SECOND EDITION

The Heart of Mathematics

An invitation to effective thinking

EDWARD B. BURGER MICHAEL STARBIRD

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The Heart of Mathematics

An invitation to effective thinking

SECOND EDITION

Edward B. Burger
WILLIAMS COLLEGE

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Key College Publishing
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About the Authors

Edward B. Burger, Michael Starbird, and some fuzzy trees.

EDWARD B. BURGER is professor of mathematics and chair at Williams College and is the author of numerous articles, books, and videos. The Mathematical Association of America has honored him on several occasions: He received the 2001 Deborah and Franklin Tepper Haimo Award for Distinguished College or University Teaching; he was named the 2001–2003 George Pólya Lecturer; and he was awarded the 2004 Chauvenet Prize. In 2001 he and Professor Starbird won a 2001 Robert W. Hamilton Book Award for the first edition of this text. In 2003, Burger received the Residence Life Academic Teaching Award from The University of Colorado at Boulder. He has also performed stand-up comedy in nightclubs and was briefly an independent contractor for Jay Leno. He has appeared on NBC-TV and National Public Radio, and he has given innumerable mathematical performances around the world.

MICHAEL STARBIRD is a University Distinguished Teaching Professor at The University of Texas at Austin and is a member of UT's Academy of Distinguished Teachers. He has won more than a dozen teaching awards, including the Minnie Stevens Piper Professorship, which is awarded to ten professors from any discipline from any college in Texas; the Jean Holloway Award for Teaching Excellence, the Chad Oliver Plan II Teaching Award, and the Friar Society Centennial Teaching Fellowship, each awarded to one professor at UT annually. Several of these were student-selected awards based on the course that uses this text—he inspires his students to discover mathematics and the joy of thought. He is a UT Recreational Sports Super Racquets Champion.

The Burger and Starbird video courses for The Teaching Company probably make this author team the most watched teachers of mathematics in the country.

About the cover

The dodecahedron (a 12-sided regular solid) reflects how mathematics allows us to see and understand our world with greater clarity. The images of urban settings, sky, and beyond remind us that mathematics is all around and can be found in everyday activities. We hope that students using this book will recognize the power of mathematics in their world.

Contents

But first, . . .

. . . a word from our sponsors: those wonderful caring folks at Key College Publishing

We are pleased and proud to bring to you the second edition of *The Heart of Mathematics: An invitation to effective thinking.* The first edition of *The Heart of Mathematics* was widely acclaimed in reviews and articles as well as by instructors and students. Its review in the June–July 2001 issue of *The American Mathematical Monthly* stated, "This is very possibly the best 'mathematics for non-mathematician' book that I have seen—and that includes popular (non-textbook) books that one would find in a general bookstore." The authors created a book that can be read and enjoyed by both faculty and students, by mathematicians and general interest readers. In fact, as the authors themselves have said,

> *We selected a few of the greatest and most interesting ideas in mathematics and tried to make them accessible and intriguing to students. We hope instructors will find some of their favorite topics among those in* The Heart of Mathematics: An invitation to effective thinking *and that instructors will enjoy introducing those mathematical vistas to their students. Our goal is to offer students the genuine ideas and modes of thinking that attracted all of us to mathematics.* EDWARD B. BURGER AND MICHAEL STARBIRD

HALLMARK FEATURES that have made *The Heart of Mathematics* the most widely adopted textbook in liberal arts and liberal studies mathematics and teacher preparation in over ten years include:

- A focus on the important ideas of mathematics and mathematical methods of investigation.
- A style of writing designed to be read and enjoyed by students and faculty, general interest readers or professionals.
- "Life Lessons," that is, effective methods of thinking that students will retain and apply beyond their college years.
- Entertaining and stimulating end-of-section Mindscape exercises for the development of application, problem-solving, and argumentation skills.

- Activities that encourage collaborative learning and group work.
- An integrated use of a variety of visualization techniques and a hands-on manipulative kit that direct students to model their thinking and to actively explore the world around them.

NEW TO THIS EDITION

- A new chapter, Chapter 8, *Deciding Wisely: Applications of Rigorous Thinking,* which presents everyday and practical applications of the thought strategies developed throughout the text to situations that students may encounter outside the mathematics classroom.
- An improved Chapter 7, *Taming Uncertainty.* Statisticians advised the authors as they crafted three sections that cover the topics generally found in an introductory statistics course. Students who read these sections will be able to identify statistical displays, understand statistical terms, and recognize statistical fallacies.
- The inclusion of five "Developing Ideas" questions at the opening of each Mindscapes section. These exercises are designed to ease students into the Mindscapes by checking reading comprehension, concept clarity, and in some cases, algebraic agility.
- An enhanced teaching package including detailed lesson plans for each section, suggested class activities, and additional mathematical background material. An instructor CD, instructor videos, and a test bank are also available to qualified adopters.
- The Student *Interactive Explorations* CD and 3D glasses are now packaged with the text.

Since its publication in 1999, *The Heart of Mathematics: An invitation to effective thinking* has won the praise and approval of instructors and students alike, quickly becoming a new standard and perennial favorite for liberal arts mathematics courses.

The communities of users have enthusiastically shared their experiences with us, and the authors have listened and responded to this feedback to provide you with this greatly enriched second edition of *The Heart of Mathematics.* You are invited to join the large number of students and instructors who have viewed important ideas of mathematics through this challenging and innovative text. We are proud to be a part of this exciting project. Please feel free to contact us with your comments and feedback at heartofmath@keycollege.com. We hope you enjoy this second edition.

Key College Publishing

Now onto the excitement of mathematics!

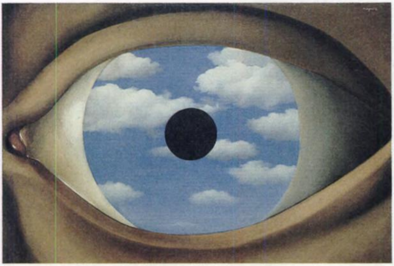
Welcome!

Of course, no one actually reads their math textbooks.
AN ANONYMOUS MATH STUDENT

We wrote this book to be read. We designed many attractions—a kit for grasping concepts hands-on, jokes (some aren't too lame), 3D pictures and glasses, and a style of presentation that we hope invites you to discover new ideas. Most of all, this book contains intriguing lessons for thinking that can change your life.

A World of Ideas

Most people do not have an accurate picture of mathematics. For many, mathematics is the torture of tests, homework, and problems, problems, problems. The very word *problems* suggests unpleasantness and anxiety. But mathematics is not "problems."



The False Mirror **(1928) by René Magritte. Discover a new world view.**

Some people view mathematics as a set of formulas to be applied to a list of problems at the ends of textbook chapters. Toss that idea into the trash. Formulas in algebra, trigonometry, and calculus are incredibly useful. But, in this book, you will see that mathematics is a network of intriguing ideas—not a dry, formal list of techniques.

We want you to discover what mathematics really is and to become a fan. However, if you are not intrigued by the romance of the subject, that's fine too, because at least you will have a firmer understanding of what it is you are judging. Mathematics is a living, breathing, changing organism with many facets to its personality. It is creative, powerful, and even artistic.

Mathematics uses penetrating techniques of thought that we can all use to solve problems, analyze situations, and sharpen the way we look at our world. This book emphasizes basic strategies of thought and analysis. These strategies have their greatest value to us in dealing with real-life decisions and situations that are completely outside mathematics. These "life lessons," inspired by mathematical thinking, empower us to better grapple with and conquer the problems and issues that we all face in our lives—from love to

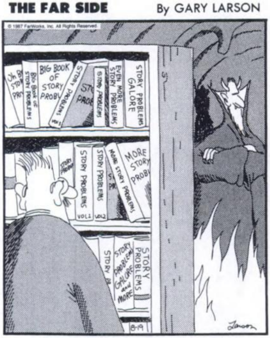
. . . mathematicians are really seeking to behold the things themselves, which can be seen only with the eye of the mind. PLATO

This, therefore, is mathematics: she reminds you of the invisible form of the soul; she gives life to her own discoveries; she awakens the mind and purifies the intellect; she brings to light our intrinsic ideas; she abolishes oblivion and ignorance which are ours by birth. PROCLUS

business, from art to politics. If you can conquer infinity and the fourth dimension, then what can't you do?

As you read this book, we hope you discover the beauty and fascination of mathematics, admire its strength, and see its value to your life. We do not have modest goals for this book. We want you to look at your life, your habits of thought, and your perception of the world in a new way. And we hope you enjoy the view.

Part of the power of mathematics lies in its inexorable quest for elegance, symmetry, order, and grace. Seeking pattern, order, and understanding is a transforming process that mathematics can help us develop.



Hell's library

Mathematics seems to endow one with something like a new sense. CHARLES DARWIN

In mathematics I can report no deficiency, except it be that men do not sufficiently understand the excellent use of Pure Mathematics. ROGER BACON

A Mathematical Journey

The realm of mathematics contains some of the greatest ideas of humankind—ideas comparable to the works of Shakespeare, Plato, and Michelangelo. These mathematical ideas helped shape history, and they can add texture, beauty, and wonder to our lives.

To make our mathematical excursion as pleasant as possible, we have tried to make it all fun—fun to read, fun to do, and fun to think about. We hope you explore some, learn some, think some, enjoy some, and add a new aspect to your view of everything. We hope you laugh at our bad jokes and silly remarks, forgive our sometimes unbridled enthusiasm, but also embrace the profound issues at hand.

The advancement and perfection of mathematics are intimately connected with the prosperity of the State. NAPOLEON I

The road through this book is not free from perils, bumps, and jolts. Sometimes you will confront issues that start beyond your comprehension, but they won't stay beyond your comprehension. The journey to true understanding can be difficult and frustrating, but stay the course and be patient. There is light at the end of the tunnel—and throughout the journey, too.

It may well be doubted whether, in all the range of science, there is any field so fascinating to the explorer—so rich with hidden treasures—so fruitful in delightful surprises—as Pure Mathematics. LEWIS CARROLL

If we do not expect the unexpected, we will never find it. HERACLITUS

What's the point of it all? Well, the bottom line is that mathematics involves profound ideas. Making these ideas our own empowers us with the strength, the techniques, and the confidence to accomplish wonders.

Travel Tips—Read the Book

We have some suggestions about how to use this book:

Answer our questions ▸ We often pose questions in the middle of a section and invite you to give an answer or a guess before continuing. Please attempt to answer these questions. If you don't know an answer for sure, guess. Don't be afraid to make lots of mistakes—that is the only way to learn. It is much better to guess wrong than not to think about the question at all.

Think ▸ This is our main goal. We want you to contemplate some of the greatest and most intriguing creations of human thought. Constantly stop and think.

Be active, not passive ▸ Our wish is for you to be an active participant. Take the concepts and make them your own. Look beyond the mathematical ideas, and don't be satisfied with mere knowledge. Challenge yourself to attain the power to figure things out on your own.

Have fun ▸ We truly believe that the ideas presented in this book are some of the most fascinating and beautiful ones around. We sincerely hope that some, if not all, of the themes will appeal to your intellect. More importantly, this journey of the imagination and of the mind should be fun. Enjoy yourself!

Finally, reading mathematics is much different from reading about many other subjects. Here's how we read mathematics. We read a sentence or two, stop reading, think about what we've read, and then realize we're completely and utterly confused. Usually we discover that we didn't really understand the previous paragraph. But, we don't get frustrated . . . it's the nature of the beast. Instead, we either reread some previous sections or just reread the previous sentence. The fuzziness slowly begins to fade ever so slightly, and the concept begins to come into focus. Then we attempt to think about the issue and work with it on our own or with friends. It is at this point that we begin to appreciate and understand the ideas presented. One of the great features of mathematics is that once we do understand an idea our grasp of it is completely solid. There is no vagueness or uncertainty. So, adopt high standards for what you view as "understanding." Be actively engaged as you read. Draw pictures, explain ideas to friends. Put yourself in the position of the discoverer of each idea. Ask questions, search for answers, and let those answers guide you to still more questions.

Shall any gazer see with mortal eyes
Or any searcher know by mortal mind—
Veil after veil will lift—but there must be
Veil upon veil behind. SIR EDWIN ARNOLD

With all good wishes,

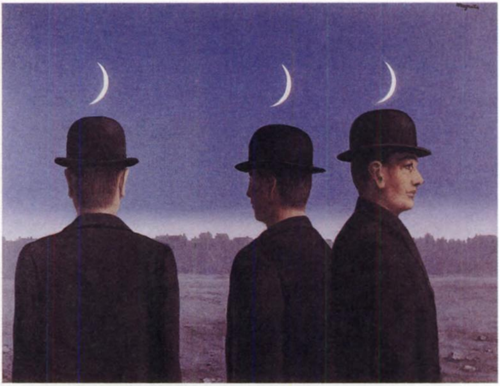
Edward B. Burger   Michael Starbird

Surfing the book

It's too early to get caught up in details. Instead, let's just surf the book and get a quick overview of what's ahead. The whole book revolves around just two basic themes:

- Effective thinking
- Some truly great ideas

What mathematical sites lie ahead? Instead of just starting in with a hot and spicy math topic, we thought it would be more fun to surf the entire book and get a quick, big-picture overview of what is on the horizon. We hope these "home pages" will pique your curiosity and tantalize your intellect. Keep an open mind; forego any previous biases and prejudices toward mathematics; and do not censor any inventive thoughts or sparks of interest you may develop toward the subject. Let's surf.



***The Masterpiece or The Mysteries of the Horizon* (1955) by René Magritte**

Fun and Games

An Introduction to Rigorous Thought

Can this book help you think more effectively, more inventively, solve life problems more creatively, and analyze issues more logically?

The short answer is "Yes."

Is there a better way to meet the powerful world of logical thought than through Fun and Games?

The short answer is "No."

Is this book strange and sometimes over the edge?

The short answer is "Absolutely."

Just hang with us and see how far we'll go.

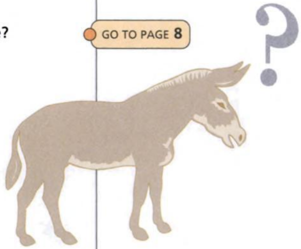
This site is an invitation to think and have fun with genies, damsels, and Dodge Ball and, in the process, develop a system of logical inquiry that we will use throughout the book and throughout our lives.

Who can better develop your thinking skills than you? As you resolve the many dilemmas in these crazy stories, you will automatically discover your own path to logical and strategic thought. Don't feel like going at it alone? Get a friend or a roommate to try some with you… it's all fun and games.

GO TO PAGE 3

GO TO PAGE 28

GO TO PAGE 8



> …the primary question was not what do we know, but how do we know it.
>
> ARISTOTLE

…are we having fun yet?

Welcome! | 1 Games | 2 Number | 3 Infinity | 4 Gems | 5 Space | 6 Chaos | 7 Chance | 8 Deciding | Farewell

http://www.heartofmath.com/NumberContemplation

Number Contemplation

> **Wherever there is a number, there is beauty.**
>
> PROCLUS

Worried about balding? How about this one: Are there two hairy people on Earth with exactly the same number of hairs on their bodies? Does Rogaine change the answer?

GO TO PAGE **41**

What do the reproductive habits of 13th-century rabbits have in common with the Parthenon?

GO TO PAGE **58**

More than you think.

Are art and music branches of mathematics?

GO TO PAGE **238**

You betcha Bach!

Don't give up! Think working on really challenging questions that others have tried to solve is fruitless? Ask Andrew Wiles. In 1994 he answered a 350-year-old question—it only took him seven years. Hey, intellectual triumphs happen—it just takes tenacity!

GO TO PAGE **75**

Can you tell time? If so, then you might have a promising career at decoding the numbers at the bottom of Universal Product Codes. Want to know how?

GO TO PAGE **86**

XQE TPS LPBE AX TZ?

GO TO PAGE **95**

So numbers are no biggie? In ancient Greece you were thrown from a ship and drowned if you told people about certain numbers. Sound irrational?

GO TO PAGE **114**

How close is 1 to 0.99999...? Closer than you might think.

GO TO PAGE **129**

Ancient questions about numbers still remain unanswered. Act now...mathematicians are standing by.

GO TO PAGE **76**

If 2 can be 1, who is the 1 to become??

Welcome! 1 Games 2 Number 3 Infinity 4 Gems 5 Space 6 Chaos 7 Chance 8 Deciding Farewell

http://www.heartofmath.com/Infinity

Infinity

Want to know about it?

GO TO PAGE 138

**Oh moment,
one and infinite.**

ROBERT BROWNING

...loading...loading...still loading...

Geometric Gems

> **Mighty is geometry; joined with art, resistless.**
>
> EURIPIDES

Good at jigsaw puzzles? Check out the Pythagorean Theorem.

GO TO PAGE 208

Want to see a picture of the sexiest rectangle? If you're 18 or over,

GO TO PAGE 232

What kind of attractive patterns can cover our floors and walls? Can special, jumbled-looking patterns have some symmetry that regular checkerboard patterns have? Probably not. . .but hey, you never know.

GO TO PAGE 257

Are straight lines really straight? Does space bend? For a free tour of the universe,

GO TO PAGE 292
(not valid in all states)

Is there a fourth dimension? Can you see it?

GO TO PAGE 316

(Warning. . .if you click on this site, you may not be able to return to this page.)

Want to get in SHAPE?

Contortions of Space

> The true spirit of delight... is to be found in mathematics as surely as in poetry.
>
> BERTRAND RUSSELL

Bend and stretch—sound advice for both aerobics and topology. GO TO PAGE 328

If you want to take off some, but not all, of your clothes... GO TO PAGE 331

Does every issue have two sides? GO TO PAGE 351

Answer: "No."

Elastic thoughts lead to solid ideas. . .the power of rubber. GO TO PAGE 363

Wondering about the mysteries of life? Want to untangle DNA? GO TO PAGE 375

You've first got to untangle knots. . .good luck skipper!

The weather and rubber—are there two places on Earth that are exactly opposite each other and yet have identical temperatures and pressures?

Either ask your local weather forecaster or... GO TO PAGE 397

Ever thought about a world of rubber?

Chaos and Fractals

Can pictures or ideas be infinitely intricate? GO TO PAGE 408

Can we predict the population, the weather, or even the positions of the planets in the future? GO TO PAGE 482

Answer: "No."

Fractals—is there anything that is not one? Probably not. . .but what is one? GO TO PAGE 446

A butterfly flaps its wings in Brazil. Two weeks later there is a tornado in Kansas—kiss Dorothy good-bye. Are these events related? GO TO PAGE 491

Nature is sheer and utter chaos. Why bother cleaning your room? GO TO PAGE 487

Can objects straddle between two dimensions? GO TO PAGE 503

> **God has put a secret art into the forces of Nature so as to enable it to fashion itself out of chaos into a perfect world system.**
>
> IMMANUEL KANT

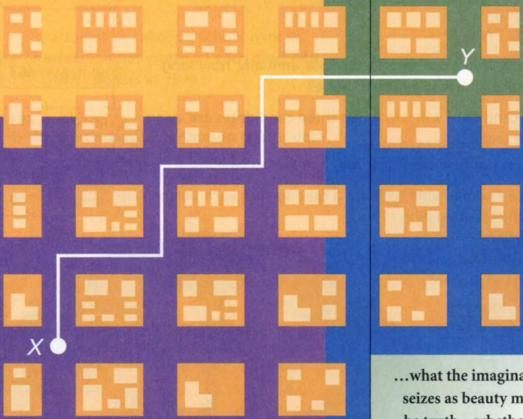
...details...details...details...

Farewell

When is the end a beginning?

GO TO PAGE **716**

How many more ideas are there for you to explore and enjoy? How long is your life?



...what the imagination seizes as beauty must be truth—whether it existed before or not

JOHN KEATS

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>
> IMMANUEL KANT

...details...details...details...

Taming Uncertainty

John and Jim are identical twins who were separated at birth. Both have married women named Jennifer who watch *Seinfeld* reruns and love ice cream. What are the odds?

GO TO PAGE 613

Answer: "Higher than you might think."

Will 2 people in a room of 30 have the same birth date? How would you bet?

GO TO PAGE 530

Why are *amazing coincidences* nearly certain to happen? Here's one: Take "amazing coincidences" and look at the letters or spaces in the prime positions: 2, 3, (skip 5 because that's the number of fingers on a hand), 7, 11, and 13. What does it spell? *m a g i c*!!! What an amazing coincidence?

GO TO PAGE 541

Surprised?

Consider all the graduates of Lakeside School over its whole history. The average net worth of each graduate increased by millions of dollars in 1996.

GO TO PAGE 586

Amazing. . .or not?

Is there an accurate method to gather data about college students' behavior in potentially sensitive situations?

GO TO PAGE 601

> Chance, too, which seems to rush along with slack reins, is bridled and governed by law.
>
> BOETHIUS

Chances are…

Deciding Wisely

Applications of Rigorous Thinking

> **Chance favors only the prepared mind.**
>
> LOUIS PASTEUR

Should you buy lottery tickets or this book? GO TO PAGE **634**

Can improved airline safety lead to more accidental deaths? GO TO PAGE **649**

Answer: "Yes."

Investing $1000, compounded at 5% annually, how long will it take to become a millionaire? GO TO PAGE **663**

Which candidate will win when three candidates are vying for one seat? GO TO PAGE **683**

The most popular candidate may not fair so well.

Are three people able to share a cake so that each person is equally satisfied with his or her portion? GO TO PAGE **700**

It's your move…think it through.

Welcome! | 1 Games | 2 Number | 3 Infinity | 4 Gems | 5 Space | 6 Chaos | 7 Chance | 8 Deciding | **Farewell**

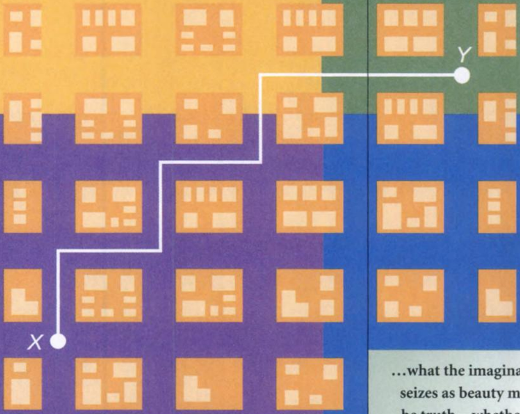
http://www.heartofmath.com/Farewell

Farewell

When is the end a beginning?

How many more ideas are there for you to explore and enjoy? How long is your life?

GO TO PAGE **716**



> **...what the imagination seizes as beauty must be truth—whether it existed before or not**
>
> JOHN KEATS

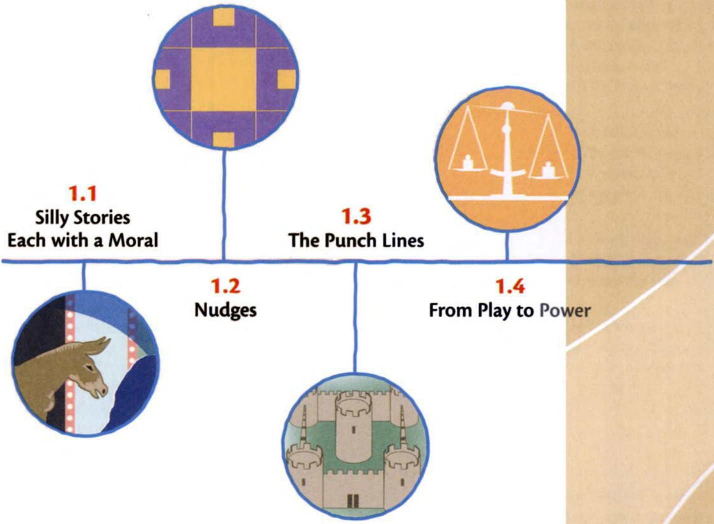
Now it's up to you...

Now that we have a sense of what's ahead, let's dig in...

CHAPTER ONE

Fun and Games

An Introduction to Rigorous Thought



understanding of a complete solution or even see how a solution will eventually fall into place. This situation is like being asked to walk through a forest in the dark. Without knowing the terrain, the natural tendency is to freeze like a deer in headlights. However, we must learn not to let this understandable fear paralyze us intellectually; we must take a step. It is only by stumbling through many small intellectual steps that we are eventually able to make any progress at all.

For example, imagine we're soccer players with the ball at midfield. In this position we can't possibly know how a goal will be achieved, and we can't stop to envision the entire progression of the future before kicking the ball. Instead, we move with the understanding that the specific goal strategy will become clear as opportunities arise.

Just try out ideas with these stories—loosen up, try to kick the ball, and don't worry if you miss. Remember:

> *Truth comes out of error more easily than out of confusion.*
> FRANCIS BACON

After you have given considerable thought to a story, move to the corresponding part in Section 1.2, "Nudges," where leading questions and suggestions provide a gentle push in the right direction, in case you need a hint. There we also identify some strategies for tackling both mathematical questions and, more importantly, questions that will arise in your life.

Section 1.3, "The Punch Lines," provides solutions and commentary about how the questions and their resolutions fit into the mathematical landscape. As you think about these stories, you will discover some profound ideas that capture the essence of some deep and beautiful mathematical concepts.

As you proceed, remember the rules on page 4, especially rule 6.

Story 1. That's a Meanie Genie

On an archeological dig near the highlands of Tibet, Alley discovered an ancient oil lamp. Just for laughs she rubbed the lamp. She quickly stopped laughing when a huge puff of magenta smoke spouted from the lamp, and an ornery genie named Murray appeared. Murray, looking at the stunned Alley, exclaimed, "Well, what are you staring at? Okay, okay, you've found me; you get your three wishes. So, what will they be?" Alley, although in shock, realized she had an incredible opportunity. Thinking quickly, she said, "I'd like to find the Rama Nujan, the jewel that was first discovered by Hardy the High Lama." "You got it," replied Murray, and instantly nine identical-looking stones appeared. Alley looked at the stones and was unable to differentiate any one from the others.

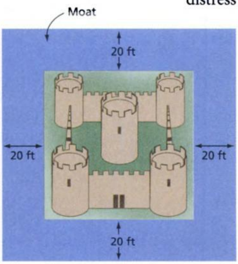
Finally she said to Murray, "So where is the Rama Nujan?" Murray explained, "It is embedded in one of these stones. You said you wished to find it. So now you get to find it. Oh, by the way, you may take only one of the stones with you, so choose wisely!" "But they look identical to me. How will I know which one has the Rama Nujan in it?" Alley questioned. "Well, eight of the stones weigh the same, but the stone containing the jewel weighs slightly more than the others," Murray responded with a devilish grin.

Alley, becoming annoyed, whispered under her breath, "Gee, I wish I had a balance scale." Suddenly a balance scale appeared. "That was wish two!" declared Murray. "Hey, that's not fair!" Alley cried. "You want to talk fair? You think it's fair to be locked in a lamp for 1729 years? You know you can't get cable TV in there, and there's no room for a satellite dish! So don't talk to me about fair," Murray exclaimed. Realizing he had gone a bit overboard, Murray proclaimed, "Hey, I want to help you out, so let me give you a tip: That balance scale may be used only once." "What? Only once?" she said, thinking out loud. "I wish I had another balance scale." ZAP! Another scale appeared. "Okay, kiddo, that was wish three." Murray snickered. "Hey, just one minute," Alley said, now regretting not having asked for one million dollars or something more standard. "Well at least this new scale works correctly, right?" "Sure, just like the other one. You may use it only once." "Why?" Alley inquired. "Because it is a 'wished' balance scale," he said, "so the rule is 'one scale, one balancing'; it's just like the rule against using one wish to wish for a hundred more wishes." "You are a very obnoxious genie." "Hey, I don't make up the rules, lady, I just follow them," he said.

So, Alley may use each of the two balance scales exactly once. Is it possible for Alley to select the slightly heavier stone containing the Rama Nujan from among the nine identical-looking stones? Explain why or why not.

Story 2. Damsel in Distress

Long ago, knights in shining armor battled dragons and rescued damsels in distress on a daily basis. Although it is not often stressed in many stories of chivalry, the rescue often involved logical thinking and creative problem solving by the damsel. Here then is a typical knightly encounter.



Once upon a time, a notorious knight captured a damsel and imprisoned her in a castle surrounded by a square moat that was infested with extraordinarily hungry alligators. The moat was 20 feet across, and no drawbridge existed because after depositing the damsel in the castle, the evil knight had taken it with him (giving his horse one major hernia).

After a time, a good knight rode up and said, "Hail sweet damsel, for I am here, and thou art there. Now what are we going to do?"

The knight, though good, was not too bright and consequently paced back and forth along the moat looking anxiously at the alligators and trying feebly to think of a plan. While doing so, he stumbled upon two sturdy beams of wood suitable for walking across but lacking sufficient length. Alas, the moat was 20 feet across, but the beams were each only 19 feet long and 8 inches wide. He tried to stretch them and then tried to think. Neither effort proved successful. He had no nails, screws, saws, Superglue, or any other method of joining the two beams to extend their length.

What to do? What to do? Fortunately, the damsel, after a suitable time to allow the good knight to attempt to solve the puzzle on his own, called to the knight and gave him a few hints that enabled him to rescue her. What was the maiden's suggestion?

This story from medieval times foreshadows our journey into the geometric and the visual.

Story 3. The Fountain of Knowledge

During an incredibly elaborate hazing stunt during pledge week, Trey Sheik suddenly found himself alone in the Sahara Desert. His desire to become a fraternity brother was now overshadowed by his desire to find something to drink (these desires, of course, are not unrelated). As he wandered aimlessly through the desert sands, he began to regret his involvement in the whole frat scene. Both hours and miles had passed and Trey was near dehydration. Only now did Trey appreciate the advantages of sobriety. Suddenly, he came upon an oasis.

There, sitting in a shaded kiosk beside a small pool of mango nectar, was an old man named Al Donte. Big Al not only ran the mango bar but was also a travel agent and could book Trey on a two-humped camel back to Michigan. At the moment, however, Trey desired nothing but a large drink of that beautifully translucent and refreshing mangoade. Al informed Trey that he sold the juice only in 8-ounce servings and the cost for one serving was $3.50. Trey frantically searched his pockets, and though he found much sand, he also discovered that he had exactly $3.50.

Trey's jubilation at the thought of liquid coating his parched throat was quickly shattered when Al casually announced that he did not have an 8-ounce glass; all he had was a 6-ounce glass and a 10-ounce glass—neither of which had any markings on it. Al, being a man of his word, would not hear of selling any more or any less than an 8-ounce serving of his libation. Trey, in desperation, wondered whether it was possible to use only the unmarked 6- and 10-ounce glasses to produce exactly 8 ounces in the 10-ounce glass. Do you think it's possible? If so, explain how, and if not, explain why. This pledge-week prank does whet our appetites for a world of numbers.

6-oz glass 10-oz glass

Story 4. Dropping Trou

Edward Burger: Exposed on April 24, 1993.

Before reading on, remember that truth is sometimes stranger than fiction. The highlight of Professor Burger's April 1993 talk to more than 300 Williams College students and their parents occurred when after removing his shoes, he tied his feet together with a stout rope, leaped onto the table, dramatically removed his belt, unzipped his zipper, and dropped his pants. The purple cows (Williams mascots) mooing about on his baggy boxer shorts completed an image not soon forgotten in the annals of mathematical talks. The more conservative parents in the audience were contemplating transferring their sons and daughters to a less "progressive" school.

But then, at the moment of maximum shock and bewilderment, Professor Burger performed the seemingly impossible feat of rehabilitating his fast-sinking reputation. Without removing the rope attached to his feet, he turned his pants inside out and pulled his trousers back to their accustomed position (though now inside out). Thus he simultaneously restored his modesty *and* his credibility by demonstrating the mathematical triumph of reversing his pants without removing the rope that was tying his feet together.

Please attempt to duplicate Professor Burger's amazing feat—in the privacy of your room, of course. You will need a rope or cord about 5 feet long. One end of the rope should be tied snugly around one ankle and the other end tied equally snugly about the other ankle. Now, without removing the rope, try to take your pants off, turn them inside out, and put them back on so that you, the rope, and your pants are all exactly as they were at the start, with the exception of your pants being inside out. While some may find this experiment intriguing, others may find it in poor taste. Everyone will agree, however, that surprising outcomes arise when we bend and contort objects and space.

Story 5. Dodge Ball

Dodge Ball is a game for two players—Player One and Player Two (although any two people can play it, even if they are not named "Player One" and "Player Two"). Each player has a special game board (shown on the next page) and is given six turns.

Player One begins by filling in the first horizontal row of his game board with a run of X's and O's. That is, on the first line of his board, he will write either an X or an O in each box. Then Player Two places either an X or an O in the first box of her board. So at this point, Player One has filled in the first row of his board with six letters, and Player Two has filled in the first box of her board with one letter.

The game continues with Player One writing down either an X or an O in each box of the second horizontal row of his board. Then Player Two writes one letter (an X or an O) in the second box of her board. The game proceeds in this fashion until all of Player One's boxes are filled with X's and O's; thus, Player One has produced six rows of six marks each, and Player Two has produced one row of six marks. All marks are visible to both players at all times.

Player One's game board

1						
2						
3						
4						
5						
6						

Player Two's game board

1	**2**	**3**	**4**	**5**	**6**

Player One wins if any of his rows exactly matches Player Two's row (Player One matches Player Two). Player Two wins if her row does not match any of Player One's rows (Player Two dodges Player One).

Would you rather be Player One or Player Two? Who has the advantage? Can you devise a strategy for either side that will always result in victory? This little game holds within it the key to understanding the sizes of infinity.

Story 6. A Tight Weave

Sir Pinsky, a famous name in carpets, has a worldwide reputation for pushing the limits of the art of floor covering. The fashion world stands agog at the clean lines and uncanny coherence of his purple and gold creations. Some call him square because his designs so richly employ that quaint quadrilateral. But squares in the hands of a master can create textures beyond the weavers' world, although not beyond human imagination.

One day Sir Pinsky began a creation with, as always, a perfect, purple square. However, one square seemed too plain, so in the exact center of it he added a gold square. He saw that the central square implicitly defined eight purple squares surrounding it. As he pondered, he realized that those eight purple squares were identical to his original large square except for two things: (1) Each was one-third the size of the whole square; and (2) none of them had gold squares in their centers.

One die has two 6's and four 2's. Another has three 5's and three 1's. The third has four 4's and two blank faces. The last die has 3's on each face. The dice are not weighted—that is, any face is just as likely to land face-up as any other.

Deep Pockets Drew strides up to the bowl to choose the winning die. Which die should Drew draw? Drew considers the die that has all 3's. Which die could Mr. Bones select that will beat the all-3's die two-thirds of the time? After finding that die, we know that the all-3's die would not be a particularly wise choice.

Next Deep Pockets Drew considers the die with four 4's and two blank faces. Why will the die with three 5's and three 1's beat it two-thirds of the time? After verifying this dicey dominance, we know that selecting the die with four 4's and two 0's would not be a smart move.

Drew next considers the die with three 5's and three 1's. Why will the die with two 6's and four 2's beat it two-thirds of the time? After confirming this superiority, we know that the die with three 5's and three 1's would not be the best die.

Only one possibility remains: The die with two 6's and four 2's. Is there a die that will beat it two-thirds of the time? Your surprising discovery will show that none of the four dice is the "best" one to select, because each one can be beaten by one of the other three dice two-thirds of the time. Amazing.

So now Drew can put the dice in a circular order where each one beats its clockwise neighbor two-thirds of the time. What is that order? After doing the math, Deep Pockets Drew chooses not to play, and as a result his pockets become deeper.

This intriguing dice game surprisingly leads to the seemingly unrelated insight that the idea of a fair and democratic voting system is impossible—so much for "a government of the people, by the people, and for the people."

Story 9. Dot of Fortune

One day three college students were selected at random from the studio audience to play the ever-popular TV game show, "Dot of Fortune." One of the students had already discovered the power and beauty of mathematical thinking, while the other two were not nearly so fortunate. The stage contained no mirrors, reflective surfaces, or television monitors. The three students were seated around a small round table and blindfolded. As Pat, the host, explained the rules of the game, Vanna affixed a conspicuous but small colored dot to each student's forehead.

"So, contestants," Pat explained, "at the sound of the bell you will remove your blindfolds. You will see your two companions sitting quietly at the table, each with a dot on his or her forehead. Each dot is either red or white. You cannot, of course, see the dot on your own forehead. After you have observed the dots on your companions' foreheads, you will raise your hand if you see at least one red dot. If you do not see a red dot, you will keep your hands on the table. The object of the game is to deduce the color of your own dot. As soon as you know the color of your dot, hit the buzzer in front of you. Do you understand the rules of the game?" All the students understood the rules, although the math fan understood them better.

"Are you ready?" asked Vanna after affixing a red dot to each student's forehead. After the contestants nodded, Vanna rang the bell and they removed their blindfolds. The studio audience quivered with anticipation. The students looked at one another's dots, and all raised their hands. After some time, the math fan hit her buzzer, knowing what color dot she had. Explain how she knew this. Why did the other students not know? This game requires creative logical reasoning—a powerful means to make discoveries whether they are in math, in life, or even (although rarely) on prime-time TV.

CAUTION!
Proceed to the next section only after you have given considerable thought to each of the stories.

1.2 Nudges

Leading Questions and Hints for Resolving the Stories

▲ *White Vertical Water* (1972) by Louise Nevelson

When we cannot use the compass of mathematics or the torch of experience . . . it is certain we cannot take a single step forward.

VOLTAIRE

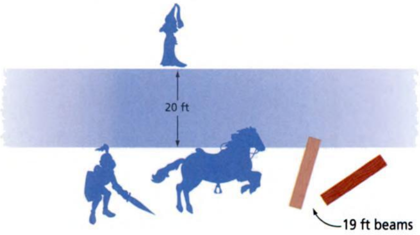
Often we discover a solution only after we move beyond what appears to be the obvious or straightforward approach.

Story 1. That's a Meanie Genie

Initially, we might think that finding the jewel is impossible because Alley is allowed to make only two comparisons. Instead of comparing stones individually, perhaps she should compare one *collection* of stones with another *collection* of stones. Now suppose Alley compares one group with another using the first scale. What can she conclude? What should she do next?

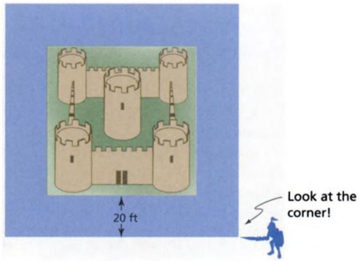
Story 2. Damsel in Distress

Thinking about variations on a situation can shed light on which features are essential and which are not. In this case we might consider a variation in which the damsel in distress is on the other side of a 20-foot river rather than surrounded by a square moat. Unfortunately for the maiden, if she were separated from bliss by a river, she would go blissless, because the two 19-foot beams, in the absence of tools, would still not enable the knight to rescue her. Could the square shape of the moat come into play in the solution?



Do not overlook small details; they often lead to tremendous discoveries.

Looking at extremes is a potent technique of analysis in many situations and may be helpful here. The extremes, either geometrical ones as in this situation or conceptual ones in other situations, frequently reveal features we might have otherwise overlooked.



Don't be afraid to experiment, especially when outcomes are uncertain.

Story 3. The Fountain of Knowledge

To solve this puzzle, combine trial and error with careful observation. As we observe the outcomes of various attempts, we can teach ourselves what may be possible. Try filling up the 10-ounce glass, and then use it to fill the 6-ounce glass. What do you have now—anything new?

Story 4. Dropping Trou

Visualization and experimentation often lead to surprising and even counterintuitive results.

We hope that you physically attempt this exercise. By actually trying a task on your own, it's often possible to discover insights that otherwise may have been hidden from view (particularly in this case).

You will notice that the rope does restrict the amount of movement of your pants. Your mission is to discover means to work around such constraints. For example, try moving parts of the pants through other parts. You may first want to try this task wearing shorts rather than long pants.

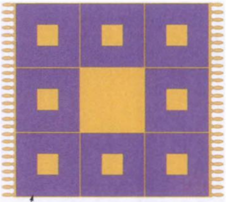
Story 5. Dodge Ball

Minor differences early on may lead to dramatically different outcomes.

Play this game a few times with a friend. Switch roles so that each of you has the opportunity to be Player One and Player Two. Remember, if you are Player One, your goal is to match one of your rows with your opponent's row. If you are Player Two, you want to dodge all six of your opponent's rows; that is, you want your row to differ in at least one spot from each of the six rows of your opponent. Who would you rather be: Player One or Player Two?

Story 6. A Tight Weave

Consider a purple square that has a smaller gold square in its center. How do each of the eight surrounding squares differ from the whole picture? They are much the same except that the whole picture has a gold square in the middle, and each of the eight surrounding squares is solid purple. How could you modify those eight one-third-size surrounding squares to make them look like smaller copies of the entire picture you see here?



We want each square to look like a smaller copy of the entire rug.

Now let's ask the question again: "In the picture you now have, is each of the eight one-third-size squares identical to smaller copies of the whole picture?" No. How would you modify each one-third-size square-with-a-gold-center to make it identical to the whole new figure? Are you done?

Don't quit.

Draw several steps of this repetitive process. At each stage, add up the areas of all the gold squares. When should you stop this process?

Story 7. Let's Make a Deal

Consider various scenarios.

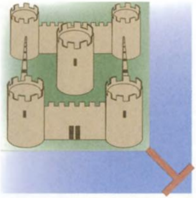
Suppose the raisin's initial guess was wrong. What would be the result if he were to change his answer?

She then takes two of these three stones and places one on each side of the second scale. If one weighs more than the other, then she knows that this stone is the one containing the jewel. If they both weigh the same, then she knows that the third stone must contain the jewel. Thus, by weighing the stones only twice, Alley is able to find the jewel.

Break a hard problem into easier ones.

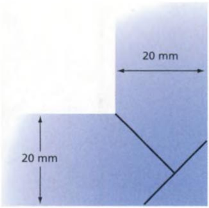
Take partial steps whenever possible. Notice that, instead of trying to identify the jewel immediately, Alley first reduces the pool of choices from nine to three. Thus she first makes the problem easier. "Divide and conquer" is an important and useful technique in both mathematics and life.

Story 2. Damsel in Distress



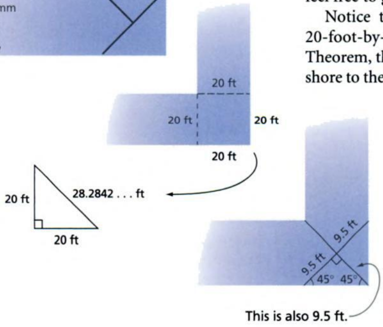
Focusing attention on the corner of the moat suggests using one of the beams to span the corner. Of course, we need to check that the two 19-foot beams are long enough to make the configuration in the picture.

There are at least two ways to verify that this picture is correct. One way is to construct a physical model. The picture shown at left is a physical model scaled down so that 1 foot in the story corresponds to 1 millimeter in the picture. You can now measure and ensure that this configuration is possible.



An alternative method would be to observe that the picture has some right triangles. This observation foreshadows our look at the Pythagorean Theorem. After we examine good old Pythagoras's theorem (Chapter 4), the following paragraphs will seem soothing and comforting. If for now you find them less so, feel free to glance through them and just move on.

Notice that the corner of the moat forms a 20-foot-by-20-foot square. By the Pythagorean Theorem, the distance from the outer corner of the shore to the inner corner of the castle island is equal to the square root of $20^2 + 20^2$. Using a calculator, we see that the distance is 28.2842 . . . feet.



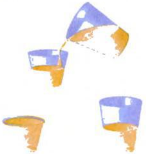
Placing the 19-foot beam diagonally across the corner of the moat as far out as it can go creates a triangle that cuts off the corner. If we draw a line from the center of the beam to the outer corner of the moat, we create two identical 45-degree right triangles, as shown. Since the length of half the beam is 9.5 feet, we learn that the center of the beam is also 9.5 feet from the outer corner of the moat.

Since the total diagonal distance from the outer corner of the moat to the corner of the castle island is 28.2842 . . . feet, the distance to the center of the beam is (28.2842 . . . feet − 9.5 feet) = 18.7842 . . . feet. Since that distance is just less than 19 feet, the other beam will just barely span the distance between the beam and the island. In gratitude for her rescue, the damsel provided the good knight with a romantic lesson in *geometry.*

Story 3. The Fountain of Knowledge

6-oz glass   10-oz glass

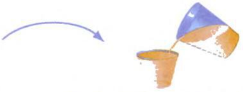
Suppose we fill up the 10-ounce glass with mango juice and slowly pour it into the 6-ounce glass, stopping at the moment the 6-ounce glass is full. Notice that what's left in the 10-ounce glass is precisely 4 ounces of mango juice. We now empty the 6-ounce glass back into the pool and refill it with the 4 ounces from the other glass. If we now refill the 10-ounce glass from the pool, we can again slowly pour its contents into the 6-ounce glass until the 6-ounce glass is full. Filling it takes exactly 2 more ounces, and now the larger glass contains exactly 8 ounces. Happily, those 8 ounces of mango juice can now be served to Trey (on a tray). If Trey had found a solution, he would have made his first discovery in an area of mathematics known as *number theory.*



Empty this   4 oz. left

Pour 4 oz. into 6-oz. glass

Refill 10-oz. glass



Pour to fill (2 oz. of mango juice)

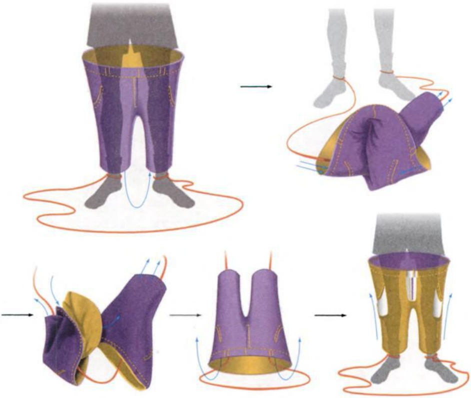
8-oz. of mango juice here!

There is more than one solution to this puzzle. For example, we could have begun by filling the 6-ounce glass and pouring its entire contents into the 10-ounce glass. See if you can use this starting point to find an alternative solution.

Story 4. Dropping Trou

The sequence of diagrams on the next page illustrates a solution to this knotty puzzle. Notice that by bending, contorting, and twisting your pants around, you can produce different configurations. Questions involving bending, contorting, and twisting lead to interesting and surprising discoveries. The notion of bending space is the fundamental notion in an area of mathematics called *topology.*

Method: With pants on rope, bring one of the ends (cuff) of the right leg through the inside of the left leg; pull all the way through. When done, pants will be right side out (still) but the rope will now go through the pants. Now reach each hand into the inside of each pants leg and grab the cuffs. Simultaneously, pull the cuffs up through the pants. The pants will be inside out and the rope will no longer be around the pants.



By doing we often discover valuable insights.

Often, thinking only in the abstract does not reveal new insights. Make the issue concrete and physical whenever possible.

Many people believe mathematical issues exist outside the realm of our life experience. In truth, many surprising and even counterintuitive mathematical discoveries can be made by freeing ourselves from old, unsubstantiated biases and experimenting with new ways of thinking and seeing.

Story 5. Dodge Ball

We want to be Player Two. Here is a strategy that will guarantee victory. Player One fills in the first row of six boxes in his table. As Player Two, we look at the first letter and ignore the last five. If his first letter is an X, we write an O; if it's an O, we write an X. Notice that, no matter what happens later, after this point, we are certain that the row we will create will definitely not be the same as Player One's first row. The two rows will differ in at least the first box. Player One now writes down his second row of six letters. We examine only the second letter in this new row. If that letter is an X, we write an O; if that letter is an O, we write an X. Now we are sure that no matter what follows, our row will not be the same as Player One's second row because the rows definitely differ in the second letter. If we repeat this process, we will have created a row of X's and O's that is different from the six rows created by Player One.

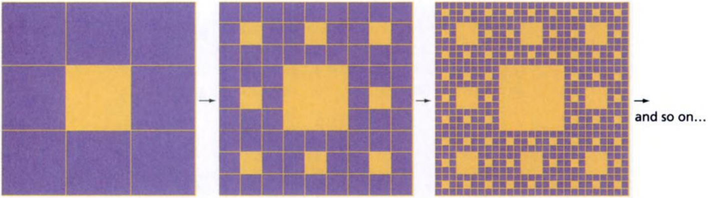
Often simple observations can have deep consequences.

Creating a row that does not match any of our opponent's rows has a powerful application in the study of *infinity.* Although this modest little game has only six steps, the concept behind it has tremendous ramifications, as we shall see in Chapter 3: "Infinity."

As a final note, we pose the following question: Suppose that we are Player One, and our opponent—who is trying to follow the strategy described above to win—makes a mistake by placing the wrong letter in the first box. Can you now describe a strategy for us, as Player One, to ensure a win? Give this new challenge a try.

Story 6. A Tight Weave

The solution is to repeat the process infinitely often. We start with a purple square. At the first stage, a single gold square of size $1/3 \times 1/3$ is placed in the center. At the next stage eight more gold squares of size $1/9 \times 1/9$ are placed in the centers of each of the eight surrounding squares. At the next stage, 8×8, or 64, more gold squares of size $1/27 \times 1/27$ are placed in the centers of each of the eight squares that surround each of the eight squares that surround the original square. At each stage, we add increasingly many gold squares, each of a smaller size. So the final picture actually has infinitely many gold squares, but each of the eight squares surrounding the central square is an exact replica, though smaller, of the whole picture. This intricate purple and gold carpet is an example of a self-similar object known as a *fractal.* In Chapter 6: "Chaos and Fractals," we will examine many such infinitely intricate objects.



What is the area of all the (infinitely many) gold squares? Since all those gold squares lie within the rug that is 1 yard square, we know the area cannot be more than 1. At the first stage, we have one gold square of size $1/3 \times 1/3$, so its area is $1/9$. At the next stage, we add eight more gold squares, each of size $1/9 \times 1/9$, so their areas total $8 \times (1/9)^2$, making the total area of gold squares at stage two equal to $1/9 + 8 \times (1/9)^2 = 0.2098\ldots$. At the third stage, we add 8^2 more squares, each of area $(1/9)^3$. Thus, the total area of gold squares at stage three equals $1/9 + 8 \times (1/9)^2 + 8^2 \times (1/9)^3 = 0.2976\ldots$. Repeating, we begin to see a pattern. The fourth stage, for example, would have a gold area equal to $1/9 + 8 \times (1/9)^2 + 8^2 \times (1/9)^3 + 8^3 \times (1/9)^4 = 0.3757\ldots$. Thus, the total area of gold squares in the final pattern would be the infinite sum:

$$\frac{1}{9} + 8 \times \left(\frac{1}{9}\right)^2 + 8^2 \times \left(\frac{1}{9}\right)^3 + 8^3 \times \left(\frac{1}{9}\right)^4 + 8^4 \times \left(\frac{1}{9}\right)^5 + 8^5 \times \left(\frac{1}{9}\right)^6 + \cdots.$$

What does it equal? Even though there are infinitely many terms, we know that the whole area must be a number not greater than 1. What number is it?

The gold area at the 5th stage is	0.4450 . . . ;
at the 10th stage it is	0.6920 . . . ;
at the 15th stage it is	0.8291 . . . ;
at the 25th stage it is	0.9474 . . . ;
at the 50th stage it is	0.9972 . . . ;
at the 100th stage it is	0.999992 . . . .

Data can help uncover surprising observations and help build intuition and understanding.

From this pattern of numbers, it becomes clear that the gold area becomes increasingly close to 1—and that is a great guess for the area.

A clever way to calculate the total area is to add up all the infinitely many terms. We start by giving a name to the total; let's call that number SUM. Below you see the infinite sum that SUM represents. Directly under that, you see what (8/9)SUM equals. Notice that multiplying each term of SUM by 8/9 just shifts that term to the right. For example, $(8/9)(1/9) = 8 \times (1/9)^2$.

$$\text{SUM} = \frac{1}{9} + 8 \times \left(\frac{1}{9}\right)^2 + 8^2 \times \left(\frac{1}{9}\right)^3 + 8^3 \times \left(\frac{1}{9}\right)^4 + 8^4 \times \left(\frac{1}{9}\right)^5 + 8^5 \times \left(\frac{1}{9}\right)^6 + \cdots .$$

$$\left(\frac{8}{9}\right)\text{SUM} = \qquad 8 \times \left(\frac{1}{9}\right)^2 + 8^2 \times \left(\frac{1}{9}\right)^3 + 8^3 \times \left(\frac{1}{9}\right)^4 + 8^4 \times \left(\frac{1}{9}\right)^5 + 8^5 \times \left(\frac{1}{9}\right)^6 + \cdots .$$

Since all the terms of (8/9)SUM are directly under an identical term of SUM, it is easy to subtract (8/9)SUM from SUM, because all the terms drop out except the first term:

$$\text{SUM} - \left(\frac{8}{9}\right)\text{SUM} = \frac{1}{9} \qquad \text{and so:}$$

$$\left(\frac{1}{9}\right)\text{SUM} = \frac{1}{9}$$

Since $(1/9)\text{SUM} = 1/9$, what is SUM? It must equal 1! In other words, the area of the gold squares is equal to the area of the entire rug. Thus, even though there are many purple threads remaining in the final pattern, as we begin to see in the illustration on page 22, the purple contributes no area to the rug. Surprise! We will see many more counterintuitive mysteries of infinity in our studies of numbers, fractals, and, of course, infinity itself.

Story 7. Let's Make a Deal

Fortunately, Warren Piece enjoys mathematics as a hobby, so he believes he can solve this conundrum. He thinks carefully, assesses the chances each way, and confidently proclaims (while still jumping up and down, of course), "I switch my guess to Door Number 1, Monty."

Monty Hall turns and says, "Okay. Let's see what deal you've made. What is behind Door Number 1?" The door swings slowly open, and the crowd gasps

Story 9. Dot of Fortune

The math fan sees a red dot on the forehead of each of the other two players. She knows she has either a white dot or a red dot on her own forehead. Let's see what happens if we suppose her dot is white.

What would her two companions at the table see? Each would see one red dot and one white dot, and each would see two arms raised. Each would be thinking, "Do I have a red dot or a white dot on my forehead? If I have a white dot, then the red-dotted person would not have her hand up. Therefore, I must have a red dot." After making this easy deduction, this person would hit the buzzer.

But what did these two people actually do? Or, more to the point, what did they *not* do? They did not hit their buzzers! If either of them had seen a white dot and a red dot and two raised hands, he or she would have been able to deduce that his or her own dot was red. Since neither person buzzed right away, neither must have seen a white dot on the math fan's forehead. Therefore, the math fan waited just long enough to know that the other two players could not deduce their own dot colors, and then she buzzed, confident that her dot was red.

There is great power to be found in logical and creative thinking.

A final question of the story is, Why did the other students not know? The answer to that question is, of course, because they had not read *The Heart of Mathematics.*

1.4 From Play to Power

Discovering Strategies of Thought for Life

> Imagination is more important than knowledge.
>
> ALBERT EINSTEIN

▲ Grandmaster Maurice Ashley, teaching chess strategies to inner-city children in Harlem.

Our stories illustrate strategies of thinking. Even in such a light-hearted setting, certain techniques of thought emerge as powerful means to illuminate the unknown—techniques applicable to any situation we may face in life. We'll encounter more "life lessons" elsewhere in *The Heart of Mathematics;* on the next page we've summarized a few. Although some may seem obvious or trivial, don't take them lightly—they can be surprisingly useful for analyzing and enjoying life's adventure.

LESSONS FOR LIFE

1. *Just do it.*
2. *Make mistakes and fail, but never give up.*
3. *Keep an open mind.*
4. *Explore the consequences of new ideas.*
5. *Seek the essential.*
6. *Understand the issue.*
7. *Understand simple things deeply.*
8. *Break a difficult problem into easier ones.*
9. *Examine issues from several points of view.*
10. *Look for patterns and similarities.*

Mindscapes INVITATIONS TO FURTHER THOUGHT

We now provide some additional stories for further amusement and enlightenment. We call them "Mindscapes" because they are vistas for the mind that encourage you to expand your way of thinking.

For each of the following situations, contemplate, analyze, and resolve the puzzle. Also, guess which branch of mathematics each situation represents: Logic, Number Theory, Infinity, Geometry, Topology, Chaos, or Probability. Of course, we haven't discussed any of these areas in depth yet, but just take a guess—being wrong is fine.

Finally, we invite you to provide an aesthetic critique of each question and your solutions. In other words, did you find either the question or your solution interesting? Which questions were the most challenging? Do you like one of your solutions better than the others? At the end of this section we provide some hints for some of the questions. Use them sparingly.

> *"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."*
> LEWIS CARROLL

1. **Late-night cash.** Suppose that David Letterman and Paul Shaffer have the same amount of money in their pockets. How much must Dave give to Paul so that Paul would have $10 more than Dave?
2. **Politicians on parade.** There were 100 politicians at a certain convention. Each politician was either crooked or honest. We are given the following two facts:

a. At least one of the politicians was honest.

b. Given any two of the politicians, at least one of the two was crooked.

Can it be determined from these facts how many of the politicians were honest and how many were crooked? If so, how many? If not, why not?

3. **The profit.** A dealer bought an item for $7, sold it for $8, bought it back for $9, and sold it for $10. How much profit did she make?

4. **The truth about . . .** Fifty-six biscuits are to be fed to 10 pets; each pet is either a cat or a dog. Each dog is to get six biscuits, and each cat is to get five. How many dogs are there? (Try to find a solution without performing any algebra.)

5. **It's in the box.** There are two boxes: one marked A and one marked B. Each box contains either $1 million or a deadly snake that will kill you instantly. You must open one box. On box A there is a sign that reads: "At least one of these boxes contains $1 million." On box B there is a sign that reads: "A deadly snake that will kill you instantly is in box A." You are told that either both signs are true or both are false. Which box do you open? Be careful, the wrong answer is fatal!

6. **Lights out.** Two rooms are connected by a hallway that has a bend in it so that it is impossible to see one room while standing in the other. One of the rooms has three light switches. You are told that exactly one of the switches turns on a light in the other room, and the other two are not connected to any lights. What is the fewest number of times you would have to walk to the other room to figure out which switch turns on the light? And the follow-up question is: Why is the answer to the preceding question "one"? (Look out, this question uses properties of real lights as well as logic.)

7. **Out of sight but not out of mind.** The infamous band Slippery Even When Dry ended their concert and checked into the Fuzzy Fig Motel. The guys in the band (Spike, Slip, and Milly) decided to share a room. They were told by Chip, the night clerk who was taking a home study course on animal husbandry, that the room cost $25 for the night.

   Milly, who took care of the finances, collected $10 from each band member and gave Chip $30. Chip handed Milly the change, $5 in singles. Milly, knowing how bad Slip and Spike were at arithmetic, pocketed two of the dollars, turned to the others, and said, "Well guys, we got $3 change, so we each get a buck back." He then gave each of the other two members a dollar and pocketed the last one for himself.

   Once the band members left the office, Chip, who witnessed this little piece of deception, suddenly realized that something strange had just happened. Each of the three band members first put in $10 so there was a total of $30 at the start. Then Milly gave each guy and himself $1 back. That means that each person put in only $9, which is a total of $27 ($9 from each of the three). But Milly had skimmed off $2, so that gives a total of $29. But there was $30 to start with. Chip wondered what happened to that extra dollar and who had it. Can you please resolve and explain the issue to Chip?

8. **The cannibals and the missionaries.** In 1853 in the wilds of central Iowa, three missionaries and three cannibals were walking in a group. The missionaries were trying to convert the cannibals to their religion, while the cannibals were looking for a chance to practice their culture on the missionaries. After a time, they all came to a river that they wished to cross. None of the six could swim, but all could row. Fortunately, on the river bank was a small rowboat available for use.

   Since the boat was small and the cannibals and the missionaries were all on the large side, it was clear that only two persons could cross at one time. It was late in the day and neither cannibals nor missionaries had eaten much recently, and the missionaries began to notice that the cannibals were indicating greater and greater appreciation for the missionaries' ample girths. The missionaries decided that being prudent was better than being a main course, so they agreed that at no time would they allow any group of missionaries to be outnumbered by cannibals during the crossing. For their part, the cannibals did not fear being outnumbered by the missionaries because they realized that an excess of missionaries would result only in more discussion among the missionaries, thus relieving the cannibals of the burden of polite conversation.

   How do the cannibals and the missionaries all cross the river using only the one boat yet at no time letting the cannibals outnumber the missionaries on either side of the river?

9. **Whom do you trust?** Congresswoman Smith opened the *Post* and saw that a bean-counting scandal had been leaked to the press. Outraged, Smith immediately called an emergency meeting with the five other members of the Special Congressional Scandal Committee, the busiest committee on Capitol Hill.

   Once they were all assembled in Smith's office, Smith declared, "As incredible as it sounds, I know that three of you always tell the truth. So now I'm asking all of you, Who spilled the beans to the press?"

   Congressman Schlock spoke up, "It was either Wind or Pocket."

   Congressman Wind, outraged, shouted, "Neither Slie nor I leaked the scandal."

   Congressman Pocket then chimed in, "Well both of you are lying!"

   This provoked Congressman Greede to say, "Actually, I know that one of them is lying and the other is telling the truth."

   Finally, Congressman Slie, with steadfast eyes, stated, "No, Greede, that is not true."

   Assuming that Congresswoman Smith's first declaration is true, can you determine who spilled the beans?

10. **A commuter fly.** A passenger train left Austin, Texas, at 12:00 p.m. bound for Dallas, exactly 210 miles away; it traveled at a steady 50 miles per hour. At the same instant, a freight train left Dallas headed for Austin on the same track, traveling at 20 miles per hour. At this same high noon, a fly leaped from the nose of the passenger train and flew along the track at 100 miles

3. **The profit.** Different people will get different answers, and each person will argue that his or hers is correct. Act out the transactions and see what happens. After you try this, go back and figure out why other answers are incorrect.

Experimentation is an effective means of resolving difficult issues.

4. **The truth about . . .** What if all the animals were cats? How many extra biscuits would you have? Consider turning some of those cats into dogs. This transformation leads to an algebra-free solution.

Often a clever idea can be more potent than conventional wisdom.

5. **It's in the box.** Consider the two possibilities carefully. You don't want to slip up on this one.

Carefully consider the outcomes of various scenarios.

6. **Lights out.** Suppose you turn on a switch, wait a half hour, and then turn the switch off. If you were then to walk into the other room, could you tell if the light had been on for half an hour? Ponder this question, and use it to resolve the original puzzle.

Don't overlook or dismiss facts that seem insignificant or irrelevant.

7. **Out of sight but not out of mind.** Don't be fooled by all the numbers. Force yourself to figure out what was paid out and what was given back.

Don't believe unsubstantiated claims, even if they sound scientific. Until you understand the issue for yourself, be skeptical!

If that doesn't help, get 30 $1 bills and act out the entire episode. Once you discover the truth, go back and find out where the problem is in the story.

Experimentation is a powerful means for discovering patterns and developing insights.

See how many different ways you can devise to understand and explain what actually happened.

Once you find an argument that resolves an issue, it is a great challenge to find a different argument. However, in attempting to find other arguments, we often gain further insight into and understanding of the situation. Also, the first argument we come up with may not be the best one.

8. **The cannibals and the missionaries.** Professor Starbird shares his grandmother's solution.

   When my grandmother was 92, I gave this problem to her along with three nickels to represent the cannibals and three Life Savers candies to represent the missionaries. We set up a line on her table to represent the river so that she could slide the cannibals and the missionaries (the nickels and the Life Savers) back and forth singly or in pairs, thereby solving the problem. When I arrived the next day for my visit, she was delighted to tell me that she had solved the problem.

   "How did you do it?" I asked.

   She replied triumphantly, "I ate the missionaries."

   We give her half credit. A useful aspect of her method was to model the problem using a concrete representation. Making a written table with two columns would also be a good way to represent the setting. One column would be one bank of the river, and the other column would be the other bank. Each row would represent the situation after a crossing. So the first row would have three C's, three M's, and a B for the boat in the left-hand box and nothing in the right. The next row might have two C's and two M's in the left box and one C, one M, and the B in the right box. Going from row to row must be obtainable by moving one or two C's or M's along with the B to the other column.

   Once you have an effective representation of this question, a little experimentation will lead to an answer.

Devising a good representation of a problem is frequently the biggest step toward finding a solution.

9. **Whom do you trust?** To find the person who leaked the story, you must determine who is telling the truth. Ask yourself whether you can determine the truthfulness or deceit of any one person.

   If Pocket is telling the truth, then Schlock and Wind are liars, and the remaining three—Pocket, Greede, and Slie—are telling the truth. Could those three all be telling the truth? If not, then you know for certain that Pocket is lying.

Since Slie contradicts Greede, you know that one of them is lying. Which one?

A rock of certainty can be the foundation of a tower of truth.

10. A commuter fly. On close inspection, notice that the fly changes directions an infinite number of times during her travels. It is possible to compute how far the fly has flown before she encounters the freight train for the first time. Once you know this, it's possible to compute the distance she travels before encountering the passenger train on the return trip. You could compute those distances and find a pattern and then solve the problem by adding up the infinite list of distances. However, there is a much easier way to solve this puzzle.

How much time will pass before the trains collide? How far will the fly fly in that length of time? Case closed.

This story is not complete without our telling an anecdote about the famous mathematician John von Neumann. Von Neumann was notorious for being extremely fast and accurate at calculating numbers in his head—oddly enough, not a skill that all mathematicians possess. One day he was walking with a friend who asked him the question of the fly between the trains. Instantly, von Neumann stated the answer. The questioner said, "Oh, you saw the trick." To which von Neumann replied, "Yes, it was an easy infinite series."

If you are not von Neumann, the fly-between-the-trains story provides a good life lesson.

Look at problems from different perspectives.

Go out of your way to think about different ways to view a problem. In this case, if you know how long the fly flies, you can compute the distance the fly travels. You have now reduced the original problem to a different, though related, problem. In this case, the different problem is much simpler to solve than the original one.

Look at related situations.

11. A fair fare. This question does not have one definitive answer. However, a look at a related problem may persuade you that one possibility is best. What if, instead of staying in one taxi the whole time, the three travelers traveled the first 10 miles together and then all got out and paid the first cabby. The first traveler then left, and the remaining two got another cab, rode 10 miles, and again got out and paid the second cabby. Then the last traveler took a cab alone for the remaining 10 miles. This rephrasing of the original problem makes the division of payment seem more obvious.

12. **Getting a pole on a bus.** It seems impossible to get the 5-foot pole on the bus, given that the largest length of an item allowed on the bus is 4 feet. Sarah gave Adam a large box to put the pole in. Now give the dimensions of the box and explain why it does the trick.

Often an inventive solution arises from looking at a situation in an unusual way.

13. **Tea time.** This question contains much unnecessary and distracting information. A close look at the story reveals that the description of the dinnerware and the names of the people are extraneous details. But what may not be quite so obvious is that the number of ounces in the teacup and the creamer and the amounts poured and spooned are also irrelevant.

Don't be distracted by this extraneous information. Suppose the problem did not contain those facts at all and instead was stated as follows:

> A creamer and a teacup each have exactly the same amount of cream and tea, respectively. An undisclosed amount of mixing of the cream and tea goes on, but after the mixing, each of the two containers still contains the same amount of liquid as the other. Is the tea more diluted than the cream, or is the cream more diluted than the tea?

Having less information might force you to look at the situation differently and consequently, to understand and solve it.

Look at problems from different perspectives.

14. **A shaky story.** Exactly one person at the party said to Sam that he or she shook eight hands. Note the obvious fact that each person with whom that person shook hands must have shaken hands with at least one person. Now determine how many hands that person's spouse shook. See if this approach leads to any insights. If it still does not, consider an easier problem: Suppose that there were just three couples, or even two couples. Search for a pattern.

If you have a hard problem, first work on a simpler, related problem to develop insight.

15. **Murray's brother.** This genie has posed a difficult challenge. It can be fun to work on, but do not work on it too long if you get frustrated. In this puzzle,

we must squeeze every ounce (or even gram) of information from every weighing.

Don't ignore information.

Each weighing must be designed to give us maximum information. After a weighing, we learn many things. Let's begin by putting four stones on each side of a scale and recording what we observe. If the scale balances, we know that all eight stones weigh the same, and the diamond is not among those eight. So the mystery stone is among the remaining four, but we still do not know whether it is heavier or lighter than the others. Can you now find the Dormant Diamond and determine whether it is heavy or light?

Suppose that the four-against-four weighing does not balance. This imbalance gives us much information. We know that the unweighed four stones all weigh the same. We know that each of the four stones on the light side of the scale are potentially light, but none of them is potentially heavier than the 11 other stones. We know similar things about the four stones on the other side of the scale. We will have to keep track of the stones and consider putting potentially light stones with potentially heavy ones to help sort things out. For example, suppose we weigh a potentially light stone with a potentially heavy stone on one side of the scale and two stones that are known to be normal on the other side. Then, depending on which way the scale tips, we can conclude which of the two stones is the Dormant Diamond.

You might think about the last step to help you find an intermediate solution. That is, you might specify what collections of stones and knowledge would allow you to find the diamond in one more weighing. For example, suppose you figure out that the diamond is among three stones that are potentially heavier than the others. Could you find the diamond in one more weighing? Or suppose you had narrowed the field to three stones, one potentially heavier than normal and two potentially lighter than normal. Could you find the diamond in one more weighing? This technique of working backward is often useful.

This balance-scale conundrum is tricky and difficult. Everyone, including experienced mathematicians, would have to think hard to solve it. It can be fun to work on if you enjoy this type of puzzle. Play with it; think carefully about what you know; carefully keep track of all the information you gather. But if you're not enjoying yourself, then just move on.

2.1 Counting

How the Pigeonhole Principle Leads to Precision Through Estimation

▲ *Numbers in Color* (1958–1959) by Jasper Johns

> The simple modes of number are of all other the most distinct; even the least variation, which is a unit, making each combination as clearly different from that which approacheth nearest to it, as the most remote; two being as distinct from one, as two hundred; and the idea of two as distinct from the idea of three, as the magnitude of the whole earth is from that of a mite.
>
> JOHN LOCKE

We begin with the numbers we first learned as children: 1, 2, 3, 4, . . . (the ". . ." indicate that there are more, but we don't have enough room to list them). These numbers are so natural to us they are actually called *natural numbers.* These numbers are familiar, but often familiar ideas lead to surprising outcomes, as we will soon see.

The most basic use of numbers is counting, and we will begin by just counting approximately. That is, we'll consider the power and the limitations of making rough estimates. In a way, this is the weakest possible use we can make of numbers, and yet we will still find some interesting outcomes. So let's just have some fun with plain old counting.

Quantitative Estimation

Make it quantitative.

One powerful technique for increasing our understanding of the world is to move from qualitative thinking to quantitative thinking whenever possible. Some people still count: "1, 2, 3, many." Counting in that fashion is effective for a simple existence but does not cut it in a world of trillion-dollar debts and gigabytes of hard-drive storage. In our modern world there are practical differences between thousands, millions, billions, and trillions. Some collections are easy to count exactly because there are so few things in them: the schools in the Big Ten Conference, the collection of letters you've written home in the past month, and the clean underwear in your dorm room. Other collections are more difficult to count exactly—such as the grains of sand in the Sahara Desert, the stars in the sky, and the hairs on your roommate's body. Let's look more closely at this last example.

It would be difficult, awkward, and frankly just plain weird to count the number of hairs on your roommate's body. Without undertaking that perverse task, we nevertheless pose the following.

Question ▸ Do there exist two nonbald people on the planet who have exactly the same number of hairs on their bodies?

It appears that we cannot answer this question since we don't know (and don't intend to find out) the body-hair counts for anyone. But can we estimate body-hair counts well enough to get some idea of what that number might be? In particular, can we at least figure out a number that we could state with confidence is larger than the number of hairs on the body of any person on Earth?

HOW HAIRY ARE WE?

Let's take the direct approach to this body-hair business. One of the authors counted the number of hairs on a 1/4-inch × 1/4-inch square area on his scalp and counted about 100 hairs—that's roughly 1600 hairs per square inch. From this modest follicle count, we can confidently say that no person on Earth has as many as 16,000 hairs in any square inch anywhere on his or her body. The author is about 72 inches tall and 32 inches around. If the author were a perfect cylinder, he would have 72-inch × 32-inch or about 2300 square inches of skin on the sides and about another 200 square inches for the top of his head and soles of his feet, for a total of 2500 square inches of skin. Since the author is not actually a perfect cylinder (he has, for example, a neck), 2500 square inches is an overestimate of his skin area. There are people who are taller and bigger than this author, but certainly there is no one on this planet who has 10 times as much skin as this author. Therefore, no body on Earth will have more than 25,000 square inches of skin. We already agreed that each square inch can have no more

A 1" square containing 2000 hairs—not too physically likely.

than 16,000 hairs on it. Thus we deduce that no person on this planet can have more than 400 million (400,000,000) hairs on his or her body.

HOW MANY ARE WE?

An almanac or a Web site would tell us that there are about 6.2 billion (6,200,000,000) people on this planet. Given this information, can we answer our question: Do there exist two nonbald people on the planet who have *exactly the same* number of hairs on their bodies? We urge you to think about this question and try to answer it before reading on.

WHY MANY PEOPLE ARE EQUALLY HAIRY

There are more than 6 billion people on Earth, but each person has many fewer than 400 million hairs on his or her body. Could it be that no two people have the same number of body hairs? What would that mean? It would mean that each of the 6 billion people would have a different number of body hairs. But we know that the number of body hairs on each person is less than 400 million. So, there are less than 400 million different possible body-hair numbers. Therefore, not all 6 billion people can have different body-hair counts.

Looking at an issue from a new point of view often enables us to understand it more clearly.

Suppose we have 400 million rooms—each numbered in order. Suppose each person did know his or her body-hair count, and we asked each person in the world to go into the room whose number is equal to his or her body-hair number. Could everyone go into a different room? Of course not! We have 6 billion people and only 400 million room choices—some room or rooms must have more than one person. In other words, there definitely exist two people, in fact many people, who have the same number of body hairs.

By using some simple estimates, we have been able to answer a question that first appeared unanswerable. The surprising twist is that in this case a rough estimate led to a conclusion about an exact equality. However, there are limitations to our analysis. For example, we are unable to name two *specific* people who have the same body-hair counts even though we know they are out there.

THE POWER OF REASONING

In spite of the silliness of our hair-raising question we see the power of reasoned analysis. We were faced with a question that on first inspection appeared unanswerable, but through creative thought we were able to crack it. When we are first faced with a new question or problem, the ultimate path of logical reasoning is often hidden from sight. When we try, think, fail, think some more, and try some more, we finally discover a path.

We solved the hairy-body question, but that question in itself is not of great value. However, once we have succeeded in resolving an issue, it is worthwhile to isolate the approach we used, because the method of thought may turn out to be far more important than the problem it solved. In this case, the key to

answering our question was the realization that there are more people on the planet than there are body hairs on any individual's body. This type of reasoning is known as the *Pigeonhole principle.* If we have an antique desk with slots for envelopes (known as *pigeonholes*) like the one shown, and we have more envelopes than slots, then certainly some slot must contain at least two envelopes. This Pigeonhole principle is a simple idea, but it is a useful tool for drawing conclusions when the size of a collection exceeds the number of possible variations of some distinguishing trait.

Once we understand the Pigeonhole principle, we become conscious of something that has always been around us—we see it everywhere. For example, in a large swim meet, some pairs of swimmers will get exactly the same times to the tenths of a second. Some days more than 100 people will die in car wrecks. With each breath, we breathe an atom that Einstein breathed before us. Each person will arrive at work during the exact same minute many times during his or her life. Many trees have the same number of leaves. Many people get the same SAT score.

Often after we learn a principle of logical reasoning, we see many instances where it applies.

Number Personalities

Srinivasa Ramanujan

G.H. Hardy

The natural numbers 1, 2, 3, . . . , besides being useful in counting, have captured the imagination of people around the world from different cultures and different eras. The study of natural numbers began several thousand years ago and continues to this day. Mathematicians who are intrigued by numbers come to know them individually. In the eye of the mathematician, individual numbers have their own personalities—unique characteristics and distinctions from other numbers. In subsequent sections of this chapter, we will discover some intriguing properties of numbers and uncover their nuances. For now, however, we wish to share a story that captures the human side of mathematicians. Of course, mathematicians, like people in other professions, display a large range of personalities, but this true story of Ramanujan and Hardy depicts almost a caricature of the "pure" mathematician. It illustrates part of the mythology of mathematics and provides insight into the personality of an extraordinary mathematician.

This interaction of two mathematicians on such an abstract plane even during serious illness is poignant. They clearly thought each number was worthy of special consideration. To affirm their special regard for each number, we now demonstrate conclusively that every natural number is interesting by means of a whimsical, though ironclad, proof.

The Intrigue of Numbers.
Every natural number is interesting.

Ramanujan and Hardy

One of the most romantic tales in the history of the human exploration of numbers involves the life and work of the Indian mathematician Srinivasa Ramanujan. Practically isolated from the world of academics, libraries, and mathematicians, Ramanujan made amazing discoveries about natural numbers.

In 1913, Ramanujan wrote to the great English mathematician G.H. Hardy at Cambridge University, describing his work. Hardy immediately recognized that Ramanujan was a unique jewel in the world of mathematics, because Ramanujan had not been taught the standard ways to think about numbers and thus was not biased by the rigid structure of a traditional education; yet he was clearly a mathematical genius. Since the pure nature of mathematics transcends languages, customs, and even formal training, Ramanujan's imaginative explorations have since given mathematicians everywhere an exciting and truly unique perspective on numbers.

Ramanujan loved numbers as his friends, and found each to be a distinct wonder. A famous illustration of Ramanujan's deep connection with numbers is the story of Hardy's visit to Ramanujan in a hospital. Hardy later recounted the incident: "I remember once going to see him when he was lying ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one and that I hoped it was not an unfavorable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.'" Notice that, indeed, $1729 = 12^3 + 1^3$, and also $1729 = 10^3 + 9^3$.

PROOF THAT NATURAL NUMBERS ARE INTERESTING

Let's first consider the number 1. Certainly 1 is interesting, because it is the first natural number and it is the only number with this property: If we pick any number and then multiply it by 1, the answer is the original number we picked. So, we agree that the first natural number is interesting.

Let us now consider the number 2. Well, 2 is the first even number, and that is certainly interesting—and, if that weren't enough, remember that 2 is the smallest number of people required to make a baby. Thus, we know that 2 is genuinely interesting.

We now consider the number 3. Is 3 interesting? Well, there are only two possibilities: Either 3 *is* interesting, or 3 *is not* interesting. Let us suppose that 3 *is not* interesting. Then notice that 3 has a spectacular property: It is the *smallest* natural number that is not interesting—which is certainly an interesting property! Thus we see that 3 is, after all, quite interesting.

Knowing now that 1, 2, and 3 are all interesting, we can make an analogous argument for 4 or any other number. In fact, suppose now that k is a certain natural number with the property that the first k natural numbers are all interesting. That is, $1, 2, 3, \ldots, k$ are all interesting. We know this fact is true if k is 1, and, in fact, it is true for larger values of k as well (2, 3, and 4, for example).

We now consider the very next natural number: $k + 1$. Is $k + 1$ interesting? Suppose it were not interesting. Then it would be the smallest natural number

11. **Many fold (S).** Suppose you were able to take a large piece of paper of ordinary thickness and fold it in half 50 times. What would the height of the folded paper be? Would it be less than a foot? About one yard? As long as a street block? As tall as the Empire State Building? Taller than Mount Everest?

12. **Only one cake.** Suppose we had a room filled with 370 people. Will there be at least two people who celebrate their birthdays on the same day?

13. **For the birds.** Years ago, before overnight delivery services and e-mail, people would send messages by carrier pigeon and would keep an ample supply of pigeons in pigeonholes on their rooftops. Suppose you have a certain number of pigeons, let's say P of them, but you have only $P - 1$ pigeonholes. If every pigeon must be kept in a hole, what can you conclude? How does the principle we discussed in this section relate to this question?

14. **Sock hop.** You have 10 pairs of socks, five black and five blue, but they are not paired up. Instead, they are all mixed up in a drawer. It's early in the morning, and you don't want to turn on the lights in your dark room. How many socks must you pull out to guarantee that you have a pair of one color? How many must you pull out to have two good pairs (each pair is the same color)? How many must you pull out to be certain you have a pair of black socks?

15. **The last one.** Here is a game to be played with natural numbers. You start with any number. If the number is even, you divide it by 2. If the number is odd, you triple it (multiply it by 3), and then add 1. Now you repeat the process with this new number. Keep going. You win (and stop) if you get to 1. For example, if we start with 17, we would have:

    17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1 → we see a 1, so we win!

    Play four rounds of this game starting with the numbers 19, 11, 22, and 30. Do you think you will always win no matter what number you start with? No one knows the answer!

III. Creating New Ideas

16. **See the three.** What proportion of the first 1000 natural numbers have a 3 somewhere in them? For example, 135, 403, and 339 all contain a 3, whereas 402, 677, and 8 do not.

17. **See the three II (H).** What proportion of the first 10,000 natural numbers contain a 3?

18. **See the three III.** Explain why almost all million-digit numbers contain a 3.

19. **Commuting.** One hundred people in your neighborhood always drive to work between 7:30 and 8:00 a.m. and arrive 30 minutes later. Why must two people always arrive at work at the same time, within a minute?

20. **RIP (S).** The Earth has 6.2 billion people and almost no one lives 100 years. Suppose this longevity fact remains true. How do you know that some year soon, more than 50 million people will die?

IV. Further Challenges

21. **Say the sequence.** The following are the first few terms in a sequence. Can you figure out the next few terms and describe how to find all the terms in the sequence?

1
11
21
1211
111221
312211
. . .

22. **Lemonade.** You want to buy a new car, and you know the model you want. The model has three options, each one of which you can either take or not take, and you have a choice of four colors. So far 100,000 cars of this model have been sold. What is the largest number of cars that you can guarantee to have the same color and the same options as each other?

V. In Your Own Words

23. **With a group of folks.** In a small group, discuss and work through the reasoning for why there are two people on Earth having the same number of hairs on their bodies. After your discussion, write a brief narrative describing your analysis and conclusion in your own words.

2.2 Numerical Patterns in Nature

Discovering the Beauty of the Fibonacci Numbers

▲ We can discover patterns by looking closely at our world.

> There is no inquiry which is not finally reducible to a question of Numbers; for there is none which may not be conceived of as consisting in the determination of quantities by each other, according to certain relations.
>
> AUGUSTE COMTE

Often when we see beauty in nature, we are subconsciously sensing hidden order—order that itself has an independent richness. Thus we stop and smell the roses—or, more accurately, count the daisies. In the previous section, we contented ourselves with estimation, whereas here we move to exact counting. The example of counting daisies is an illustration of discovering numerical patterns in nature through direct observation. The pattern we find in the daisy appears elsewhere in nature and also gives rise to issues of aesthetics that touch such diverse fields as architecture and painting. We begin our investigation, however, firmly rooted in nature.

Have you ever examined a daisy? Sure, you've picked off the white petals one at a time while thinking: "Loves me . . . loves me not," but have you ever

Look for patterns.

taken a good hard look at what's left once you've finished plucking? A close inspection of the yellow in the middle of the daisy reveals unexpected structure and intrigue. Specifically, the yellow area contains clusters of spirals coiling out from the center. If we examine the flower closely, we see that there are, in fact, two sets of spirals—a clockwise set and a counterclockwise set. These two sets of spirals interlock to produce a hypnotic interplay of helical form.

Interlocking spirals abound in nature. The cone flower and the sunflower both display nature's signature of dual, locking spirals. Flowers are not the only place in nature where spirals occur. A pinecone's exterior is composed of two sets of interlocking spirals. The rough and prickly facade of a pineapple also contains two collections of spirals.

Be Specific: Count

In our observations we should not be content with general impressions. Instead, we move toward the specific. In this case we ponder the quantitative quandary: How many spirals are there? An approximate count is: lots. Is the number of clockwise spirals the same as the number of counterclockwise spirals? You can physically verify that the pinecone has 5 spirals in one direction and 8 in the other. The pineapple has 8 and 13. The daisy and cone flower both have 21 and 34. The sunflower has a staggering 55 and 89. In each case, we observe that the number of spirals in one direction is nearly twice as great as the number of spirals in the opposite direction. Listing all those numbers in order we see

5, 8, 13, 21, 34, 55, 89.

Is there any pattern or structure to these numbers?

Suppose we were given just the first two numbers, 5 and 8, on that list of spiral counts. How could we use these two numbers to build the next number? How can we always generate the next number on our list?

We note that 13 is simply 5 plus 8, whereas 21, in turn, is 8 plus 13. Notice that this pattern continues. What number would come after 89? Given this pattern, what number should come before 5? How about before that? How about before that? And before that?

Leonardo's Legacy: The Fibonacci Sequence

The rule for generating successive numbers in the sequence is to add up the previous two terms. So the next number on the list would be $55 + 89 = 144$.

Through spiral counts, nature appears to be generating a sequence of numbers with a definite pattern that begins

1 1 2 3 5 8 13 21 34 55 89 144 . . . .

Leonardo of Pisa, or Fibonacci

This sequence is called the *Fibonacci sequence,* named after the mathematician Leonardo of Pisa (better known as Fibonacci—a shortened form of Filius Bonacci, *son of Bonacci*), who studied it in the 13th century. After seeing this surprising pattern, we hope you feel compelled to count for yourself the spirals in the previous pictures of flowers. In fact, you may now be compelled to count the spirals on a pineapple every time you go to the grocery store.

Why do the numbers of spirals always seem to be consecutive terms in this list of numbers? The answer involves issues of growth and packing. The yellow florets in the daisy begin as small buds in the center of the plant. As the plant grows, the young buds move away from the center toward a location where they have the most room to grow—that is, in the direction that is least populated by older buds. If one simulates this tendency of the buds to find the largest open area as a model of growth on a computer, then the spiral counts in the geometrical pattern so constructed will appear in our list of numbers. The Fibonacci numbers are an illustration of surprising and beautiful patterns in nature. The fact that nature and number patterns reflect each other is indeed a fascinating concept.

A powerful method for finding new patterns is to take the abstract patterns that we directly observe and look at them by themselves. In this case, let's move beyond the vegetable origins of the Fibonacci numbers and just think about the Fibonacci sequence as an interesting entity in its own right. We conduct this investigation with the expectation that interesting relationships that we find among Fibonacci numbers may also be represented in our lives.

FIBONACCI NEIGHBORS

Unexpected patterns are often a sign of hidden, underlying structure.

We observed that flowers, pinecones, and pineapples all display consecutive pairs of Fibonacci numbers. These observations point to some natural bond between adjacent Fibonacci numbers. In each case, the number of spirals in one direction was not quite twice as great as the number of spirals in the other direction. Perhaps we can find richer structure and develop a deeper understanding of the Fibonacci numbers by moving from an estimate ("not quite twice") to a precise value. So, let's measure the relative size of each Fibonacci

$$\varphi = 1 + \cfrac{1}{\boxed{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}$$

The answer is: The number in the frame is φ again. Why? Well, suppose we were just shown the number inside the frame without any of that other stuff around it. We'd look at that new number and realize that the 1's go on forever, and thus that number is just φ. Stay with this picture until you see the idea behind it. Therefore, we just discovered that

$$\varphi = 1 + \frac{1}{\varphi}.$$

SOLVING FOR φ

Now we have an equation involving just φ, and this will allow us to solve for the exact value of φ. First, we can subtract 1 from both sides to get

$$\varphi - 1 = \frac{1}{\varphi}.$$

Multiplying through by φ we get

$$\varphi^2 - \varphi = 1$$

or just

$$\varphi^2 - \varphi - 1 = 0.$$

This "quadratic equation" can be solved using the quadratic formula, which states that

$$\varphi = \frac{1 \pm \sqrt{5}}{2}.$$

But since φ is bigger than 1, we must have

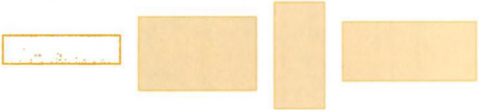
$$\varphi = \frac{1 + \sqrt{5}}{2}.$$

Using a calculator, express $(1 + \sqrt{5})/2$ as a decimal and compare it with the data from our previous calculator experimentation on the quotients of consecutive Fibonacci numbers. Well, there we have it—our goal was to find the exact value of φ, and through a process of observation and thought we succeeded.

The Golden Ratio

At the moment, we have no reason to consider the number φ to be especially interesting; however, it is somewhat curious that the quotients of consecutive Fibonacci numbers do seem to approach this fixed value. We started with simple observations of flowers and pinecones. We saw a numerical pattern among our observations. The pattern led us to the number $(1 + \sqrt{5})/2$.

The number $\varphi = (1 + \sqrt{5})/2$ is called the *Golden Ratio* and, besides its connection with nature's spirals, it captures the proportions of some especially pleasing shapes in art, architecture, and geometry. Just to foreshadow what is to come when we revisit the Golden Ratio in the geometry chapter, here is a question: What are the proportions of the most attractive rectangle? In other words, when someone says "rectangle" to you, and you think of a shape, what is it? Light some scented candles, put on a Yanni CD, close your eyes, and dream about the most attractive and pleasing rectangle you can imagine. Once that image is etched in your mind, open your eyes, put out the candles, and pick from the four choices below the rectangle that you think is most representative of that magical rectangle dancing in your mind.



Many people think that the second rectangle from the left is the most aesthetically pleasing—the one that captures the notion of "rectangleness." That rectangle is called the *Golden Rectangle,* and we will examine it in detail in Chapter 4. The ratio of the dimensions of the sides of the Golden Rectangle is a number rich with intrigue. If we divide the length of the longer side by the length of the shorter side, we get φ: the Golden Ratio. A 3-inch × 5-inch index card is close to being a Golden Rectangle. Notice that its dimensions, 3 and 5, are consecutive Fibonacci numbers. In the geometry chapter we will consider the aesthetic issues involving φ and make some interesting connections between the Fibonacci numbers and the Golden Rectangle in art.

To Be or Not to Be Fibonacci

After finding Fibonacci numbers hidden in the spirals of nature, it saddens us to realize that not all numbers are Fibonacci. However, we are delighted to announce that in fact every natural number is a neat *sum* of Fibonacci numbers. In particular, *every* natural number is either a Fibonacci number or it is expressible uniquely as a sum of Fibonacci numbers whereby no two are adjacent Fibonacci numbers. Here is one way to find the sum:

1. Write down a natural number.
2. Find the largest Fibonacci number that does not exceed your number. That Fibonacci number is the first term in your sum.
3. Subtract that Fibonacci number from your number and look at this new number.

4. Find the largest Fibonacci number that does not exceed this new number. That Fibonacci number is the second number in your sum.
5. Continue this process.

For example, consider the number 38. The largest Fibonacci number not exceeding 38 is 34. So consider $38 - 34 = 4$. The largest Fibonacci number not exceeding 4 is 3, and $4 - 3 = 1$, which is a Fibonacci number. Therefore, $38 = 34 + 3 + 1$. Similarly, we can build any natural number just by adding Fibonacci numbers in this manner. In one sense, Fibonacci numbers are building blocks for the natural numbers through addition.

Fun and Games with Fibonacci

Fibonacci numbers not only appear in nature; they can also be used to accumulate wealth. (Moral: Math pays.) We can see this moral for ourselves in a game called *Fibonacci nim,* which is played with two people. All we need is a pile of sticks (toothpicks, straws, or even pennies will do). Person One moves first by taking any number of sticks (at least one but not all) away from the pile. After Person One moves, it is Person Two's move, and the moves continue to alternate between them. Each person (after the first move) may take away as many sticks as he or she wishes; the only restriction is that he or she must take at least one stick but no more than two times the number of sticks the previous person took. The player who takes the last stick wins the game.

Suppose we start with ten sticks, and Person One removes three sticks, leaving seven. Now Person Two may take any number of the remaining sticks from one to six (six is two times the number Person One took). Suppose Person Two removes five, leaving two in the pile. Now Person One is permitted to take any number of sticks from one to ten ($10 = 2 \times 5$), but because there are only two sticks left, Person One takes the two sticks and wins. Play Fibonacci nim with various friends and with different numbers of starting sticks. Get a feel for the game and its rules—but don't wager quite yet.

If we are careful and use the Fibonacci numbers, we can always win. Here is how. First, we make sure that the initial number of sticks we start with is *not* a Fibonacci number. Now we must be Person One, and we find some poor soul to be Person Two. If we play it just right, we will always win. The secret is to write the number of sticks in the pile as a sum of nonconsecutive Fibonacci numbers. Figure out the *smallest* Fibonacci number occurring in the sum, and remove that many sticks from the pile on the first move. Now it is your luckless opponent's turn. No matter what he or she does, we will repeat the preceding procedure. That is, once he or she is done, we count the number of sticks in the pile, express the number as a sum of nonconsecutive Fibonacci numbers, and then remove the number of sticks that equals the smallest Fibonacci number in the sum. It is a fact that, no matter what our poor opponent does, we will always be able to remove that number of sticks without breaking the rules. Experiment with this game and try it. Wager at will—or not.

a look back

WE DEFINE THE FIBONACCI NUMBERS successively by starting with 1, 1, and then adding the previous two terms to get the next term. These numbers are rich with structure and appear in nature. The numbers of clockwise and counterclockwise spirals in flowers and other plants are consecutive Fibonacci numbers. The ratio of consecutive Fibonacci numbers approaches the Golden Ratio, a number with especially pleasing proportions. While not all numbers are Fibonacci, every natural number can be expressed as the sum of distinct, nonconsecutive Fibonacci numbers.

The story of Fibonacci numbers is a story of pattern. As we look at the world, we can often see order, structure, and pattern. The order we see provides a mental concept that we can then explore on its own. As we discover relationships in the pattern, we frequently find that those same relationships refer back to the world in some intriguing way.

Understand simple things deeply.

Mindscapes INVITATIONS TO FURTHER THOUGHT

In this section, Mindscapes marked (H) have hints for solutions at the back of the book. Mindscapes marked (S) have solutions.

I. Developing Ideas

1. **Fifteen Fibonaccis.** List the first 15 Fibonacci numbers.
2. **Born φ.** What is the precise number that the symbol φ represents? What sequence of numbers approaches φ?
3. **Tons of ones.** Verify that $1 + \frac{1}{1+\frac{1}{1}}$ equals 3/2.
4. **Twos and threes.** Simplify the quantities $2 + \frac{2}{2+\frac{2}{2}}$ and $3 + \frac{3}{3+\frac{3}{3}}$.
5. **The family of φ.** Solve the following equations for x: $x = 2 + \frac{1}{x}$, $x = 3 + \frac{1}{x}$.

II. Solidifying Ideas

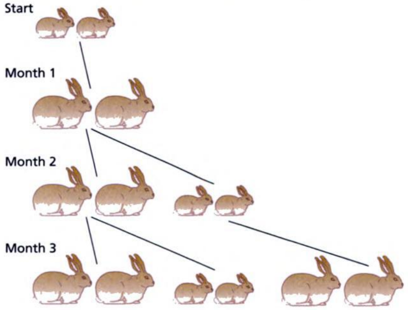
6. **Baby bunnies.** This question gave the Fibonacci sequence its name. It was posed and answered by Leonardo of Pisa, better known as Fibonacci.

   Suppose we have a pair of baby rabbits: one male and one female. Let us assume that rabbits cannot reproduce until they are one month old and that they have a one-month gestation period. Once they start reproducing, they produce a pair of bunnies each month (one of each sex). Assuming that no pair ever dies, how many pairs of rabbits will exist in a particular month?

During the first month, the bunnies grow into rabbits. After two months, they are the proud parents of a pair of bunnies. There will now be two pairs of rabbits: the original, mature pair and a new pair of bunnies. The next month, the original pair produces another pair of bunnies, but the new pair of bunnies is unable to reproduce until the following month. Thus we have:

Time in Months	Start	1	2	3	4	5	6	7
Number of Pairs	1	1	2					

Continue to fill in this chart and search for a pattern. Here is a suggestion: Draw a family tree to keep track of the offspring.



We'll use the symbol F_1 to stand for the first Fibonacci number, F_2 for the second Fibonacci number, F_3 for the third Fibonacci number, and so forth. So $F_1 = 1$, and $F_2 = 1$, and, therefore, $F_3 = F_2 + F_1 = 2$, and $F_4 = F_3 + F_2 = 3$, and so on. In other words, we write F_n for the nth Fibonacci number where n represents any natural number. So, the rule for generating the next Fibonacci number by adding up the previous two can now be stated symbolically as:

$$F_n = F_{n-1} + F_{n-2}.$$

7. **Discovering Fibonacci relationships (S).** By experimenting with numerous examples in search of a pattern, determine a simple formula for $(F_{n+1})^2 + (F_n)^2$—that is, a formula for the sum of the squares of two consecutive Fibonacci numbers.

8. **Discovering more Fibonacci relationships.** By experimenting with numerous examples in search of a pattern, determine a simple formula for

carefully and beat your friend. Play again with another (non-Fibonacci) number of sticks to start. Record the number of sticks removed at each stage of each game. Finally, reveal the secret strategy and record your friend's reaction.

III. Creating New Ideas

26. **Discovering still more Fibonacci relationships.** By experimenting with numerous examples in search of a pattern, determine a formula for $F_{n+1} \times F_{n-1} - (F_n)^2$—that is, a formula for the product of a Fibonacci number and the Fibonacci number that comes after the next one, minus the square of the Fibonacci number in between them. (*Hint:* The answer will be different depending on whether n is even or odd. Consider examples of different cases separately.)

27. **Finding factors (S).** By experimenting with numerous examples, find a way to factor F_{2n} into the product of two natural numbers that are from famous sequences. That is, consider every other Fibonacci number starting with the second 1 in the sequence, and factor each in an interesting way. Discover a pattern. (*Hint:* Mindscape II.10 may be relevant.)

28. **The rabbits rest.** Suppose we have a pair of baby rabbits—one male and one female. As before, the rabbits cannot reproduce until they are one month old. Once they start reproducing, they produce a pair of bunnies (one of each sex) each month. Now, however, let us assume that each pair dies after three months, immediately after giving birth. Create a chart showing how many pairs we have after each month from the start through month nine.

29. **Digging up Fibonacci roots.** Using the square root key on a calculator, evaluate each number in the top row and record the answer in the bottom row.

Number	$\sqrt{\left(\frac{F_3}{F_1}\right)}$	$\sqrt{\left(\frac{F_4}{F_2}\right)}$	$\sqrt{\left(\frac{F_5}{F_3}\right)}$	$\sqrt{\left(\frac{F_6}{F_4}\right)}$	$\sqrt{\left(\frac{F_7}{F_5}\right)}$	$\sqrt{\left(\frac{F_8}{F_6}\right)}$	$\sqrt{\left(\frac{F_9}{F_7}\right)}$
Computed Value							

Looking at the chart, make a guess as to what special number $\sqrt{F_{n+2}/F_n}$ approaches as n gets larger and larger.

30. **Tribonacci.** Let's start with the numbers 0, 0, 1, and generate future numbers in our sequence by adding up the previous three numbers. Write out the first 15 terms in this sequence, starting with the first 1. Use a calculator to evaluate the value of the quotients of consecutive terms (dividing the smaller term into the larger one). Do the quotients seem to be approaching a fixed number?

31. **Fibonacci follies.** Suppose you are playing a round of Fibonacci nim with a friend. You start with 15 sticks. You start by removing two sticks; your friend

then takes one; you take two; your friend takes one. What should your next move be? Can you make it without breaking the rules of the game? Did you make a mistake at some point? If so, where?

32. **Fibonacci follies II.** Suppose you are playing a round of Fibonacci nim with a friend. You start with 35 sticks. You start by removing one stick; your friend then takes two; you take three; your friend takes six; you take three, your friend takes two. What should your next move be? Can you make it without breaking the rules of the game? Did you make a mistake at some point? If so, where?

33. **Fibonacci follies III.** Suppose you are playing a round of Fibonacci nim with a friend. You start with 21 sticks. You start by removing one stick; your friend then takes two. What should your next move be? Can you make it without breaking the rules of the game? What went wrong?

34. **A big fib.** Suppose we have a natural number that is not a Fibonacci number—let's call it N. Suppose that F is the largest Fibonacci number that does not exceed N. Show that the number $N - F$ must be smaller than the Fibonacci number that comes right before F.

35. **Decomposing naturals (H).** Use the result of Mindscape III.34, together with the notion of systematically reducing a problem to a smaller problem, to show that every natural number can be expressed as a sum of distinct, nonconsecutive Fibonacci numbers.

IV. Further Challenges

36. **How big is it?** Is it possible for a Fibonacci number greater than 2 to be exactly twice as big as the Fibonacci number immediately preceding it? Explain why or why not. What would your answer be if we removed the phrase "greater than 2"?

37. **Too small.** Suppose we have a natural number that is not a Fibonacci number—let's call it N. Let's write F for the largest Fibonacci number that does not exceed N. Show that it is impossible to have a sum of two distinct Fibonacci numbers each less than F add up to N.

38. **Beyond Fibonacci.** Suppose we create a new sequence of natural numbers starting with 0 and 1. Only this time, instead of adding the two previous terms to get the next one, let's generate the next term by adding 2 *times* the previous term to the term before it. In other words: $F_{n+1} = 2F_n + F_{n-1}$. Such a sequence is called a *generalized Fibonacci sequence.* Write out the first 15 terms in this generalized Fibonacci sequence. Adapt the methods that were used in this section to figure out that the quotient of consecutive Fibonacci numbers approaches $(1 + \sqrt{5})/2$ to discover the exact number that F_{n+1}/F_n approaches as n gets large.

39. **Generalized sums.** Let F_n be the generalized Fibonacci sequence defined in Mindscape IV.38. Can every natural number be expressed as the sum of

distinct, nonconsecutive generalized Fibonacci numbers? Show why, or give several counterexamples. What if you were allowed to use consecutive generalized Fibonacci numbers? Do you think you could do it then? Illustrate your hunch with four or five specific examples.

40. **It's hip to be square (H).** Adapt the methods of this section to prove that the numbers $\sqrt{F_{n+2}/F_n}$ approach φ as n gets larger and larger. (Here, F_n stands for the usual Fibonacci number. See Mindscape IV. 39.)

V. In Your Own Words

41. **Personal perspectives.** Write a short essay describing the most interesting or surprising discovery you made in exploring the material in this section. If any material seemed puzzling or even unbelievable, address that as well. Explain why you chose the topics you did. Finally, comment on the aesthetics of the mathematics and ideas in this section.

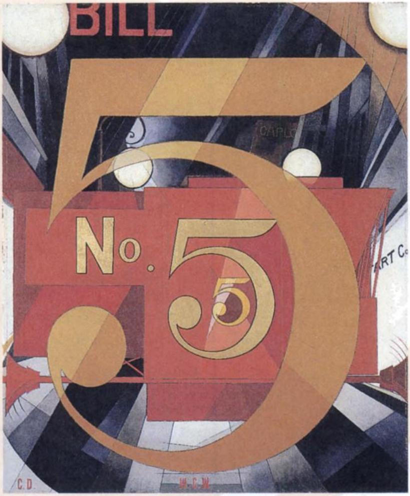
42. **With a group of folks.** In a small group, discuss and work through the reasoning for how the quotients of consecutive Fibonacci numbers approach the Golden Ratio. After your discussion, write a brief narrative describing the rationale in your own words.

43. **Creative writing.** Write an imaginative story (it can be humorous, dramatic, whatever you like) that involves or evokes the ideas of this section.

44. **Power beyond the mathematics.** Provide several real-life issues—ideally, from your own experience—for which some of the strategies of thought presented in this section would provide effective methods for approaching and resolving them.

2.3 Prime Cuts of Numbers

How the Prime Numbers Are the Building Blocks of All Natural Numbers



▲ *I Saw the Figure 5 in Gold* (1928) by Charles Demuth

> . . . number is merely the product of our mind.
>
> KARL FRIEDRICH GAUSS

The natural numbers, 1, 2, 3, . . . , help us describe and understand our world. They in turn form their own invisible world filled with abstract relationships, some of which can be revealed through simple addition and multiplication. These basic operations lead to subtle insights about our familiar numbers.

Our strategy for uncovering the structure of the natural numbers is to break down complex objects and ideas into their fundamental components. This simple yet powerful technique recurs frequently throughout this book, and throughout our lives. As we become accustomed to using this strategy, we will see that complicated situations are often best analyzed first by investigating the building blocks of an idea or an object and then by understanding how these building blocks combine to create a complex whole. The natural numbers are a good arena for observing this principle in action.

Building Blocks

What are the fundamental components of the natural numbers? How can we follow the suggestion of breaking down numbers into smaller components?

There are many ways to express large natural numbers in terms of smaller ones. For example, we might first think of addition: Every natural number can be constructed by just adding $1 + 1 + 1 + 1 + \cdots + 1$ enough times. This method demonstrates perhaps the most fundamental feature of natural numbers: They are simply a sequence of counting numbers, each successive one bigger than its predecessor. However, this feature provides only a narrow way of distinguishing one natural number from another.

Divide and Conquer

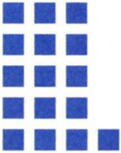
How can one natural number be expressed as the product of smaller natural numbers? This innocent-sounding question leads to a vast field of interconnections among the natural numbers that mathematicians have been exploring for thousands of years. Our adventure begins by recalling the arithmetic from our youth and looking at it afresh.

One method of writing a natural number as a product of smaller ones is first to divide and then to see if there is a remainder. We were introduced to division a long time ago in third or fourth grade—we weren't impressed. Somehow it paled in comparison to, say, recess. The basic reality of long division is that either it comes out even or there is a remainder. If the division comes out even, we then know that the smaller number divides evenly into the larger number and that our number can be factored. For example, 12 divided by 3 is 4, so $3 \times 4 = 12$ and 3 and 4 are factors of 12. More generally, suppose that n and m are any two natural numbers. We say that n divides evenly into m if there is an integer q such that

$$nq = m.$$

The integers n and q are called *factors* of m.

If the division does not come out even, the remainder is less than the number we tried to divide. For example, 16 divided by 5 is 3 with a remainder of 1.



This whole collection of elementary school flashbacks can be summarized in a statement that sounds far more impressive than "long division," namely, the *Division Algorithm.*

there? Are there more than a million primes? Are there more than a billion primes? Are there infinitely many primes?

Since there are infinitely many natural numbers, it seems reasonable to think that we would need bigger and bigger primes in order to build up the bigger and bigger natural numbers. Therefore, we might conclude that there are infinitely many primes. This argument, however, isn't valid. Do you see why? It's not valid because we can build natural numbers as large as we wish just by multiplying a couple of small prime numbers. For example, using just 2 and 3, we are able to make huge numbers:

$$45{,}349{,}632 = 2^8\, 3^{11}.$$

The point is that, since we are allowed to use tons of 2's and 3's in our product, we can construct numbers as large as we wish. Therefore, at this moment it seems plausible that there may be only finitely many prime numbers, even though every large (and small) natural number is a product of primes.

INFINITELY MANY PRIMES

Euclid

Although the previous argument was invalid, it turns out that there are infinitely many prime numbers, and thus there is no largest prime number. Euclid discovered the following famous valid proof more than 2000 years ago. It is beautiful in that the idea is clever and uncomplicated. In fact, on first inspection we may think there is some sleight of hand going on, as if some fast-talking salesman in a polyester plaid sports jacket is trying to sell us a used car—before we know it, we're the not-so-proud owner of a 1973 Gremlin. But this is not the case. Think along with us as we develop this argument. The ideas fit together beautifully, and, if you stay with it, the argument will suddenly "click." Let's now examine one of the great triumphs of human reasoning.

The Infinitude of Primes.
There are infinitely many prime numbers.

THE STRATEGY BEHIND THE PROOF

The strategy for proving that there are infinitely many prime numbers is to show that, for each and every given natural number, we can always find a prime number that is larger than that natural number. Since we can consider larger and larger numbers as our natural number, this claim would imply that there are larger and larger prime numbers. Thus we would show that there must be infinitely many primes, because we could find primes as large as we want without bound.

Before moving forward with the general idea of the proof, let's illustrate the key ingredient with a specific example. Suppose we wanted to find a prime number that is greater than 4. How could we proceed? Well, we could just say

"5" and be done with it, but that is not in the spirit of what we are trying to do. Our goal is to discover a method that can be generalized and used to find a prime number that exceeds an *arbitrary* natural number, not just the pathetically small number 4. So we seek a systematic means of finding a prime that exceeds, in this case, 4. What should we do? Our challenge is to:

1. find a number bigger than 4 that is not evenly divisible by 2;
2. find a number bigger than 4 that is not evenly divisible by 3; and
3. find a number bigger than 4 that is not evenly divisible by 4.

Each of these tasks individually is easy. To satisfy (1), we just pick an odd number. To satisfy (2), we just pick a number that has a remainder when we divide by 3. To satisfy (3), we just pick a number that has a remainder when we divide by 4. We now build a number that meets all those conditions simultaneously. Let's call this new number N for *new*. Here is an N that meets all three conditions simultaneously:

$$N = (1 \times 2 \times 3 \times 4) + 1.$$

So, N is really just the number 25, but, since we are trying to discover a general strategy, let's not think of N as merely 25 but instead think of N as the more impressive $(1 \times 2 \times 3 \times 4) + 1$.

We notice that N is definitely larger than 4. By the Prime Factorization of Natural Numbers, we know that N is either a prime number or a product of prime numbers. In the first possibility, if N is a prime number, then we have just found a prime number that is larger than 4—which was our goal. We now must consider the only other possibility: that N is not prime. If N is not prime, then N is a product of prime numbers. Let's call one of those prime factors of N "*OUR-PRIME*." So *OUR-PRIME* is a prime that divides evenly into N.

Now what can we say about *OUR-PRIME*? Is *OUR-PRIME* equal to 2? Well, if we divide 2 into N we see that, since

$$N = 2 \times (1 \times 3 \times 4) + 1,$$

we have a remainder of 1 when 2 is divided into N. Therefore, 2 does not divide evenly into N, and so *OUR-PRIME* is not 2. Is *OUR-PRIME* equal to 3? No, for the same reason:

$$N = 3 \times (1 \times 2 \times 4) + 1,$$

so we get a remainder of 1 when 3 is divided into N; therefore, 3 does not divide evenly into N and hence is not a factor of N. Likewise, we will get a remainder of 1 when 4 is divided into N, and, therefore, 4 is not a factor of N. So 2, 3, and 4 are not factors of N. Hence we conclude that all the factors of N must be larger than 4 (there are no factors of N that are 4 or smaller). But that means that, since *OUR-PRIME* is a prime that is a factor of N, *OUR-PRIME* must be a prime number greater than 4. Therefore, we have just found a prime number greater than 4. Mission accomplished!

Following this same strategy, show that there must be a prime greater than 5. Can you use the preceding method to show there must be a prime greater than 10,000,000? Try it now.

FINALLY, THE PROOF

Now let's use the method we developed in the specific example to prove our theorem in general. Remember that we wish to demonstrate that, for any particular natural number, there is a prime number that exceeds that particular number. Let m represent an arbitrary natural number. Our goal now is to show that there is a prime number that exceeds m. To accomplish this lofty quest, we will, just as before, construct a new number using all the numbers from 1 to m. We'll call this new number N (for *new* number) and define it to be 1 plus the product of all the natural numbers from 1 to m—in other words, (or more accurately in other symbols):

$$N = (1 \times 2 \times 3 \times 4 \times \cdots \times m) + 1.$$

It is fairly easy to see that N is larger than m. By the Prime Factorization of Natural Numbers we know that there are only two possibilities for N: Either N is prime or N is a product of primes. If the first possibility is true, then we have found what we wanted since N is larger than m. But if it is not true, then we must consider the more challenging possibility: that N is a product of prime numbers.

If N is a product of primes, then we can choose one of those prime factors and call it *BIG-PRIME*. So, *BIG-PRIME* is a prime factor of N. Let's now try to pin down the value of *BIG-PRIME*. We'll start off small.

Does *BIG-PRIME* equal 2? Well, if we divide 2 into N we see that, since

$$N = 2 \times (1 \times 3 \times 4 \times \cdots \times m) + 1,$$

we have a remainder of 1 when 2 is divided into N. Therefore, 2 does not divide evenly into N, and so *BIG-PRIME* cannot equal 2.

Does *BIG-PRIME* equal 3? No, for the same reason:

$$N = 3 \times (1 \times 2 \times 4 \times \cdots \times m) + 1,$$

so we get a remainder of 1 when 3 is divided into N. In fact, what is the remainder when any number from 2 to m is divided into N? The remainder is always 1 by the same reasoning that we used for 2 and 3.

Okay, so we see that none of the numbers from 2 through m divides evenly into N. That fact means that none of the numbers from 2 through m is a factor of N. Therefore, what can be said about the size of the factor *BIG-PRIME*? Answer: It must be BIG since we know that N has no factors between 2 and m. Hence any factor of N must be larger than m. Therefore, *BIG-PRIME* is a prime number that is larger than m.

Well, we did it! We just showed that there is a prime that exceeds m. Since this procedure works for any value of m, this argument shows that there are arbitrarily large prime numbers. Therefore, we must have infinitely many prime numbers, and we have completed the proof.

THE CLEVER PART OF THE PROOF

In the proof, each step by itself isn't too hard, but the entire argument, taken as a whole, is subtle. What is the most clever part of the proof? In other words, where is the most imagination required? Which step in the argument would have been hardest to think up on your own?

We believe that the most ingenious part of the proof is the idea of constructing the auxiliary number N (one more than the product of all the numbers from 1 to m). Once we have the idea of considering that N, we can finish the proof. But it took creativity and contemplation to arrive at that choice of N. We might well say to ourselves, "Gee, I wouldn't have thought of making up that N." Generally, slick proofs such as this are arrived at only after many attempts and false starts—just as Euclid no doubt experienced before he thought of this proof. Very few people can understand arguments of this type on first inspection, but once we can hold the whole proof in our minds, we will regard it as straightforward and persuasive and appreciate its aesthetic beauty. Ingenuity is at the heart of creative mathematical reasoning, and therein lies the power of mathematical thought.

Prime Demographics

Now that we know for sure that there are infinitely many prime numbers, we wonder how the primes are distributed among the natural numbers. Is there some pattern to their distribution? There are infinitely many primes, but how rare are they among the numbers? What proportion of the natural numbers are prime numbers? Half? A third? To explore these questions, let's start by looking at the natural numbers and the primes among them. Here are the first few with the primes printed in bold:

1, **2**, **3**, 4, **5**, 6, **7**, 8, 9, 10, **11**, 12, **13**, 14, 15, 16, **17**, 18, **19**, 20, 21, 22, **23**, 24, 25, 26, 27, 28, **29**, . . .

Out of the first 24 natural numbers, nine are primes. We see that $9/24 = 0.375$ of the first 24 natural numbers are primes—that's just a little over one-third. Extrapolating from this observation would we guess that just over one-third of *all* natural numbers are prime numbers? We could try an experiment; namely, we could continue to list the natural numbers and find the proportions of primes and see whether that proportion remains about one-third of the total number. If we do this experiment, we will learn an important lesson in life: Don't be too hasty to generalize based on a small amount of evidence.

Before high-speed computers were available, calculating (or just estimating) the proportion of prime numbers in the natural numbers was a difficult task. In fact, years ago "computers" were people who did computations. Such people were amazingly accurate, but they required a great deal of time and dedication to accomplish what today's electronic computers can do in seconds. An 18th-century Austrian arithmetician, J.P. Kulik, spent 20 years of his life creating, by hand, a table of the first 100 million primes. His table was never published, and sadly the volume containing the primes between 12,642,600 and 22,852,800 has disappeared.

Today, software computes the number of primes less than n for increasingly large values of n, and the computer prints out the proportion: (number of primes less than n)/n. Computers have no difficulty producing such a table for values of n up to the billions, trillions, and beyond. If we examine the results, we notice that the proportion of primes slowly goes downward. That is, the percentage of numbers less than a million that are prime is smaller than the percentage of numbers less than a thousand that are prime. The primes, in some sense, get sparser and sparser as the numbers get bigger and bigger.

A CONJECTURE ABOUT PATTERNS

In the early 1800s, Karl Friedrich Gauss (*left*), one of the greatest mathematicians ever—known by many as the Prince of Mathematics—and A.M. Legendre (*right*), another world-class mathematician, made an insightful observation about primes. They noticed that, even though primes do not appear to occur in any predictable pattern, the proportion of primes is related to the so-called natural logarithm—a function relating to exponents that we may or may not remember from our school daze.


Karl Friedrich Gauss


A.M. Legendre

Years ago, one needed to interpolate huge tables to find the logarithm. Today, we have scientific calculators that compute logarithms instantly and painlessly. Get out a scientific calculator and look for the LN key. Type "3" and then hit LN. You should see 1.09861. . . . We encourage you to try some natural-logarithm experiments on your calculator. How does the size of the natural logarithm of a number compare with the size of the number itself?

Gauss and Legendre conjectured that the proportion of the number of primes among the first n natural numbers is approximately $1/\text{Ln}(n)$. The following chart, constructed with the aid of a computer (over which Gauss and Legendre would drool), shows the number of primes up to n, the proportions of primes, and a comparison with $1/\text{Ln}(n)$.

n	Number of Primes up to n	Proportion of Primes up to n (Number of Primes $\leq n$)/n	$1/\text{Ln}(n)$	Proportion − $1/\text{Ln}(n)$
10	4	0.4	0.43429...	0.03429...
100	25	0.25	0.21714...	0.03285...
1000	168	0.168	0.14476...	0.02323...
10,000	1229	0.1229	0.10857...	0.01432...
100,000	9592	0.09592	0.08685...	0.00906...
1,000,000	78,498	0.078498	0.07238...	0.00611...
10,000,000	664,579	0.0664579	0.06204...	0.00441...
100,000,000	5,761,455	0.05761455	0.05428...	0.00332...
1,000,000,000	50,847,534	0.050847534	0.04825...	0.00259...

Andrew Wiles

We will never know whether Fermat had discovered a correct proof of his Last Theorem, but we do know one thing. He did not discover the proof that Andrew Wiles (*left*) of Princeton University produced in 1994, 357 years after Fermat wrote his tantalizing marginal note. If Fermat had somehow conceived of Wiles's deep and complicated proof, he would not have written, "The margin of this book is not large enough to contain it." He would have written, "The proof would require a moving van to carry it." Wiles's proof drew on entirely new branches of mathematics and incorporated ideas undreamed of in the 17th century.

Some of the greatest minds in mathematics have worked on Fermat's Last Theorem. The statement of Fermat's Last Theorem does not strike us as intrinsically important or interesting—it just states that a certain type of equation never can be solved with natural numbers. What *is* interesting and important are all the deep mathematical ideas that arose during attempts to prove Fermat's Last Theorem, many of which led to new branches of mathematics. Although Fermat's Last Theorem has not yet been used for practical purposes, the new theories developed to attack it turned out to be valuable in many practical technological advances. From an aesthetic perspective, it is difficult to determine which questions are more important than others. When problems resist attempts by the best mathematical minds over many years, the problems gain prestige. Fermat's Last Theorem resisted all attacks for 357 years, but it finally succumbed.

Andrew Wiles's complete proof of Fermat's Last Theorem is over 130 pages long, and it relies on many important and difficult theorems, including some new theorems from geometry (although it appears surprising that geometry should play a role in solving this problem involving natural numbers). When mathematicians expose connections between seemingly disparate areas of mathematics, they feel an electric excitement and pleasure. In mathematics, as in nature, elements fit together and interrelate. As we will begin to discover, deep and rich connections weave their way through the various mathematical topics, forming the very fabric of truth.

The Vast Unknown

Every answer allows us to recognize and formulate new questions.

Many people think mathematics is a static, ancient body of facts, formulas, and techniques. In reality, much of it is a wondrous mystery with many questions unanswered and more still yet to be asked. Many people think we will soon know all there is to know. But this impression is not the case, even though such thinking has persisted throughout history.

> *Everything that can be invented has been invented.*
> CHARLES H. DUELL, COMMISSIONER U.S. PATENT OFFICE, 1899

> *We don't know a millionth of one percent about anything.*
> THOMAS EDISON

Human thought is an ever-expanding universe—especially in mathematics. We know a small amount, and our knowledge allows us to glimpse a small part of what we do not know. Vastly larger is our *ignorance* of what we do not know. An important shift in perspective on mathematics and other areas of human knowledge occurs when we move from the sense that we know most of the answers to the more accurate and comforting realization that we will not run out of mysteries.

So, after celebrating, as we have, some of the great mathematical achievements that were solved only after many decades of human creativity and thought, we close this section by gazing forward to questions that remain unsolved—*open* questions. From among the thousands of questions on which mathematicians are currently working, here are two famous ones about prime numbers that were posed hundreds of years ago and are still unsolved. Fermat's Last Theorem has been conquered; but somehow the mathematical force is not ready to let go its hold and let these two fall.

The Twin Prime Question.
Are there infinitely many pairs of prime numbers that differ from one another by two? (11 and 13, 29 and 31, and 41 and 43 are examples of some such pairs.)

The Goldbach Question.
Can every positive, even number greater than 2 be written as the sum of two primes? (Pick some even numbers at random, and see whether you can write them each as a sum of two primes.)

Computer analysis allows us to investigate a tremendous number of cases, but the results of such analyses do not provide ironclad proof for all cases.

We have seen how to decompose natural numbers into their fundamental building blocks, and we have discovered further mysteries and structures in this realm. Can we use these antique, abstract results about numbers in our modern lives? The amazing and perhaps surprising answer is a resounding *yes!*

a look back

THE PRIME NUMBERS are the basic multiplicative building blocks for natural numbers, since every natural number greater than 1 can be factored into primes. We can prove that there are infinitely many primes by showing that we can always find a prime number larger than any specified number. The strategy is to take the product of all numbers up to the specified number and then add 1. This new large integer must have a prime factor greater than the original specified number.

The study of primes goes back to ancient times. Some questions remained unanswered for a long time before being resolved. And others remain unanswered.

We discovered proofs of the Prime Factorization of Natural Numbers and the Infinitude of Primes by carefully exploring specific examples and searching for patterns. Considering specific examples while thinking about the general case guided us to new discoveries.

Understanding a specific case well is a major step toward discovering a general principle.

Mindscapes INVITATIONS TO FURTHER THOUGHT

In this section, Mindscapes marked (H) have hints for solutions at the back of the book. Mindscapes marked (S) have solutions.

I. Developing Ideas

1. **Primal instincts.** List the first 15 prime numbers.
2. **Fear factor.** Express each of the following numbers as a product of primes: 6, 24, 27, 35, 120.
3. **Odd couple.** If n is an odd number greater than or equal to 3, can $n + 1$ ever be prime? What if n equals 1?
4. **Tower of power.** The first four powers of 3 are $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, and $3^4 = 81$. Find the first 10 powers of 2. Find the first five powers of 5.
5. **Compose a list.** Give an infinite list of natural numbers that are not prime.

II. Solidifying Ideas

6. **A silly start.** What is the smallest number that looks prime but really isn't?
7. **Waiting for a nonprime.** What is the smallest natural number n, greater than 1, for which $(1 \times 2 \times 3 \times \cdots \times n) + 1$ is *not* prime?
8. **Always, sometimes, never.** Does a prime multiplied by a prime ever result in a prime? Does a nonprime multiplied by a nonprime ever result in a prime? Always? Sometimes? Never? Explain your answers.
9. **The dividing line.** Does a nonprime divided by a nonprime ever result in a prime? Does it ever result in a nonprime? Always? Sometimes? Never? Explain your answers.
10. **Prime power.** Is it possible for an extremely large prime to be expressed as a large integer raised to a very large power? Explain.

11. **Nonprimes.** Are there infinitely many natural numbers that are not prime? If so, prove it.

12. **Prime test.** Suppose you are given a number n and are told that 1 and the number n divide into n. Does that mean n is prime? Explain.

13. **Twin primes.** Find the first 15 pairs of twin primes.

14. **Goldbach.** Express the first 15 even numbers greater than 2 as the sum of two prime numbers.

15. **Odd Goldbach (H).** Can every odd number greater than 3 be written as the sum of two prime numbers? If so, prove it; if not, find the smallest counterexample and show that the number given is definitely not the sum of two primes.

16. **Still the 1 (S).** Consider the following sequence of natural numbers: 1111, 11111, 111111, 1111111, 11111111, ... Are all these numbers prime? If not, can you describe infinitely many of these numbers that are definitely not prime?

17. **Zeros and ones.** Consider the following sequence of natural numbers made up of 0's and 1's: 11, 101, 1001, 10001, 100001, 1000001, 10000001, ... Are all these numbers prime? If not, find the first such number that is not prime and express it as a product of prime numbers.

18. **Zeros, ones, and threes.** Consider the following sequence of natural numbers made up of 0's, 1's, and 3's: 13, 103, 1003, 10003, 100003, 1000003, 10000003, ... Are all these numbers prime? If not, find the first such number that is not prime and express it as a product of prime numbers.

19. **A rough count.** Using results discussed in this section, estimate the number of prime numbers that are less than 10^{10}.

20. **Generating primes (H).** Consider the list of numbers: $n^2 + n + 17$, where n first equals 1, then 2, 3, 4, 5, 6, ... What is the smallest value of n for which $n^2 + n + 17$ is not a prime number? (*Bonus:* Try this for $n^2 - n + 41$. You'll see an amazingly long string of primes!)

21. **Generating primes II.** Consider the list of numbers: $2^n - 1$, where n first equals 2, then 3, 4, 5, 6, ... What is the smallest value of n for which $2^n - 1$ is not a prime number?

22. **Floating in factors.** What is the smallest natural number that has three distinct prime factors in its factorization?

23. **Lucky 13 factor.** Suppose a certain number when divided by 13 yields a remainder of 7. What is the smallest number we would have to subtract from our original number to have a number with a factor of 13?

24. **Remainder reminder (S).** Suppose a certain number when divided by 13 yields a remainder of 7. If we add 22 to our original number, what is the remainder when this new number is divided by 13?

25. **Remainder roundup.** Suppose a certain number when divided by 91 yields a remainder of 52. If we add 103 to our original number, what is the remainder when this new number is divided by 7?

III. Creating New Ideas

26. **Related remainders (H).** Suppose we have two numbers that both have the same remainder when divided by 57. If we subtract the two numbers, are there any numbers that we know will definitely divide evenly into this difference? What is the largest number that we are certain will divide into the difference? Use this observation to state a general principle about two numbers that have the same remainder when divided by another number.

27. **Prime differences.** Write out the first 15 primes all on one line. On the next line, underneath each pair, write the difference between the larger number and the smaller number in the pair. Under this line, below each pair of the previous line, write the difference between the larger number and the smaller number. Continue in this manner. Your "triangular" table should begin with:

```
2    3    5    7    11    13    17    ...
 \  / \  / \  / \  /  \  /  \  /
  1    2    2    4     ...
   \  / \  / \  / \  /
    1    0    2    ...
```

and so on . . .

Once your chart is made, imagine that all the primes were listed on that first line. What would you guess is the pattern for the sequence of numbers appearing in the first entry of each line? The actual answer is not known. It remains an open question! What do you think?

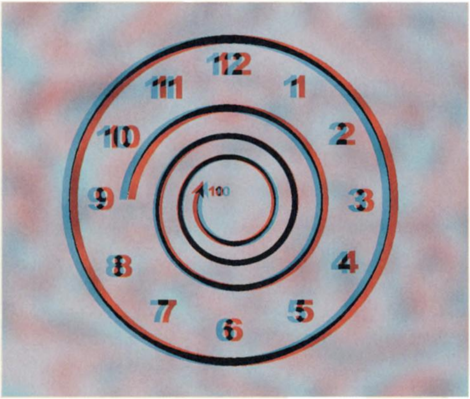
28. **Minus two.** Suppose we take a prime number greater than 3 and then subtract 2. Will this new number always be a prime? Explain. Are there infinitely many primes for which the answer to the question is yes? How does this last question relate to a famous open question?

29. **Prime neighbors.** Does there exist a number n such that both n and $n + 1$ are prime numbers? If so, find such an n; if not, show why not.

30. **Perfect squares.** A perfect square is a number that can be written as a natural number squared. The first few perfect squares are 1, 4, 9, 16, 25, 36. How many perfect squares are less than or equal to 36? How many are less than or equal to 144? In general, how many perfect squares are less than or equal to n^2? Using all these answers, estimate the number of perfect squares less than or equal to N. (*Hint:* Your estimate may involve square roots and should be the exact answer whenever N is itself a perfect square.)

2.4 Crazy Clocks and Checking Out Bars

Cyclical Clock Arithmetic and Bar Codes



> A rule to trick
> th' arithmetic.
>
> RUDYARD KIPLING

▲ Use the 3D glasses from the back of your book to view this picture.

Cycles are familiar parts of life. The seasons, phases of the moon, day and night, birth and death—all are among the most powerful natural forces that define our lives, and all are cycles. Whole cultural traditions revolve around this cyclic reality of life; consider, for example, the notion of reincarnation, unless you already considered it in a previous life. We can use these cycles as models to develop analogous constructs in the realm of numbers. Such explorations create yet another kind of cycle, because the abstract mathematical insights refer back to the world, and we find applications of these abstractions in our daily lives.

Our strategy for examining cycles in the world of numbers is to find a phenomenon in nature (in this case cyclicity) and to develop a mathematical model that captures some features of the natural processes. This method of reasoning by analogy is a powerful way to develop new ideas, because we use existing ideas, events, and phenomena to guide us in creating new insights.

Most people would not believe that there is deep and powerful number theory going on when they glance at their watch or check out at the grocery store: "Sure, numbers are involved. The time of day is expressed in numbers, as is the price of an item—but these numbers are neither deep nor powerful anything!" Most people, however, are sadly mistaken. The fabulous world of exotic number theory lies hidden in everyday objects.

Time

What time is it? Suppose your watch says it is 9:00 (9 o'clock), and you are to meet the love of your life in 37 hours. What time will your watch read when you fall into the arms of your soul mate? Careful—the answer is not 46 o'clock. The answer is 10:00. So some type of strange arithmetic must be required, since 9 plus 37 equals 10—wacky! How does one perform arithmetic in the context of telling time by a clock? Unlike the natural numbers, which get larger and larger when we add them together, a clock cycles around, and in 12 hours the clock returns to its original position (assuming we're using a 12-hour clock). Counting with a clock in some sense is easier than standard counting, because the numbers never get too large. For example, to add 37 to 9 we could count as follows:

9, 10, 11, 12,
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,
1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Notice how, once we get to 12, we start all over again and cycle back to 1. This procedure involves a kind of arithmetic different from standard arithmetic.

CLOCK ARITHMETIC

For the moment, let's refer to this arithmetic—where we return to 1 after 12—as *clock arithmetic*. Let's look at a few examples of clock arithmetic. We have already seen that $9 + 37 = 10$. What is $6 + 12$? The answer is 6, since adding 12 just spins us back around to where we started. So 12 is just like 0 in this clock arithmetic. This observation allows us to perform this new arithmetic in a different way. For example, with $9 + 37$, we could notice that 37 is equal to $12 + 12 + 12 + 1$. But remember that adding 12 is just like adding 0, so really 37 is equivalent to 1. Therefore, $9 + 37 = 9 + 1 = 10$.

Let's consider a different kind of question: What does $(4 \times 7) + 20$ equal in clock arithmetic? Well, $4 \times 7 = 28$, but 28 is equivalent to 4, since $28 = 12 + 12 + 4$. Now 20 is equivalent to 8, since $20 = 12 + 8$. Therefore, $(4 \times 7) + 20 = 4 + 8 = 12$. So the answer is 12.

What would happen to our arithmetic if we had a crazy clock that looked like this:

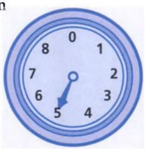
Notice now that adding 7 spins us back to where we started, and thus adding 7 is the same as adding 0. So now $6 + 4 = 10$ is equivalent to 3, since $10 = 7 + 3$. In other words, with this crazy clock, 4 hours after 6 o'crazy clock is actually 3 o'crazy clock. Why would anyone ever bother with such a crazy clock? Well, actually we use this crazy clock, not for telling time, but for telling days in a week. Once again we see that the notion of cycles is natural and important. In fact, as we will now discover, this kind of crazy-clock arithmetic helps us find errors in grocery prices, our checking accounts, UPS package deliveries, airline tickets, and driver's license numbers; it even helps us check out Shakespeare—*read on MacDuff.*

EQUIVALENCE

As we look at cycles, we are developing an idea of *equivalence.* The notion of equivalence occurs in clock arithmetic, for example, when we note that 37 is equivalent to 1. As we develop the idea of cyclical arithmetic, this concept of equivalence will become a central theme.

Identifying similarities among different objects is often the key to understanding a deeper idea.

Let's carefully define a type of arithmetic that will capture the spirit of our previous observations and generalize the notion of clock arithmetic. First we'll explore the notion of a new hip or "mod" clock. Suppose we are given a number—for example, 9, and we have an unusual clock that has 9 hours marked on it: 0, 1, 2, 3, 4, 5, 6, 7, 8. Suppose the hour hand is on the 5. Then 9 hours later the hand returns to the 5, and thus according to this clock, adding 9 doesn't change what the clock reads. So, we can now perform arithmetic using this clock by remembering that 9 is equivalent to 0. We'll write the fact that 9 is equivalent to 0 as $9 \equiv 0$, where the symbol "$\equiv$" means "equivalent." Of course, 9 is not *equal* to 0, but using this clock we see that 9 is *equivalent* to 0. Let's call this arithmetic *mod 9 clock arithmetic.* The key is that we can perform arithmetic as usual with the understanding that a 9 may be replaced by a 0. For example:



$$13 + 25 = (9 + 4) + (9 + 9 + 7) \equiv (0 + 4) + (0 + 0 + 7) =$$
$$4 + 7 = 11 = 9 + 2 \equiv 0 + 2 = 2 \bmod 9,$$

so $13 + 25 \equiv 2 \bmod 9$. We write the phrase "mod 9" at the end to remind us and indicate to others what kind of mod clock we are using to perform the arithmetic. In terms of the clock itself, we could have computed the preceding sum by placing the hour hand on the 0 and then moving the hand around 13 hours (which brings us to 4) and then moving from that 4 another 25 hours, which brings us to 2.

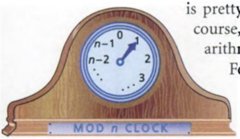
Once we see how mod 9 clock arithmetic works, we can abstract the idea by dispensing with the visual aid of the clock and calling it *mod 9 arithmetic.* This notation is convenient, since in mathematical jargon "mod" actually stands for "modulo" or "modular," which means just doing arithmetic using the equivalence of $9 \equiv 0 \bmod 9$ (translation: "9 is equivalent to 0 modulo 9" or "9 is equivalent to 0 in mod 9 arithmetic").

Explore ideas systematically.

In this next example, notice how we are able to replace large numbers with smaller equivalent numbers by just writing them in terms of 9's (also notice how remainders are making an appearance in our work):

$$(3 \times 5) + (7 \times 100) = (9 + 6) + (7 \times ((9 \times 11) + 1)) \equiv$$
$$(0 + 6) + (7 \times ((0 \times 11) + 1)) = 6 + 7 = 13 = 9 + 4 \equiv$$
$$0 + 4 = 4 \bmod 9,$$

so, $(3 \times 5) + (7 \times 100) \equiv 4 \bmod 9$. Once we get the hang of it, this arithmetic is pretty easy. We just pull out multiples of 9 and replace them by 0's. Of course, we can now think about other mod clocks. We can do this modular arithmetic with any clock, as long as we know how many hours it has on it. For any particular natural number n, we write "mod n" to mean that we are thinking about arithmetic on a mod clock that has n hours on it (marked $0, 1, 2, \ldots, n-1$), and so adding n to any number just brings us back to where we started—thus n is equivalent to 0.



PRACTICE MAKES PERFECT

Let's really make this new arithmetic our own. Below we ask a few questions. Some present true equivalences, and others do not. Check each one, and determine which are correct and which are wrong. For the ones that are incorrect, figure out a correct answer. Notice that in each case we are using a different mod number, so first have a look at the "mod n" part to see what kind of arithmetic to use.

THE HOT QUESTIONS

As a warm up, we'll answer the first question.

1. Is $26 + 31^5 \equiv 0 \bmod 29$?

This statement is true since we are considering a mod 29 clock, so 29 is equivalent to 0. Therefore, $26 + 31^5 = 26 + (29 + 2)^5 \equiv 26 + 2^5 = 26 + 32 = 26 + (29 + 3) \equiv 26 + 3 = 29 \equiv 0 \bmod 29$. (Notice how we did not need to figure out that $31^5 = 28{,}629{,}151$.) Now it's your turn.

2. Is $7^2 + (5 \times 57) \equiv 40 \bmod 48$?

3. Is $2^4 + 5^{301} + (6 \times 31) \equiv 3 \bmod 5$?

4. Is $9^{2000} \equiv 1 \bmod 80$? (*Hint:* Write 2000 as 2×1000.)

THE COOL ANSWERS

2. is incorrect: $7^2 + (5 \times 57) \equiv 49 + (5 \times 9) \equiv 1 + 45 = \mathbf{46} \bmod 48$.

3. is also incorrect: $2^4 + 5^{301} + (6 \times 31) \equiv 16 + 0^{301} + (1 \times 1)$
$\equiv 1 + 1 = \mathbf{2} \bmod 5$.

4. is correct: $9^{2000} = 9^{2\times 1000} = (9^2)^{1000} = (81)^{1000} \equiv 1^{1000}$
$= \mathbf{1} \text{ mod } 80.$

Notice how we can work with enormous powers of numbers without even breaking a sweat by just carefully reducing the numbers in clever ways to smaller equivalent numbers in the modular arithmetic. Now let's see how we use modular arithmetic in our daily lives without our even realizing it.

The Mod World of Modular Arithmetic

THE CLAIRVOYANT KLEENEX CONSULTANT

You have the flu and feel awful. You moan, you groan, you sneeze, you wheeze—let's face it, you're sick. As you sit up in bed, you feel lightheaded—not because you have a fever, but because you have been watching too many hours of mind-numbing daytime TV. You notice in your boredom that on the bottom of your Kleenex box there is a toll-free number for consumer service—1-800-KLEENEX, which amuses you (because of your lightheadedness). Although you are extremely satisfied with the tissues, you decide to dial the number and talk to somebody because you are feeling lonely. The perky Kleenex representative on the other end of the telephone line asks you to read the 12-digit bar code appearing on the bottom of the box. You look and see those thin, fat, and medium lines that make up the Universal Product Code (UPC), which is now tattooed on nearly every product. As all those lines dance in your head you read:

0 3 6 0 0 0 2 8 1 5 0 9

The chipper voice immediately responds by saying, "I think you made a mistake, could you please read them again?" You glance back at the bar code, still bleary-eyed, and realize that you indeed made a mistake. In fact, the numbers appearing under the bar code are 0 3 6 0 0 0 2 8 5 1 0 9; you reversed the 5 and the 1. But how did your telephone partner immediately know you made a mistake? Perhaps Kleenex reps are clairvoyant, but they definitely use modular arithmetic.

CHECK DIGITS

A bar code and its associated numbers (usually 12 or 13 digits) make up the UPC. The first six digits encode information about the manufacturer, and the next five digits encode information about the product. That leaves us with the last digit, which is called the *check digit.* The check digit provides a means of detecting if a UPC number is incorrect. Here is how the check digit works with 12 digits: We line up the first 11 digits of the UPC—let's call them: d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}. We now combine them in an unusual way. We

Our world contains many examples of cycles. One way to develop mathematical ideas is to look at natural phenomena, model them using mathematics, and then explore the abstract ideas contained in the model. We can take general notions and refine them to develop ideas. We can then explore our new world of the mind without referring back to nature. In the process, however, we often find that our thought experiments are useful in the real world.

Create abstract ideas by modeling nature.

•

Explore ideas systematically, investigate consequences, and formulate general principles.

Mindscapes INVITATIONS TO FURTHER THOUGHT

In this section, Mindscapes marked (H) have hints for solutions at the back of the book. Mindscapes marked (S) have solutions.

I. Developing Ideas

1. **A flashy timepiece.** You own a very expensive watch that is currently flashing "3:00." What time will it read in 12 hours? In 14 hours? In 25 hours? In 240 hours? What time is it when an elephant sits on it?
2. **Living in the past.** Your watch currently reads "8:00." What time did it read 24 hours earlier? Ten hours earlier? Twenty-five hours earlier? What time did it read 2400 hours earlier?
3. **Mod prods.** Which number from 0 to 6 is equivalent to 16 mod 7? Which number from 0 to 6 is equivalent to 24 mod 7? Which number from 0 to 6 is equivalent to 16×24 mod 7? What number is equivalent to $(16 \bmod 7) \times (24 \bmod 7) \bmod 7$? What do you notice about the last two quantities you computed?
4. **Mod power.** Reduce 7 mod 3. Reduce 7^2 mod 3. Reduce $[7 \bmod 3]^2$ mod 3. Would you rather find 7 mod 3 first and then square it, or square 7 and then find 7^2 mod 3? What if you had to reduce 7^{1000} mod 3? Okay, you guessed it, now go ahead and reduce 7^{1000} and mod 3.
5. **A tower of mod power.** Reduce 13 mod 11. Reduce 13^2 mod 11. Compare $(13 \bmod 11)^2$ with 13^2 mod 11. Now reduce 13^3 mod 11 and 13^4 mod 11 without raising 13 to the power 3 or 4.

6. **Hours and hours.** The clock now reads 10:45. What time will the clock read in 96 hours? What time will the clock read in 1063 hours? Suppose the clock reads 7:10. What did the clock read 23 hours earlier? What did the clock read 108 hours earlier?

7. **Days and days.** Today is Saturday. What day of the week will it be in 3724 days? What day of the week will it be in 365 days?

8. **Months and months (H).** It is now July. What month will it be in 219 months? What month will it be in 120,963 months? What month was it 89 months ago?

9. **Celestial seasonings (S).** Which of the following is the correct UPC for Celestial Seasonings Ginseng Plus Herb Tea? Show why the other numbers are not valid UPCs.

0	7 1 7 3 4	0 0 0 2 1	8
0	7 0 7 3 4	0 0 0 2 1	8
0	7 0 7 4 3	0 0 0 2 1	8

10. **SpaghettiOs.** Which of the following is the correct UPC for Franco-American SpaghettiOs? Show why the other numbers are not valid UPCs.

0	5 1 0 0 0	0 2 5 6 2	4
0	5 1 0 0 0	0 2 5 2 6	4
0	5 1 0 0 0	0 2 5 2 6	5

11. **Progresso.** Which of the following is the correct UPC for Progresso minestrone soup? Show why the other numbers are not valid UPCs.

0	4 1 1 9 6	0 1 0 1 2	1
0	5 2 0 1 0	0 0 1 2 1	2
0	0 5 0 5 5	0 0 5 0 5	3

12. **Tonic water.** Which of the following is the correct UPC for Canada Dry tonic water? Show why the other numbers are not valid UPCs.

0	1 6 9 0 0	0 0 3 0 3	4
0	2 4 0 0 1	1 0 6 9 1	3
0	1 0 0 1 0	2 0 1 1 0	5

13. **Real mayo (H).** The following is the UPC for Hellmann's 8-oz. Real Mayonnaise. Find the missing digit.

0	4 8 0 0 1	2 6 ■ 0 4	2

14. **Applesauce.** The following is the UPC for Lucky Leaf Applesauce. Find the missing digit.

0	2 8 5 0 0	1 1 0 7 0	■

15. **Grand Cru.** The following is the UPC for Celis Ale Grand Cru. Find the missing digit.

    ■ 35888 41201 9

16. **Mixed nuts.** The following is the UPC for Planter's 6.5-oz. Mixed Nuts. Find the missing digit.

    0 29■00 07367 8

17. **Blue chips.** The following is the UPC for Garden of Eatin' 10-oz. Blue Corn Chips. Find the missing digit.

    0 15839 ■0001 5

18. **Lemon.** The following is the UPC for RealLemon Lemon Juice. Find the missing digit.

    0 53000 15108 ■

19. **Decoding (S).** A friend with lousy handwriting writes down a UPC. Unfortunately, you can't tell his 4's from his 9's or his 1's from his 7's. If the code looks like 9 0 3 0 6 8 8 2 3 5 1 7, is there any way to deal with the ambiguity? If so, what is the actual UPC? If it is not possible to determine the correct UPC, explain why.

20. **Check your check.** Find the bank code on your check. Verify that it is a valid bank code.

21. **Bank checks.** Determine the check digits for the following bank codes:

    3 1 0 6 1 4 8 3 ■ 0 2 5 7 1 1 0 8 ■

22. **More bank checks.** Determine the check digits for the following bank codes:

    6 2 9 1 0 0 2 7 ■ 5 5 0 3 1 0 1 1 ■

23. **UPC your friends.** Have a friend find a product that has a 12-digit UPC. Ask your friend to carefully read aloud the digits but to skip one digit and say "blank" in its place. Figure out the missing digit. Do this with several different products if you wish. Explain to your friend how you did it. Record the UPCs, the missing digit, and your friend's reactions.

24. **Whoops.** A UPC for a product is

    0 51000 02526 5

    Explain why the errors in the following misread versions of this UPC would not be detected as errors:

    0 51000 02625 5
    0 50000 05526 5

25. **Whoops again.** A bank code is

    0 1 1 7 0 1 3 9 8

Explain why the errors in the following misread versions of this bank code would not be detected:

7 1 1 0 0 1 3 9 8

0 1 1 7 0 8 3 9 1

III. Creating New Ideas

26. **Mod remainders (S).** Where would 129 be on a mod 13 clock (clock goes from 0 to 12)? What is the remainder when 129 is divided by 13?

27. **More mod remainders.** Where would 2015 be on a mod 7 clock? What is the remainder when 2015 is divided by 7? Generalize your observations and state a connection between mod clocks and remainders.

28. **Money orders.** U.S. Postal Money Orders have a 10-digit serial number and a check digit. The check digit is the number between 0 and 6 that represents what the 10-digit serial number is equivalent to using a mod 7 clock. This check digit is the same as the remainder when the serial number is divided by 7. What is the check digit for a money order with serial number 6830910275?

29. **Airline tickets.** An airline ticket identification number is a 14-digit number. The check digit is the number between 0 and 6 that represents what the identification number is equivalent to using a mod 7 clock. Thus, the check digit is just the remainder when the identification number is divided by 7. What is the check digit for the airline ticket identification number 1 006 1559129884?

30. **UPS.** United Parcel Service uses the same check digit method used on U.S. Postal Money Orders and airline tickets for its package-tracking numbers. What would be the check digit for UPS tracking number 84200912?

31. **Check a code.** U.S. Postal Money Order serial numbers, airline ticket identification numbers, UPS tracking numbers, and Avis and National rental car identification numbers all use the mod 7 check digit procedure. Find an example and check the check digit. For instance, get a copy of an airline ticket and check the identification number.

32. **ISBN.** The 10-digit book identification number, called the International Standard Book Number (ISBN), has its last digit as the check digit. The check digit works on a mod 11 clock. If the ISBN has digits d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8 d_9 d_{10}, then to check if this number is valid, we compute the following number:

$$\mathbf{1}d_1 + \mathbf{2}d_2 + \mathbf{3}d_3 + \mathbf{4}d_4 + \mathbf{5}d_5 + \mathbf{6}d_6 + \mathbf{7}d_7 + \mathbf{8}d_8 + \mathbf{9}d_9 + \mathbf{10}d_{10}$$

If the ISBN is correct, this new calculated number should be equivalent to 0 mod 11. We use the digit X to stand for 10 on the mod 11 clock. For example, consider the ISBN 0-387-97993-X. To check this ISBN, we compute (mod 11) (remember that $X = 10$):

$$(\mathbf{1} \times 0) + (\mathbf{2} \times 3) + (\mathbf{3} \times 8) + (\mathbf{4} \times 7) + (\mathbf{5} \times 9) +$$
$$(\mathbf{6} \times 7) + (\mathbf{7} \times 9) + (\mathbf{8} \times 9) + (\mathbf{9} \times 3) + (\mathbf{10} \times 10) =$$
$$6 + 24 + 28 + 45 + 42 + 63 + 72 + 27 + 100 \equiv$$
$$6 + 2 + 6 + 1 + 9 + 8 + 6 + 5 + 1 \equiv$$
$$44 \equiv 0 \bmod 11.$$

Therefore, this number is a valid ISBN. Verify this check method for the ISBN on the copyright page of this book.

33. ISBN check (H). Find the check digits for the following ISBNs:

0-219-60512-■; 1-101-38216-■.

34. ISBN error. The ISBN 3-540-06395-6 is incorrect. Two adjacent digits have been transposed. The check digit is not part of the pair of reversed digits. What is the correct ISBN?

35. Brush up your Shakespeare. Find a book containing a play by Shakespeare and check its ISBN.

IV. Further Challenges

36. Mods and remainders. Use the Division Algorithm (see Section 2.3) to show that the remainder when a number n is divided by m is equal to the position n would be on a mod m clock (a mod m clock goes from 0 to $m - 1$).

37. Catching errors (H). Give some examples in which the UPC check digit does not detect an error of two switched adjacent digits. Try to determine a general condition whereby a switching error in those digits would not be detected. (*Hint:* Consider the difference of the digits.)

38. Why three? In the UPC, why is 3 the number every other digit is multiplied by rather than 6? (*Hint:* Multiply every digit from 0 to 9 by 3 and look at the answers mod 10. Do the same with 6 and compare your results.) Are there other numbers besides 3 that would function effectively? What number might you try?

39. A mod surprise. For each number n from 1 to 4, compute n^2 mod 5. Then for each n, compute n^3 mod 5 and finally n^4 mod 5. Do you notice anything surprising?

40. A prime magic trick. Pick a prime number and call it p. Now pick any natural number smaller than p and call it a. Compute a^{p-1} mod p. What do you notice? You can use this observation as the basis for a magic trick. Have a friend think of a natural number less than p (but keep it to him- or herself). Tell that person that you will predict and write what the remainder will be when a^{p-1} is divided by p. Write your answer and seal it in an envelope, and then ask what the person's number was. Now, to your friend's amazement, compute the remainder when a^{p-1} is divided by p and reveal the hidden prediction. Record your friend's reaction. The next section uses this observation in a powerful way. Check it out.

is possibly the most powerful example of the unforeseen applicability of abstract mathematical ideas—in this case to the digital world. Who would have thought that *cryptography*—the study of secret codes—would become an important part of daily life and that the exploration of numbers would be central to coding?

This section is difficult. To master every part of the mathematics involved requires a significant effort. Luckily for us, the idea of public key cryptography is interesting even without fully delving into the mathematical details that make it work. This challenging section is evidence that as the world changes, ideas that seem marginal today may become central tomorrow. Good luck.

Coding and Decoding

How can we code and decode messages? One possibility is to replace one letter by another letter. For example, suppose we created the following coding scheme:

Message	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Coded As	T	H	E	Q	U	I	C	K	B	R	O	W	N	F	X	J	M	P	D	V	L	A	Z	Y	G	S

If you wanted to send the message:

YOUR JOKES ARE LAME,

then you could send the coded message:

GXLP RXOUD TPU WTNU.

The major problem with this code is that breaking it is easy even without knowing the key. That is, any enemy who captured a sufficiently long encrypted message could figure out the original message. More elaborate coding methods are harder to break but still can be deciphered if the codes are shared. That is, suppose you are receiving messages from both Bill and Hillary, and each encodes his or her message using the same scheme. If Hillary captures Bill's encrypted message, couldn't she decode it by simply reversing the encoding procedure? It seems that we must trust our friends—a grave drawback to any shared coding scheme. Ideally, we would prefer a code by which people are able to encode messages to us but are at the same time unable to decode other messages that have been encoded by the same process. Is such a coding scheme possible? If someone could encode a message, then all he or she would have to do to decode messages in the same type of coding scheme is reverse the coding procedure. However, this plausible statement turns out to be false, and therein lies the core of modern coding methods.

Attractive ideas in one realm often have unexpected uses elsewhere.

In this section we will look at a coding technique invented during the last few decades that uses a 350-year-old theorem about modular arithmetic to encrypt and decode secret messages.

PUBLIC KEY CODES

The method uses an encrypting and decoding scheme that is fundamentally new in the coding business. The new wrinkle is the invention of the public key code. *Public key codes* are codes that allow us to encode any message but prevent us from decoding other messages encrypted by the same technique. Such codes are called *public key codes* because we can tell the entire world how to encode messages to us. We can even tell our enemies. We can take out an ad in the newspaper telling everyone how to encode messages to us; it's no secret; it's public. The key is that we and only we can *decode* an encrypted message. Isn't this notion counterintuitive? How can such a coding scheme work? We'll take a look at one such scheme known as the RSA public key code.

Before jumping into the technical details of this coding scheme, let's try to make the basic idea of the encoding and decoding aspects of the RSA code plausible. For this purpose, we journey to Carson City.

THE CARSON CITY KID AND THE PERFECT SHUFFLE CODE

The Carson City Kid was the master of cards (actually he was no kid, but it sounds better than "the Carson City Yuppie"). His hands were quicker than the eye, and his morals were just as fast. One thing the Kid could do without fail was what is known in the trade as a *perfect shuffle.* That is, he would cut the deck of 52 cards precisely in half and then shuffle them perfectly—one card from the top half, then one from the bottom half, and so on, intermixing the cards exactly—one from one side, one from the other. For the larcenous among our readers, the advantage of a perfect shuffle is that, contrary to typical random shuffles, perfect shuffles only *appear* to bring disorder to the deck. The original ordering is restored after exactly 52 perfect shuffles.

The Kid was an enterprising soul who did not want to spend his life in casinos, rolling in money and surrounded by glamorous and attractive people.

The Basic Theme of the "Public" Aspect of the RSA Coding Scheme in Ten Sentences

Let's select two enormous prime numbers—and we mean enormous—say each having about 300 digits—and multiply those numbers together. How can we multiply them together? Computers are whizzes at *multiplying* natural numbers—even obscenely long ones. *Factoring* large numbers, however, is hard—even for computers. Computers are smart but not infinitely smart—there are limits to the size of natural numbers that they can factor. In fact, our product is much too large for even the best computers to factor. So if we announce that huge product to the world, even though it can be factored in theory, in practice it cannot. Thus we are able to announce the gigantic number to everyone, and yet no one but we would know its two factors. This huge product is the *public* part of the RSA public key code. Somehow, the fact that only we know how to factor that number allows us to decode messages while others cannot. It's not obvious why this factoring fact is helpful in making secret codes, but we'll see that it's really at the heart of the matter.

Instead, he decided to go into the secret message biz and be surrounded by glabrous and atrocious people. His method was simple. He knew that most people could not execute perfect shuffles. They could do only five or six shuffles before messing up. The Kid's method was straight-forward: The code sender would take a deck of 52 blank playing cards and write the message using one letter per card. Then the sender spy would carefully do five perfect shuffles—leaving the deck of cards in an apparently random order. The spy receiving the shuffled deck would then hand the coded message (the shuffled deck) to the Kid.

The Kid knew exactly what to do. He quickly shuffled the deck with 47 more perfect shuffles and voilà! The cards had rearranged themselves exactly into their original order, and so the message could be read.

Of course, the Kid's technique is too simple to use in practice. With determination, a person who captured the five-shuffled deck could do the reverse of those perfect shuffles. However, the Kid's technique demonstrates a mathematical fact that revolutionized the coding business.

The RSA Coding Scheme

Rivest (top) and Adleman (below), wondering where Shamir is.

In 1977, Ronald Rivest, Adi Shamir, and Leonard Adleman discovered a public key coding scheme that uses modular arithmetic. This public key coding method is referred to as the *RSA Coding Scheme* and is now used millions of times each day. Kid Carson's 47 perfect shuffles that return the deck to its original order captures the spirit of this RSA public key coding scheme. A shuffling procedure encodes a message, and only the receiver knows how to continue to shuffle the message further in a way that unshuffles the message—no one besides the receiver can perform that additional shuffling. So, now there are two basic questions we hope you are wondering: (1) What are we shuffling? and (2) How do we keep shuffling to get back to where we started?

HOW THE SCHEME WORKS: SHUFFLING NUMBERS

We will shuffle numbers. That is, we will first convert our message to numbers and then shuffle those numbers. How do we shuffle the numbers? Let's take a prime number, say 5. Pick any number that does not have 5 as a factor, for example, 8. To shuffle 8, let's just raise 8 to higher powers and look at the remainders of those powers of 8 when we divide by 5. In other words, raise 8 to higher powers and look at those numbers mod 5.

Powers of 8	Powers of 8 mod 5
$8^1 = 8$	3
$8^2 = 64$	4
$8^3 = 512$	2
$8^4 = 4096$	1
$8^5 = 32{,}768$	3

Look for patterns.

The second column represents a type of shuffling of a four-card deck where the "shuffling" is accomplished by multiplying by 8. Notice that after five shuffles of multiplication by 8 mod 5 we get back to 3.

Let's try this again with a different-size deck. Suppose we pick the prime 7 and choose a number that does not have 7 as a factor—say 10. Let's shuffle 10 by raising it to higher powers and considering those powers mod 7.

Powers of 10	Powers of 10 mod 7
$10^1 = 10$	3
$10^2 = 100$	2
$10^3 = 1000$	6
$10^4 = 10{,}000$	4
$10^5 = 100{,}000$	5
$10^6 = 1{,}000{,}000$	1
$10^7 = 10{,}000{,}000$	3

Here we notice that after seven shuffles of powers mod 7 we get back to 3. Now it's your turn. Try this shuffling yourself. Let's set the prime number to be 5. Now pick some numbers that have no factor of 5 and shuffle them by raising them to powers mod 5. Try this shuffling with at least two different numbers. What do you notice? How many shuffles get us back to where we started mod 5? Let's look for patterns.

By experimenting, we discover that, if we shuffle 5 times mod 5, we get back to where we started. We also notice that, after we shuffle 4 times mod 5, we always get 1 as the answer. This observation turns out to be a mathematical fact—known as *Fermat's Little Theorem.*

Fermat's Little Theorem.

If p is a prime number and n is any integer that does not have p as a factor, then n^{p-1} is equal to 1 mod p. In other words, n^{p-1} will always have a remainder of 1 when divided by p.

It is Fermat's Little Theorem, proved more than 350 years ago, that is the basis of our shuffling procedure. Now let's tackle the RSA public key code scheme.

AN ILLUSTRATIVE, CRYPTIC EXAMPLE

We introduce the RSA public key code method by considering a specific example. Using a diabolically clever idea that will be explained later, we construct and publicize a pair of numbers to the world—in this example the numbers 7 and 143. In real life the numbers would be much larger, perhaps having several hundred digits each. At the same time we construct the public numbers, we also construct and keep secret a decoding number, in this case 103. The public part of the public key code does not contain the key to unlock

Ground your understanding in the specific.

the code; instead, the key is the secret decoding number that is kept only by the receiver of encrypted messages. It never needs to be transmitted to anyone else. We'll explain later how all these numbers were created.

We not only publicize the numbers 7 and 143 but also explain exactly how to use them to encrypt a message. Here are the instructions, which could be published in the newspaper.

Encoding Messages

Suppose that W is a secret Swiss bank account number (less than 143) that the sender wants to encrypt and send to someone. The sender computes W^7 (remember that 7 is the first public number) and then computes the remainder when W^7 is divided by 143, the second public number. That is,

$$W^7 = 143q + C,$$

where the remainder C is an integer between 0 and 142. Or, expressed in modular arithmetic,

$$W^7 \equiv C \bmod 143.$$

The number C is now the coded version of W.

Decoding Messages

The receiver receives the coded message C and now must decode it. This decoding process requires the receiver to compute C^{103} (recall that 103 is the secret number that no one but the receiver knows) and then to compute the remainder when C^{103} is divided by 143. That is,

$$C^{103} = 143q + D,$$

where the remainder D is an integer between 0 and 142. Or, expressed in modular arithmetic,

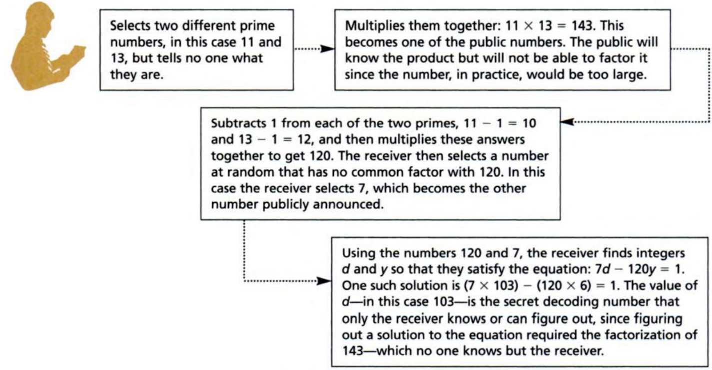
$$C^{103} \equiv D \bmod 143.$$

The amazing fact is that D (the decoded message) will always be identical to W (the original, uncoded message). Thus the receiver decoded the coded message C to produce the original message W.

Suppose someone sets up the public key code described above, announces the public numbers (7 and 143) and the coding method, and keeps the number 103 secret. Now let's further suppose that a friend wishes to secretly send her Swiss bank account number, 71. The table shows the sequence of events to code the message 71.

To encode the number 71, the sender computes 71^7, which equals 9,095,120,158,391, and then computes the remainder when this number is divided by 143. The remainder turns out to be 124. So 124 is the encoded version of 71, and that is what the sender sends to the receiver. Now the receiver has to decode 124.

Receiver



The number d (d for *decoding*), which in this case equals 103, is the secret decoding number that we keep to ourselves. That's it! Whenever we select the numbers p, q, e, and d in the manner described above, the coding scheme will always work.

WHY DO THOSE NUMBERS WORK?

Let's see why this coding and decoding scheme always works. Before we give an overview of why the RSA coding scheme works, we have to confess that what follows is difficult. What makes it difficult? The answer is that there are many steps. Although each step on its own is no great intellectual feat, when we string them together, one after another, the logic and modular arithmetic can get out of hand. These details are more interesting to some than to others. So readers who decide to invest the energy to learn what follows must expect to struggle and to reread the information several times. Other readers may decide to limit their investment in this topic and move on—remember Vietnam.

We begin our explanation by a quick recap: First we picked two primes,

$$p = 11 \quad \text{and} \quad q = 13.$$

Their product is 143, and that is one of the public numbers. We next computed

$$(p - 1)(q - 1)$$

or, in this case,

$$10 \times 12 = 120$$

The Heart of Mathematics

Authors Edward B. Burger, Williams College, and Michael Starbird, The University of Texas at Austin, listened and responded to users' feedback to provide you with this greatly enriched second edition.

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