that one number is one-third the other. Ratios are frequently used to make comparisons of quantities. One ratio stands out among the rest, and that is the ratio of the lengths of the two parts of a line segment which allows us to make the following equality of two ratios (the equality of two ratios is called a proportion): that the longer segment (*L*) is to the shorter segment (*S*) as the entire original segment (*L*+*S*) is to the longer segment (*L*). Symbolically, this is written as $\frac{L}{S}=\frac{L+S}{L}$. Geometrically, this may be seen in figure 1-1:

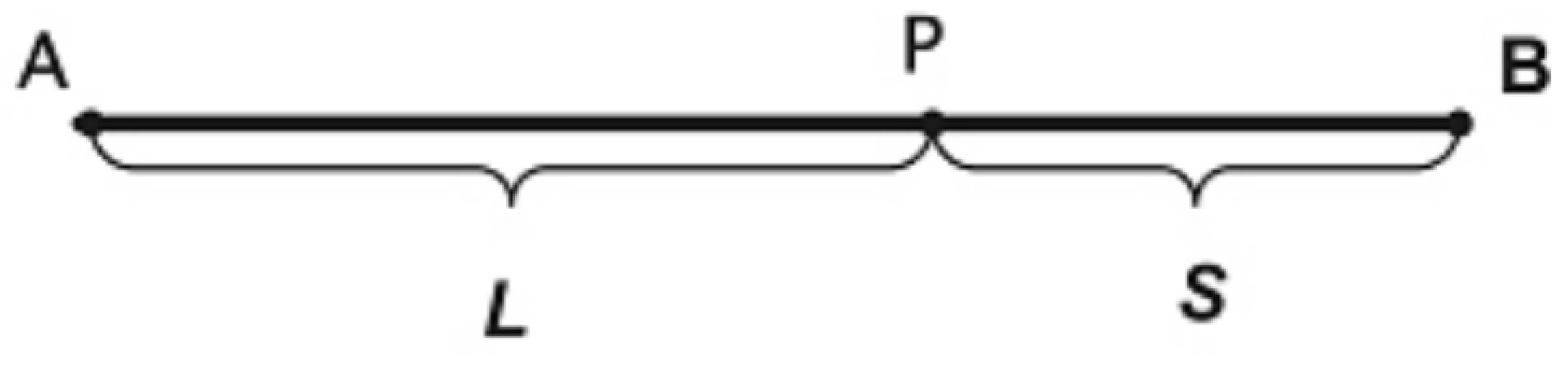

Figure 1-1

This is called the *golden ratio* or the *golden section*—in the latter case we are referring to the "sectioning" or partitioning of a line segment. The terms *golden ratio* and *golden section* were first introduced during the nineteenth century. We believe that the Franciscan friar and mathematician Fra Luca Pacioli (ca. 1445–1514 or 1517) was the first to use the term *De Divina Proportione* (*The Divine Proportion*), as the title of a book in 1509, while the German mathematician and astronomer Johannes Kepler (1571–1630) was the first to use the term *sectio divina* (divine section). Moreover, the German mathematician Martin Ohm (1792–1872) is credited for having used the term *Goldener Schnitt* (golden section). In English, this term, *golden section*, was used by James Sully in 1875.[1]

You may be wondering what makes this ratio so outstanding that it deserves the title "golden." This designation, which it richly deserves, will be made clear throughout this book. Let's begin by seeking to find its numerical value, which will bring us to its first unique characteristic.

To determine the numerical value of the golden ratio $\frac{L}{S}$ we will change this equation $\frac{L}{S}=\frac{L+S}{L}$ or $\frac{L}{S}=\frac{L}{L}+\frac{S}{L}$ to its equivalent, when $x=\frac{L}{S}$, to get[2]: $x=1+\frac{1}{x}$.

We can now solve this equation for x using the quadratic formula, which you may recall from high school. (The quadratic formula for solving for x in the general quadratic equation $ax^2 + bx + c = 0$ is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. See the appendix for a derivation of this formula.) We then obtain the numerical value of the golden ratio:

$$\frac{L}{S}=x=\frac{1+\sqrt{5}}{2},$$

which is commonly denoted by the Greek letter, phi[3]: ϕ.

$$\phi=\frac{L}{S}=\frac{1+\sqrt{5}}{2}\approx\frac{1+2.2360679774997896964091736687312762354406}{2}$$

$$\approx\frac{3.2360679774997896964091736687312762354406}{2}$$

$$\approx 1.61803.$$

Notice what happens when we take the reciprocal of $\frac{L}{S}$, namely $\frac{S}{L}=\frac{1}{\phi}$:

$$\frac{1}{\phi}=\frac{S}{L}=\frac{2}{1+\sqrt{5}},$$

which when we multiply by 1 in the form of $\frac{1-\sqrt{5}}{1-\sqrt{5}}$, we get

$$\frac{2}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}=\frac{2\cdot(1-\sqrt{5})}{1-5}=\frac{2\cdot(1-\sqrt{5})}{-4}=\frac{1-\sqrt{5}}{-2}=\frac{\sqrt{5}-1}{2}=\frac{\sqrt{5}+1}{2}-1=\phi-1$$
$$\approx 0.61803.$$

But at this point you should notice a very unusual relationship. The value of ϕ and $\frac{1}{\phi}$ differ by 1. That is, $\phi-\frac{1}{\phi}=1$. From the normal relationship of reciprocals, the product of ϕ and $\frac{1}{\phi}$ is also equal to 1, that is, $\phi \cdot \frac{1}{\phi} = 1$. Therefore, we have two numbers, ϕ and $\frac{1}{\phi}$, whose difference and product is 1—these are the only two numbers for which this is true! By the way, you might have noticed that

$$\phi + \frac{1}{\phi} = \sqrt{5}, \text{ since } \frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2} = \sqrt{5}.$$

We will often refer to the equations $x^2 - x - 1 = 0$ and $x^2 + x - 1 = 0$ during the course of this book because they hold a central place in the study of the golden ratio. For those who would like some reinforcement, we can see that the value ϕ satisfies the equation $x^2 - x - 1 = 0$, as is evident here:

$$\phi^2 - \phi - 1 = \left(\frac{\sqrt{5}+1}{2}\right)^2 - \frac{\sqrt{5}+1}{2} - 1 = \frac{5+2\sqrt{5}+1}{4} - \frac{2\left(\sqrt{5}+1\right)}{4} - \frac{4}{4}$$

$$= \frac{5+2\sqrt{5}+1-2\sqrt{5}-2-4}{4} = 0.$$

The other solution of this equation is

$$\frac{1-\sqrt{5}}{2} = -\frac{\sqrt{5}-1}{2} = -\frac{1}{\phi},$$

while $-\phi$ satisfies the equation $x^2 + x - 1 = 0$, as you can see here:

$$(-\phi)^2 + (-\phi) - 1 = \phi^2 - \phi - 1 = \left(\frac{\sqrt{5}+1}{2}\right)^2 - \frac{\sqrt{5}+1}{2} - 1 = 0.$$

The other solution to this equation is $\frac{1}{\phi}$.

Having now defined the golden ratio numerically, we shall *construct* it geometrically. There are several ways to construct the golden section of a line segment. You may notice that we appear to be using the terms *golden ratio* and *golden section* interchangeably. To avoid confusion, we will use the term *golden ratio* to refer to the numerical value of ϕ and the term *golden section* to refer to the geometric division of a segment into the ratio ϕ.

GOLDEN SECTION CONSTRUCTION 1

Our first method, which is the most popular, is to begin with a unit square *ABCD*, with midpoint *M* of side *AB*, and then draw a circular arc with radius *MC*, cutting the extension of side *AB* at point *E*. We now can claim that the line segment *AE* is partitioned into the golden section at point *B*. This, of course, has to be substantiated.

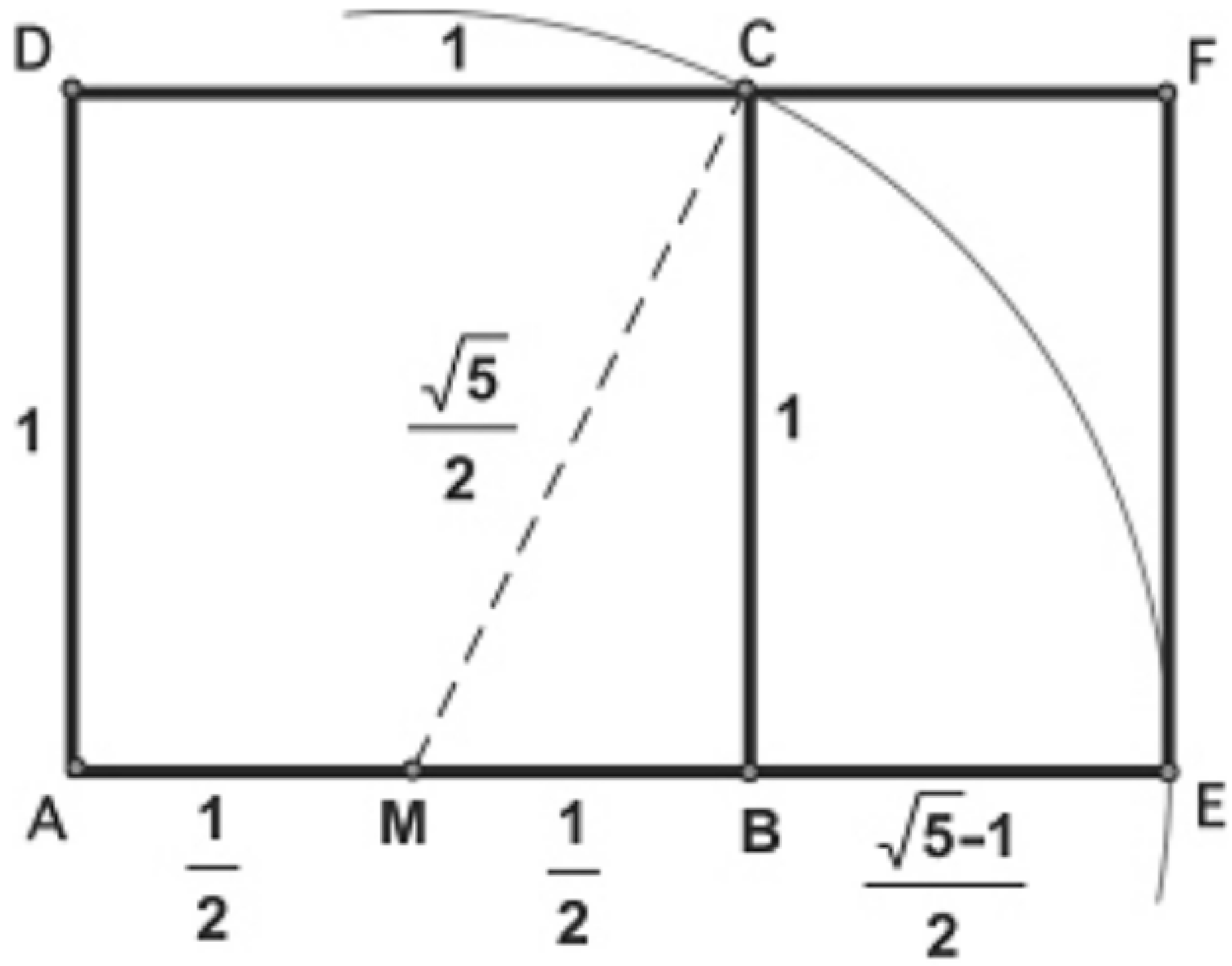

Figure 1-2

To verify this claim, we would have to apply the definition of the golden section: $\frac{AB}{BE} = \frac{AE}{AB}$, and see if it, in fact, holds true. Substituting the values obtained by applying the Pythagorean theorem to ΔMBC as shown in figure 1-2, we get the following:

$$MC^2 = MB^2 + BC^2 = \left(\frac{1}{2}\right)^2 + 1^2 = \frac{1}{4} + 1 = \frac{5}{4}\text{; therefore, } MC = \frac{\sqrt{5}}{2}.$$

It follows that

$$BE = ME - MB = MC - MB = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}\text{, and}$$

$$AE = AB + BE = 1 + \frac{\sqrt{5}-1}{2} = \frac{2}{2} + \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2}.$$

We then can find the value of $\frac{AB}{BE} = \frac{AE}{AB}$, that is,

$$\frac{1}{\frac{\sqrt{5}-1}{2}} = \frac{\frac{\sqrt{5}+1}{2}}{1},$$

which turns out to be a true proportion, since the cross products are equal. That is,

$$\left(\frac{\sqrt{5}-1}{2}\right)\cdot\left(\frac{\sqrt{5}+1}{2}\right) = 1\cdot 1 = 1.$$

We can also see from figure 1-2 that point *B* can be said to divide the line segment *AE* into an inner golden section, since

$$\frac{AB}{AE} = \frac{1}{1+\frac{\sqrt{5}-1}{2}} = \frac{1}{\frac{\sqrt{5}+1}{2}} = \frac{\sqrt{5}-1}{2} = \frac{1}{\phi}.$$

Meanwhile, point *E* can be said to divide the line segment *AB* into an outer golden section, since

$$\frac{AE}{AB} = \frac{1+\frac{\sqrt{5}-1}{2}}{1} = \frac{\sqrt{5}+1}{2} = \phi.$$

You ought to take notice of the shape of the rectangle *AEFD* in figure 1-2. The ratio of the length to the width is the golden ratio:

$$\frac{AE}{EF} = \frac{\frac{\sqrt{5}+1}{2}}{1} = \frac{\sqrt{5}+1}{2} = \phi.$$

This appealing shape is called the *golden rectangle*, which will be discussed in detail in chapter 4.

GOLDEN SECTION CONSTRUCTION 2

Another method for constructing the golden section begins with the construction of a right triangle with one leg of unit length and the other twice as long, as is shown in figure 1-3.[4] Here we will partition the line segment *AB* into the golden ratio. The partitioning may not be obvious yet, so we urge readers to have patience until we reach the conclusion.

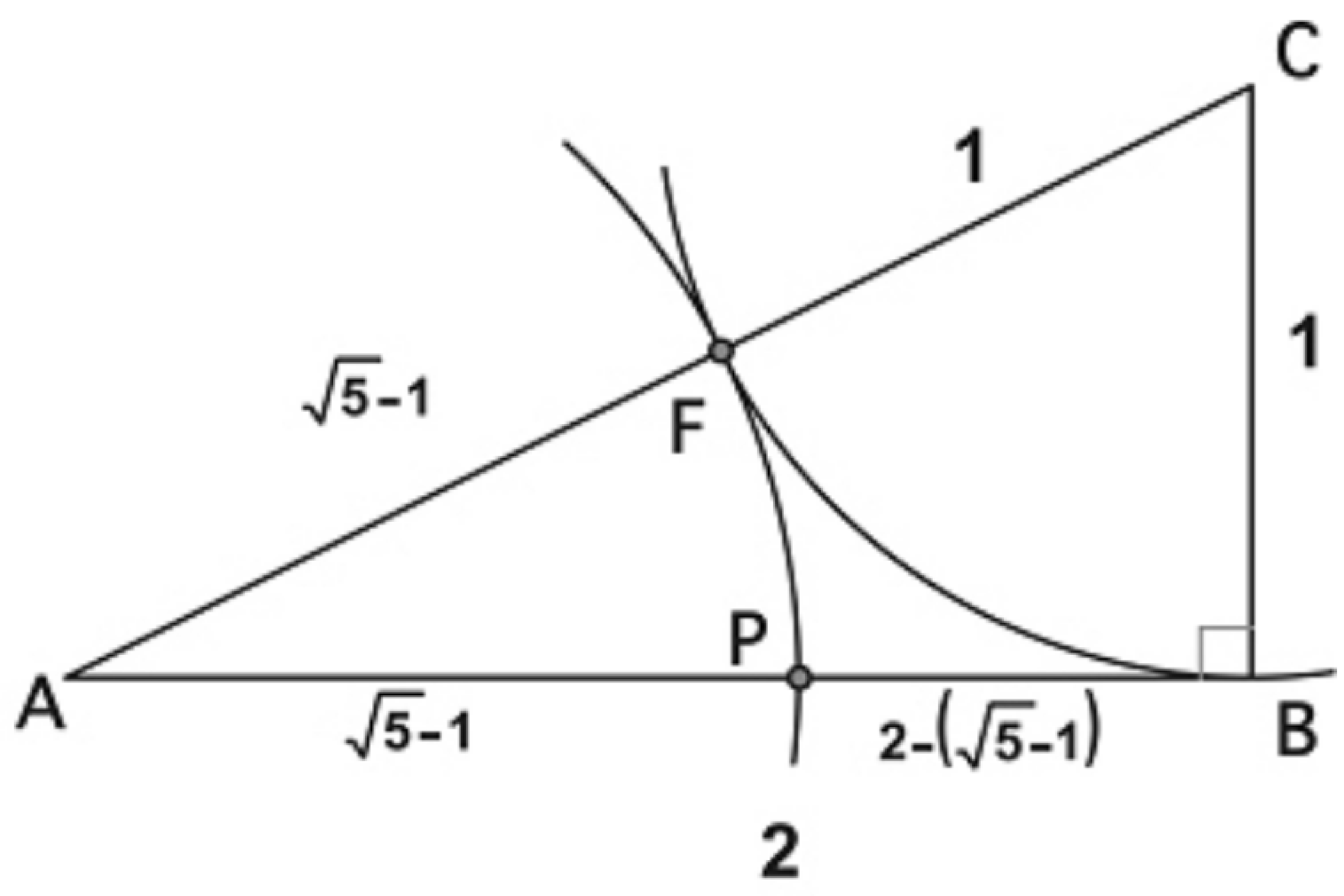

Figure 1-3

With *AB* = 2 and *BC* = 1, we apply the Pythagorean theorem to ΔABC. We then find that $AC=\sqrt{2^2+1^2}=\sqrt{5}$. With the center at point *C*, we draw a circular arc with radius 1, cutting line segment *AC* at point *F*. Then we draw a circular arc with the center at point *A* and the radius *AF*, cutting *AB* at point *P*.

Because $AF=\sqrt{5}-1$, we get $AP=\sqrt{5}-1$. Therefore, $BP=2-(\sqrt{5}-1)=3-\sqrt{5}$.

To determine the ratio $\frac{AP}{BP}$, we will set up the ratio $\frac{\sqrt{5}-1}{3-\sqrt{5}}$, and then to make some sense of it, we will rationalize the denominator by multiplying the ratio by 1 in the form of $\frac{3+\sqrt{5}}{3+\sqrt{5}}$.

We then find that

$$\frac{\sqrt{5}-1}{3-\sqrt{5}}\cdot\frac{3+\sqrt{5}}{3+\sqrt{5}}=\frac{3\sqrt{5}+5-3-\sqrt{5}}{3^2-(\sqrt{5})^2}=\frac{2\sqrt{5}+2}{9-5}=\frac{2\left(\sqrt{5}+1\right)}{4}=\frac{\sqrt{5}+1}{2}=\phi\approx 1.61803,$$

which is the golden ratio! Therefore, we find that point *P* cuts the line segment *AB* into the golden ratio.

GOLDEN SECTION CONSTRUCTION 3

We have yet another way of constructing the golden section.

Consider the three adjacent unit squares shown in figure 1-4,. We construct the angle bisector of $\angle BHE$. There is a convenient geometric relationship that will be very helpful to us here; that is, that the angle bisector in a triangle divides the side to which it is drawn proportionally to the two sides of the angles being bisected.[5] In figure 1-4 we then derive the following relationship: $\frac{BH}{EH}=\frac{BC}{CE}$. Applying the Pythagorean theorem to ΔHFE, we get $HE=\sqrt{5}$. We can now evaluate the earlier proportion by substituting the values shown in figure 1-4:

$\frac{1}{\sqrt{5}}=\frac{x}{2-x}$, from which we get $x=\frac{2}{\sqrt{5}+1}$, which is the reciprocal of $\frac{\sqrt{5}+1}{2}=\phi$.

Therefore, $x=\frac{1}{\phi}\approx 0.61803$

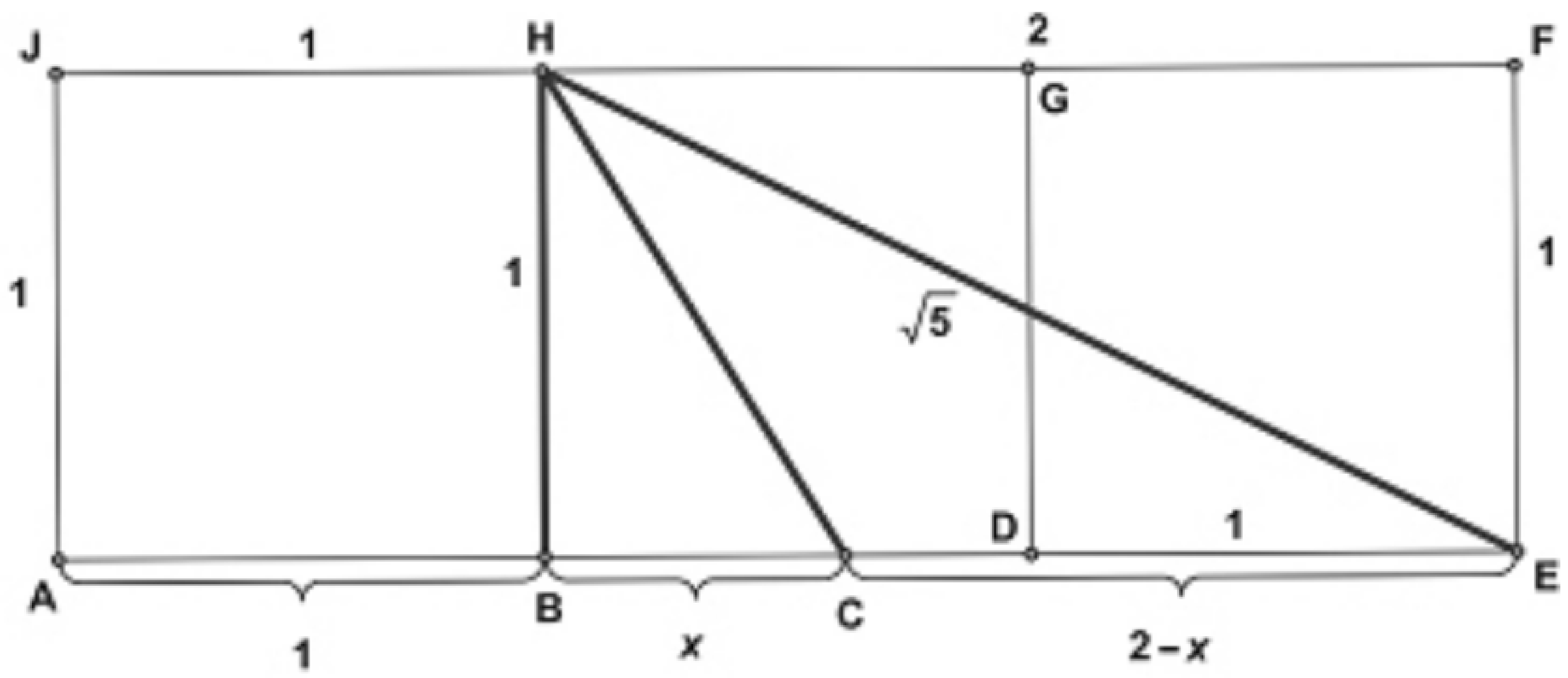

Figure 1-4

Thus, we can then conclude that point B divides the line segment AC into the golden section, since

$\frac{AB}{BC}=\frac{1}{x}=\frac{\sqrt{5}+1}{2}\approx 1.61803$, the recognized value of the golden ratio.

GOLDEN SECTION CONSTRUCTION 4

Analogous to the previous construction is one that begins with two congruent squares as shown in figure 1-5. A circle is drawn with its center at the midpoint, M, of the common side of the squares, and a radius half the length of the side of the square. The point of intersection, C, of the circle and the diagonal of the rectangle determines the golden section, AC, with respect to a side of the square, *AD*.

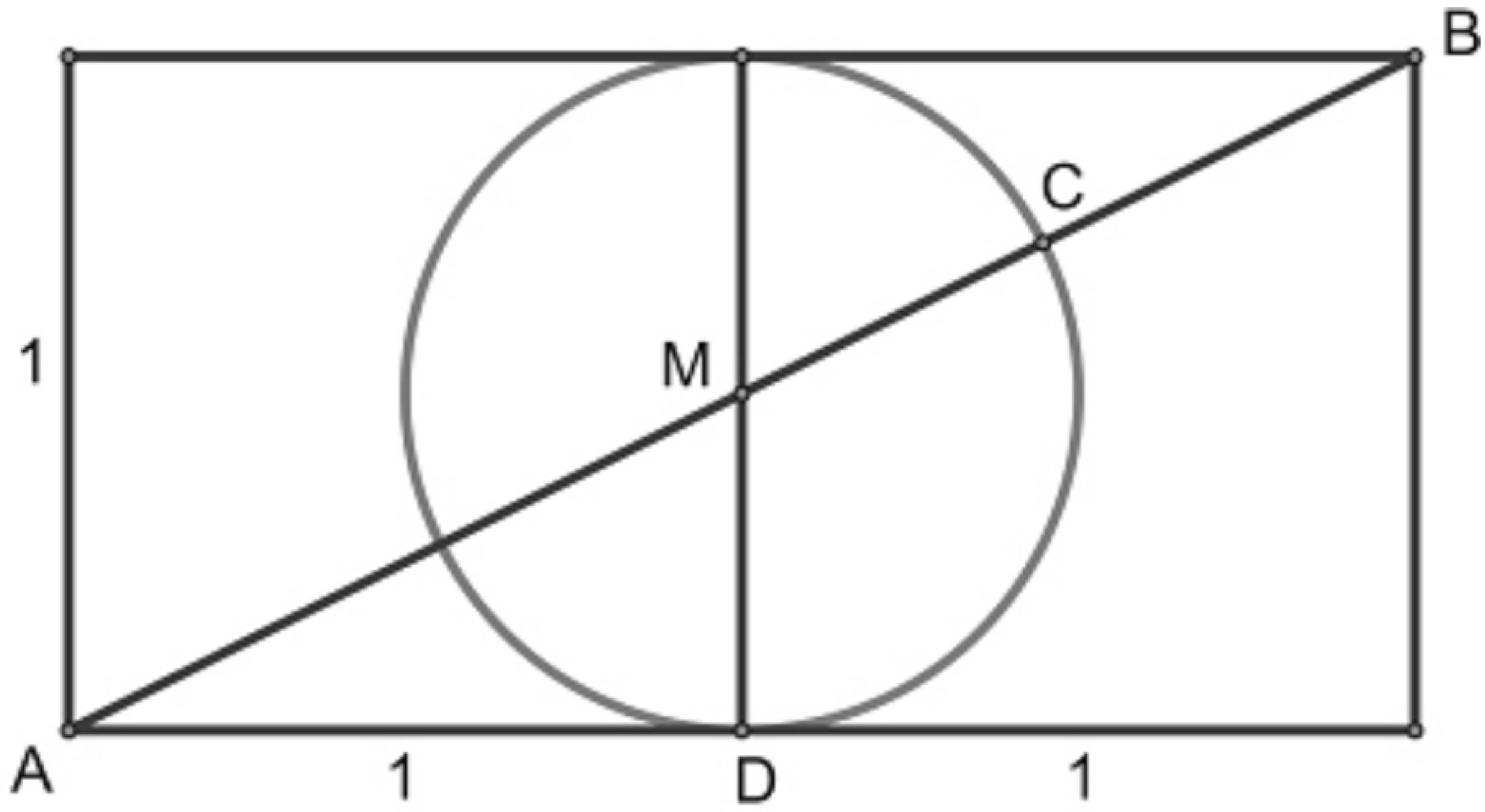

Figure 1-5

With *AD* = 1 and *DM* = $\frac{1}{2}$, we get *AM* = $\frac{\sqrt{5}}{2}$ by applying the Pythagorean theorem to triangle *AMD*. (See fig. 1-6.) Since *CM* is also a radius of the circle, *CM* = *DM* = $\frac{1}{2}$. We can then conclude that

$$AC = AM + CM = \frac{\sqrt{5}}{2} + \frac{1}{2} = \frac{\sqrt{5}+1}{2} = \phi.$$

Furthermore,

$$BC = AB - AC = \sqrt{5} - \frac{\sqrt{5}+1}{2} = \frac{\sqrt{5}-1}{2} = \frac{1}{\phi}.$$

We have thus constructed the golden section and its reciprocal.

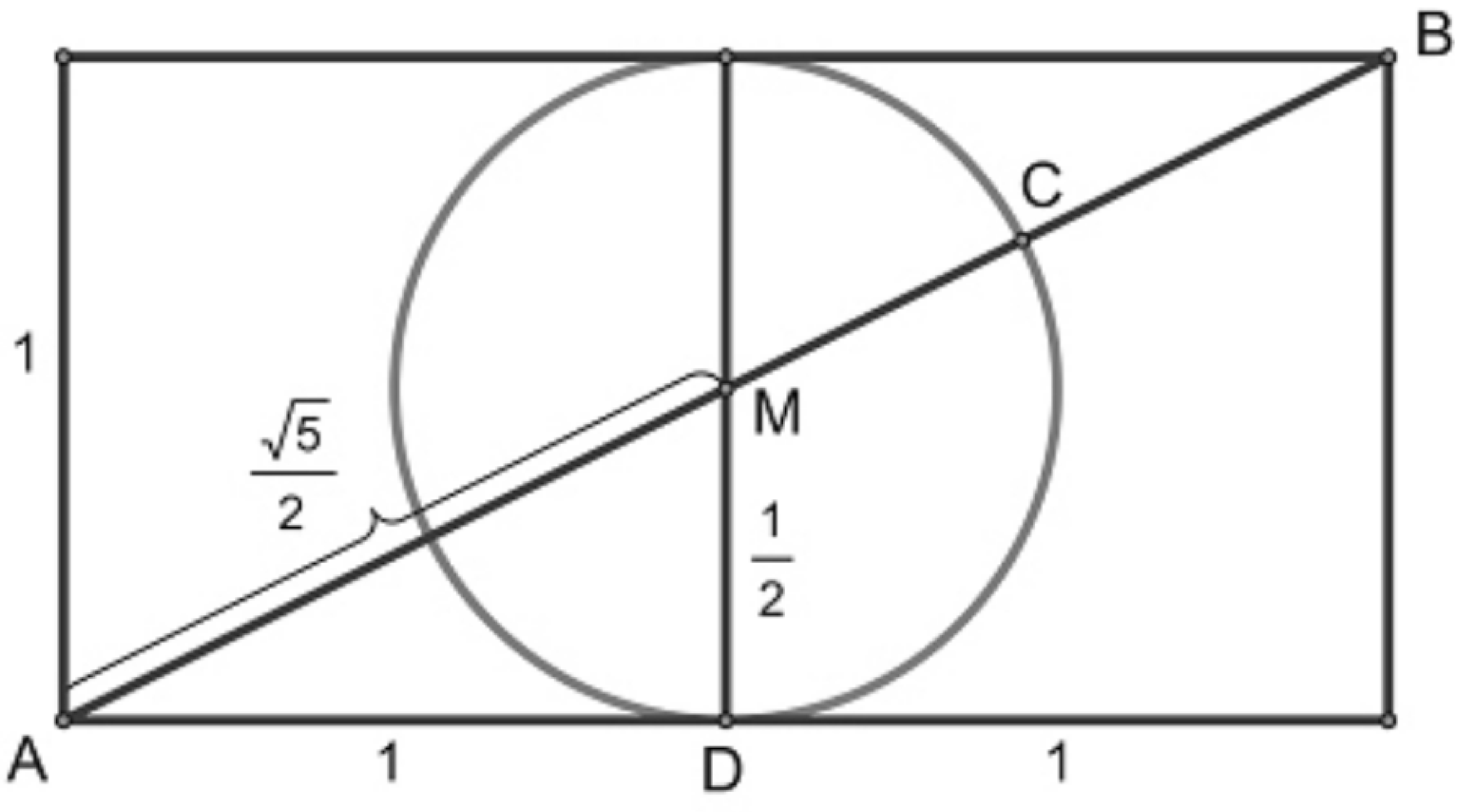

Figure 1-6

GOLDEN SECTION CONSTRUCTION 5

In this rather simple construction we will show that the semicircle on the side (extended) of a square, whose radius is the length of the segment from the midpoint of the side of the square to an opposite vertex, creates a line segment where the vertex of the square determines the golden ratio. In figure 1-7, we have square *ABCD* and a semicircle on line *AB* with center at the midpoint *M* of *AB* and radius *CM*. We encountered a similar situation with Construction 1, where we concluded that $\frac{AB}{BE} = \phi$ and $\frac{AE}{AB} = \phi$.

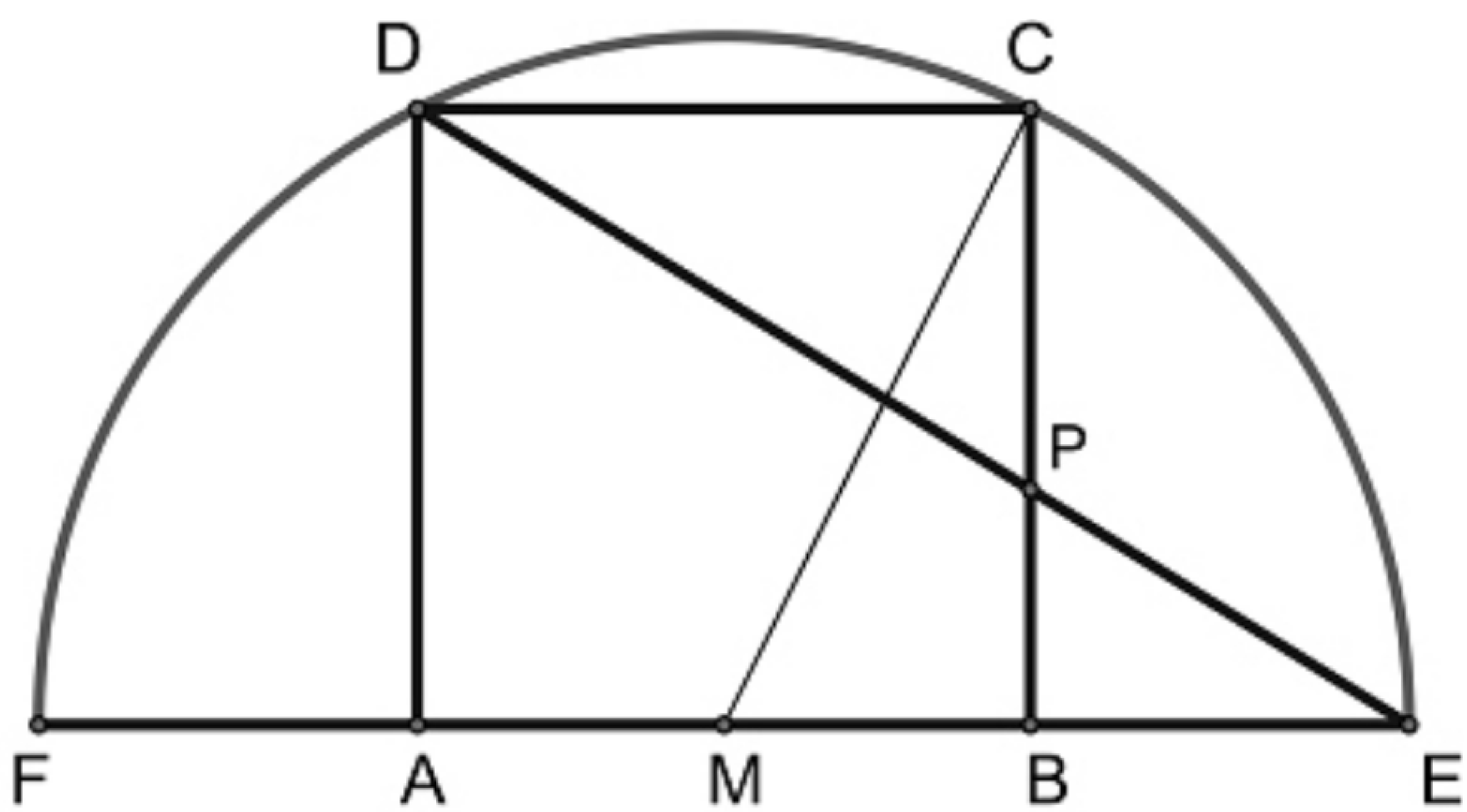

Figure 1-7

However, here we have an extra added attraction: *DE* and *BC* partition each other into the golden section at point *P*. This is easily justified in that triangles *DPC* and *EBP* are similar and their corresponding sides, *DC* and *BE*, are in the golden ratio. Hence, all the corresponding sides are in the golden ratio, which here is $\frac{CP}{PB}=\frac{DP}{PE}=\phi$.

GOLDEN SECTION CONSTRUCTION 6

Some of the constructions of the golden section are rather creative.[6] Consider the inscribed equilateral triangle *ABC* with line segment *PT* bisecting the two sides of the equilateral triangle at points *Q* and *S* as shown in figure 1-8.

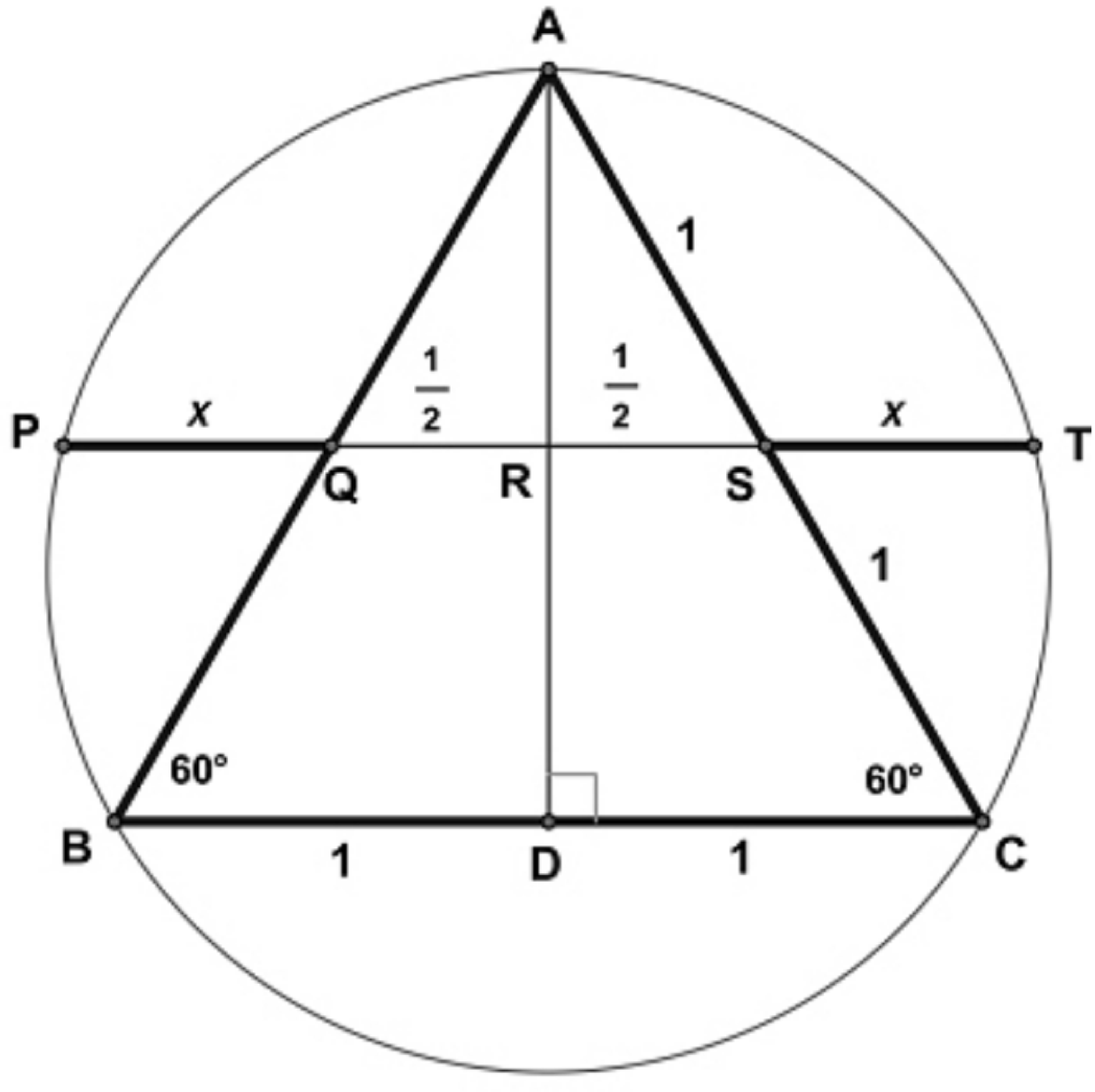

Figure 1-8

We will let the side length of the equilateral triangle equal 2, which then provides us with the segment lengths as shown in figure 1-8. The proportionality there gives us $\frac{RS}{CD}=\frac{AS}{AC}$, which then by substituting appropriate values yields $\frac{RS}{1}=\frac{1}{2}$, and so RS = $\frac{1}{2}$.

A useful geometric theorem will enable us to find the length of the segments *PQ* = *ST* = *x* due to the symmetry of the figure. The theorem states that the products of the segments of two intersecting chords of a circle are equal. From that theorem, we find

$$PS \cdot ST = AS \cdot SC$$

$$(x+1) \cdot x = 1 \cdot 1$$

$$x^2 + x - 1 = 0$$

$$x = \frac{\sqrt{5}-1}{2}.$$

Therefore, the segment *QT* is partitioned into the golden section at point *S*, since

$$\frac{QS}{ST} = \frac{1}{x} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2} \approx 1.61803,$$

which we recognize as the value of the golden ratio. We can generalize this construction by saying that the midline of an equilateral triangle extended to the circumcircle is partitioned into the golden section by the sides of the equilateral triangle.

GOLDEN SECTION CONSTRUCTION 7

This is a rather easy construction of the golden ratio in that it simply requires constructing an isosceles triangle inside a square as shown in figure 1-9. The vertex *E* of ΔABE lies on side *DC* of square *ABCD*, and altitude *EM* intersects the inscribed circle of ΔABE at point *H*. The golden ratio appears in two ways here. First, when the side of the square is 2, then the radius of the inscribed circle $r = \frac{1}{\phi}$, and second when the point *H* partitions *EM* into the golden ratio as $\frac{EM}{HM} = \phi$.

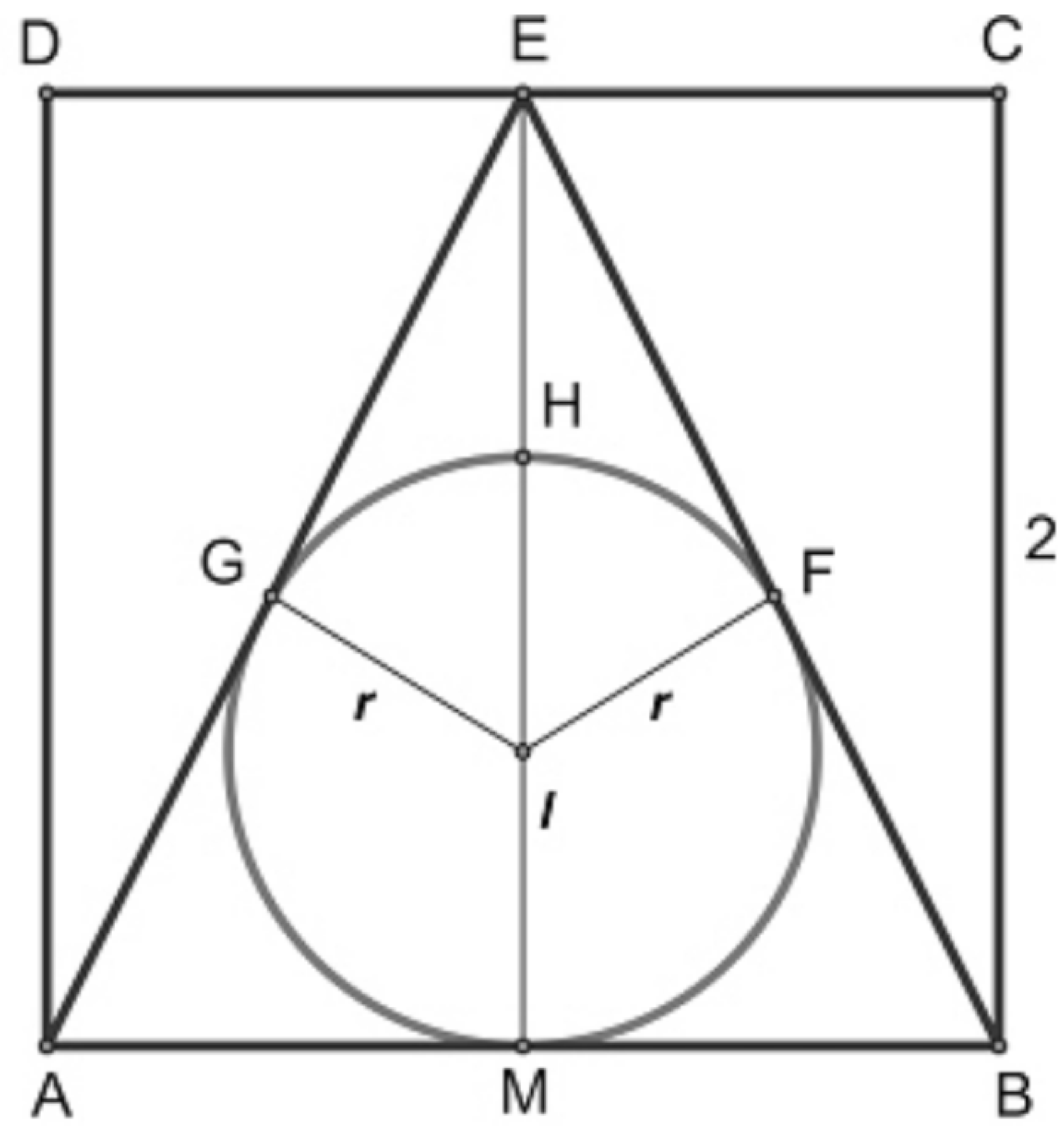

Figure 1-9

To justify this construction, we will let the side of the square have length 2. This gives us *BM* = 1 and *EM* = 2. Then, with the Pythagorean theorem applied to triangle *MEB*, we derive $AE = BE = \sqrt{5}$, whereupon we recognize that $GE = \sqrt{5} - 1$ (fig. 1-10).[7]

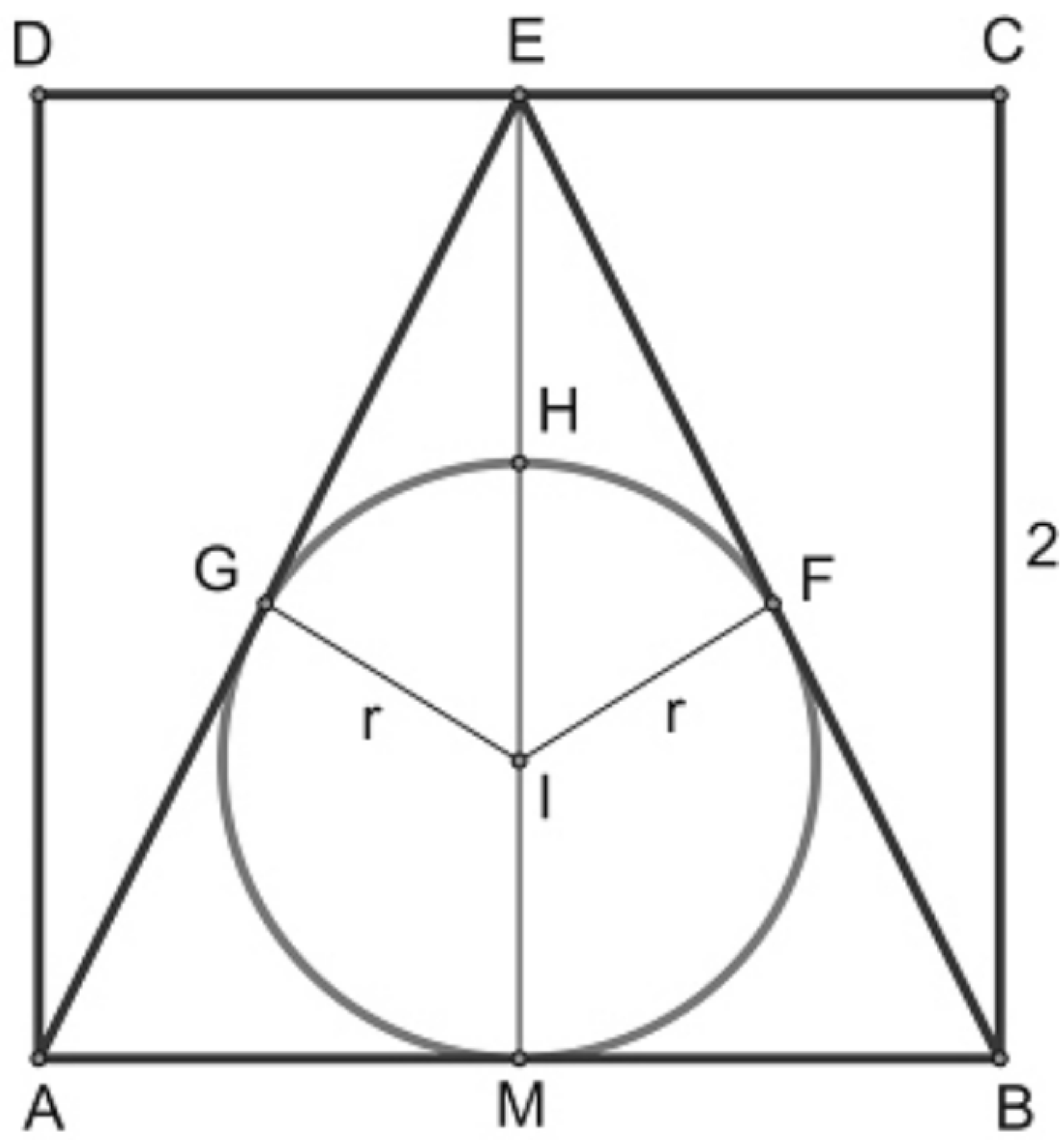

Figure 1-10

For the second appearance, again we apply the Pythagorean theorem, this time to ΔEGI, giving us $EI^2 = GI^2 + GE^2$. Put another way, $(2-r)^2 = r^2 + (\sqrt{5}-1)^2$; therefore, $4-4r+r^2 = r^2+5-2\sqrt{5}+1$. This determines the length of the radius of the inscribed circle

$$r = \frac{\sqrt{5}-1}{2} = \frac{1}{\phi}.$$

Now, with some simple substitution, we have $EM = 2$ and $HM = 2r$, yielding the ratio $\frac{EM}{HM} = \frac{2}{2r} = \frac{1}{r} = \phi$.

GOLDEN SECTION CONSTRUCTION 8

A somewhat more contrived construction also yields the golden section of a line segment. To do this, we will construct a unit square with one vertex placed at the center of a circle whose radius is the length of the diagonal of the square. On one side of the square we will construct an equilateral triangle. This is shown in figure 1-11.

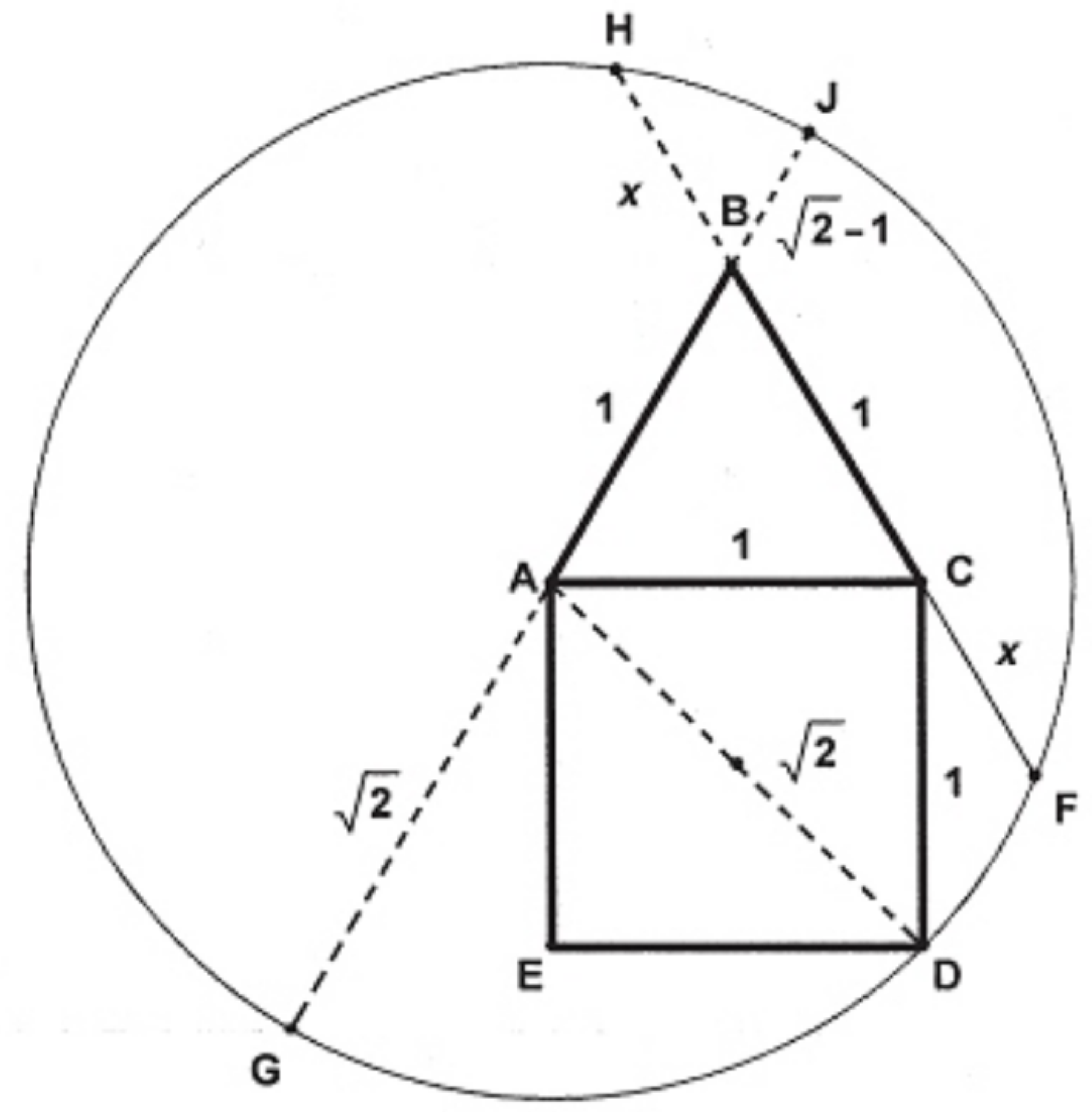

Figure 1-11

Again applying the Pythagorean theorem to triangle *ACD*, we get the radius of the circle as $\sqrt{2}$, which gives us the lengths of *AD*, *AG*, and *AJ*. Because of symmetry, we have *BH* = *CF* = *x*. Again applying the theorem involving intersecting chords of a circle (as in Construction 6), we get the following:

$$GB \cdot BJ = HB \cdot BF$$

$$(\sqrt{2}+1)(\sqrt{2}-1) = x(x+1)$$

$$x = \frac{\sqrt{5}-1}{2}.$$

Once again we find the segment *BF* is partitioned into the golden section at point *C*, since

$$\frac{BC}{CF} = \frac{1}{x} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2} \approx 1.61803$$

, which we recognize as the value of the golden ratio.

GOLDEN SECTION CONSTRUCTION 9

We can derive the equation $x^2 + x - 1 = 0$, the so-called *golden equation*, in a number of other ways, one of which involves constructing a circle with a chord *AB*, which is extended to a point *P* so that when a tangent from *P* is drawn to the circle, its length equals that of *AB*. We can see this in figure 1-12, where $PT = AB = 1$.

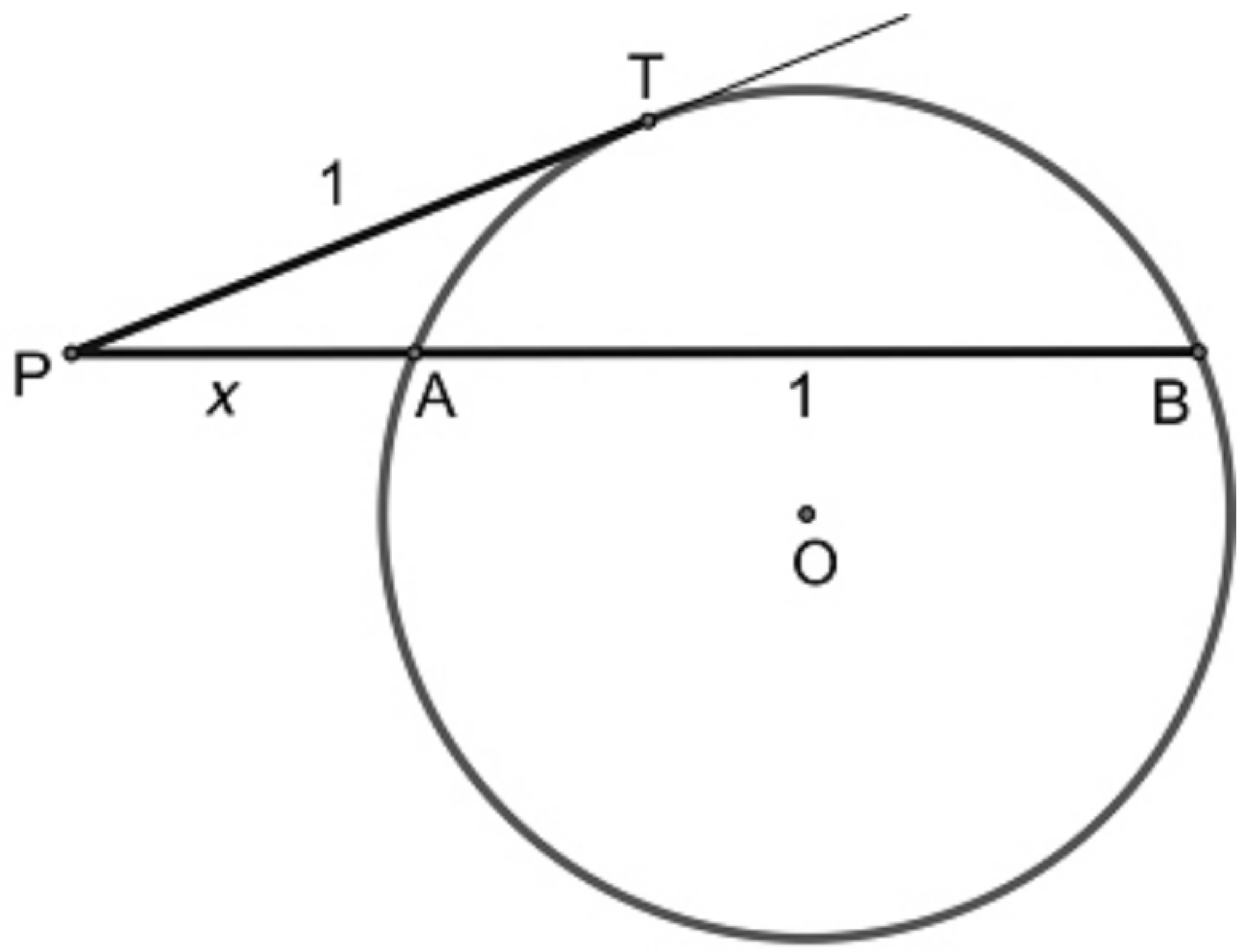

Figure 1-12

Here we will apply a geometric theorem which states that when, from an external point, *P*, a tangent (*PT*) and a secant (*PB*) are drawn to a circle, the tangent segment is the mean proportional between the entire secant and the external segment, that is, $\frac{PB}{PT} = \frac{PT}{PA}$. This yields $PT^2 = PB \cdot PA$, or $PT^2 = (PA + AB) \cdot PA$. If we let $PA = x$, then $1^2 = (x + 1)x$, or $x^2 + x - 1 = 0$, and, as before, we can conclude that point *A* determines the golden section of line segment *PB*, since the solution to this equation is the golden ratio.

The next method we present is a bit convoluted. Yet, it begins with the famous 3-4-5 right triangle — probably one of the earliest to be recognized as a true right triangle, going back to the so-called *rope-stretchers* of ancient Egypt.[8]

GOLDEN SECTION CONSTRUCTION 10

In figure 1-13 we have the 3-4-5 right triangle *ABC*. The bisector of $\angle ABC$ intersects side *AC* at point *G*. With *G* as its center, a circle of radius *GC* is drawn and can be shown to be tangent to both *BC* and *AB*.

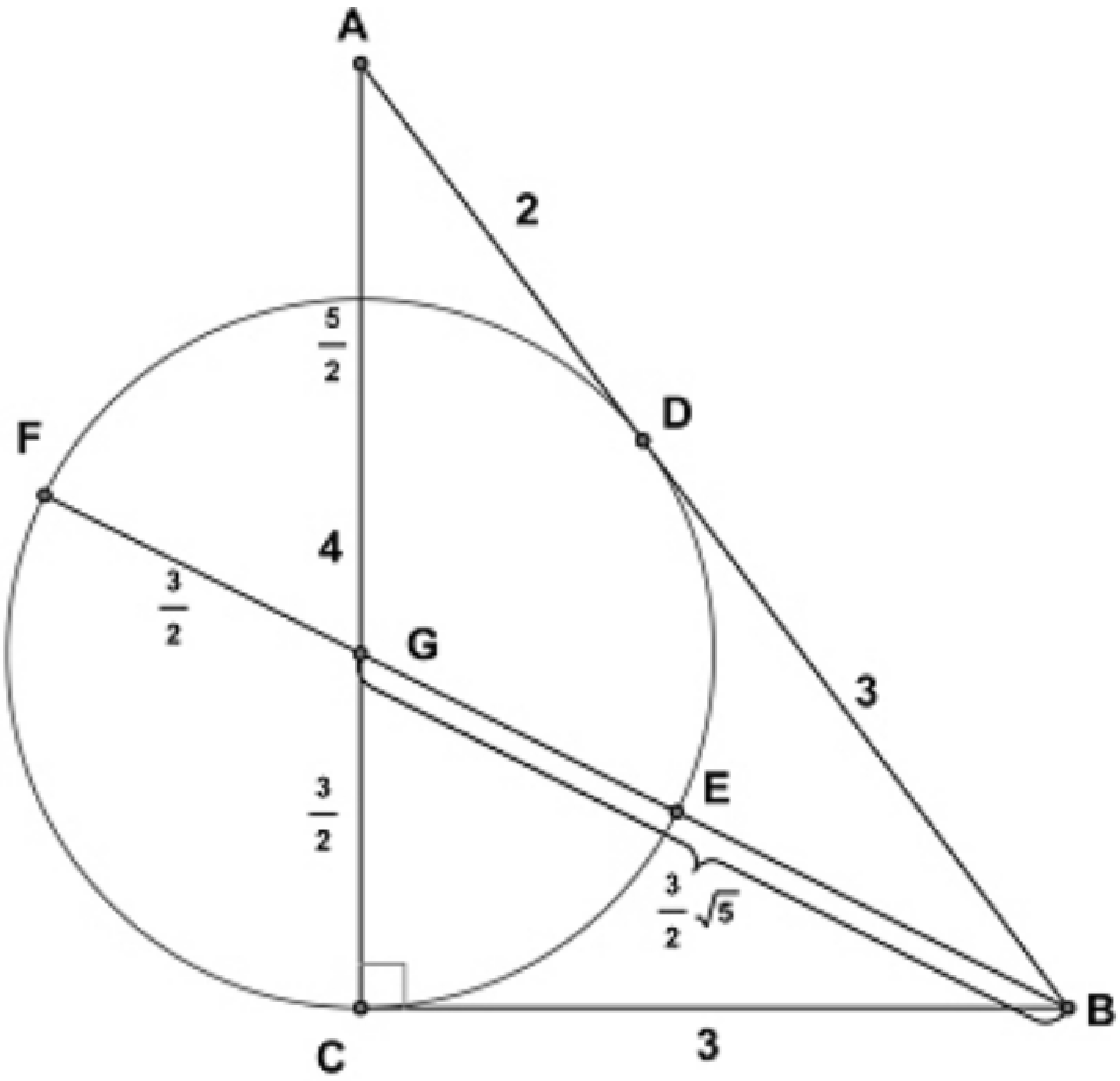

Figure 1-13

As we noted earlier, the bisector of an angle of a triangle divides the side to which it is drawn proportionally to the angle's two sides. Therefore, $\frac{AG}{GC}=\frac{AB}{BC}=\frac{5}{3}$ or $AG=\frac{5}{3}GC$.

With $AG + GC = 4$, we get $\frac{5}{3}GC+GC=\frac{8}{3}GC=4$, or $GC=\frac{3}{2}$. So we can determine that $AG=\frac{5}{2}$.

$GC = GD = GE = GF$ are radii of the circle, so we then have $FG=\frac{3}{2}$, and $GE=\frac{3}{2}$. Applying the Pythagorean theorem to ΔGBC, we $GB^2=BC^2+GC^2=9+\frac{9}{4}=\frac{45}{4}$. Therefore, $GB=\frac{3}{2}\sqrt{5}$.

We are now ready to show that the point E partitions the line segment BF into the golden ratio:

$$\frac{BF}{FE}=\frac{GF+GB}{GF+GE}=\frac{\frac{3}{2}+\frac{3}{2}\sqrt{5}}{\frac{3}{2}+\frac{3}{2}}=\frac{\sqrt{5}+1}{2}\approx 1.61803,$$

which by now is easily recognizable as the golden ratio.

A similar construction with a 3-4-5 right triangle was discovered by Gabries Bosia while pondering the knight's moves in chess.[9]

GOLDEN SECTION CONSTRUCTION 11

In figure 1-14, we see three concentric circles with radii of lengths 1, 2, and 4 units, respectively. *PR* is tangent to the inner circle at *T* and cuts the other circles at points *P*, *Q*, and *R*.

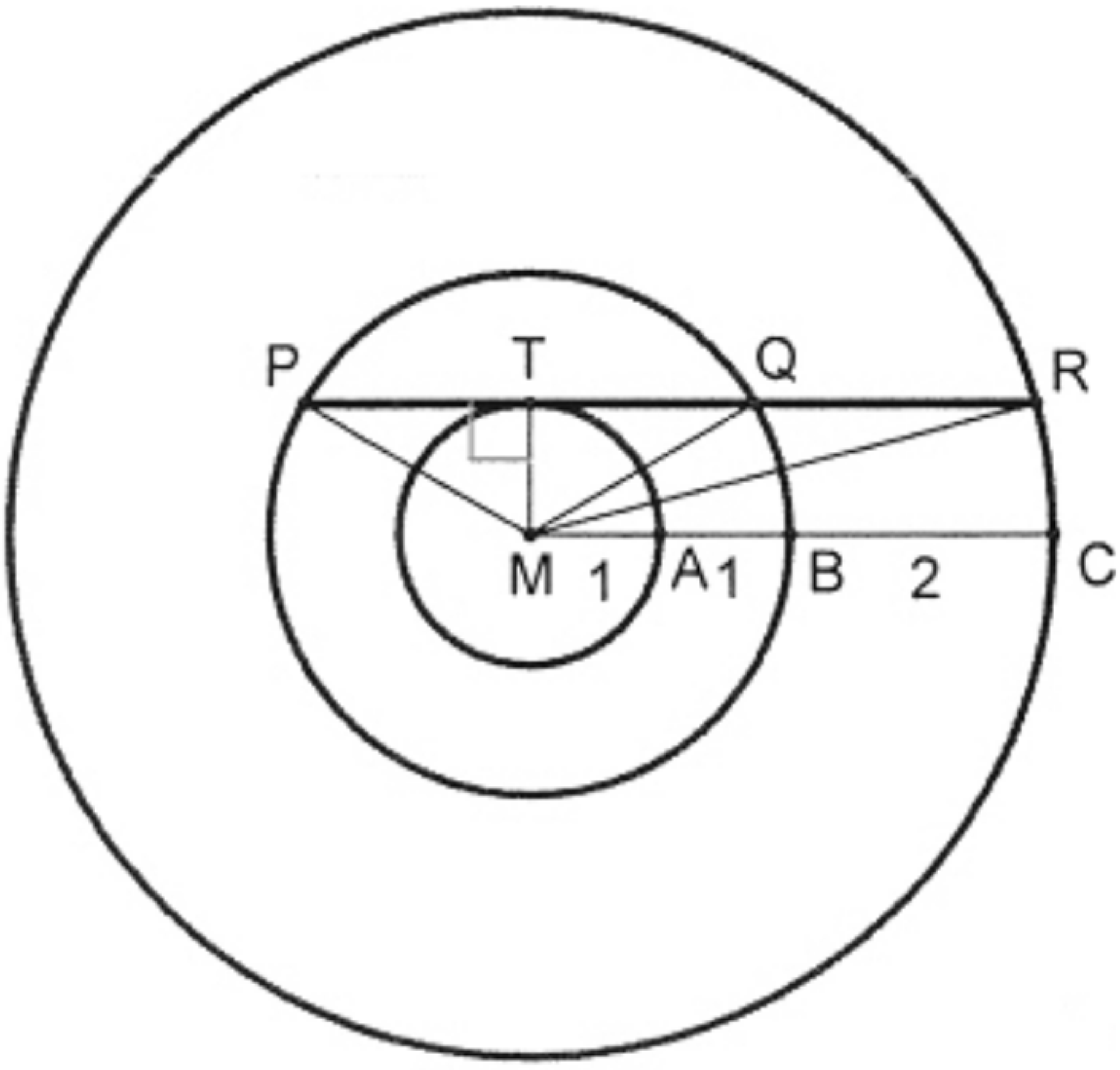

Figure 1-14

With $AM = AB = 1$ and $BC = 2$, we apply the Pythagorean theorem to ΔMQT and ΔMRT, and get $QT = \sqrt{2^2 - 1^2} = \sqrt{3}$, and $RT = \sqrt{4^2 - 1^2} = \sqrt{15}$.

As we have $PR = RT + PT = RT + QT = \sqrt{15} + \sqrt{3} = \sqrt{3}(\sqrt{5}+1)$, and $PQ = PT + QT = 2\sqrt{3}$, we derive

$$\frac{PR}{PQ} = \frac{\sqrt{3}\left(\sqrt{5}+1\right)}{2\sqrt{3}} = \frac{\sqrt{5}+1}{2} \approx 1.61803,$$

which is again recognizable as the golden ratio.

GOLDEN SECTION CONSTRUCTION 12

We have yet another way of constructing the golden section, this time with three circles. Consider the three adjacent congruent circles with radius $r = 1$, as shown in figure 1-15.

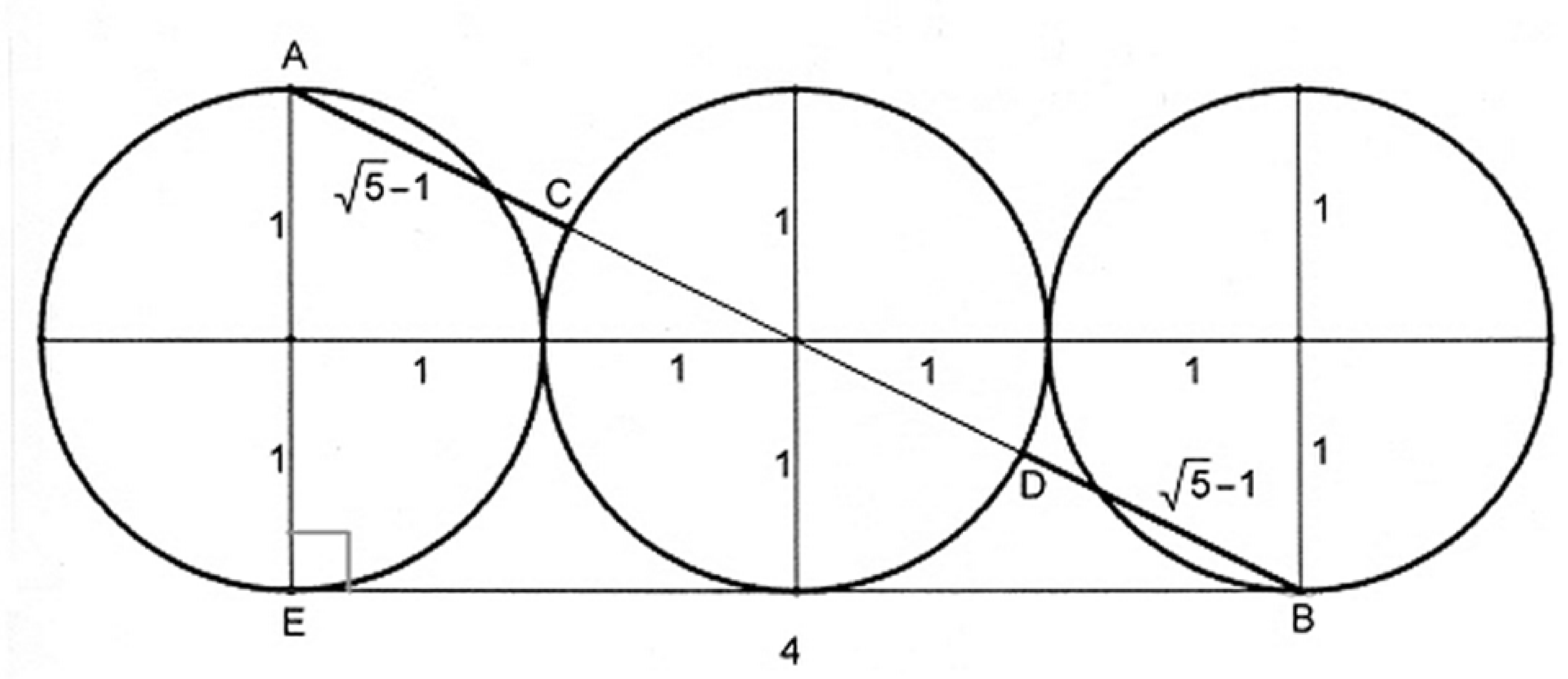

Figure 1-15

In figure 1-15, we have $AE = 2$ and $BE = 4$. We apply the Pythagorean theorem to ΔABE to get $AB = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. Because of the symmetry, $AC = BD$ and $CD = 2$, we then have $AB = AC + CD + BD = 2AC + BD = 2AC + 2$. Therefore, $2AC + 2 = 2\sqrt{5}$. It then follows that $AC = \sqrt{5} - 1$ and $AD = AB - BD = AB - AC = 2\sqrt{5} - (\sqrt{5} - 1) = \sqrt{5} + 1$.

The ratio $\frac{AD}{CD} = \frac{\sqrt{5}+1}{2} \approx 1.61803$ again denotes the golden ratio.

You may notice that each time we have been using a unit measure as our basis. We could have used a variable, such as x, and we would have gotten the same result; however, using 1 rather than x is just a bit simpler.

GOLDEN SECTION CONSTRUCTION 13

When we place the three equal unit circles tangent to each other and tangent to the semicircle, as shown in figure 1-16, we have the makings for another construction of the golden section.

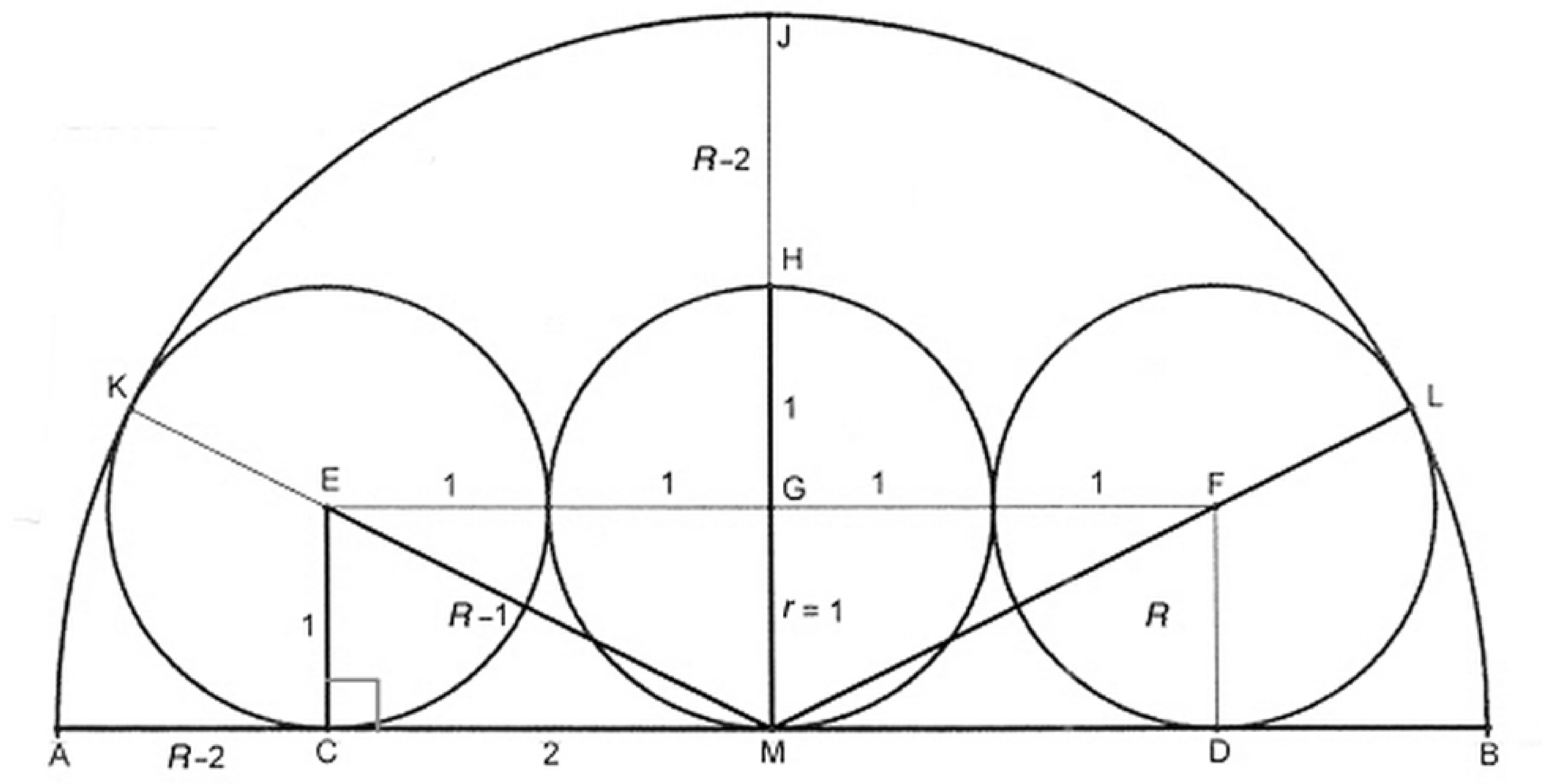

Figure 1-16

First, we note that $AM = BM = JM = KM = LM = R$, and $GH = GM = CE = DF$ (= r) = 1 (and also $CM = DM = EG = FG = 2$) and $EM = R - r = R - 1$. When we apply the Pythagorean theorem to ΔCEM in figure 1-16, we get $EM^2 = CM^2 + CE^2$, or $(R - 1)^2 = 2^2 + 1^2$.

When we solve this equation for R, we get

$$R^2 - 2R + 1 = 5$$
$$R^2 - 2R - 4 = 0$$
$$R = 1 \pm \sqrt{5}.$$

Since a radius cannot be negative, we only use the positive root of R; therefore, $R = 1 + \sqrt{5}$.

We then take the ratio $\frac{R}{r} = \sqrt{5} + 1$. Yet, half this ratio will give us the golden ratio:

$$\frac{1}{2}\left(\frac{R}{r}\right) = \frac{\sqrt{5}+1}{2}.$$

Therefore, $\frac{LM}{HM} = \frac{R}{2r} = \frac{R}{2} = \frac{\sqrt{5}+1}{2} \approx 1.61803.$

Additionally, the ratios $\frac{HM}{HJ}$ and $\frac{CM}{AC}$ also produce the golden ratio, since with $R - 2r = R - 2 = 1 + \sqrt{5} - 2 = \sqrt{5} - 1$, which then gives us

$$\frac{HM}{HJ}=\frac{CM}{AC}=\frac{2r}{R-2r}=\frac{2}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{2}.$$

GOLDEN SECTION CONSTRUCTION 14

Another construction of the golden section was popularized by Hans Walser,[10] who placed the three circles on a coordinate grid as shown in figure 1-17. This construction can be further expanded as we show here. A circle with radius length 1 is enclosed by two circles of radius length 3.

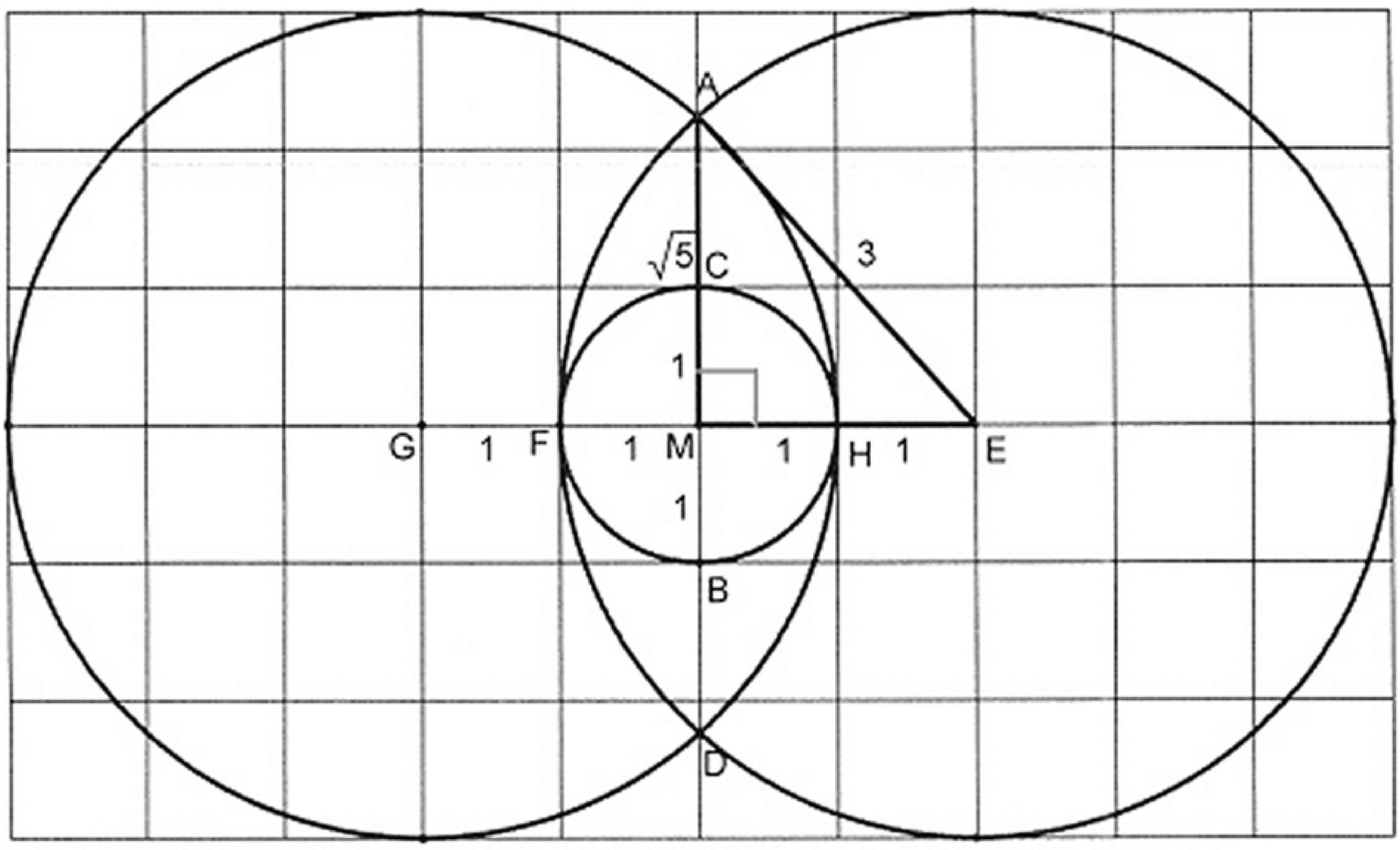

Figure 1-17

With $AE = EF = GH = 3$ and $BC = 2$, we can find the length of AM by applying the Pythagorean theorem to ΔAEM, whereupon $AM = \sqrt{3^2-2^2}=\sqrt{5}$. Since $AB=AM+BM=\sqrt{5}+1$, then we can establish

$$\frac{AB}{BC}=\frac{\sqrt{5}+1}{2}=\phi\approx 1.61803,$$

which is again recognizable as the golden ratio.

Also, the ratio $\frac{BC}{AC}$ demonstrates the golden section:

$$\frac{BC}{AC}=\frac{BC}{AM-CM}=\frac{2}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{2}.$$

We now present the classic construction of the golden section based on the work of Euclid, which is a pleasant variation of the first construction we offered. Perhaps one of the greatest contributions to our knowledge of mathematics is *Elements* by Euclid, a work divided into thirteen books that covers plane geometry, arithmetic, number theory, irrational numbers, and solid geometry. It is, in fact, a compilation of the knowledge of mathematics that existed up to his time, approximately 300 BCE. We have no records of the dates of Euclid's birth and death, and little is known about his life, though we do know that he lived during the reign of Ptolemy I (305–285 BCE) and taught mathematics in Alexandria, now Egypt. We conjecture that he attended Plato's Academy in Athens, studying mathematics from Plato's students, and later traveled to Alexandria. At the time, Alexandria was the home to a great library created by Ptolemy, known as the Museum. It is believed that Euclid wrote his *Elements* there since that city was also the center of the papyrus industry and book trade. To date, *Elements*, after over one thousand editions, presents synthetic proofs for his propositions and thereby set a standard of logical thinking that impressed many of the greatest minds of our civilization. Notable among them is Abraham Lincoln, who carried a copy of *Elements* with him as a young lawyer and would study the presented propositions on a regular basis to benefit from its logical presentations.

GOLDEN SECTION CONSTRUCTION 15

So now we come to Euclid's construction of the golden section. In figure 1-18, a right triangle, ΔABC, is constructed with legs of length 1 and $\frac{1}{2}$. An arc is drawn with center C and radius of length BC, and AC is extended to point D. A second arc is drawn with center A and tangent to the first arc, naturally passing through point D. Using the Pythagorean theorem, we can see that $BC = \frac{\sqrt{5}}{2}$; we will let the length of AD be x.

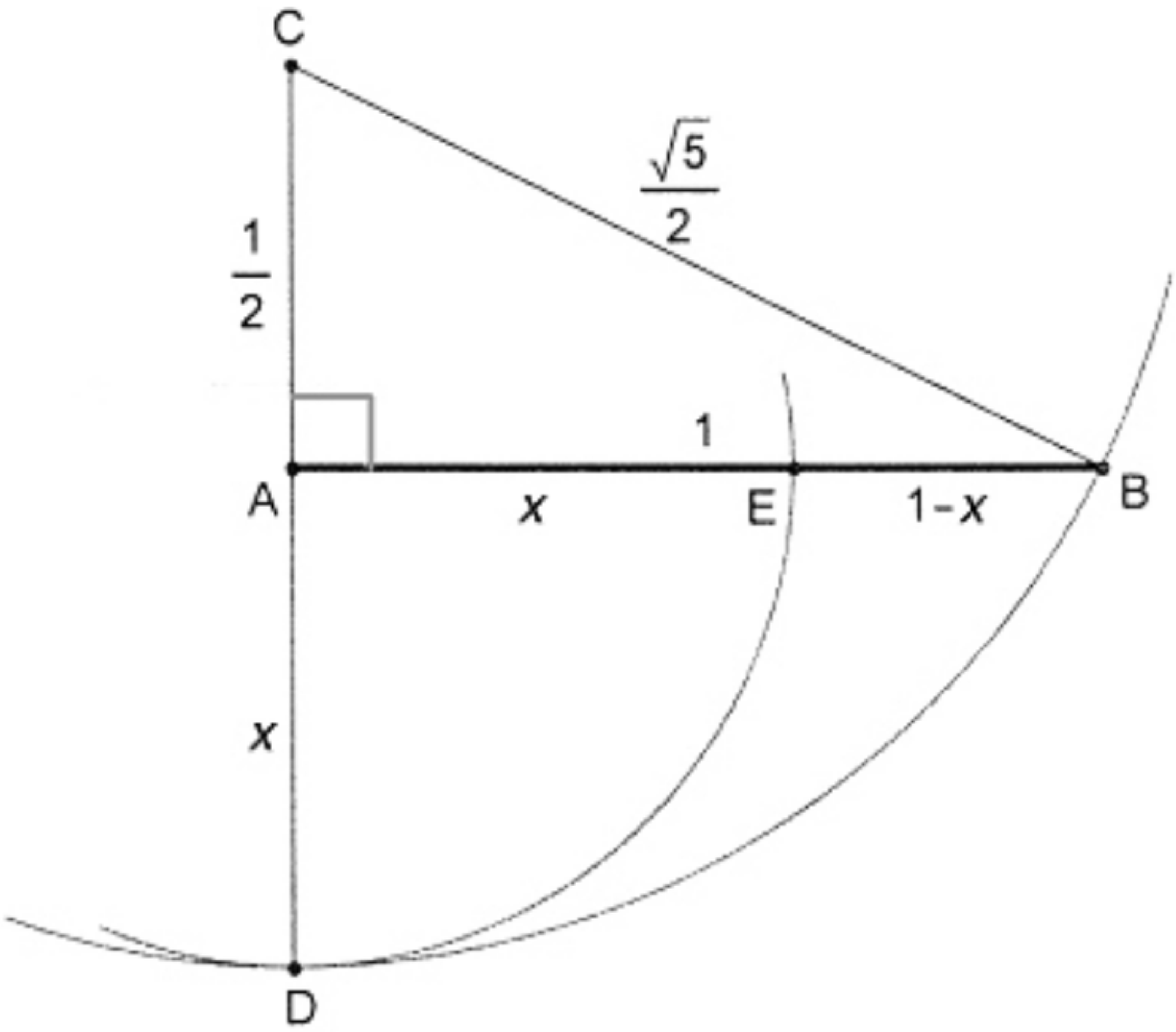

Figure 1-18

$$x = AE = AD = CD - AC = BC - AC = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}, \text{ and}$$

$$BE = AB - AE = 1 - x = 1 - \frac{\sqrt{5}-1}{2} = \frac{3-\sqrt{5}}{2}.$$

This sets up the ratio

$$\frac{AE}{BE} = \frac{\frac{\sqrt{5}-1}{2}}{\frac{3-\sqrt{5}}{2}} = \frac{\sqrt{5}-1}{3-\sqrt{5}} = \frac{\sqrt{5}-1}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}}$$

$$= \frac{3\sqrt{5}+5-3-\sqrt{5}}{9-5} = \frac{2\sqrt{5}+2}{4} = \frac{\sqrt{5}+1}{2} = \phi \approx 1.61803,$$

which is again recognizable as the golden ratio.

GOLDEN SECTION CONSTRUCTION 16

The last in our collection of constructions of the golden section is one that may look a bit overwhelming but actually is very simple, as it uses only a compass! All we need is to draw five circles.[11]

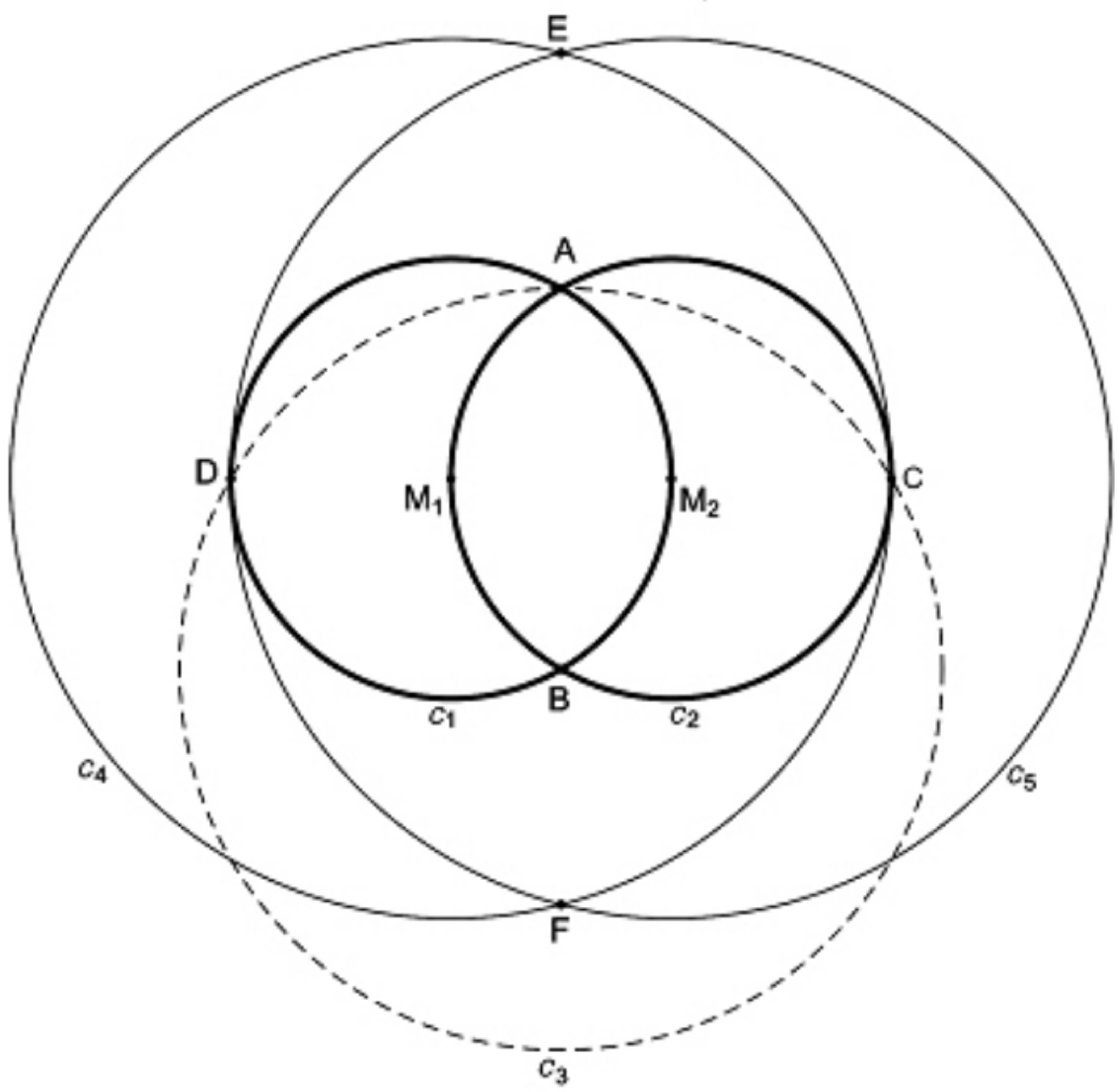

Figure 1-19

In figure 1-19, we begin by constructing circle c_1 with center M_1 and radius $r_1 = r$. Then, with a randomly selected point M_2 on circle c_1, we construct a circle, c_2, with center M_2 and radius $r_2 = r$; naturally $M_1M_2 = r$. We indicate the points of intersection of the two circles, c_1 and c_2, as *A* and *B*. Constructing circle c_3 with center *B* and radius $AB = r_3$ will intersect circles c_1 and c_2 at points *C* and *D*. (Note that the points *D*, M_1, M_2, and *C* are collinear.) We now construct circle c_4 with center at M_1 and radius $M_1C = r_4 = 2r$. Finally, circle c_5 with center M_2 and radius $M_2 D = r_5 = r_4 = 2r$ is constructed so that it intersects circle c_4 at points *E* and *F*.

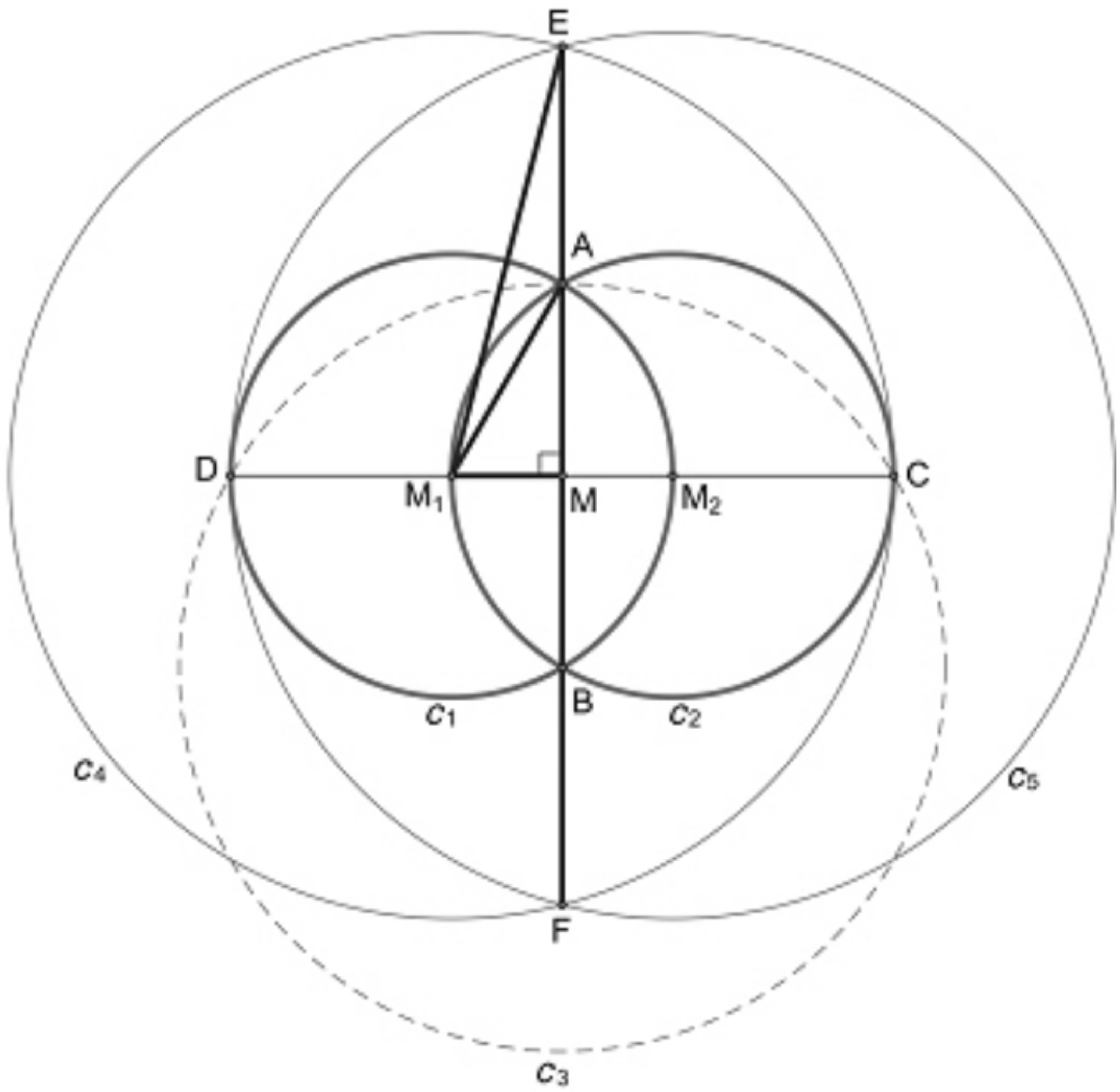

Figure 1-20

From figure 1-20, as a result of obvious symmetry, $AE = BF$, $AF = BE$, $AM = BM$, $EM = FM$, and $CM = DM$, $MM_1 = MM_2$. We can then get $\frac{AB}{AE} = \frac{BE}{AB} = \phi$ (or analogously, $\frac{AB}{BF} = \frac{AF}{AB} = \phi$).

This can be justified rather simply by inserting a few line segments. The radius of the first circle is $r_1 = r = AM_1$, and the radius of the fourth circle is $r_4 = 2r = CM_1 = EM_1$. We can apply the Pythagorean theorem to ΔAMM_1 to get $AM_1^2 = AM^2 + MM_1^2$, or

$$r^2 = AM^2 + \left(\frac{r}{2}\right)^2,$$

which then determines $AM = \frac{r}{2}\sqrt{3}$. Then, applying the Pythagorean theorem to ΔEMM_1, we get EM_1^2 ($= CM_1^2$) $= EM^2 + MM_1^2$, or

$$(2r)^2 = EM^2 + \left(\frac{r}{2}\right)^2,$$

whereupon $EM = \frac{r}{2}\sqrt{15}$.

We now seek to show that the ratio we asserted above is in fact the golden ratio.

$$\frac{AB}{AE} = \frac{AM + BM}{EM - AM} = \frac{2AM}{EM - AM} = \frac{2 \cdot \frac{r}{2}\sqrt{3}}{\frac{r}{2}\sqrt{15} - \frac{r}{2}\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{\sqrt{3}(\sqrt{5}-1)} = \frac{2}{\sqrt{5}-1} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{\sqrt{5}+1}{2} = \phi.$$

Now the second ratio that we must check is

$$\frac{BE}{AB} = \frac{EM + BM}{AM + BM} = \frac{EM + AM}{2AM}$$

$$= \frac{\frac{r}{2}\sqrt{15} + \frac{r}{2}\sqrt{3}}{2 \cdot \frac{r}{2}\sqrt{3}} = \frac{\sqrt{3}(\sqrt{5}+1)}{2\sqrt{3}} = \frac{\sqrt{5}+1}{2} = \phi.$$

In both cases we have shown that the golden ratio is in fact determined by the five circles we constructed.

We should not want to give the impression that we have covered all possible constructions of the golden section. There currently exist about forty such constructions of the golden section—with new methods being developed continually. As we mentioned, there exist a host of curious geometric configurations where the golden section can be found, but we shall leave these hiding places for later in the book. Notice, however, that our goal for construction of the golden section is to somehow get a length equal to $\sqrt{5}$. For now, we simply want to introduce the numerical value of the golden ratio and its sightings algebraically and geometrically, as it can be seen partitioning a line segment.

Chapter 2
The Golden Ratio in History

One can never say with certainty where the golden ratio first appeared in the civilized world. To the best of our knowledge, the earliest use of the golden ratio occurred in the ancient Egyptians' construction of the Great Pyramid at Giza, the only one of the "seven wonders of the ancient world" that still exists today. We might someday yet discover older examples where this ratio may be seen. What is still unknown to this day is whether the architect of this structural wonder, Hemiunu (ca. 2570 BCE), consciously chose the dimensions that yield the golden ratio, as he strove to achieve beauty in this structure, or it arose simply by chance. This and other questions about the structure of the pyramid have prompted the writing of numerous books, and yet the issue is still without a definitive conclusion.

This colossal structure, built about 2560 BCE, is the oldest and largest of three pyramids in the Giza Necropolis near modern-day Cairo, Egypt.

The Great Pyramid of Giza. Photo courtesy of Wolfgang Randt.

For years archaeologists have studied this famous pyramid inside and out. Yet for our purposes, we shall focus on its outer dimensions. We will use the cubit as the unit measure, since that was what was used in the time of the construction. (A *cubit* is the first recorded unit of length used in ancient times. It is the measure from the elbow to the tip of the middle finger, and was assumed to be a length of 52.25 cm.) The diagram of the pyramid (fig. 2-1) shows its height to be 280 cubits, half its base length as 220 cubits, and its slant height as 356 cubits.[1]

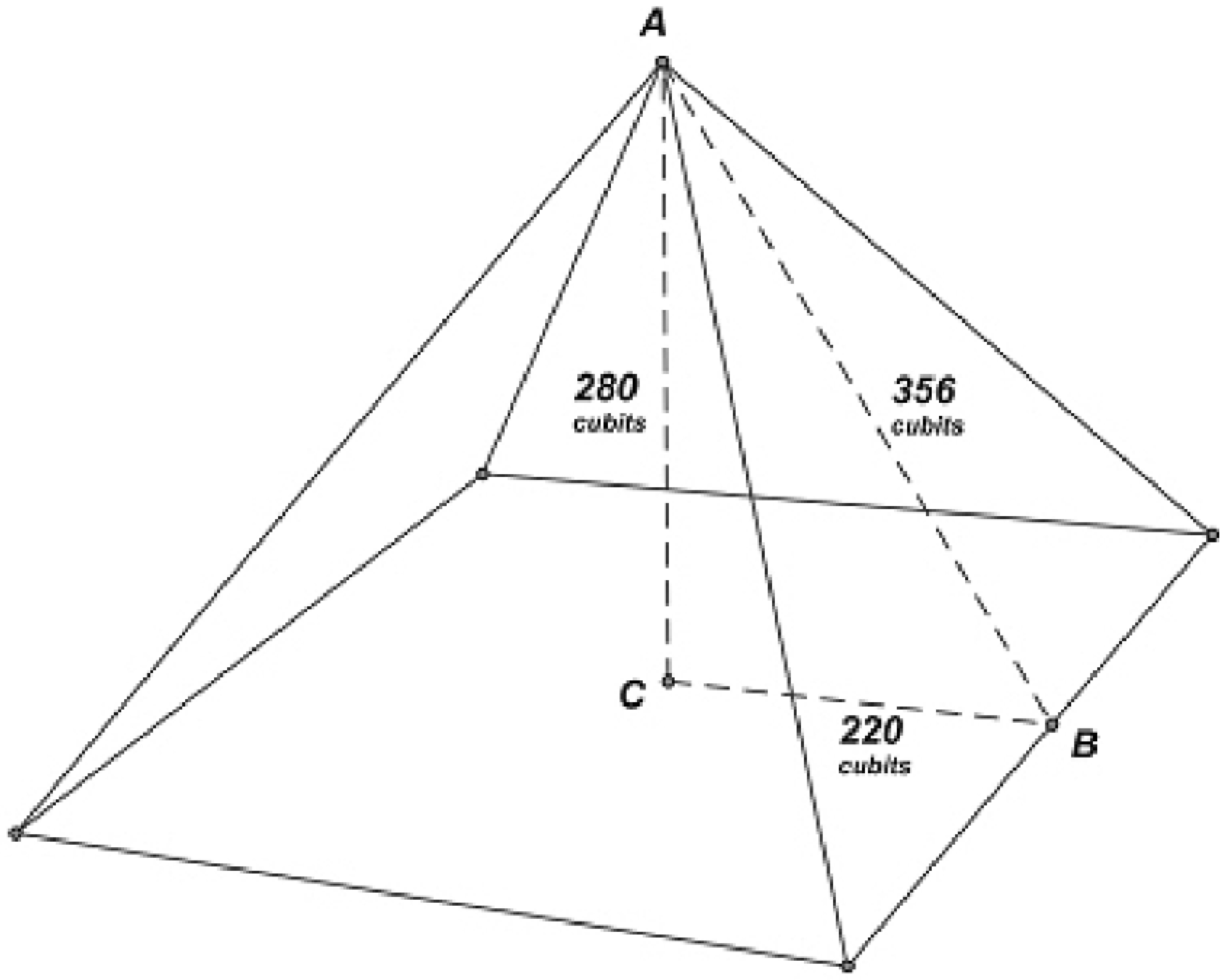

Figure 2-1

The ratio of the slant height to half the base length is $\frac{AB}{BC}=\frac{h_{\Delta}}{a/2}=\frac{356}{220}=\frac{89}{55}$ ≈1.61818, which is approximately equal to the golden ratio of 1.61803.

Furthermore, as if that isn't enough to convince you that this marvelous pyramid is based on the golden ratio, consider the ratio of the height of the pyramid to half the base length, namely $\frac{280}{220}=\frac{14}{11}=1.27272$…, which is very close to the square root of ϕ, or approximately 1.2720196…. Were we to divide each of the dimensions of triangle *ABC* (fig. 2-1) by 220, we would get a triangle with the dimensions shown in figure 2-2:

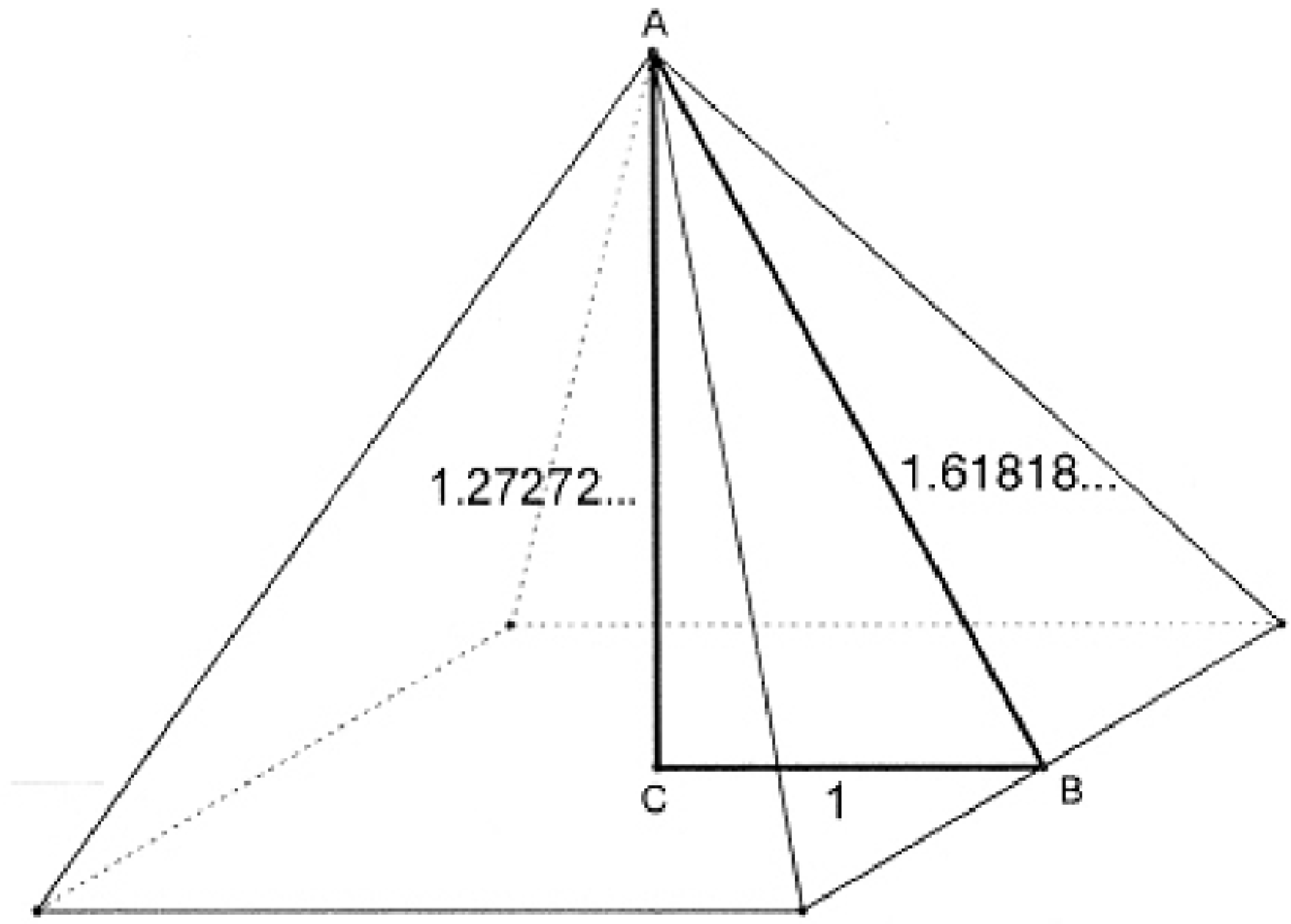

Figure 2-2

These values approximate the golden ratio in various forms, as we can see in figure 2-3a, where we have in terms of ϕ the dimensions of a similar right triangle.

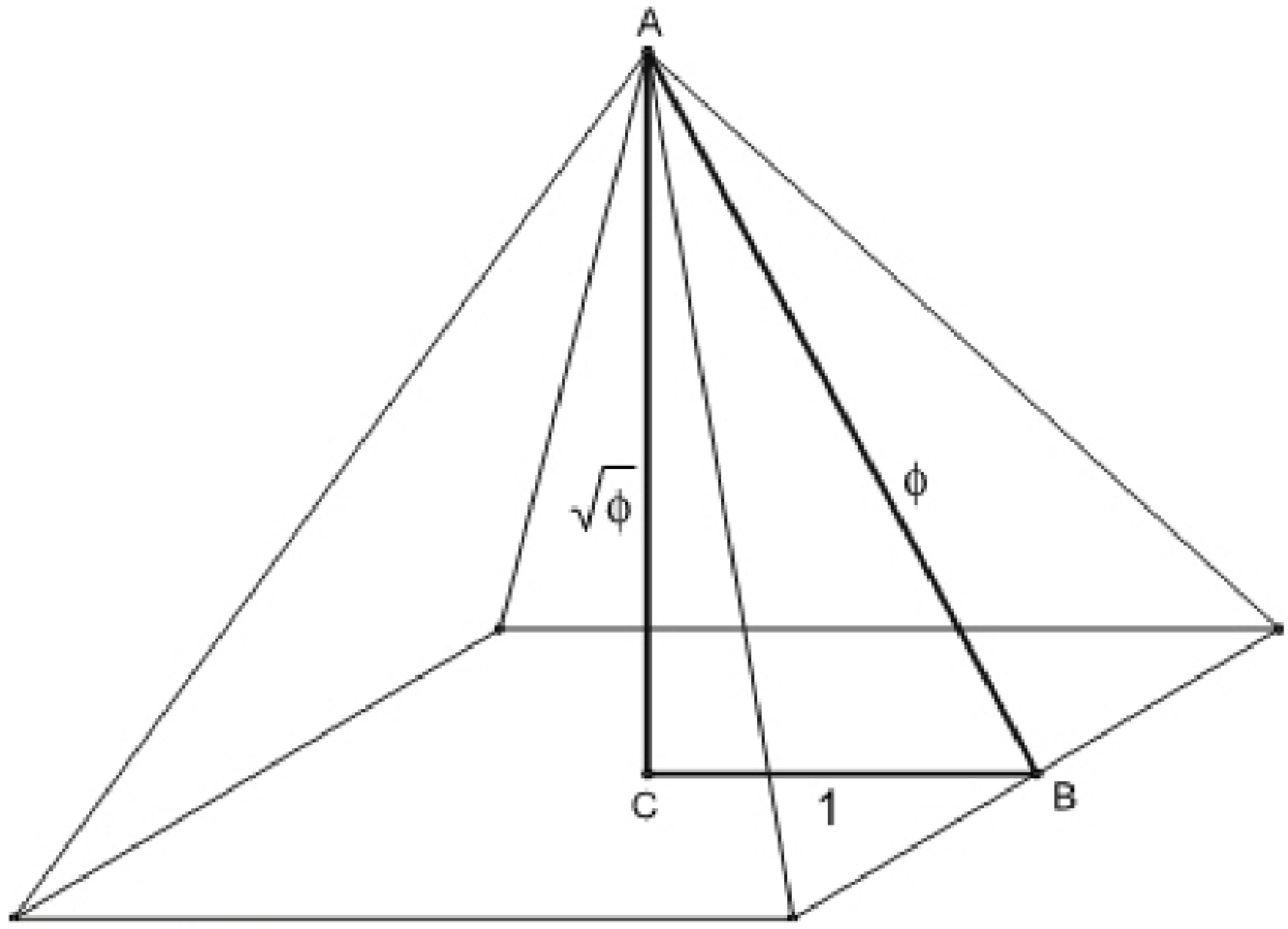

Figure 2-3a

According to the Greek historian Herodotus (ca. 484–424 BCE), the Khufu (Cheops) Pyramid at Giza was constructed in such a way that the square of the height of the pyramid is equal to the area of one of the lateral sides.[2] Once we analyze this curious relationship, we will get some rather surprising results.

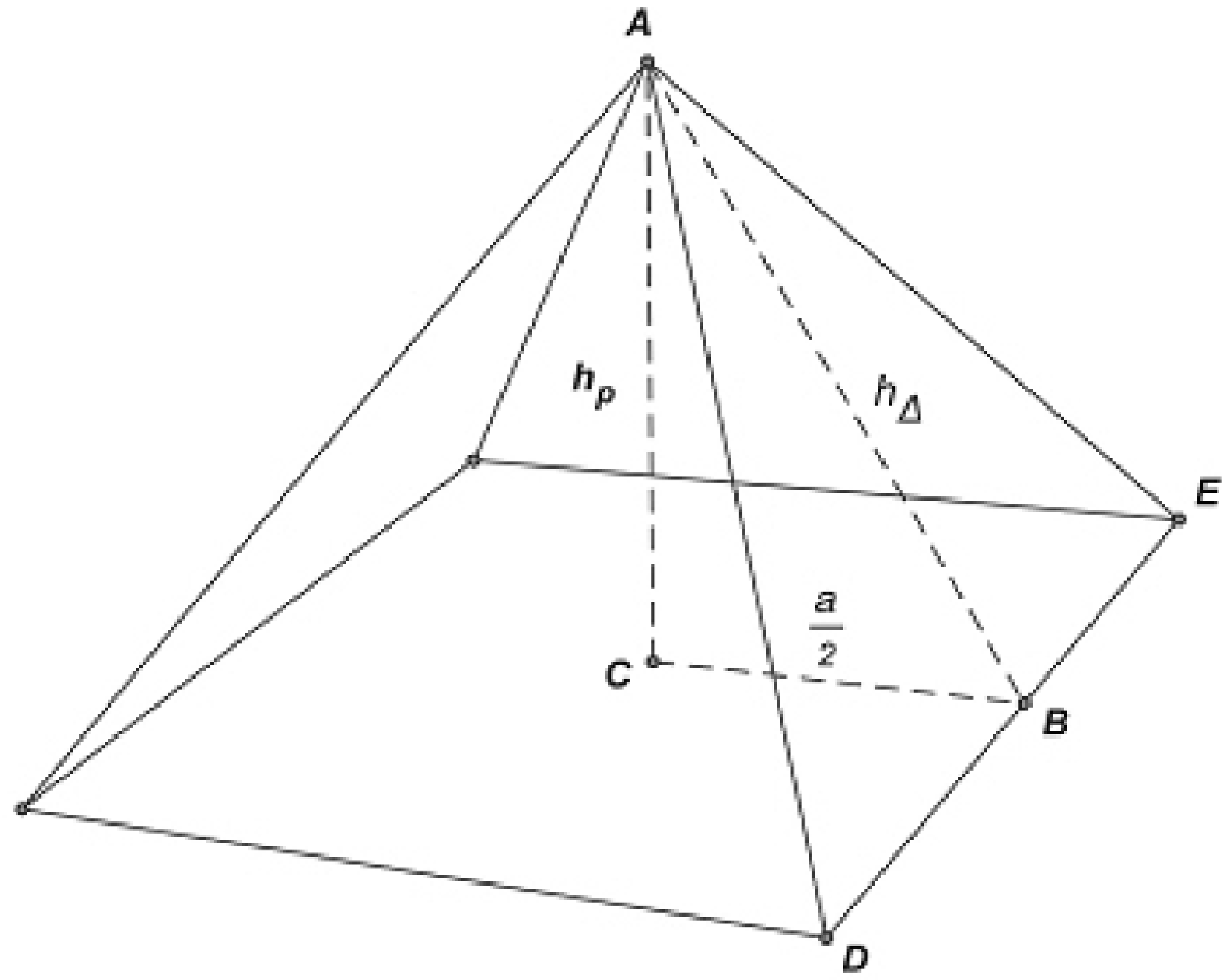

Figure 2-3b

We begin by applying the Pythagorean theorem to triangle *ABC* in figure 2-3b to arrive at the following: $h_\Delta^2 = \frac{a^2}{4} + h_P^2$.

The area of one of the lateral triangles is $A = \frac{a}{2} \cdot h_\Delta$.

Using the curious relationship Herodotus ascribed to the Giza Pyramid, we get: $h_P^2 = h_\Delta^2 - \frac{a^2}{4} = A = \frac{a}{2} \cdot h_\Delta$.

If we divide both sides of the equation $\frac{a}{2} \cdot h_\Delta = h_\Delta^2 - \frac{a^2}{4}$ by $\frac{a}{2} \cdot h_\Delta$, we get $1 = \frac{h_\Delta}{\frac{a}{2}} - \frac{\frac{a}{2}}{h_\Delta}$. By letting $x = \frac{h_\Delta}{\frac{a}{2}}$, and then taking the reciprocal, $\frac{\frac{a}{2}}{h_\Delta} = \frac{1}{x}$, and then substituting in the equation above, we get a simplified equation: $1 = x - \frac{1}{x}$, which just happens to generate the (golden) equation $x^2 - x - 1 = 0$, whose solution is the golden ratio $x_1 = \phi$ and $x_2 = -\frac{1}{\phi}$. By now you realize that x_2 is negative and holds no real meaning for us geometrically; so we won't consider it here.

Using today's measurement capabilities, this great pyramid has the following dimensions (see fig. 2-1):

Cheops pyramid	Length of the side of the base: a	Height of lateral triangle: h_Δ	Pyramid height h_p	$\frac{h_\Delta}{\frac{a}{2}}$	$\frac{C}{2h_p}$
measurements	230.56 m	186.54 m	146.65 m	1.61813471 ($\approx\phi$)	3.144357313 ($\approx\pi$)

Lo and behold, the ratio of the height of the lateral triangle to half its base is

$$\frac{h_\Delta}{\frac{a}{2}} = 1.61813471$$

Was this done intentionally by the design of a genius architect? No one knows. We can only point out what has been found by measurement and historical clues.

In the history of the golden ratio, the next significant sighting would be that of the Pythagoreans, who, among other applications, used it in their music investigations. The first recorded direct reference to this famous ratio is found in Euclid's *Elements*, which as we mentioned earlier, was a compilation of everything that was known about mathematics at the time of its writing, which was about 300 BCE. This monumental work consisted of thirteen books, in which there are two references made to the golden ratio: In book 2, proposition 11, he constructs a straight line (segment), which is cut so that a rectangle is formed by the whole segment and one of its parts (segment), that is equal to the square on the remaining segment. This can be demonstrated pictorially, as shown in figure 2-4. There we begin with the line segment *ACB*, where point *C* cuts the segment so that *CB* is used to form rectangle *ABHF*, where *CB* = *HB*, and square *ACGD* has the same area as the rectangle *ABHF*. This equality of areas can be expressed as $AC^2 = AB \cdot CB$, which then can be converted to $\frac{AB}{AC} = \frac{AC}{CB}$, and which then is referred to in a second citation in *Elements*.[3]

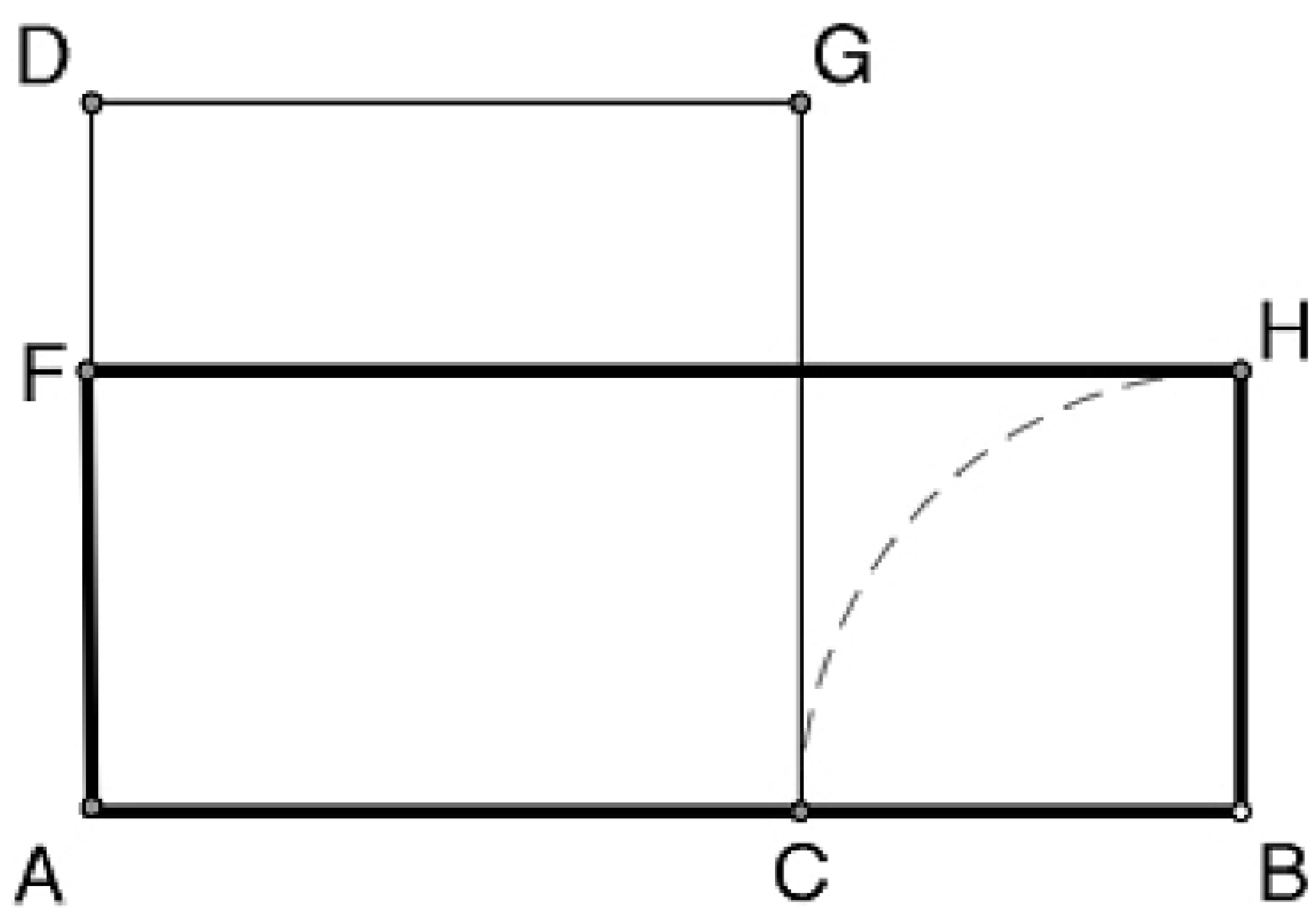

Figure 2-4

Here Euclid refers to a given straight line that is cut—or sectioned — into a mean and extreme ratio,[4] namely for the line segment *AB*, containing point *C*, thus we get $\frac{AB}{AC}=\frac{AC}{CB}$. This is precisely our definition of the golden section.

As we survey the history of the golden ratio, we find its next prominent display in the works of the great Greek sculptor Phidias (ca. 490–ca. 430 BCE). His design for the construction of the Parthenon in Athens, Greece (fig. 2-5), as well as the sculptures he made to adorn this structure, such as the famous statue of Zeus, are said to be reflective of this beautiful ratio. As a matter of fact, the Greek letter ϕ is used by many mathematicians today (as well as in this book) to represent the golden section, as it is the first letter of Phidias's name when written in Greek as *Φειδίας*.[5] As you can see in figure 2-5, the Parthenon in Athens, Greece, fits nicely into a golden rectangle—that is, a rectangle where the quotient of the sides is the golden ratio (see chapter 4). Furthermore, in figure 2-5a you will notice a number of additional golden ratios. Yet even today, no one can say with certainty that Phidias had the golden ratio in mind when he designed the structure.

Figure 2-5. Parthenon in Athens, Greece.

And some more:

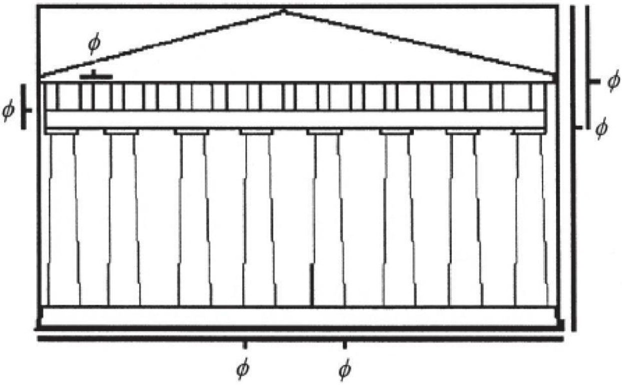

Figure 2-5a

As we continue to trace the history of the golden ratio, we find the next significant sighting in a three-volume book titled *De Divina Proportione* (*The Divine Proportion*), written in 1509 by the Franciscan friar and mathematician Fra Luca Pacioli (ca. 1445–1514 or 1517). The book contains drawings by the Italian painter, sculptor, architect, and also mathematician[6] Leonardo da Vinci (1452–1519) of the five Platonic solids. DaVinci also drew the Vitruvian Man (fig. 2-6), in about 1487. This is a picture of a man's body, which clearly exhibits a very close approximation to the golden ratio.

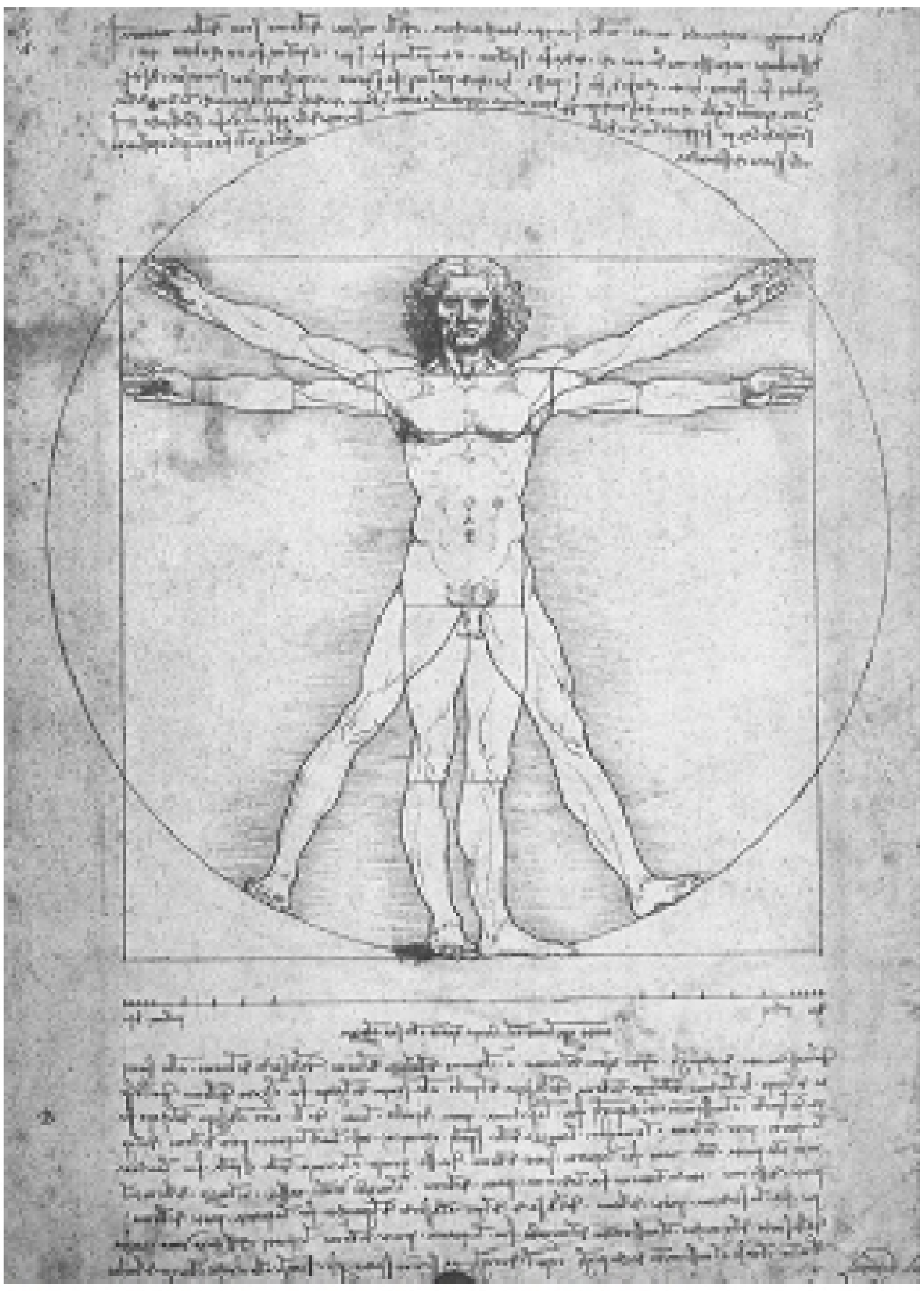

Figure 2-6. The Vitruvian Man.

Da Vinci provided notes based on the work of Marcus Vitruvius Pollio (ca. 84–ca. 27 BCE), an ancient Roman writer, architect, and engineer. The drawing, which is in the possession of the Gallerie dell' Accademia in Venice, Italy, is often considered one of the early breakthroughs of pictorially depicting a perfectly proportioned human body. Apparently da Vinci derived these geometric proportions from Vitruvius's treatise *De Architectura*, book 3.

The drawing shows a male figure in two superimposed positions with his arms and legs apart and inscribed in a circle and square, which are tangent at only one point. The golden ratio is exhibited in that the distance from the navel to the top of his head divided by the distance from the soles of the man's feet to his navel (which appears to be at the center of the circle, as shown in figure 2-7), which is about 0.656, approximates the golden ratio (which we know is 0.618…).

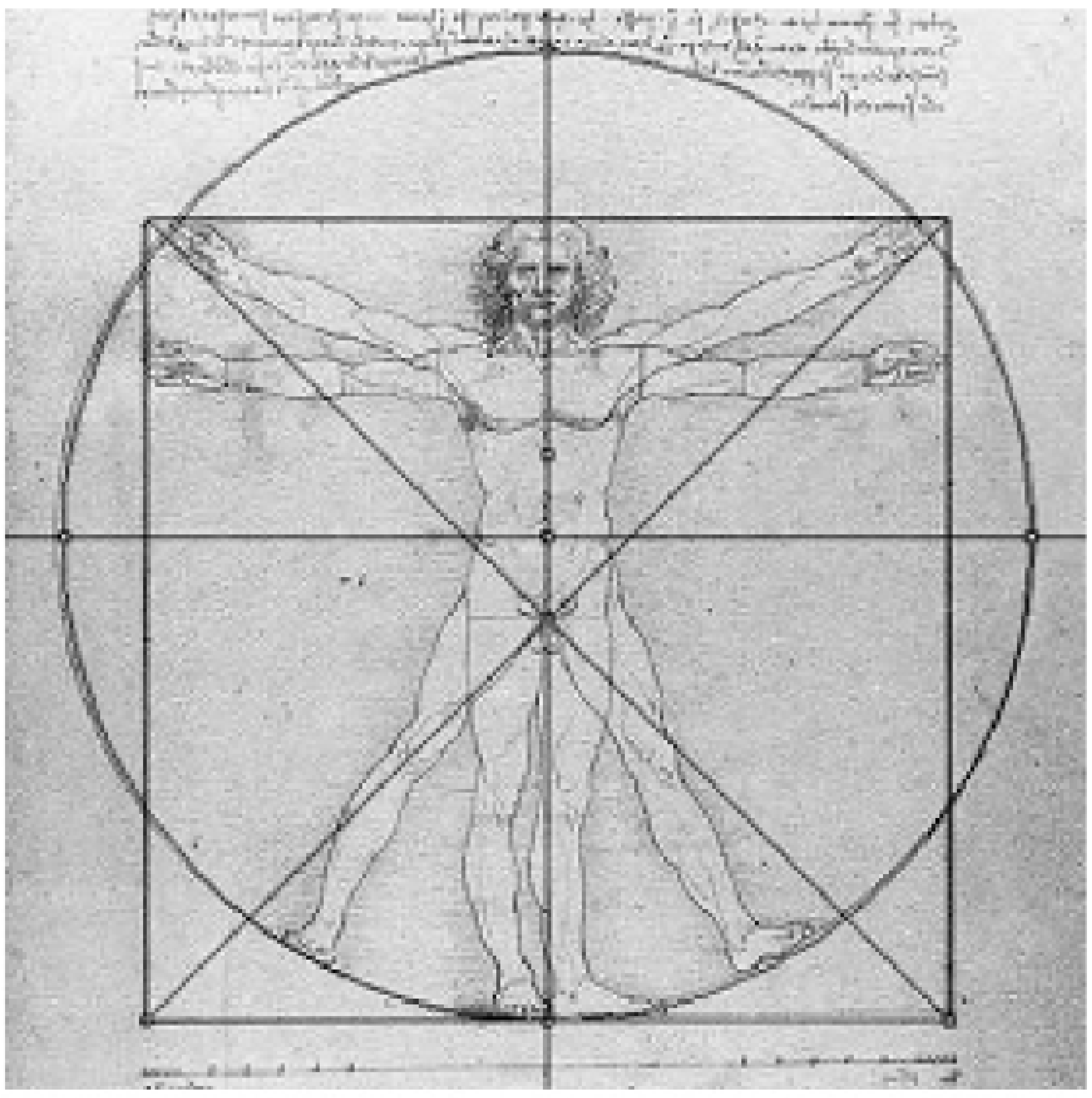

Figure 2-7

Had the square's upper vertices been somewhat closer to the circle, then the golden ratio would have been attained. This can be seen in figure 2-8, where the radius of the circle is selected to be 1, and the side of the square is 1.618, approximately equal to ϕ, the golden ratio.

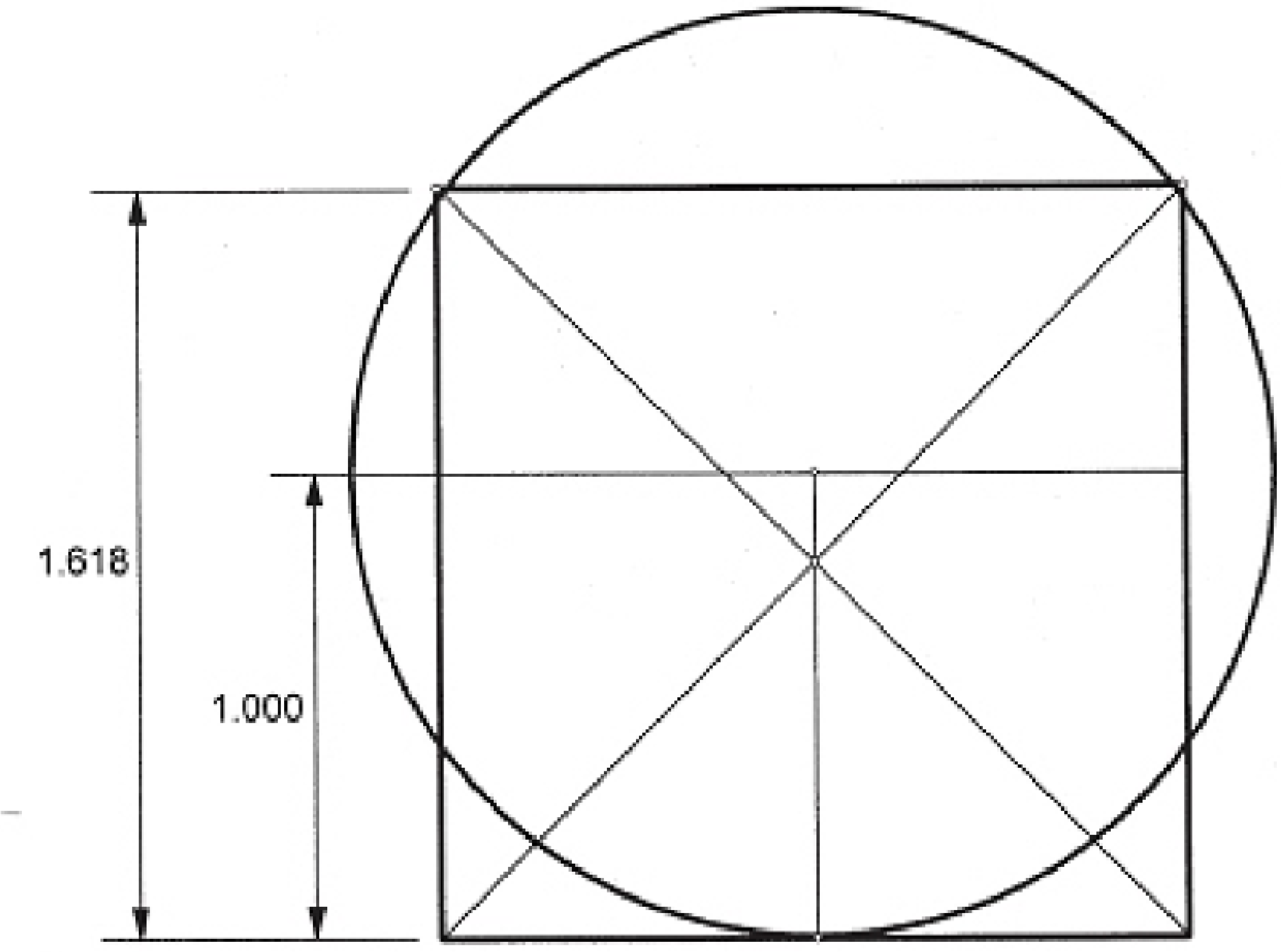

Figure 2-8

To what extent architects and artists consciously used the golden section in their work will remain a mystery because there are no documents that clearly establish its use. Our desire to find examples of the golden ratio might also play into the many sightings. Still, there are countless examples of where it is believed that the golden ratio appears in art and architecture. Many such examples appear throughout the Internet[7] There are also those who are skeptical about the authenticity of these sightings, such as Dan Pedoe,[8] George Markowsky,[9] Marguerite Neveux,[10] and Roger Herz-Fischler.[11] On the other hand, there are structures (for example, by Le Corbusier),[12] sculptures (such as by Étienne Béothy),[13] or paintings and graphics (for example, by Jo Niemeyer)[14] that document the use of the golden section in their design.[15]

In chapter 3 we will explore the extraordinary relationship between the golden ratio and the Fibonacci sequence, which was popularized by the French mathematician Edouard Lucas (1842–1891) and named after the Italian mathematician Fibonacci (or Leonardo of Pisa, ca. 1175–after 1240). Not only is Lucas credited with discovering characteristics of the numbers based on the regeneration of rabbits problem found in chapter 12 of Fibonacci's book *Liber Abaci* (1202), but he is also responsible for establishing an analogous sequence of numbers that carries his name. Among the relationships he discovered is that which relates the *Lucas numbers* and the *Fibonacci numbers* to the golden ratio. These relationships and much more will be explored in chapter 3.

Chapter 3

The Numerical Value of the Golden Ratio and Its Properties

In the previous chapters, we have established that the golden ratio between *a* and *b* is $\frac{a+b}{b}=\frac{b}{a}$, where *a* and *b* are positive real numbers. As with all ratios, this one has a very specific numerical value. To get the numerical value of this ratio, we first must set up the equation that we get from this ratio by equating the product of the means and extremes, namely $b^2 = a(a + b) = a^2 + ab$. This equation can be written as $b^2 - ab - a^2 = 0$ and can be solved for either *a* or *b*; say, we solve for *b*. Using the formula for solving quadratic equations,[1] we find that

$$b=\frac{a\left(1+\sqrt{5}\right)}{2}=a\frac{\sqrt{5}+1}{2}.$$

Since a length (*a*, *b*) cannot be negative, we ignored the negative root

$$\frac{a\left(1-\sqrt{5}\right)}{2}=-a\frac{\sqrt{5}-1}{2}.$$

Therefore, by dividing both sides of this equation by *a*, we get

$$\frac{b}{a}=\frac{\sqrt{5}+1}{2},$$

which is then the value of the golden ratio, ϕ. Numerically, this is approximately[2] equal to:

$$\phi=\frac{\sqrt{5}+1}{2}$$

≈1.618033988749894848204586834365638117720309179805762862135448622705260462818902449707207204189391137484754088075386891752126633862223536931793180060766726354433389086595939582905638322661319928290267880675208766892501711696207 03

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9615617002504643382437764861028383126833037242926752631165
3392473167111211588186385133162038400522216579128667529465
4906811317159934323597349498509040947621322298101726107059
6116456299098162905552085247903524060201727997471753427775
9277862561943208275051312181562855122248093947123414517022
3735805772786160086883829523045926478780178899219902707769
0389532196819861514378031499741106926088674296226757560523
172777520353613936, which we approximate to 1.61803.

Now if we take the reciprocal of $\phi=\frac{1+\sqrt{5}}{2}$ to get $\frac{1}{\phi}=\frac{2}{1+\sqrt{5}}$ and then multiply this fraction by 1 in the form of $1=\frac{1-\sqrt{5}}{1-\sqrt{5}}$, we get $\frac{2}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}=\frac{\sqrt{5}-1}{2}=\frac{1}{\phi}$, which then gives us the approximate value[3]:

$$\frac{1}{\phi}=\frac{\sqrt{5}-1}{2}$$

≈.61803398874989484820458683436563811772030917980576286213
5448622705260462818902449707207204189391137484754088075386
8917521266338622235369317931800607667263544333890865959395
8290563832266131992829026788067520876689250171169620703222
1043216269548626296313614438149758701220340805887954454749
2461856953648644492410443207713449470495658467885098743394
4221254487706647809158846074998871240076521705751797883416
6256249407589069704000281210427621771117778053153171410117
0466659914669798731761356006708748071013179523689427521948
4353056783002287856997829778347845878228911097625003026961
5617002504643382437764861028383126833037242926752631165339
2473167111211588186385133162038400522216579128667529465490
6811317159934323597349498509040947621322298101726107059611
6456299098162905552085247903524060201727997471753427775927
7862561943208275051312181562855122248093947123414517022373
5805772786160086883829523045926478780178899219902707769038
9532196819861514378031499741106926088674296226757560523172
777520353613936, which we approximate to 0.61803.

We see that the value of ϕ has a unique characteristic. Aside from the usual fact that the product of a number and its reciprocal is 1, which, here, gives us $\phi \cdot \frac{1}{\phi} = 1$, the difference of ϕ and its reciprocal, $\frac{1}{\phi}$, is surprisingly also 1, that is, $\phi - \frac{1}{\phi} = 1$. This is the only number for which this is true!

The not-too-well-known mathematician Michael Maestlin (1550–1631), who happened to be one of Johannes Kepler's teachers and later his friend, is credited with the first expansion of the value of ϕ to a five-place accuracy, as $\phi \approx$ **1.6180**340, in 1597, while at the University of Tübingen (Germany). As with most famous numbers in mathematics, there is always a desire to seek greater accuracy of a value. This means calculating the value to a larger number of decimal places. Naturally, today we can use computers to facilitate this goal; here is a short history of these milestones of the recent past.

Year	**Number of places of the value of ϕ**	**Mathematician**
1966	4,599	M. Berg
1976	10,000	J. Shallit
1996	10,000,000	G. J. Fee and S. Plouffe
2000	1,500,000,000	X. Gourdon and P. Sebah
2007	5,000,000,000	A. Irlande
2008	17,000,000,000	A. Irlande
2008	31,415,927,000	X. Gourdon and P. Sebah
2008	100,000,000,000	S. Kondo and S. Pagliarulo
2010	1,000,000,000,000	A. Yee

Having now established the numerical value of the golden ratio, let us inspect some of the properties of this most unusual number. We begin by considering the irrationality of ϕ. To do this, we will embark on a nifty little excursion through some simple number theory. The realm of *real* numbers is composed of *rational* and *irrational* numbers. They can be either positive or negative. When expressed in decimal form, the rational numbers are either terminating decimals or repeating decimals, while the irrational numbers do not repeat with any repeating pattern and continue indefinitely. Another way of distinguishing these numbers is that only the rational numbers can be expressed as quotients of integers.

Here are some examples:

Rational numbers: 3;

$$-\frac{1}{2} = -0.5000 \ldots = -0.5\overline{0} = 0.5;$$

$$\frac{2}{3} = 0.666 \ldots = 0.\overline{6}.$$

Irrational numbers: $\sqrt{2} = 1.414213562 \ldots;$

$$\pi = 3.141592653 \ldots;$$

$$e = 2.718281828 \ldots.$$

We claim that the number ϕ is an irrational number—one that has an unending decimal value—one that has no repeating pattern. We can establish that $\sqrt{5}$ is irrational and therefore

$$\frac{\sqrt{5}+1}{2} = \phi$$

would also be irrational. To prove that $\sqrt{5}$ is irrational, we begin by supposing the contrary, namely that $\sqrt{5}$ is rational, implying that $\sqrt{5} = \frac{p}{q}$, a fraction that may be assumed to be in lowest terms. Squaring both sides and clearing denominators, we get $5q^2 = p^2$. Thus the left-hand side is divisible by 5 and therefore so is the right-hand side. But 5 is a prime. Therefore, since 5 divides p^2, it must also divide p. Thus, $p = 5r$, for some r. Then we have $5q^2 = p^2 = 25r^2$, so that $q^2 = 5r^2$. Repeating the previous argument, we find that q is also divisible by 5, contradicting our assumption that the fraction was in lowest terms. Therefore, $\sqrt{5}$ is not rational. Thus, ϕ must then be an irrational number.

As we will see in chapter 4, the irrationality of ϕ will be realized by the fact that the diagonal of a regular pentagon is incommensurate with a side of the pentagon, which means they have no common measure. Similarly, the irrationality of π is seen by the fact that the diameter of a circle and its circumference have no common measure.

We continue by considering the powers of ϕ. To do so, we first must find the value of ϕ^2 in terms of ϕ.
Since

$$\phi = \frac{\sqrt{5}+1}{2},$$

$$\phi^2 = \left(\frac{\sqrt{5}+1}{2}\right)^2 = \frac{5+2\sqrt{5}+1}{4} =$$

$$\frac{2\sqrt{5}+6}{4} = \frac{\sqrt{5}+3}{2} = \frac{\sqrt{5}+1}{2}+1 = \phi+1.$$

You may find this equation $\phi^2 = \phi + 1$ somewhat familiar, as we have encountered it at several points already. From this equation, we can generate an interesting mathematical expression to express the value of ϕ in a very unusual fashion. By taking the square root of both sides, this equation can be rewritten as $\phi = \sqrt{\phi+1}$, or $\phi = \sqrt{1+\phi}$. We will now replace ϕ under the radical sign with its equivalent $\phi = \sqrt{1+\phi}$ to get

$$\phi = \sqrt{1+\left(\sqrt{1+\phi}\right)} = \sqrt{1+\sqrt{1+\phi}}.$$

Then, repeating this process (i.e., replacing the last ϕ with $\phi = \sqrt{1+\phi}$),
we get

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\phi}}}.$$

Continuing this process gives us

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\phi}}}},$$

and so on, until you realize that this will go on to infinity and look something like

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}}}}}}}}}}}}$$

Suppose we now consider the following analogous nest of radicals:

$$x=\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\ldots}}}}}}}}}}}}}}}}$$

We can evaluate this value for x by using the following technique: There is an infinite number of radicals in this nest. Without loss of accuracy, we can temporarily "ignore" the outermost radical and see that the remaining expression is actually the same as the original one:

$$x=\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\ldots}}}}}}}}}}}}}}}}$$

So if we substitute this value of x into the original expression, we get $x=\sqrt{1-x}$. By squaring both sides, we get the following quadratic equation: $x^2 = 1 - x$, or $x^2 + x - 1 = 0$. When we apply the quadratic formula to this equation, we get (ignoring the negative root)

$$x=\frac{\sqrt{5}-1}{2}\approx 0.61803,$$

which is $\frac{1}{\phi}$. Again, we have a most unusual relationship between ϕ and $\frac{1}{\phi}$:

$$\phi=\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}}}}}}}}}}}}}$$

and

$$\frac{1}{\phi}=\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\ldots}}}}}}}}}}}}}}}}$$

Let us now investigate powers of ϕ. In order to inspect the successive powers of ϕ, we will break them down into their component parts. It may at first appear more complicated than it really is. You should try to follow each step (it's really not difficult—and yet very rewarding!) and then extend it to further powers of ϕ.

$\phi=\phi$
$\phi^2=\phi+1$
$\phi^3=\phi\cdot\phi^2=\phi(\phi+1)=\phi^2+\phi=(\phi+1)+\phi=2\phi+1$
$\phi^4=\phi^2\cdot\phi^2=(\phi+1)(\phi+1)=\phi^2+2\phi+1=(\phi+1)+2\phi+1=3\phi+2$
$\phi^5=\phi^3\cdot\phi^2=(2\phi+1)(\phi+1)=2\phi^2+3\phi+1=2(\phi+1)+3\phi+1=5\phi+3$
$\phi^6=\phi^3\cdot\phi^3=(2\phi+1)(2\phi+1)=4\phi^2+4\phi+1=4(\phi+1)+4\phi+1=8\phi+5$
$\phi^7=\phi^4\cdot\phi^3=(3\phi+2)(2\phi+1)=6\phi^2+7\phi+2=6(\phi+1)+7\phi+2=13\phi+8$
$\phi^8=\phi^4\cdot\phi^4=(3\phi+2)(3\phi+2)=9\phi^2+12\phi+4=9(\phi+1)+12\phi+4=21\phi+13$
$\phi^9=\phi^5\cdot\phi^4=(5\phi+3)(3\phi+2)=15\phi^2+19\phi+6=15(\phi+1)+19\phi+6=34\phi+21$
$\phi^{10}=\phi^5\cdot\phi^5=(5\phi+3)(5\phi+3)=25\phi^2+30\phi+9=25(\phi+1)+30\phi+9=55\phi+34$
and so on….

By this point, you should be able to see a pattern emerging. As we take further powers of ϕ, the end result of each power of ϕ is actually equal to a multiple of ϕ plus a constant. Further inspection shows that the coefficients of ϕ and the constants follow the pattern 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,…. This sequence of numbers is famous and is known as the *Fibonacci sequence*.[4] Beginning with two 1s, each successive number is the sum of the two preceding numbers. The Fibonacci numbers are perhaps the most ubiquitous numbers in all of mathematics; they come up in just about every field of the subject. Yet, as we mentioned earlier, they only made their “debut” in the Western world in chapter 12 of a 1202 publication, *Liber Abaci*, by Leonardo of Pisa, most commonly known today as Fibonacci (ca. 1175–after 1240), in the solution of a simple problem about the breeding of rabbits.

We recently discovered that the Fibonacci numbers were described in early Indian mathematics writings.[5] The earliest appearance can be found under the name *mātrāmeru* (Mountain of Cadence), which appeared in *Chandahsūtras* (*Art of Prosody*) by the Sanskrit grammarian Pingala (between the fifth and second century BCE). In a more complete fashion were the writings of Virahānka (sixth century CE) and Ācārya Hemacandra (1089–1172), who cites the Fibonacci numbers. It is speculated that Fibonacci may have come to these numbers from his Arabic sources, which exposed him to these Indian writings.

Sometime before his death in 1564, the German calculation master Simon Jacob[6] made the first published connection between the golden ratio and the Fibonacci series, but it appears to have been something of a side note.[7] Jacob had published a numerical solution for the golden ratio. In the margin of the page discussing the Euclidean algorithm from the second proposition of book 7 of Euclid's *Elements*, he wrote the first twenty-eight terms of the Fibonacci sequence and stated:

> In following this sequence one comes nearer and nearer to that proportion described in the 11th proposition of the 2nd book and the 30th of the 6th book of Euclid, and though one comes nearer and nearer to this proportion it is impossible to reach or to overcome it.

We will use the symbol F_7 to represent the seventh Fibonacci number, and F_n to represent the nth Fibonacci number, or as we say, the general Fibonacci number, that is, any Fibonacci number. Therefore, in general terms, we would write the rule of the Fibonacci numbers as $F_{n+2} = F_n + F_{n+1}$ with $n \geq 1$, and $F_1 = F_2 = 1$.

Let us look at the first thirty Fibonacci numbers.

$F_1 = 1$	$F_{11} = 89$	$F_{21} = 10,946$
$F_2 = 1$	$F_{12} = 144$	$F_{22} = 17,711$
$F_3 = 2$	$F_{13} = 233$	$F_{23} = 28,657$
$F_4 = 3$	$F_{14} = 377$	$F_{24} = 46,368$
$F_5 = 5$	$F_{15} = 610$	$F_{25} = 75,025$
$F_6 = 8$	$F_{16} = 987$	$F_{26} = 121,393$
$F_7 = 13$	$F_{17} = 1,597$	$F_{27} = 196,418$
$F_8 = 21$	$F_{18} = 2,584$	$F_{28} = 317,811$
$F_9 = 34$	$F_{19} = 4,181$	$F_{29} = 514,229$
$F_{10} = 55$	$F_{20} = 6,765$	$F_{30} = 832,040$

The list of powers of ϕ can easily be extended by using the Fibonacci numbers directly in the pattern we developed above.

$$\begin{aligned}
\phi &= 1\phi + 0\\
\phi^2 &= 1\phi + 1\\
\phi^3 &= 2\phi + 1\\
\phi^4 &= 3\phi + 2\\
\phi^5 &= 5\phi + 3\\
\phi^6 &= 8\phi + 5\\
\phi^7 &= 13\phi + 8\\
\phi^8 &= 21\phi + 13\\
\phi^9 &= 34\phi + 21\\
\phi^{10} &= 55\phi + 34\\
\phi^{11} &= 89\phi + 55\\
\phi^{12} &= 144\phi + 89\\
\phi^{13} &= 233\phi + 144\\
\phi^{14} &= 377\phi + 233\\
&\ldots
\end{aligned}$$

Since the Fibonacci numbers appear as the coefficients of ϕ, as well as the constants, we can write all powers of ϕ in a linear form: $\phi^n = a\phi + b$, where a and b are consecutive Fibonacci numbers. In the general case, we can write this as: $\phi^n = F_n\phi + F_{n-1}$, with $n \geq 1$ and $F_0 = 0$. (See the appendix for a proof of this statement.)

You should also take note that each power of ϕ is the sum of the two immediately preceding powers of ϕ. We can develop another amazing pattern involving the Fibonacci numbers and the golden ratio. This will involve a structure that is called a *continued fraction*. We will begin with a brief introduction to continued fractions.[8] A continued fraction is a fraction in which the denominator contains a mixed number (a whole number and a proper fraction). We can take an improper fraction such as $\frac{13}{7}$ and express it as a mixed number: $1\frac{6}{7} = 1 + \frac{6}{7}$. Without changing the value, we could then write this as

$$1 + \frac{6}{7} = 1 + \cfrac{1}{\frac{7}{6}},$$

which in turn could be written (again without any value change) as

$$1+\cfrac{1}{1+\cfrac{1}{6}}.$$

This is a continued fraction. We could have continued this process, but when we reach a unit fraction[9] (as in this case, the unit fraction is $\frac{1}{6}$), we are essentially finished.

To enable you to get a better grasp of this technique, we will create another continued fraction. We will convert $\frac{12}{7}$ to a continued fraction form. Notice that, at each stage, when a proper fraction is reached, we take the reciprocal of the reciprocal (e.g., change

$$\frac{2}{5} \text{ to } \cfrac{1}{\cfrac{5}{2}}$$

as we will do in the example that follows), which does not change its value:

$$\frac{12}{7}=1+\frac{5}{7}=1+\cfrac{1}{\cfrac{7}{5}}=1+\cfrac{1}{1+\cfrac{2}{5}}=1+\cfrac{1}{1+\cfrac{1}{\cfrac{5}{2}}}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2}}}.$$

If we break up a continued fraction into its component parts (called *convergents*),[10] we get closer and closer to the actual value of the original fraction.

First convergent of $\frac{12}{7}$: $\quad 1.$

Second convergent of $\frac{12}{7}$: $\quad 1+\frac{1}{1}=2.$

Third convergent of $\frac{12}{7}$: $\quad 1+\cfrac{1}{1+\cfrac{1}{2}}=1+\frac{2}{3}=1\frac{2}{3}=\frac{5}{3}.$

Fourth convergent of $\frac{12}{7}$: $\quad 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2}}}=\frac{12}{7}.$

The above examples are all *finite* continued fractions, which are equivalent to rational numbers (those that can be expressed as simple fractions). It would then follow that an irrational number would result in an *infinite* continued fraction. And that is exactly the case. A simple example of an infinite continued fraction is that of $\sqrt{2}$.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}}}}}.$$

We have a short way to write a long (in this case infinitely long) continued fraction: [1; 2,2,2,2,2,2,2,…], or when there are these endless repetitions, we can write it in an even shorter form as $[1;\overline{2}]$, where the bar over the 2 indicates that the 2 repeats endlessly.

In general, we can represent a continued fraction as

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}},$$

where a_i are real numbers and $a_i \neq 0$ for $i > 0$. We can write this in short fashion as $[a_0; a_1, a_2, a_3, \ldots, a_{n-1}, a_n]$.

Now that the concept of a continued fraction has been described, we can apply it to the golden ratio. We begin with the equation of the golden ratio: $\phi = 1 + \frac{1}{\phi}$. If we substitute $1 + \frac{1}{\phi}$ for the ϕ in the denominator of the fraction of this equation, we get

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{\phi}} = [1;\, 1, \phi],$$

and then continue this process by substituting the value $\phi = 1 + \frac{1}{\phi}$ in each case for the last denominator of the previous equation, we will get the following:

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{\left(1 + \cfrac{1}{\phi}\right)}} = [1;\, 1, 1, \phi].$$

Repeating this procedure, we get an *infinite* continued fraction that looks like this:

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}},$$

or $\phi = [1;\, 1, 1, 1, \ldots] = [\bar{1}]$. (See the appendix.)

This gives our now already famous ϕ another unique characteristic, namely that it is equal to the most primitive infinite continued fraction—one with all 1s.

Let us take the value of this continued fraction in successive parts (which are called *convergents*), each of which will successively bring us closer to the value of the infinite continued fraction. The successive convergents are as follows:

$$1 = \frac{F_2}{F_1} = \frac{1}{1} = [1] = 1$$

$$1 + \frac{1}{1} = 2$$

$$= \frac{F_3}{F_2} = \frac{2}{1} = [1;\ 1] = 2$$

$$1 + \cfrac{1}{1 + \cfrac{1}{1}} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$= \frac{F_4}{F_3} = \frac{3}{2} = [1;\ 1,1] = 1.5$$

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = 1 + \cfrac{1}{1 + \cfrac{1}{2}} = 1 + \cfrac{1}{\cfrac{3}{2}} = \frac{5}{3}$$

$$= \frac{F_5}{F_4} = \frac{5}{3} = [1;\ 1,1,1] = 1.\overline{6}$$

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}} = 1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{3}{2}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{2}{3}} = 1 + \cfrac{1}{\cfrac{5}{3}} = \frac{8}{5}$$

$$=\frac{F_6}{F_5}=\frac{8}{5}=[1;\ 1,1,1,1]=1.6$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{3}{2}}}}$$

$$=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{3}}}=1+\cfrac{1}{1+\cfrac{1}{\cfrac{5}{3}}}=1+\cfrac{1}{\cfrac{8}{5}}=\frac{13}{8}$$

$$=\frac{F_7}{F_6}=\frac{13}{8}=[1;\ 1,1,1,1,1]=1.625$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{3}{2}}}}}$$

$$=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{3}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{5}{3}}}}=1+\cfrac{1}{1+\cfrac{1}{\cfrac{8}{5}}}=1+\cfrac{1}{\cfrac{13}{8}}=\frac{21}{13}$$

$$=\frac{F_8}{F_7}=\frac{21}{13}=[1;\ 1,1,1,1,1,1]=1.\overline{615384}$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{3}{2}}}}}}$$

$$=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{3}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{5}{3}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{8}{5}}}}=1+\cfrac{1}{1+\cfrac{1}{\cfrac{13}{8}}}$$

$$=1+\cfrac{1}{\cfrac{21}{13}}=\frac{34}{21}=\frac{F_9}{F_8}=\frac{34}{21}=[1;\,1,1,1,1,1,1,1]=1.\overline{619047}$$

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}}}}}$$

$$=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{3}{2}}}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{3}}}}}}$$

$$=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{5}{3}}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{8}{5}}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{13}{8}}}}=1+\cfrac{1}{1+\cfrac{1}{\frac{21}{13}}}$$

$$=1+\cfrac{1}{\frac{34}{21}}=\frac{F_{10}}{F_9}=\frac{55}{34}=[1;1,1,1,1,1,1,1,1]=1.6\overline{1764705882352941}$$

As they progress, you will notice how these convergents seem to "sandwich in," or converge, to the value of ϕ, with which we are now quite familiar, approximately 1.618034. What also emerges from these continually increasing convergents is that the final simple fractional values of these convergents happen to be composed of the Fibonacci numbers.

Aside from the continued fraction

$$\phi=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}}}}}$$

getting ever closer to the value of ϕ as we increase its length, we shall now see another surprising relationship of ϕ and the Fibonacci numbers.

In the following chart, we can see that the ratios of consecutive members of the Fibonacci sequence also approach the value of ϕ. In mathematical terms, we say that the limit of the quotient/ratio of two consecutive Fibonacci numbers,

$$\frac{F_{n+1}}{F_n}$$

is the value of ϕ. Mathematicians typically write this as

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi .$$

The famous Scottish mathematician Robert Simson (1687–1768), who wrote an English-language book based on Euclid's *Elements*, which is largely responsible for the development of the foundation of the high school geometry course taught in the United States, was the first to popularize the notion that the ratio

$$\frac{F_{n+1}}{F_n}$$

of two consecutive Fibonacci numbers will approach the value ϕ of the golden ratio. Yet it was Johannes Kepler (1571–1630) whom we credit with discovering that the reciprocal quotient

$$\frac{F_n}{F_{n+1}}$$

of two consecutive Fibonacci numbers approaches the reciprocal of the golden ratio $\frac{1}{\phi}$.

We can see this in the left column of the following table, where F_n represents the nth Fibonacci number and F_{n+1} the next, or (n+1)st, Fibonacci number.

The Ratios of Consecutive Fibonacci Numbers[11]

$\frac{F_{n+1}}{F_n}$	$\frac{F_n}{F_{n+1}}$
$\frac{1}{1} = 1.000000000$	$\frac{1}{1} = 1.000000000$
$\frac{2}{1} = 2.000000000$	$\frac{1}{2} = 0.500000000$
$\frac{3}{2} = 1.500000000$	$\frac{2}{3} \approx 0.666666667$
$\frac{5}{3} \approx 1.666666667$	$\frac{3}{5} = 0.600000000$
$\frac{8}{5} = 1.600000000$	$\frac{5}{8} = 0.625000000$
$\frac{13}{8} = 1.625000000$	$\frac{8}{13} \approx 0.615384615$
$\frac{21}{13} \approx 1.615384615$	$\frac{13}{21} \approx 0.619047619$
$\frac{34}{21} \approx 1.619047619$	$\frac{21}{34} \approx 0.617647059$
$\frac{55}{34} \approx 1.617647059$	$\frac{34}{55} \approx 0.618181818$
$\frac{89}{55} \approx 1.618181818$	$\frac{55}{89} \approx 0.617977528$
$\frac{144}{89} \approx 1.617977528$	$\frac{89}{144} \approx 0.618055556$
$\frac{233}{144} \approx 1.618055556$	$\frac{144}{233} \approx 0.618025751$
$\frac{377}{233} \approx 1.618025751$	$\frac{233}{377} \approx 0.618037135$
$\frac{610}{377} \approx 1.618037135$	$\frac{377}{610} \approx 0.618032787$
$\frac{987}{610} \approx 1.618032787$	$\frac{610}{987} \approx 0.618034448$

By taking the reciprocals of each of the fractions on the left side, we get the column on the right side—also, as expected, approaching the value of $\frac{1}{\phi} \approx 0.618034$.[12] Once again we notice this most unusual relationship between ϕ and $\frac{1}{\phi}$, namely that $\phi = \frac{1}{\phi} + 1$—this time via the Fibonacci numbers.

THE BINET FORMULA

Until now, we accessed the Fibonacci numbers as members of their sequence. If we wish to find a specific Fibonacci number without listing all of its predecessors, we have a general formula to do just that. In other words, if you would like to find the thirtieth Fibonacci number without writing the sequence of Fibonacci numbers up to the twenty-ninth member (F_{29}) (which is a procedure that is somewhat cumber-some), you would use the Binet formula. In 1843 the French mathematician Jacques-Philippe Marie Binet[13] (1786-1856) developed this formula, which allows us to find any Fibonacci number without actually listing the sequence as we would otherwise have to do.

The *Binet formula*[14] is as follows:

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - \left(-\frac{1}{\phi}\right)^n\right],$$

or without using the ϕ, we have

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

which will give us the Fibonacci number (F_n) for any natural number n (a proof of this formula can be found in the appendix).

As is often the case in mathematics when a formula is named after a mathematician, controversies arise as to who was actually the first to discover it. Even today, when a mathematician comes up with what appears to be a new idea, others are usually hesitant to attribute the work to that person. They often say something like: “It looks original, but how do we know it wasn't done by someone else earlier?” Such is the case with the Binet formula. When he publicized his work, there were no challenges to Binet, but in the course of time, some claims have surfaced that Abraham de Moivre (1667–1754) was aware of it in 1718, Nicolaus Bernoulli (1687–1759) knew it in 1728, and his cousin Daniel Bernoulli (1700–1782)[15] also seems to have known the formula before Binet. Also, the prolific mathematician Leonhard Euler (1707–1783) is said to have known it in 1765. Nevertheless, it is still known today as the *Binet formula*.

Let's stop and marvel at this wonderful formula. For any natural number n, the irrational numbers in the form of $\sqrt{5}$ seem to disappear in the calculation, and a Fibonacci number appears. In other words, the Binet formula delivers the possibility of obtaining any Fibonacci number, and can also be expressed in terms of the golden ratio, ϕ.

So, now we shall use this formula. Let's try using it to find a Fibonacci number, say, the 128th Fibonacci number. We would ordinarily have a hard time getting to this Fibonacci number—that is, by writing out the Fibonacci sequence with 128 terms until we arrive at it.

Applying the Binet formula, and using a calculator of course, for $n = 128$ we get:

$$F_{128} = \frac{1}{\sqrt{5}}\left(\phi^{128} - \left(-\frac{1}{\phi}\right)^{128}\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{128} - \left(\frac{1-\sqrt{5}}{2}\right)^{128}\right)$$

$$= 251{,}728{,}825{,}683{,}549{,}488{,}150{,}424{,}261.$$

As we claimed earlier, we can also express the Fibonacci numbers (in Binet form) exclusively in terms of the golden ratio, ϕ, as

$$F_n = \frac{\phi^n - \left(-\frac{1}{\phi}\right)^n}{\phi + \frac{1}{\phi}}, \text{ where } n \geq 1.$$

Familiarity with the Fibonacci numbers reminds us of their recursive definition: $F_{n+2} - F_{n+1} - F_n = 0$, which comes from the original definition of the Fibonacci numbers: $F_{n+2} = F_n + F_{n+1}$, where $n \geq 1$ and $F_1 = F_2 = 1$. Recall the Fibonacci number sequence:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

Rather than beginning with 1 and 1, suppose we were to begin with 1 and 2. Then we would still generate a similar sequence, except we would be missing the first 1. Edouard Lucas[16] (1842–1891), the French mathematician who is largely responsible for bringing the Fibonacci numbers to light in recent years, suggested an analogous sequence; however, this time beginning with 1 and 3. That is, for the (now-called) *Lucas numbers:* $L_{n+2} = L_n + L_{n+1}$, when $n \geq 1$, and $L_1 = 1$ and $L_2 = 3$. The sequence looks like this:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
L_n	1	3	4	7	11	18	29	47	76	123	199	322	521	843	1,364	2,207

Once again, our golden ratio comes into play in that we can also express the Lucas numbers in terms of the golden ratio:

$$L_n = \phi^n + \left(-\frac{1}{\phi}\right)^n, \text{ where } n \geq 1.$$

Let's admire the continued fraction development of $\frac{L_{n+1}}{L_n}$, and notice how it differs from that of $\frac{F_{n+1}}{F_n}$.

Only the last denominator is different. It is a 3 instead of a 1—this is also the difference in the beginning of the Lucas sequence of numbers: The second number is a 3 instead of a 1, as with the Fibonacci numbers.

For example, consider the following two examples:

$$\frac{F_7}{F_6}=\frac{13}{8}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}} \text{ and } \frac{L_7}{L_6}=\frac{29}{18}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}}}.$$

In general and in the shortened format we have the following:

$$\frac{L_2}{L_1}=\frac{3}{1}=[3]=3$$

$$\frac{L_3}{L_2}=\frac{4}{3}=[1;\ 3]=1.\overline{3}$$

$$\frac{L_4}{L_3}=\frac{7}{4}=[1;\ 1,3]=1.75$$

$$\frac{L_5}{L_4}=\frac{11}{7}=[1;\ 1,1,3]=1.\overline{571428}$$

$$\frac{L_6}{L_5}=\frac{18}{11}=[1;\ 1,1,1,3]=1.\overline{63}$$

$$\frac{L_7}{L_6}=\frac{29}{18}=[1;\ 1,1,1,1,3]=1.6\overline{1}$$

$$\frac{L_8}{L_7}=\frac{47}{29}=[1;\ 1,1,1,1,1,3]=1.\overline{6206896551724137931034482758}$$

$$\frac{L_9}{L_8}=\frac{76}{47}=[1;\ 1,1,1,1,1,1,3]$$
$$=1.\overline{6170212765957446808510638297872340425531914893}$$

$$\frac{L_{10}}{L_9}=\frac{123}{76}=[1;\ 1,1,1,1,1,1,1,3]=1.61\overline{8421052631578947361}, \text{ etc.}$$

One might then ask if this can be extended to any starting pair of numbers. That is, were we to begin a Fibonacci-like sequence with other starting numbers, would we also be able to express the numbers in terms of ϕ?

Suppose we choose the starting numbers of such a sequence to be $f_1 = 7$ and $f_2 = 13$ with the same recursive relationship as before, where $f_{n+2} = f_n + f_{n+1}$ (with $n \geq 1$). We would then have the following sequence, which does not have a particular name, as the Fibonacci or Lucas sequences do:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_n	7	13	20	33	53	86	139	225	364	589	953	1,542	2,495	4,037	6,532

Yet the big surprise is that the ratio of consecutive members of the sequence will tend toward the golden ratio as the numbers increase—as was the case with the Fibonacci and the Lucas numbers before. In the chart below, notice how the ratio of $\frac{f_{n+1}}{f_n}$ approaches ϕ =1.6180339887498948482…as a limit. It is believed that the Fibonacci numbers provide the best approximation of ϕ, though this is not easily seen from the chart.[17]

n	$\frac{F_{n+1}}{F_n}$	$\frac{L_{n+1}}{L_n}$	$\frac{f_{n+1}}{f_n}$
1	1	3	1.857142857
2	2	1.333333333	1.538461538
3	1.5	1.75	1.65
4	1.666666666	1.571428571	1.606060606
5	1.6	1.636363636	1.622641509
6	1.625	1.611111111	1.616279069
7	1.615384615	1.620689655	1.618705035
8	1.619047619	1.617021276	1.617777777
9	1.617647058	1.618421052	1.618131868
10	1.618181818	1.617886178	1.617996604
11	1.617977528	1.618090452	1.618048268
12	1.618055555	1.618012422	1.618028534
13	1.618025751	1.618042226	1.618036072
14	1.618037135	1.618030842	1.618033192
15	1.618032786	1.618035190	1.618034292
16	1.618034447	1.618033529	1.618033872
17	1.618033813	1.618034164	1.618034033
18	1.618034055	1.618033921	1.618033971
19	1.618033963	1.618034014	1.618033995
20	**1.6180339**98	**1.6180339**78	**1.6180339**86
…			
100	**1.618033988**	**1.618033988**	**1.618033988**

We can see this better when we take the fifty-place approximation of the value of

ϕ = 1.6180339887498948482045868343656381177203091798057….

Now compare this value to the approximations below for n = 100:

$$\frac{F_{n+1}}{F_n} = \mathbf{1.61803398874989484820458683436563811772031}27439637\ldots$$

$$\frac{L_{n+1}}{L_n} = \mathbf{1.618033988749894848204586834365638117720305}6156477\ldots$$

$$\frac{f_{n+1}}{f_n} = \mathbf{1.61803398874989484820458683436563811772030}82783971\ldots$$

Curiously enough, if we take the ratio of the Lucas numbers to the Fibonacci numbers, $\frac{L_n}{F_n}$, it seems to approach $\sqrt{5} = 2.236067977$ …, as shown in the chart below.

n	$\frac{L_n}{F_n}$
1	1
2	3
3	2
4	2.333333333
5	2.2
6	2.25
7	2.230769230
8	2.238095238
9	2.235294117
10	2.236363636
11	2.235955056
12	2.236111111
13	2.236051502
14	2.236074270
15	2.236065573
16	2.236068895
17	2.236067626
18	2.236068111
19	2.236067926
20	2.236067997
...	
100	**2.236067977**

You may be impressed further by observing that if we take the ratio of alternating Fibonacci numbers, the limit as the numbers increase will approach the value ϕ+1. Another way of saying this is that by taking increasing Fibonacci numbers for $\frac{F_{n+2}}{F_n}$, we gradually approach the value of ϕ+1 as shown in the chart below:

n	$\frac{F_{n+2}}{F_n}$	Approximation of $\frac{F_{n+2}}{F_n}$	$\sqrt{\frac{F_{n+2}}{F_n}}$	Approximation of $\sqrt{\frac{F_{n+2}}{F_n}}$
1	$\frac{F_3}{F_1}=\frac{2}{1}$	2	$\sqrt{\frac{F_3}{F_1}}=\sqrt{\frac{2}{1}}$	1.414213562
2	$\frac{F_4}{F_2}=\frac{3}{1}$	3	$\sqrt{\frac{F_4}{F_2}}=\sqrt{\frac{3}{1}}$	1.732050807
3	$\frac{F_5}{F_3}=\frac{5}{2}$	2.5	$\sqrt{\frac{F_5}{F_3}}=\sqrt{\frac{5}{2}}$	1.581138830
4	$\frac{F_6}{F_4}=\frac{8}{3}$	2.666666666	$\sqrt{\frac{F_6}{F_4}}=\sqrt{\frac{8}{3}}$	1.632993161
5	$\frac{F_7}{F_5}=\frac{13}{5}$	2.6	$\sqrt{\frac{F_7}{F_5}}=\sqrt{\frac{13}{5}}$	1.612451549
6	$\frac{F_8}{F_6}=\frac{21}{8}$	2.625	$\sqrt{\frac{F_8}{F_6}}=\sqrt{\frac{21}{8}}$	1.620185174
7	$\frac{F_9}{F_7}=\frac{34}{13}$	2.615384615	$\sqrt{\frac{F_9}{F_7}}=\sqrt{\frac{34}{13}}$	1.617215080
8	$\frac{F_{10}}{F_8}=\frac{55}{21}$	2.619047619	$\sqrt{\frac{F_{10}}{F_8}}=\sqrt{\frac{55}{21}}$	1.618347187
9	$\frac{F_{11}}{F_9}=\frac{89}{34}$	2.617647058	$\sqrt{\frac{F_{11}}{F_9}}=\sqrt{\frac{89}{34}}$	1.617914416
10	$\frac{F_{12}}{F_{10}}=\frac{144}{55}$	2.618181818	$\sqrt{\frac{F_{12}}{F_{10}}}=\sqrt{\frac{144}{55}}$	1.618079669
...				
100	$\frac{F_{102}}{F_{100}}$	2.618033988	$\sqrt{\frac{F_{102}}{F_{100}}}$	1.618033988

Yet, if we consider the series

$$\sqrt{\frac{F_{n+2}}{F_n}},$$

it approaches the golden ratio, ϕ, as compared to the value reached by the series $\frac{F_{n+2}}{F_n}$, which tends toward $\phi+1$. If you consider that we already established that $\phi+1=\phi^2$, the relationship above should not be completely unexpected.[18]

One more little tidbit relating the Fibonacci numbers to the golden ratio can be seen by taking the series of reciprocals of Fibonacci numbers in the position of powers of 2.

$$\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_4}+\frac{1}{F_8}+\frac{1}{F_{16}}+\ldots+\frac{1}{F_{2^k}}+\ldots=4-\phi\approx 2.381966011250105151 7,$$

or written another way,