The Joy of Finite Mathematics

The Language and Art of Math

Chris P. Tsokos and Rebecca D. Wooten

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About the Authors

Chris P. Tsokos is Distinguished University Professor of Mathematics and Statistics at the University of South Florida. Dr. Tsokos received his B.S. in Engineering Sciences/Mathematics, his M.A. in Mathematics from the University of Rhode Island, and his Ph.D. in Statistics and Probability from the University of Connecticut. Professor Tsokos has also served on the faculties at Virginia Polytechnic Institute and State University and the University of Rhode Island.

Dr. Tsokos' research has extended into a variety of areas, including stochastic systems, statistical models, reliability analysis, ecological systems, operations research, time series, Bayesian analysis, and mathematical and statistical modeling of global warming, both parametric and nonparametric survival analysis, among others. He is the author of more than 300 research publications in these areas.

For the past four years Professor Tsokos' research efforts have been focused on developing probabilistic models, parametric and nonparametric statistical models for cancer and GLOBAL WARMING data. Specifically, his research aims are data driven and are oriented toward understanding the behavior of breast, lung, brain, and colon cancers. Information on the subject matter can be found on his website.

Professor Tsokos has more than 300 publications in his research areas of interest. He is the author of several research monographs and books, including *Random Integral Equations with Applications to Life Sciences and Engineering*, *Probability Distribution: An Introduction to Probability Theory with Applications, Mainstreams of Finite Mathematics with Applications*, *Probability with the Essential Analysis, Applied Probability Bayesian Statistical Methods with Applications to Reliability*, and *Mathematical Statistics with Applications*, among others.

Dr. Tsokos is the recipient of many distinguished awards and honors, including Fellow of the American Statistical Association, USF Distinguished Scholar Award, Sigma Xi Outstanding Research Award, USF Outstanding Undergraduate Teaching Award, USF Professional Excellence Award, URI Alumni Excellence Award in Science and Technology, Pi Mu Epsilon, election to the International Statistical Institute, Sigma Pi Sigma, USF Teaching Incentive Program, and several humanitarian and philanthropic recognitions and awards.

Professor Tsokos is a member of several academic and professional societies. He is serving as Honorary Editor, Chief-Editor, Editor or Associate Editor of more than twelve academic research journals.

Rebecca D. Wooten is Assistant Professor of Mathematics and Statistics at the University of South Florida. She received her M.A./B.A. in Mathematics and her Ph.D. in Statistics from the University of South Florida. She has worked for 15 years in teaching and has been recognized for her excellence in teaching; teaching courses such as Liberal Arts Math, Finite Mathematics, Basic Statistics, Introduction to Statistics, and Applied Statistics Methods.

Her research interests are concentrated in Applied Statistics with emphasis on Environmental Studies. Her research publications span a variety of areas such as Global Warming (carbon dioxide and temperature), Atmospheric Sciences and Geography (hurricanes), Geology (volcanic ash fall), Marine Biology (red tide), among others.

Professor Wooten is extensively involved in activities to improve education not only in Mathematics and Statistics, but Education in general. She is the Academic Coordinator for two free-educational assistance program which offer opportunities for students to volunteer and the local community to get the assistance in their studies that they would otherwise be unable to afford.

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Preface

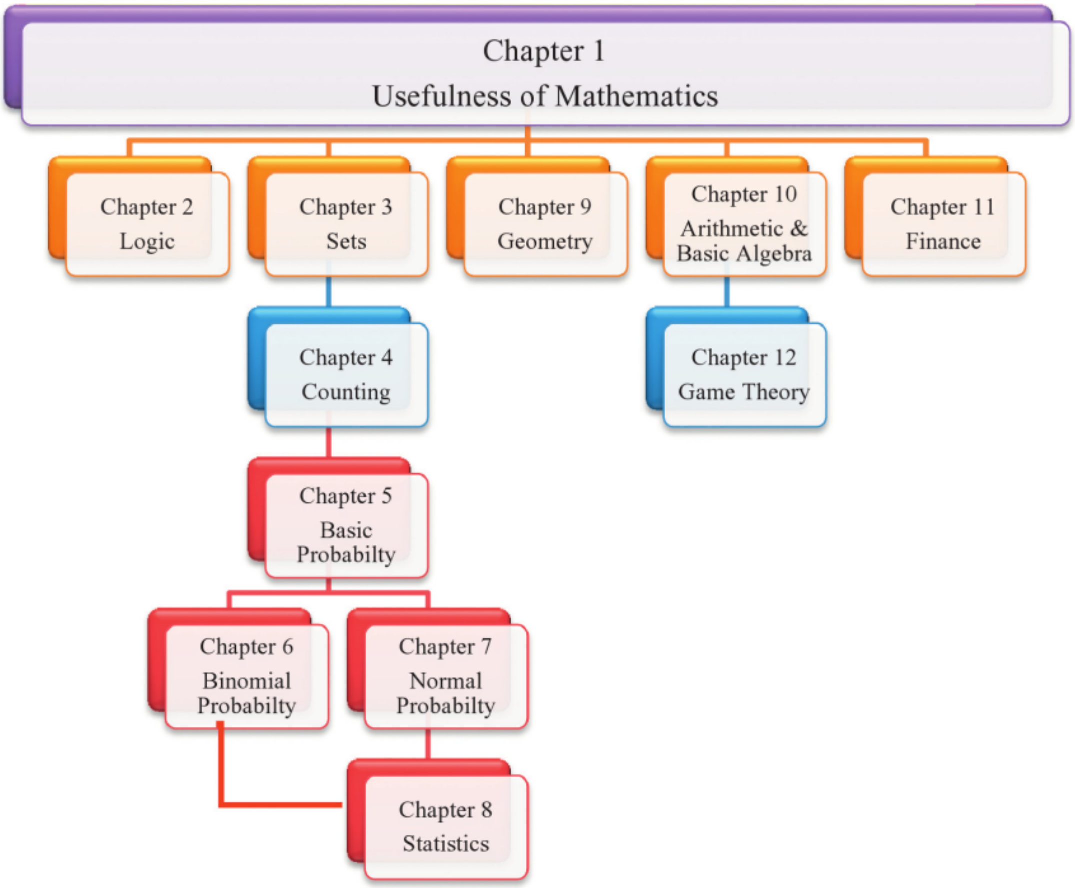
This book has been written to present certain aspects of modern finite mathematics from an elementary point of view, with emphasis on relevance to real-world problems. The objective is to create a positive attitude toward mathematics for the non-science-orientated college student and to demonstrate its usefulness in solving problems that we frequently encounter in our complex society.

Throughout the text, the aim has been to de-emphasize difficult theoretical concepts; thus, an intuitive treatment leads to practical applications of the various subject area topics. We believe that with such an approach, the modern college student will complete this course with the good feeling that mathematics is not only useful but enjoyable to work with.

The ***Joy of Finite Mathematics*** has several distinguishing features:

- The text has been written for students with only high school mathematics.
- Diagrams and graphs are used to illustrate mathematical concepts or thoughts.
- Step-by-step directions are given for the implementation of mathematical methods to problem solving.
- Emphasis has been placed on usefulness of mathematics to real-world problems.
- To provide motivation to the reader, most chapters are preceded by a short biography of a scientist who made important contributions to the subject area under consideration.
- Mathematical concepts are introduced as clearly and as simply as possible, and they are followed by one or more examples as an aid to thorough understanding.
- Each chapter ends with a complete summary that includes the definitions, properties, and rules of the chapter, followed by a Review Test.
- Each chapter contains numerous critical thinking and basis exercises with problems that reflect on the mainstream of the chapter.

The book has been designed to give the instructor wide flexibility in structuring a one or two-semester course, or a full-year course. Although some chapters are dependent on other others, many options are allowed (see accompanying diagraph).


Chapter 1
Usefulness of Mathematics
Chapter 2
Logic
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Very Special Acknowledgment

We wish to express our appreciation to the following academic educators for having reviewed our book and expressed their opinions.

An outstanding book for students to obtain basic knowledge of the usefulness of mathematics. Excellent motivation strategies throughout the book. It will inspire the student to learn the importance of mathematics.

Dr. Ram Kafle, Department of Mathematics and Statistics, Sam Houston University.

A very constructive and motivating book of finite mathematics. Special emphasis on the applications of math to real-world problems. The interactive approach of presenting their material is excellent. The student will acquire a very good understanding of what mathematics is all about.

Dr. Bong-jin Choi, Lineberger Comprehensive Cancer Center, The University of North Carolina at Chapel Hill.

This book is a masterful treatment of finite mathematics for undergraduate students who are afraid of mathematics. It will enlighten the student of the interdisciplinary use of mathematics at the very basic level. The book provides excellent illustrations of the use of mathematics/statistics to solve important problems.

Dr. Yong Xu, Department of Mathematics and Statistics, Radford University.

The Joy of Finite Mathematics *provides an excellent treatment of the subject. Unique emphasis on the importance of mathematical sciences to our society. The non-mathematics-oriented undergraduate student will find the contents of the book easy to read and very inspiring to learn more of the subject matter.*

Dr. K. Pokhrel, Department of Mathematics & Computer Systems, Mercyhurst University.

This is an excellent book of finite mathematics. It offers a justifiable, useful, and motivating approach to what mathematics is all about to the undergraduate student with minimum prior knowledge of the subject. The selection of the contents of the book, examples, and exercises is outstanding.

Dr. N. Khanal, Department of Mathematics, University of Tampa.

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Several Options for a Semester Course in Finite Mathematics

Five possible options in designing a basic course in **finite mathematics** are given below, along with some remarks for each selection.

Options 1 and 2 offer a detailed coverage of specific topics in **math**, each spanning seven chapters:

Option 1:

Chapters Covered	Title
Chapter 1	The Usefulness of Mathematics
Chapter 2	Logic
Chapter 3	Sets
Chapter 4	Counting Techniques
Chapter 5	Probability
Chapter 8	Statistics
Chapter 9	Geometry

Covering materials necessary for the **CLAST** (**College Level Academic Skill Test**) exam, excluding algebra, these six topics are often taught collectively. In addition to the necessary high school algebra, these topics prepare a student well for the **CLAST** exam.

Option 2:

Chapters Covered	Title
Chapter 1	The Usefulness of Mathematics
Chapter 2	Logic
Chapter 3	Sets
Chapter 5	Probability
Chapter 6	Bernoulli Trials
Chapter 7	The Bell-shaped Curve
Chapter 8	Statistics

Option 2 provides the materials necessary for a comprehensive understanding of basic **probability** and **statistics**. This option is a broad introduction, including the underlying probabilities necessary to compute basic descriptive statistics, as well as inferential statistics in terms of interval estimates and tests of hypothesis.

Options 3-5 offer a more detailed coverage of specific topics in **math, each spanning** six chapters:

Option 3:

Chapters Covered	Title
Chapter 1	The Usefulness of Mathematics
Chapter 3	Sets
Chapter 4	Counting Techniques
Chapter 5	Probability
Chapter 6	Bernoulli Trials
Chapter 7	The Bell-shaped Curve

These topics enhance the study of **probability**. Option 3 begins with the basic concepts of categorization into **sets**, **counting** sets, and measuring **basic probabilities** empirically. It then continues with measuring basic probabilities hypothetically using either the discrete **binomial probability distribution, or** the **continuous normal probability** distribution.

Option 4:

Chapters Covered	Title
Chapter 1	The Usefulness of Mathematics
Chapter 4	Counting Techniques
Chapter 5	Probability
Chapter 6	Bernoulli Trials
Chapter 7	The Bell-shaped Curve
Chapter 8	Statistics

Option 4 covers materials necessary for the study of the basic aspects of **statistics**. This option includes **counting basic** empirical and hypothetical **probabilities** empirically. It also includes the basic necessities of **statistics**, descriptively and inferentially, for means and proportions.

Option 5:

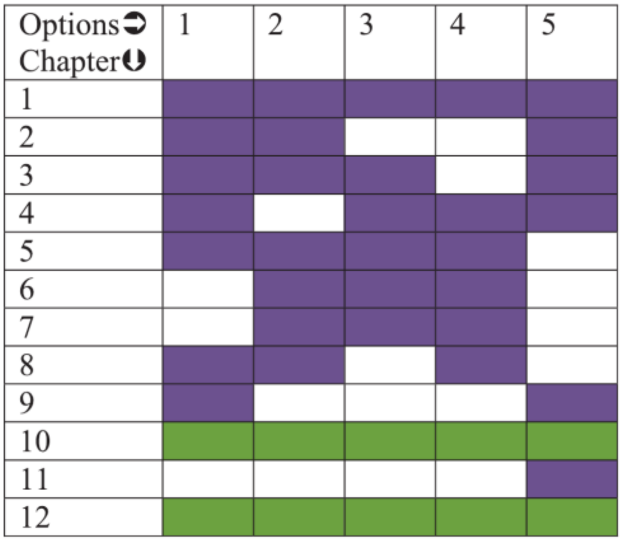
Chapters Covered	Title
Chapter 1	The Usefulness of Mathematics
Chapter 2	Logic
Chapter 3	Sets
Chapter 4	Counting Techniques
Chapter 9	Geometry
Chapter 11	Arithmetic and Algebra

Option 5 covers materials necessary to gain a basic understanding of the language of deterministic **math**. This option provides a basic understanding of **logic**, **sets**, **counting**, **geometry**, and **algebra**.

Note: Game theory can be included in any scheme that includes the algebra and arithmetic.

A SUMMARY OF THE PROPOSED OPTIONS

Depending on which option you choose (1, 2, 3, 4, or 5), the purple indicates which chapters should be included; the green indicates optional chapters in each scheme.



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Joy of Finite Mathematics

Special Features

Motivation

- The usefulness of mathematics, especially those branches that constitute **finite math**, is illustrated both from a historical perspective, and by the role it plays in our daily lives.
- We emphasize an interactive approach to teaching **finite mathematics**.

The Language

- Teaching any student basic finite mathematics requires a basic understanding of the underlying symbolic language. Mathematics has many dialects: **logic**, **set theory**, **combinatorics (counting)**, **probability**, **statistics**, **geometry**, **algebra**, and **finance**, for example. Learning through relevance and interpretation of symbolism is vital.

The Relevant Questions

- A complete introduction of mathematics in a finite world, the notation used, and the underlying interpretation is presented. Relevant and useful questions associated with each dialect are posed, which will be answered through the process of learning finite mathematics.

The Review

- Reviews of each basic concept are given at the end of each chapter. The reviews enhance the learning of the basic aspects of each topic and their usefulness.

Step-by-Step

- Clear and concise **step-by-step** procedures are used in the development of various methodologies. Procedures are easy to follow, comprehend, and use to solve problems.

Highlights

- **Definitions**, **rules**, **methods** and **procedures** are highlighted with boldface and their meanings and usefulness follow with an abundance of relevant examples and applications.

Graphs and Tables

- Throughout the book, emphasis is placed on the extensive use of **tables**, **diagrams**, and **graphs** to clearly illustrate definitions, outlined methods, comparisons, etc. These visual aids invite clear interpretation of what they represent and their relevance to the text.

Applications and Interpretation

- We utilize a **step-by-step** approach in the illustrated examples (applications) that relate to the various dialects and their interpretations that have been introduced. Emphasis is placed on properly denoting the problems symbolically, interpreting the argument, outlining the defined set, measuring the probability, or, in general, finding the solution. Then, we encourage the student to clearly state any conclusions that can be drawn from the application.

Critical Reviews

- Each chapter ends with a review of: the new mathematical vocabulary, the most important concepts and methods, an abundance of review exercises, and a practice test that is based on the material from the preceding chapter.

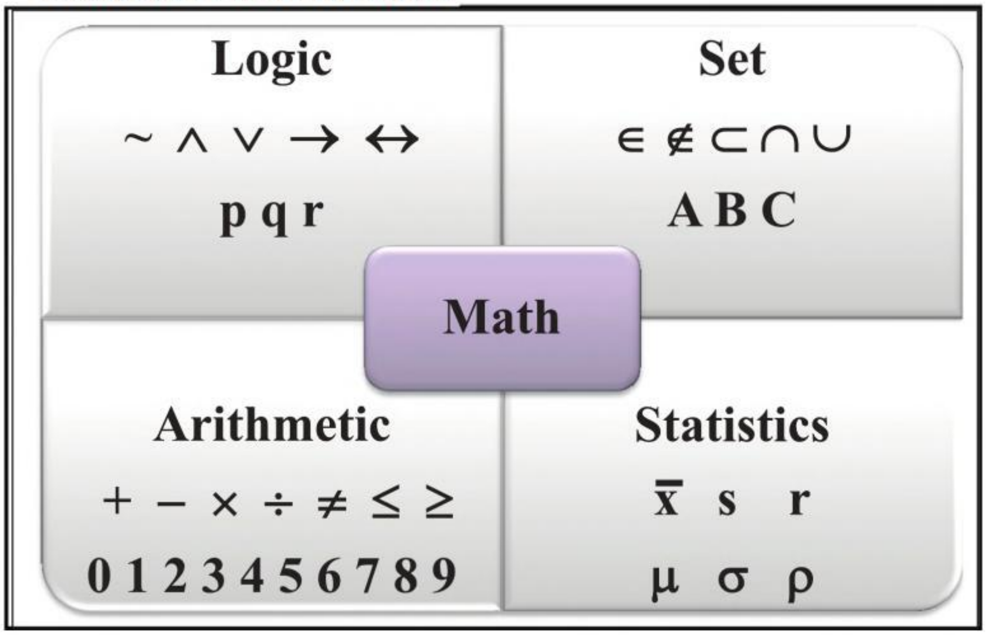
Inspiration

- Throughout the book, we utilize important **historical facts** and **pose interesting** and **relevant questions**. We also include **humorous events**, **pictures**, **graphs**, **tables**, **biographical sketches of famous scientists**, **popular** and **classical quotes**, and more. These are all tools to **challenge**, **inspire** and **motivate** students to learn the mathematical thinking and to illustrate the absolute relevance of math to our society.

Challenging Problems

- Throughout the book, there are sections and challenging problems that are somewhat more advanced for a basic course in **finite mathematics** and are left to the discretion of the instructor.

Dialects of Mathematics



Number is the within of all things.

PYTHAGORAS

Chapter 1 The Usefulness of Mathematics

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

John Louis von Neumann

The essence of mathematics is not to make simple things complicated, but to make complicated things simple.

S. Gudder

Go down deep enough into anything and you will find mathematics.

Dean Schlicter

The man ignorant of mathematics will be increasingly limited in his grasp of the main forces of civilization.

John Kemeny

Pure mathematics is, in its way, the poetry of logical ideas.

Albert Einstein

Mathematics is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency.

Descartes

Mathematics is the Queen of the Sciences.

Carl Friedrich Gauss

Mathematics is the science of definiteness, the necessary vocabulary of those who know.

W.J. White

Mathematics is the science which uses easy words for hard ideas.

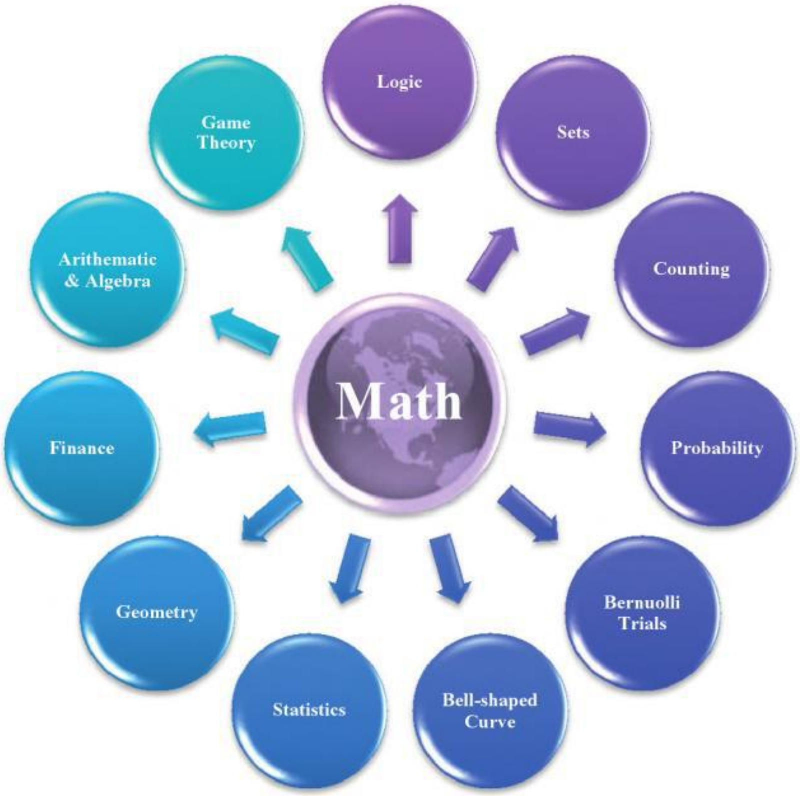
Edward Kasner and James R. Newman

***Philosophy** is a game with objectives and no rules.*

***Mathematics** is a game with rules and no objectives.*

Goals and Objectives

The main objective of this chapter is to give an overview and motivate the non-mathematically oriented student about the usefulness of mathematics in several important fields. We begin with a brief historical perspective of the subject and proceed to discuss the importance and usefulness of all the areas that we believe constitute a course in Finite Mathematics. The diagram below illustrates the areas covered. Although not all the chapters of the textbook need to be covered in a one semester or two quarter course, we believe that the student can gain some basic knowledge by studying this chapter.



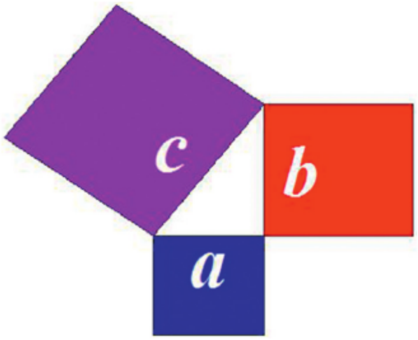
Thus, our goal here is to familiarize you with different areas and "dialects" of Math and:

- Learn about the history of Math
- Learn about the Math that is the foundation of logic
- Learn about the interplay of Math and sets
- Learn about counting techniques
- Learn about Math used to obtain probabilities of events
- Learn about binomial trials that leads to the Bernoulli probabilities
- Learn about the paramount importance of the Bell-Shaped Curve
- Learn about using Math to develop useful statistical methods

- Learn how Math is used to obtain measures of the earth or geometry
- Learn the Math that is arithmetic and algebra
- Learn how we use Math to answer basic financial questions
- Learn how we use Math in Game Theory, solving systems of equations to optimize strategies.

Pythagoras of Samos was the first to call himself a philosopher, Greek for "lover of wisdom." Pythagorean ideas greatly influenced western philosophy. Best known for the theorem which carries his name, Pythagoras was also a mathematician, scientist, musician, and mystic.

He founded the religious movement called Pythagoreanism. The Pythagoreans first applied themselves to mathematics, a science which they improved, and penetrated within; they fancied that the principles of mathematics were the principles of all things. A younger contemporary, Eudemus, shrewdly remarked that "they changed geometry into a literal science; they diverted arithmetics from the service of commerce"… Aristotle.



1.1 INTRODUCTION TO MATH

Mathematics played a very significant role in all our technological, scientific, medical, educational and economic accomplishments in our global society. However, just as important is the fact that mathematics indirectly interweaves every aspect of our daily lives; **mathematics** is the most powerful interdisciplinary language in almost all fields of **engineering**, every aspect of **health sciences**, **education**, **social** and **physical sciences**, **economics**, **finance**, **environmental sciences**, **Global Warming**, and of course **music** and **art**, among many other disciplines.

Math is the language of thought. We think faster than we speak and we speak faster than we write… therefore, to convey our thoughts quickly, Mathematicians abbreviate everything.
Rebecca D. Wooten

The word mathematics comes from the Greek word ***matheno*** which means ***I learn***. Historically mathematics has its origin in the Orient when the Babylonians, in about 2000 BC, collected a lot of materials on the subject that we identify as **elementary algebra**. However, the modern concept of mathematics started in Greece around the fifth and fourth centuries BC. At this time, mathematics was subjected to philosophical discussion that was a unique priority in the Greek city states. The Greek philosophers were quite aware of the mathematical difficulties involved in understanding continuity, infinity, motion and the problems of making measurements of arbitrary quantities. Eudoxus' theory was very significant in geometrically understanding these concepts that were later significantly improved by Euclid's elements. Thus, the Greeks have an enormous influence on the tremendous development of today's mathematics.

Mathematics is the "brain" for

- Engineering
- Health Sciences
- Education
- Social Science
- Physical Sciences
- Economics
- Finance
- Environmental Sciences
- Global Warming
- Music
- Art
- Among others…

Discrete
Apart or detached from others; separate; distinct

Absolute
Not mixed or adulterated; pure

Relative
Something having, or standing in, some relation to something else

Continued
To go on with or persist in; to continue an action

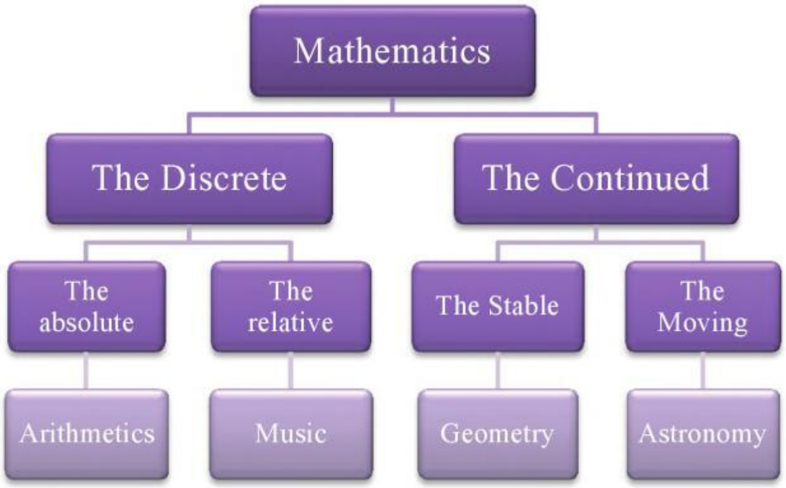
Stable
Not likely to fall or give way; firm, steady

Moving
To pass from one place or position to another

H. Weyl, one of the truly great mathematicians of the twentieth century, stated "***without the concepts, methods, and results found and developed by previous generations right down to the Greek antiquity, one cannot understand either the aim or the achievements of mathematics in the last 50 years***" (American Math Monthly, Vol. 102, 1995).

Historically mathematics was defined as the logical study of shape, arrangement, and quantity. Furthermore, attempts have been made to think of mathematics as two branches: **Applied Mathematics** and **Pure** or **Abstract Mathematics**. The branches of applied mathematics are concerned with the study of physical, biological, medical or sociological worlds. Pure or Abstract Mathematics is concerned with the study and development of the principles of mathematics as such and is not concerned with their immediate usefulness.

In addition, we also had a divide of mathematics in the **discrete** and the **continuous**. Herbert W. Turnbull, in his essay on the "**World of Mathematics**" states: ***To Pythagoras we owe the very word mathematics and its double fold branches***; that is,



This double fold of mathematics played a major role in the development and usefulness of mathematics. In fact, **Aristotle** summarizes this historical divide as follows:

> ***"The Pythagoreans first applied themselves to mathematics, a science which they improved; and, penetrated with it, they fancied that the principles of mathematics were the principles of all things."*** And a younger contemporary, Eudemus, shrewdly remarked that ***"they changed geometry into a literal science; they diverted arithmetic from the service of commerce"***.

The ***Joy of Finite Mathematics*** is written to show at a very basic level, that mathematics is useful to virtually everyone, especially those students who do not like mathematics as we approach mathematics as a language used to describe simple and complex problems that we encounter in our daily lives.

In the essay on "***The Nature of Mathematics***," by Philip E B Jourdain, he begins with "***An eminent mathematician once remarked that he was never satisfied with his knowledge of a mathematical theory until he could explain it to the next man that he met in the street.***" This is so very true and we believe it is our responsibility in writing this text to explain to our students the usefulness of mathematical methods and theories using real world problems. Thus, the student has the right to ask "***what is the usefulness of mathematics***?"

Intelligence is the ability to adapt to change.
Stephen Hawking

We have taken that aspect of the student asking such questions as our responsibility in positively responding. We proceed to address this important issue by raising several relative questions in the interdisciplinary structure of mathematics that constitute the areas of the subject that we have identified as "Finite Mathematics." Thus, in what follows is the main thrust of the basic dialects of mathematics for students whose primary interest is not the subject matter, but how to enhance their understanding of the usefulness of mathematics. For motivating the students we begin each branch of mathematics by stating several real world questions, the answers to which will lead to the importance and usefulness of mathematics. We believe that this interactive approach will motivate the learning process and take our students on a very "joyful ride" to learn finite mathematics.

Number is the within of all things
Pythagoras

My goal is simple. It is a complete understanding of the universe, why it is as it is and why it exists at all.
Stephen Hawking

1.2 WHAT IS LOGIC?

Logic is derived from the Greek ***λογική*** meaning ***conforming to laws of reasoning***. The branch of philosophy that treats forms of **thinking**, **reasoning** or **arguing** is also referred to as **Logic**. Averroes defines logic as "***the tool for distinguishing between the true and the false.***" **Logic** is divided into two parts: inductive and deductive reasoning. **Inductive reasoning** draws conclusions based on specific examples whereas **deductive reasoning** draws conclusions from definitions and axioms.

Archival Note
Averroes as he is known in Greek is **Abū 'l-Walīd Muḥammad bin Aḥmad bin Rushd**, and he defined **logic** and is the founder of **Algebra**.

Thus, our goal in learning **Logic** is to be in a position to make logical decisions regarding such questions as:

- □ **Politics**: A politician claims "*if you don't vote for me, then you will not get the tax cuts*"—does this imply that if you do vote for him, that you will get the tax cuts?
- □ **Health**: *If you work out more, then you will lose weight and tone your muscles, and if you watch your calorie intake, you will lose weight*. Does this mean that *if you lost weight that you must have both worked out and watched your calorie intake*?
- □ **Travel**: *If Athens is in Greece and Berlin is in Germany*, then *when I visit Germany and not Athens*, then does it follow that *I went to Berlin*?
- □ **Lottery**: *If Frank wins the lottery, then Frank will take you to dinner. Frank did not win the lottery and did not take you to dinner*. Did Frank lie?
- □ **Law**: *If you are 17, then you are a minor. Jordan is not 17*; therefore, can we conclude that *Jordan is not a minor*?

Debate between Averroes and Porphyry
Monfredo de Monte Imperiali Liber de herbis, *14th century*

It is the aim to teach logical reasoning to enable students to reason using the art of deduction and to draw correct conclusions when confronted with facts.

1.3 USEFULNESS OF SETS

The word "**set**" has more definitions than any other word in the English language due to its many origins. One origin of the word set is from the Old English ***settan*** meaning ***cause to sit***, ***put in some place***, ***fix firmly***, and another is from the Old French ***setta***, meaning ***collection of things***. The branch of mathematics which deals with the study of sets is called **Set Theory**. The modern study of Set Theory was begun by **Georg Cantor** and **Richard Dedekind** in the 1870s. The language of sets can be used to define nearly all mathematical "**objects**" such as functions.

Set Theory begins with a fundamental binary relationship (similar to logic) between objects $\mathcal{O}$ and a set S, namely that of membership. Either an object (element) belongs to a set, or it does not.

Usefulness of Sets: To present data, relevant information in a systematic manner so that it will be visually attractive and easily understood and so that it can be used effectively to address various questions of interest. Thus, our goal in learning about sets is to be in a position to make categorical decisions regarding such questions as:

- ☐ Business: A store owner notes that more people like chocolate muffins than blueberry muffins. With this information, how many of each should be made? How many customers are expected to purchase both?
- ☐ Cancer: If survival is a function on the type of treatment(s) received, then which treatment is better or is a combination of treatments better?
- ☐ Meteorology: Given 20 readings of temperatures over a period of 14 days taken at two relatively close stations, when comparing these temperatures; do they appear to fall in the same temperature range?

It is our aim in learning **Set Theory** that students will be able to describe information **categorically** as well as to be able to display information graphically in **Venn diagrams** and use these graphics to support any inferences made regarding relationships among the various sets.

Archival Note
Georg Ferdinand Ludwig Philipp Cantor was a German mathematician, the inventor of **Set Theory**, and the first to establish the importance of one-to-one correspondence between sets.

Archival Note
Julius Wilhelm Richard Dedekind was another German mathematician who worked with Cantor and is also well known for his work in **abstract algebra**.

Categorical

Unambiguously explicit and direct

Data consisting of nominal information

Qualitative data organized in a contingency table

Venn Diagram

A set diagram that shows all possible relations between finite collections

1.4 COUNTING TECHNIQUES

Count is from the Old French counter meaning ***add up***, but also ***tell a story***. Some of the first known use of counting was with shepherds who, when tending their sheep, would tie **knots in a rope** as they sent their sheep out to graze. In the evening, when the sheep returned to the fold for safety during the night, the shepherd would untie a knot and if there were any knots in the rope, they knew there were sheep that needed to be found. The branch of **mathematics** dealing with counting can extend from **tally marks**—making a mark for each number and then tallying these marks, **enumeration**—counting aloud

Archival Note
Counting only involves the whole (counting) numbers, 0, 1, 2, 3… It first started with the natural counting numbers, and then we introduced the number zero to represent "nothing" or the number of elements in the empty set. It was **Jiu zhange suan-shu** who first used red rods to denote "**positive**" values and black rods to denote "**negative**" values in his writing ***Nine Chapters on the Mathematical Art***. For a long period of time, negative solutions to problems where considered "false."

Diophantus, in the third century AD, referred to the idea of "$\mathbf{2x+10=0}$" as absurd.

or on your fingers to more complex counting techniques such as **combinations** and **permutations**.

Thus, our goal in **Counting Techniques** is to be in a position to count and discern the implication of such questions as:

- ☐ **Social**: At a social get-together of ten individuals, how many introductions will be needed to ensure everyone has met face-to-face?
- ☐ **Coding**: When coding a confidential letter using only the letters in the English alphabet, how many distinct codings are there? How many are needed such that no letter is mapped to itself in the coding?
- ☐ **Civics**: A board of directors consisting of ten women and 15 men need to form a five member committee to oversee next year's fund raiser. How many possible committees consist of exactly three men and two women?
- ☐ **Job Assignment**: A real estate agency has ten realtors and only nine new property listings. How many possible assignments of realtors to a house?
- ☐ **Diet**: There is a list of ten fruits you are willing to eat and your goal is to eat four fruits a day. To mix it up each day you create a meal plan that covers all possible combinations of four out of ten fruits. How many options are there for fruits in this meal plan?

The purpose in learning various **counting techniques** is to enable the student to determine the logistics necessary in such detailed coordination of a complex operation. This ranges from counting people or supplies to organizing committees and daily life.

Combination

When r out of n objects are taken without replacement and without distinction in ordering

Permutation

When r out of n objects are taken without replacement and with distinction in ordering

Counting is the religion of this generation. It is its hope and its salvation.

Gertrude Stein

It's not the voting that's democracy, it's the counting.

Alfred Emanuel Smith

Innumerable actions are going on through us all the time. If we started counting them, we should never come to an end.

Vinoba Bhave

Music is the pleasure the human mind experiences from counting without being aware that it is counting.

Gottfried Leibniz

Probability is the very guide to life.

Cicero

1.5 PROBABILITY

Probability is from the French ***probabilite'*** meaning ***quality of being probable*** or ***something likely to be true***. **Probability** is the branch of mathematics which is a way of expressing knowledge (or belief) that an event will or will not occur numerically, a form of empirical inductive reasoning leading to statistical inferences.

The idea of **probability** originated with games of chance in the seventeenth century. The earliest writings in the area were the result of the collaboration of the eminent mathematicians Blaise Pascal and Pierre Fermat, and a gambler, Chevalier de Mere. To them, there seem to be contradictions between mathematical calculations and the events of actual games of chance involving throwing dice, tossing a coin, spinning a roulette wheel, or playing cards.

Thus, our goal in learning about **Probability** is to be in a position to compute and interpret the relevance of probabilities and address such questions as:

- ☐ **Breast Cancer**: A patient goes to the doctor with a lump in her breast. What is the probability that it is a tumor? What is the probability that it is cancerous?
- ☐ **Finance**: What is the probability that the value of the Dollar will be higher than the Euro in 2015?
- ☐ **Sports**: What is the probability that the USF quarterback will complete half of his passes in a given game?
- ☐ **Engineering**: What is the probability that a computer software package will fail?

Probability is expectation founded upon partial knowledge.

George Boole

- □ **Sociology**: What is the probability that a disadvantaged child in an urban area will pass the Florida Comprehensive Assessment Test (FCAT)?
- □ **Meteorology**: What is the probability that a hurricane will obtain hurricane status category 3 or more?
- □ **Statistics**: What is the probability that the mean number of accidents during New Year's Eve will exceed that of the previous year's number of accidents?
- □ **Physiology**: What is the probability that an experimental animal will convulse upon administration of a certain pharmacological agent?
- □ **Education**: What is the probability that an individual's score on an intelligence test will show significant improvement following a refresher course in verbal skills?

It is our aim to learn some of the very basic aspects of probability so that we not only answer questions such as those given above, but also to understand the role the subject plays in our daily lives. Learning **probability** is intended to put the student in a position to apply probability to any area of study that they are interested in: **statistics**, **engineering**, **operations research**, **physics**, **medicine**, **business**, **economics**, **accounting**, **education**, **sociology**, **physiology**, **agriculture**, **meteorology**, **linguistics** and **political science**, among others, and to use this information to make knowledgeable decisions.

1.6 BERNOULLI TRIALS

Trial is Anglo-French meaning ***act or process of testing***. A **Bernoulli trial** is an experiment whose outcome is random, but has one of only two possible outcomes: ***success*** or ***failure***. The discrete probability distribution that we use to answer such questions, among others, is the **binomial** or **Bernoulli probability distribution;** a mathematical expression that generates the actual probability for specific inputs that relate to a given question. We encounter many important situations that can be characterized by a **discrete random variable** with this developed distribution.

It is our goal in studying **Bernoulli trials** to put ourselves in a position to compute binomial probabilities and address such questions as:

- □ **Births**: A baby born less than 36 weeks is consider premature. What is the probability that a baby will be born premature?
- □ **Medicine**: What is the probability that a given drug will be effective to cure a specific disease?
- □ **Politics**: What is the probability that Candidate A will be elected president of the US?
- □ **Gambling**: What is the probability that I will obtain an odd number in a single roll of a fair die?
- □ **Computers**: What is the probability that the computer you purchased online will be operable (non-defective)?

We will learn how to use this very important probability distribution to answer the above questions, among others.

Discrete

A type of measure such that the outcomes are separate and distinct

Random

Taken such that each individual is equally likely to be selected

Variable

A distinct characteristic of an individual to be observed or measured

How dare we speak of the laws of chance? Is not chance the antithesis of all law?

Joseph Bertrand

1.7 THE BELL-SHAPED CURVE

The **Bell-shaped Curve** is commonly called the **normal curve** and is mathematically referred to as the **Gaussian probability distribution**.

Unlike **Bernoulli trials** which are based on discrete counts, the **normal distribution** is used to determine the probability of a continuous random variable.

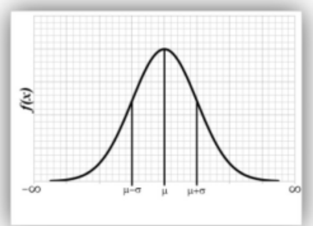
The **normal** or **Gaussian probability distribution** is the most popular and important distribution because of its unique mathematical properties, which facilitate its application to practically any physical problem in the real world; if not for the data's distribution directly, then in terms of the distribution associated with sampling. It constitutes the basis for the development of many of the statistical methods that we will learn in the following chapters.

Thus, our goal in studying the **Bell-Shaped Curve** is to put ourselves in a position to compute and interpret probabilities associated with continuous random variables and address such questions as:

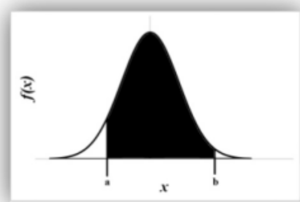
- ☐ **Cancer**: What is the probability that in a given group of lung cancer patients, an individual selected at random is Asian?
- ☐ **Education**: What is the probability that a student will have a final grade in finite mathematics between 85 and 95?
- ☐ **Sports**: What is the probability that a given lineman's weight on the USF football team will be between 275 and 325 pounds?
- ☐ **Rainfall**: What is the probability that the average rainfall in the State of Rhode Island in the year 2012 will be between 16 and 24 inches?
- ☐ **Chemistry**: What is the probability that an acid solution made by a specific method will satisfactorily etch a tray?

The objective in learning the mathematical properties of the **normal probability distribution** is to realize its usefulness in characterizing the behavior of continuous random variables that frequently occur in daily experience.

Continuous
✍ *A type of measure such that the outcomes are dense, that is, between any two outcomes, other possible outcomes exist.*



The graph of the normal probability distribution is a "**bell-shaped**" curve, as shown in the figure above. The constants **μ** and **σ** are the parameters.



The area under the curve represents the underlying probability of the situation.

1.8 STATISTICS

The branch of ***Statistics***, meaning ***quantitative fact or statement***, is becoming more widely accepted as a necessity for understanding all aspects that influence our daily lives. In almost every field of study, **statistics** is used to estimate the unknown, a characteristic of the individual we would like to know about in a given population. It is similar to the **Scientific Method**, in that we must first understand and clearly state the problem, gather the relevant information, formulate a hypothesis and test this hypotheses by recording and analyzing the data, before we can interpret the data and state our conclusion.

The basic idea behind **descriptive statistics** is to reduce a set of data down to one piece of information that describes some aspect of the data—an estimate of the population mean, or its central tendency, deviation, range, extremes, etc. Thus, our goal in studying **Statistics** is to be able to analyze and interpret real world data so that we will better understand the phenomenon that we are studying and address such questions as:

- ☐ **Business**: What is the mean profit made per hours of production time as a function of employees on the floor?
- ☐ **Politics**: What percentage of the people truly desire a tax increase given only 40% of individuals vote?
- ☐ **Chemistry**: What is the point of saturation for carbon dioxide in the atmosphere?
- ☐ **Medicine**: What is the mean tumor size in a patient with brain cancer?

Statistics
✍ *The art of decision making in the presence of uncertainty*
*As opposed to **statistic**—*
a numerical datum

Hypothesis
✍ *Greek meaning "**to suppose**"*

A statistical analysis, properly conducted, is a delicate dissection of uncertainties, a surgery of suppositions.
M.J. Moroney

Statistics may be defined as "a body of methods for making wise decisions in the face of uncertainty."
W.A. Wallis

- ☐ **Engineering**: What is the mean maximum load (kN) for a fishing line? What is the mean elongation?
- ☐ **Astronomy**: What is the mean temperature fluctuation in the Sea of Tranquility on the moon?
- ☐ **Agriculture**: What is the mean yield of corn per acre given the number of acres planted?
- ☐ **Sociology**: What is the mean number of texts sent by a cellular phone user in a given month? Is there a difference in usage between teens and adults?

Euclid is the **Father of Geometry** best known for his book ***Elements*** *which consist of 13 books covering* ***Euclidian Geometry***.

The point in learning basic statistics is to be able to efficiently gather, organize, analyze and interpret data in order to address questions that arise from every field of study and that apply to everyday living in a growing global society.

1.9 GEOMETRY

Geometry is from the Ancient Greek word ***γεωμετρια*** meaning ***measurement of earth or land***. This branch of mathematics is concerned with questions regarding the **shape**, **size**, **relative positions** and properties of **space**. **Euclidean geometry** is a mathematical system that assumes a small set of axioms and deductive propositions and theorems that can be used to make accurate measurement of unknown values based on their geometric relation to known measures.

Title page of Sir Henry Billingsley's first English version of **Euclid's Elements**, 1570

Thus, our goal in studying **Geometry** is to be able to accurately measure the world around us, perform basic calculations that address such questions as:

I've always been passionate about geometry and the study of three-dimensional forms.

Erno Rubik

There is geometry in the humming of the strings; there is music in the spacing of the spheres.

Pythagoras

Give me a lever long enough and a fulcrum on which to place it and I shall move the world.

Archimedes

- ☐ **Agriculture**: Using similar triangles, given the height of a stick and the length of its shadow at 2:00 PM, measuring the shadow of the tree at the same time, determine the height of the tree.
- ☐ **Carpentry**: If two boards, mitered at a 60° angle, are reversed and attached to create a frame, what is the angle formed by the joint?
- ☐ **Playground**: How large should a sandbox be if only 5 ft of wood is available and how much sand is needed if the wood is 6 in. tall?
- ☐ **Rubik Cube**: How many squares are there on the surface of a Rubik Cube?
- ☐ **Business**: If a showroom has 10,000 square feet of space to be converted into offices, but must leave 5000 square feet for the showroom floor and each office must be 200 square feet of space, how many offices can be created at most?
- ☐ **Chemistry**: What is the shape of a sugar molecule? How does this differ between mono-dextrose and poly-dextrose sugars? What is the difference in volume?

The intention behind learning **Geometry** is to enable the student to be proficient in both the art and science of **geometry**. **Geometry** is used in areas ranging from **graphic design** to **Einstein's theory of general relativity**; when a surveyor plots land, a manufacturer determines the best packaging for a stack of spherical oranges to be shipped or a car manufacturer redesigns a parabolic headlight, for example.

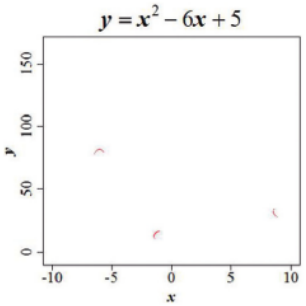
1.10 ARITHMETIC AND ALGEBRA

Arithmetic means ***the art of counting***, and ***Algebra*** means ***reunion of broken pieces***. **Arithmetic** is the oldest and most elementary branch of mathematics and deals with the study of quantity such as those that result from combining other quantities, which leads directly to **Algebra**. **Algebra** is a branch of mathematics outlining arithmetics, the rules of **operations** such as **addition**, **subtraction**, **multiplication** and **division**, but also **relations** such as **equalities**, **inequalities** and **functions**.

Arithmetic and **Algebra** are the building blocks of most areas in **mathematics**, usually taught as part of the curriculum in primary and secondary education. However, even at university level, these topics are extremely useful allowing general formulations to be the first step in the systematic exploration of more complex problems that can be solved using **Math**. Thus, our goal in studying **Arithmetic & Algebra** is to be able answer such questions as:

Diophantus is traditionally known as the **Father of Algebra**, but this has recently put up to debate in that **Al-Jabr**, the author of **Arithmetic** gives the elementary algebra before **Diophantus** in 200-214 CE.

- ☐ **Health**: Based on the nutritional information for three dog foods based on three required nutrients, how much of each type of dog food should be included in a single serving to optimize the nutritional intake?
- ☐ **Farming**: Given 100 feet of fencing, how should the length of a pen be related to the width, if the fence is to create two adjacent pens sharing a common size with maximum area?
- ☐ **Business**: If you sell tickets for $20 each and you sell as many as you can, which beforehand is an unknown quantity, x, how does your profit relate to this unknown value x?
- ☐ **Social**: If you know that you have x adults coming for dinner and one child, and each adult eats three manicotti shells and the child eats one, how many shells must be made, y, as a function of the number of adults invited, x.



How does y relate to x?

The aim of learning **Arithmetic** and **Algebra** is to refresh the student's understanding of the subject matter and to introduce more relevant uses of this dialect of **Math**. Remember: ***we think faster than we speak and we speak faster than we write***. Therefore, to address large complex problems such as building a bridge, we need a very short handed language. This universal language is **Math**, and **Arithmetic** and **Algebra** are a large part of this language.

1.11 FINANCE

Finance means ***to ransom***, or **to *manage money***. This science of funds management includes **business finance**, **personal finance** and **public finance**. Our goal is to use mathematics to teach the student to have a better understanding of **basic personal finance**; such finances will include savings and loans in terms of time, money, risk and how they are interrelated in addition to spending and budget.

Thus, our goal in studying the **Basics of Finance** is to be able to understand and manage personal finances and address such questions as:

- ☐ **Personal Budget**: How much do you spend each month on Rent, Electricity, Phone, Internet, Food, Gas, Insurance, etc.
- ☐ **Wedding**: How much can I afford to spend? If I finance a wedding on credit, how long will it take to pay off this debt and how much will it eventually cost?
- ☐ **Transportation**: What should be the maximum payment I should agree to in order to ensure my vehicle is not repossessed.
- ☐ **Housing**: Can you afford to move out of your apartment and into a house? What is the expected down payment? Inspection fees? The expected property taxes?

In the business world, the rearview mirror is always clearer than the windshield.

Warren Buffett

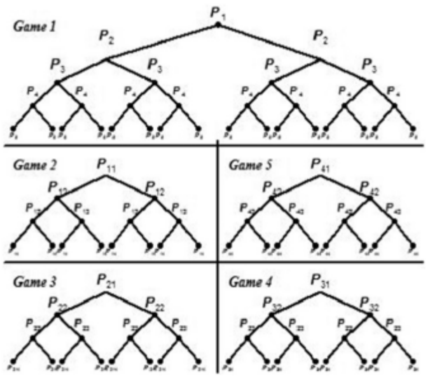
Budget

✍ *An estimate, often itemized, of expected income and expense for a given period of time in the future*

✍ *A sum of money set aside for a specific purpose*

In the absence of the gold standard, there is no way to protect savings from confiscation through inflation. There is no safe store of value.

Alan Greenspan



If thou dost play with him at any game,
Thou art sure to lose, and, of that natural luck,
He beats thee 'gainst the odds.

Shakespeare

- □ **Home Repairs**: How long would it take to save enough to replace an air conditioning unit? How much can be saved be investing into a sinking fund versus using credit?
- □ **Credit**: If you make the minimum required payment and have a minimum purchase each month, how long will you be indebted to the creditor?
- □ **Christmas Funds**: How much needs to be put into a sinking fund in order for you to have saved up $1000 in a Christmas fund over a period of 11 months starting in January?

In is imperative in today's economy that everyone has a basic understanding of finance. Many individuals are overwhelmed when confronted with mounting bills or credit; however, it is important that they budget, even if the final amount is negative. Once we are aware of the problem, we can begin to work out the solution. Understanding the basic **mathematics** behind **Basic Finance**, we will be in a position to make positive changes in our present financial state and better plan for the future.

1.12 GAME THEORY

Game Theory is a study of strategic decision making between two rational decision-makers. Here, we address **two-person zero-sum games**; games designed such that one players gain equals the second player's loss.

- □ Strictly Determined Games: The Saddle Point
- □ Games with Mixed Strategies
- □ Reducing Matrix Games to a System of Linear Equations

CRITICAL THINKING AND BASIC EXERCISE

1.1. Who was the first to call himself a philosopher?
1.2. What is Pythagoreanism?
1.3. What does the word "matheno" mean?
1.4. Mathematics can be divided into what two branches?
1.5. Who wrote "The Nature of Mathematics"?
1.6. What word is derived from the Greek meaning conforming to laws of reason?
1.7. Distinguish between the two types of reasoning.
1.8. In Logic, which type of reasoning is used to draw correct conclusions when confronted with facts?
1.9. What does the word "setta" mean?
1.10. The modern study of sets began with two mathematicians; name them and state where they are from.
1.11. Sets are used for what type of measure: numerical or categorical?
1.12. What diagram is used to graph categorical information and to support any inferences made regarding relationships among the various sets?
1.13. What are some of the first known uses of counting?
1.14. Name three counting techniques.
1.15. Name the area of study that is a form of empirical inductive reasoning leading to statistical inferences.
1.16. In what area of study are Blaise Pascal, Pierre Fermat and Chevalier de Mere known to have collaborated?
1.17. Who said "Probability is expectation founded upon partial knowledge"?
1.18. A binomial experiment is also known by what other name?
1.19. In a Bernoulli trial, there are exactly how many possible outcomes?
1.20. The binomial probability distribution is characterized by what type of random variable? Continuous or Discrete.
1.21. The normal probability distribution is also known by what other name?
1.22. Outline the steps associated with the Scientific Method.
1.23. Distinguish between Descriptive and Inferential Statistics.

1.24. List the points you need to learn in basic statistics to be an efficient researcher.
1.25. What branch of mathematics is concerned with questions regarding shape, size, relative positions and properties of space?
1.26. Name the mathematical system that assumes a small set of axioms and deductive propositions and theorems that can be used to make accurate measurement,
1.27. The art of counting is better known by what name?
1.28. Name the rules of operations in Algebra.
1.29. Why are Arithmetic and Algebra important?
1.30. Name the science of funds management.
1.31. Name the study of strategic decision making between two rational decision-makers.

SUMMARY OF IMPORTANT CONCEPTS

The first chapter, ***The Usefulness of Mathematics***, introduces Mathematics and its history. This motivational chapter answers the question "what is logic"; outlining the usefulness of Sets; the start of Counting Techniques and how counts are the foundation of empirical probabilities. The first chapter also includes the usefulness of Mathematics in basic Probability and Statistics, Geometry, and Finance; an overview of basic Arithmetic and Algebra along with Game Theory.

The second chapter on ***Logic*** covers statements and their truth values; outlines the symbolisms used to express statements in the short hand language of Math including logical operators: conjunction, disjunction, negation and implication; how to construct truth tables and determine equivalent statements. This chapter helps the student understand logical reasoning by interpreting logical symbolism by giving their English translation. Properties of Logic covered include Tautologies, Self-Contradictions, Paradox, Equivalence, and Algebra of Statements; Variations on the Conditional Statement; Quantified Statements; Testing the Validity of an Argument and Applications of Logic.

The third chapter on ***Sets*** gives an introduction to Set Theory, covers collections of objects, the symbolisms used to express these collections (sets) in the short hand language of Math including set operators: intersection, union, complement and subset; and how they relate to logical operators. This chapter covers the Algebra of Sets, some basic counting principles applied to sets.

The fourth chapter on ***Counting Techniques*** introduces counting principles beyond that of simple sets to that of the Multiplication Principle, Permutations and Combinations, Distinct Orderings, and other counting techniques such as the Binomial Theorem, and Pascal's Triangle.

The fifth chapter on ***Basic Probabilities*** gives an introduction to probability and various definitions: personal probability, empirical and theoretical. This chapter covers the experimental probabilities using sample spaces, the basic laws of probability, conditional probability and Bayes rule.

The sixth chapter on ***Binomial Probability*** introduces discrete random variables, discrete probability distribution in general including expected value and variance followed by the Binomial Probability Distribution and the expected value and variance for the Binomial random variable.

The seventh chapter on ***Normal Probability*** introduces continuous random variables and the Normal probability distribution. This chapter also ties back in with discrete random variables covering Normal Approximation to the Binomial.

The eighth chapter on ***Descriptive Statistics*** covers gathering and organizing data; graphical representations of qualitative information and quantitative information; and measuring central tendencies and deviations from the center.

The ninth chapter on ***Geometry*** covers rounding and types of measurement; properties of lines: linear, linear pairs, two lines and three lines; properties of angles: categorization and additive principles; properties of triangles: categorization and similar/equivalent triangles; and properties of quadrilaterals and polygons. This chapter also covers area, surface area and volume.

The tenth chapter on ***Arithmetic and Basic Algebra*** covers the real number system, basic arithmetic: addition, subtraction, multiplication and division; pattern recognition: sequences and series; algebraic expressions and relationships; equations: equalities and systems of equations; and functions: linear and quadratic equation.

The eleventh chapter on ***Finance*** covers basic financing including sinking funds and amortization: various savings situations and comparison shopping: credit versus cash, leasing versus purchasing, and renting versus owning. This chapter also covers effective rates and uses them to compare CD versus credit and comparisons of credit cards. There is also a section on personal finance: how to create a monthly budget; insurance: what every homeowner should know and your credit report.

The twelfth chapter on ***Game Theory*** covers two-person zero-sum games. This chapter covers the Matrix Game; strictly determined games, games with mixed strategies and instructions on how to reduce Matrix Games to Systems of Equations.

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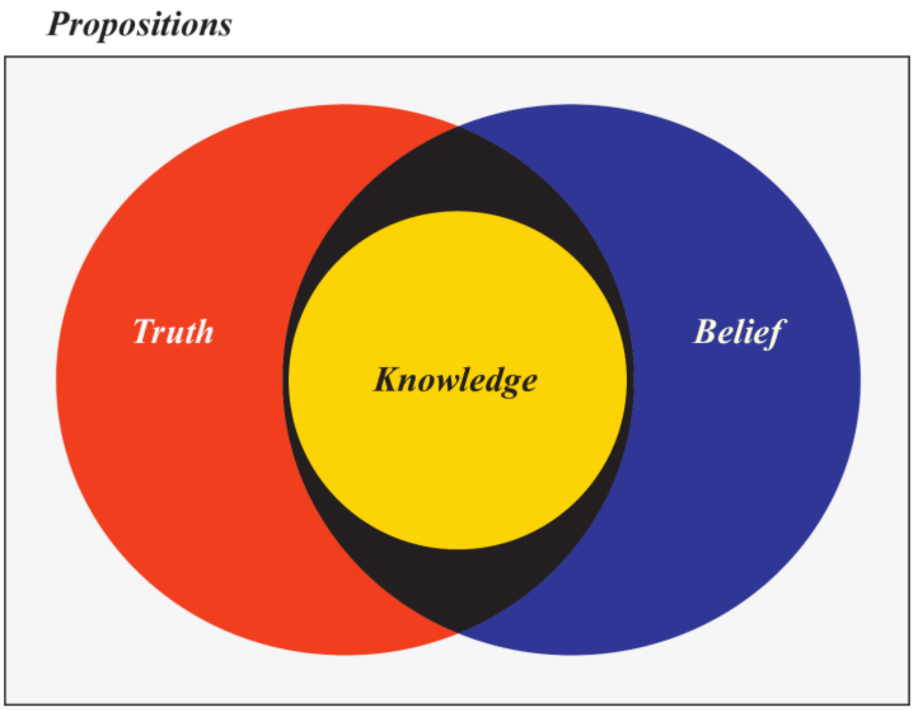


Number is the within of all things.

PYTHAGORAS

Chapter 2 Logic

In the field of mathematics, Aristotle is probably best noted for his contributions to the methods of proofs. He was the first to provide clear distinctions between axioms, postulates, and definitions. In addition, he contributed theorems of geometry, infinity, and continuity. He was considered to be a philosopher, but the philosophy of his time included what is now classified as natural sciences. Among Aristotle's writings on logic (called later the Organon) are Prior Analytics, Posterior Analytics, and Sophisms. In these and in his other works, he systematized the formal rules of logic and introduced syllogism, a form of deductive reasoning.

Aristotle was a student of Plato's Academy and later became a teacher. When he was forty-one, he began to supervise the education of Alexander the Great, for which he received the beginnings of his fortune. He later taught in the Lyceum in Athens and began amassing a book and map collection for a museum of natural history. This arrangement eventually led to a "school" after his death at the age of sixty-two.

Aristotle's influence was so encompassing and pervasive that many of his contributions were not even questioned until the middle of the nineteenth century even though many of his theories were incorrect. His writings, including accurate as well as misdirected ideas, were accepted as the ultimate authority during the medieval period and were upheld by the Roman Catholic Church beyond the time of Galileo (Aristotle had believed in the geocentric; i.e., earth-centered solar system, which Galileo unsuccessfully argued against in the Inquisition). Although his dogmatic followers deterred further advances for many centuries, Aristotle did much to advance science in his time. His many fields of study included biology (he devised classifications for all kinds of plants and animals), metaphysics and logic, ethics and politics, rhetoric and poetics, weather, and the other physical sciences.

2.1 LOGIC

Why should we have chosen to begin the study of **finite mathematics** with a chapter on logic? The following argument is offered by way of illustration: Mathematics must be based on logic. This is a basic course in mathematics with emphasis on its usefulness. Therefore this course must be based on logic. An argument of this form is known as a **syllogism**. A **syllogism** is a typical **Aristotelian** argument. **Aristotle** gave the first systematic treatment of the principles of logical reasoning which earlier **Greeks** had begun to formulate. **Aristotelian logic** is the fundamental form of logical reasoning which is still utilized today.

Logic

Derived from the Greek word logos which means reason or discourse

Syllogism

An argument supported by two premises; deductive reasoning

An extremely subtle, but sophisticated argument

Aristotelian

Of, pertaining to, based on, or derived from Aristotle or his theories.

The assertion that mathematics must be based on logic is justifiable because virtually all mathematical results are obtained by logical deductions from other previously obtained results now generally accepted as true, or from assertions which have been assumed without proof. Sometimes, making logical deductions is not as straightforward and simple as we might like; thus, it is true here as in many other situations that possession of a set of rather specific rules makes the task much easier.

George Boole (1815-1864), an English mathematician, integrated logic into algebra and essentially founded the field of mathematical logic. He introduced the use of symbols to represent statements or assertions, which greatly increases the ease and speed of manipulation of concepts in deductive logic. **Mathematical logic** is also known as **symbolic logic** for this reason. In this chapter some of the fundamental concepts of mathematical logic will be discussed with a view toward enabling the reader:

1. *To be able to apply logic to analyze problems, and*
2. *To obtain proficiency in the correct methods of logical reasoning.*

> *... no general method for the solution of questions in the theory of probabilities can be established which does not explicitly recognize ... those universal laws of thought which are the basis of all reasoning ...*
>
> **George Boole**

To this end, we must consider that we think faster than we speak, we speak faster than we write, therefore to think quickly and communicate these ideas, we must learn to abbreviate almost everything. A summary of modern symbolic logic can be found in the summary, at the end of the chapter.

2.2 STATEMENTS AND THEIR TRUTH VALUES

In this section we shall discuss one of the basic concepts of logic; namely, that of a statement. We shall also introduce some other important terms and symbols. We begin with a definition.

True Statements:

- *Monday is a day of the week*
- **5** *is a natural number*

False Statements:

- *January is a day of the week*
- **5** *is a negative integer*

Definition 2.2.1 Statement

A **statement** is a declarative sentence which is either true or false, but not both. We shall denote statements symbolically by lower case letters p, q, r...

We judge statements with respect to their truth value. That is,

Definition 2.2.2 Truth Value

The **truth value** of a statement is the truth or falsity of the statement. We shall denote *true* by **T** and *false* by **F**.

Example 2.2.1 Classify Sentences

Consider the following sentences; classify each as statement, question or command:

(a) London is in France
(b) $\mathbf{3+5=8}$
(c) Who is here?
(d) Put the book on the shelf.
(e) Sometimes it rains.

Facts:

(a) London is, in fact, not in France
(b) **3 + 5** is equal to **8**
(c) Not a statement
(d) Not a statement
(e) Sometimes it does rain.

Solution

Sentence (a) is a false statement, and sentences (b) and (e) are true statements. However, sentences (c) and (d) are not statements because neither can be assigned a truth value of true or false. Sentence (c) is a question and sentence (d) is a command.

Simple Statements:

- *Today is Monday*
- *Tomorrow is Christmas*
- *The sun is shining*
- *There are rain clouds in the sky*
- *I will study English*
- *I will study Math*

The statements given in the preceding example are composed of terms in a certain relation to each other.

Compound Statements:

- *Today is Monday and it is the day before Christmas*
- *The sun is not shining and there are rain clouds in the sky.*
- *I will study English or Math*

Definition 2.2.3 Compound Statement

A statement consisting of a single such relationship is called a **simple statement**.

However, consider the statement "The square root of thirty-six is six and six is an even number." This statement is a combination of two components; namely, "The square root of thirty-six is six" and "six is an even number." Thus, we have a compound statement.

***Note: iff** reads "**if and only if**"*

Definition 2.2.4 Compound Statement

A **compound statement** is a statement composed of two or more statements connected by the logical connectives, "and," "or," "if then," "not," and "if and only if." A statement which is not compound is said to be a **simple statement**.

*Let **b** represent* **Deb *is beautiful*** and ***s** represent* **Deb *is smart***.

(a) $b \wedge s$

$\wedge$ *reads "and"*

Let ***h*** represent **Matthew is here**, ***w*** represent **Washington is in the United States** and ***s*** represent **Sugar is sweet**.

(b) $(h \rightarrow w) \vee s$

$\rightarrow$ *reads "implies"*

$\vee$ *reads "or"*

Example 2.2.2 Simple/Compound Statement

Consider the following statements:

(a) "**Deb is beautiful and Deb is smart**." This statement is a compound statement composed of the simple statements "**Deb is beautiful**" and "**Deb is smart**" linked by means of the logical connective "**and**."

(b) "**If Matthew is here, then Washington is in the United States, or sugar is sweet**." This is a compound statement composed of the compound statement "**If Matthew is here, then Washington is in the United States**" and the simple statement "**Sugar is sweet**" by means of the logical connective "or."

Good, too, Logic, of course; in itself, but not in fine weather

A. H. Clough
English Poet (1848)

The truth value of a compound statement is completely determined by the truth values of the simple statements that form the compound statement.

We shall now study some of the most important ***connectives*** and illustrate their meanings by various examples. In logic, connectives are referred to as ***operators***.

$\wedge$
reads
"and"

"and"
means
"both"

Definition 2.2.5 Conjunction: $p \wedge q$

The **conjunction** of two statements ***p*** and ***q*** is the compound statement "***p*** and ***q***"; written symbolically, $p \wedge q$, where the symbol $\wedge$ is read "and" or "but."

The truth value of $p \wedge q$ is determined using property 2.2.1.

Property 2.2.1 Conjunction: $p \wedge q$

If p is true and q is true, then $p \wedge q$ is true; otherwise $p \wedge q$ is false.

Row 1: *When both **p** and **q** are true, the statement "**p** and **q**" is true.*
Row 2: *When **p** is true, but **q** is not true, the statement "**p** and **q**" is false.*
Row 3: *When **p** is false, and only **q** is true, the statement "**p** and **q**" is false.*
Row 4: *When both **p** and **q** are false, the statement "**p** and **q**" is false.*

Columns 1 and 2 give all possible truth value combinations of the statements p and q. Column 3 gives the truth value of the statement $p \wedge q$ for each of the four combinations of the individual truth values of p and q. We observe that p and q is true only when p and q are both true; otherwise $p \wedge q$ is false. This table defines the truth value of the compound statement $p \wedge q$ as determined by the truth values of p and q separately. The representation shown by Table 2.1 is called the truth table for the conjunction $p \wedge q$.

Example 2.2.3 Truth Values

Determine the truth value of each of the following conjunctions:
(a) Athens is in Greece and Rhodes is an island.
(b) Berlin is in Germany and Casablanca is in Williamsburg.
(c) Roses are red and violets are blue.

Facts:
(a) *Athens is in Greece Rhodes is an island*
(b) *Berlin is in Germany Casablanca is in Morocco*
(c) *Roses are red Violets are blue*

Solution

By Property 1, "$p \wedge q$" is true only when p and q are both true. Thus, we have (a) true, (b) false, and (c) true.

Definition 2.2.6 Disjunction: $p \vee q$

The **disjunction** of two statements p and q is the compound statement "p or q"; written symbolically is, $p \vee q$, where the symbol $\vee$ is read "or."

$\vee$
reads
"or"

Example 2.2.4 Symbolism in Logic

Let p and q represent the statements "George teaches mathematics" and "George lives in Greece"; then $p \vee q$ denotes the disjunction "George teaches mathematics or George lives in Greece."

TABLE 2.1 Truth Table for the conjunction $p \wedge q$

Column	1	2	3
	p	q	$p \wedge q$
	T	T	T
	T	F	F
	F	T	F
	F	F	F

The truth value of $p \vee q$ is determined by:

Property 2.2.2 Disjunction: $p \vee q$ (Inclusive)

When p is true or q is true, or if both p and q are true, then $p \vee q$ is true; otherwise $p \vee q$ is false. Thus, the disjunction $p \vee q$ of the two statements p and q is false only when both p and q are false.

A good decision is based on knowledge and not on numbers.

Plato

That is, the standard disjunction $\vee$ is the **inclusive disjunction** which is true if at least one of the statements is true. The **exclusive disjunctive**, symbolized $\underline{\vee}$ is only true if one or the other statements are true, but not both. Thus the truth value of $p \underline{\vee} q$ is determined by: p is true and q is false, or p is false and q is true. In this text "or" will be inclusive. The truth value of $p \vee q$ is determined from the truth values of p and q by the following truth table (Table 2.2):

Column 3 tells us that the disjunction, $p \vee q$, is false only when both p is false and q is false; in other words, when p and q are *both false*.

Inclusive ***"or"*** means ***"at least one"***

Row 1: *When both* p *and* q *are true, the statement "*p *or* q*" is true.*
Row 2: *When* p *is true, but* q *is not true, the statement "*p *or* q*" is true.*
Row 3: *When* p *is false, and only* q *is true, the statement "*p *or* q*" is true.*
Row 4: *However, when both* p *and* q *are false, the statement "*p *and* q*" is false.*

Example 2.2.5 Disjunctions

Obtain the truth value of each of the following disjunctions:

(a) $\mathbf{3 + 5 = 9}$ or $\mathbf{4 + 10 = 14}$.
(b) Atlanta is in Florida or Armstrong landed on the moon.
(c) The earth is square or football is a gentle game.

Solution

By Property 2, "$p \vee q$" is false only when p and q are both false. Thus, we have (a) true, (b) true, and (c) false.

A mind all ***logic*** *is like a knife all blade, it makes the hand bleed that uses it's*

Tagore

TABLE 2.2 Truth Table for the inclusive disjunction $p \vee q$

Column	1	2	3
	p	q	$p \vee q$
	T	T	T
	T	F	T
	F	T	T
	F	F	F

TABLE 2.3 Truth Table for the exclusive disjunction $p \underline{\vee} q$

Column	1	2	3
	p	q	$p \underline{\vee} q$
	T	T	F
	T	F	T
	F	T	T
	F	F	F

Row 1: *When both **p** and **q** are true, the statement "**p** or **q**" is false.*
Row 2: *When **p** is true, but **q** is not true, the statement "**p** or **q**" is true.*
Row 3: *When **p** is false, and only **q** is true, the statement "**p** or **q**" is true.*
Row 4: *However, when both **p** and **q** are false, the statement "**p** and **q**" is false.*

Exclusive "or" means ***"exactly one"***

Not true is false
Not false is true

For the exclusive "or," that is "p or q, but *not* both," the truth value is determined by:

Property 2.2.3 Disjunction: $p \underline{\vee} q$ (Exclusive)

When p is true and q is false, or when p is false and q is true, then $p \underline{\vee} q$ is true; otherwise $p \underline{\vee} q$ is false. Thus, this alternative disjunction $p \underline{\vee} q$ of the two statements; p and q are false only when both p and q are false and when both p and q are true.

In Table 2.3, Column 3 tells us that the exclusive disjunction $p \underline{\vee} q$ is true only when exactly one of the statements p and q are true. If p and q have the same truth value then $p \underline{\vee} q$ is false.

Many times it is necessary to negate a given statement, p, forming the "negation of p." This is accomplished by writing "It is false that" before p, or, if possible, using the word "**not**" in the statement p.

Example 2.2.6 Negations

Following is the statement p, give its corresponding negation $\sim p$:
(a) $\mathbf{4 + 6 = 10}$.
(b) London is in England.
(c) Maria is pretty.
(d) Jonathan is not here.

Solution

(a) $\sim(\mathbf{4+6=10})$ is $\mathbf{4+6 \neq 10}$, (b) It is not the case that London is in England *or* London is not in England, (c) Maria is not pretty, (d) Jonathan is here.

Definition 2.2.7 Negation: $\sim p$

The **negation** of p is denoted by $\sim p$. The truth value is obtained from the property: If p is true, then $\sim p$ is false and if p is false, then $\sim p$ is true. The symbol $\sim$ is read as not; that is, $\sim p$ is the statement "not p." Other common denotations of the negation of p are $\bar{p}$, $\neg p$ or even $-p$; however, we shall use the notation $\sim p$.

TABLE 2.4 Truth Table for the negation $\sim p$

Column	1	2
	p	$\sim p$
	T	F
	F	T

Statements:

- *I like vanilla ice cream*
- *I am not a Leo*
- *My favorite color is green*

Their Negations:

- *I do not like vanilla ice cream*
- *I am a Leo*
- *My favorite color is not green*

$\rightarrow$
reads
*"**implies**"*

Hypothesis

- *From the Greek—basis or supposition*
- *An assumption or concession made for the sake of argument*
- *A preceding event, condition or cause*

Conclusion

- *The necessary consequence of a preceding event, condition or cause*
- *Something necessarily following from a set of conditions*

The truth value of the negation $\sim p$ is determined by the following property:

Property 2.2.4 Negation: $\sim p$

When p is true, then $\sim p$ is false; if p is false, then $\sim p$ is true.

We observe that the truth value of the negation of any statement is always the opposite of the truth value of the original statement.

The following truth table defines the truth value of $\sim p$ as it depends on the truth values of p, Table 2.4.

It is clear that the first column in Table 2.4 gives the original statement while Column 2 gives the corresponding negation.

Just as in English, a double negative statement is the positive statement; that is, not "not p" is p; symbolically $\sim(\sim p)$ is p.

Definition 2.2.8 Conditional Statement: $p \rightarrow q$

Statements of the form "if p then q" are called **conditional statements** and are denoted by $p \rightarrow q$. The statement p is called the **hypothesis** (**condition/antecedent**) while statement q is called the **conclusion (consequence)**. The truth value of $p \rightarrow q$ is given by the property: $p \rightarrow q$ is false only when p is true and q is false; otherwise $p \rightarrow q$ is true.

It is not the statements' truth value we are concerned with, but whether the implication "$\rightarrow$" fails to be true. The key word here is "if," this condition is often misinterpreted. A mother say to the child, "if you misbehave, you will not get to go outside and play" and the child is horrible but the mother says, "Just go and play." This teaches the child that either mom is lying or "if" has no meaning. Unfortunately, children often assume the latter. However, this is not true, "if" is a very powerful word; only "if" the premise is true does the implication have to lead to the conclusion. Without the condition, the implication is vacuously true. Given the statement "if I win the lottery, then I will buy you dinner" and then this person does not win the lottery, but does buy me dinner, this does not make the original statement a lie (false), it simply has not come to pass.

Example 2.2.7 Conditional Statements

Consider again the following statements:

p: I win the lottery.
q: I will buy you dinner.

Example 2.2.7 Conditional Statements—cont'd

Hence, $p \rightarrow q$ represents "If I win the lottery, then I will buy you dinner." If I win the lottery and I buy you dinner, then I have kept my word. Thus, $p \rightarrow q$ is true when p is true and q is true. However, if I did not win the lottery and do not buy you dinner, or I did not win the lottery but do take you to dinner, I did *not* break my word. Hence, $p \rightarrow q$ is true when p is false whether q is true or false. Only when I win the lottery and do not take you to dinner has my word been broken. Hence, the only time that $p \rightarrow q$ is false is when p is true, but (and) q is false.

Row 1: When both **p** *and* **q** *are true, the statement "if* **p**, *then* **q**" *is true.*
Row 2: When **p** *is true, but* **q** *is not true, the statement "if* **p**, *then* **q**" *is false.*
Row 3: When **p** *is false, and only* **q** *is true, the statement "if* **p**, *then* **q**" *is not false; that is, true.*
Row 4: However, when both **p** *and* **q** *are false, the statement "if* **p**, *then* **q**" *is not false; that is, true.*

(a) **T→T** is **T**
(b) **F→T** is **T**
(c) **F→F** is **T**
(d) **T→F** is **F**

TRUE WHEN
T→T
F→T
F→F

If p true, it is suffice to say, then q is true. However, q only needs to be true when p is true.

Under what conditions is this conditional statement $p \rightarrow q$ is true.

This example leads us to the following property for the conditional statement $p \rightarrow q$:

Property 2.2.5 Conditional Statement: $p \rightarrow q$

The **conditional statement** $p \rightarrow q$ is false only when p is true and q is false; otherwise $p \rightarrow q$ is true.

Table 2.5 defines the truth value of $p \rightarrow q$, dependent on the truth values of p and q.

Column 3 tells us that the conditional statement $p \rightarrow q$ is false only when p is true and q is false; in other words, when the hypothesis is true and the conclusion is false.

Example 2.2.8 Conditional Truth Values

Determine the truth value of each of the following conditional statements:

(a) If **2 + 3 = 5**, then **6 + 9 = 15**.
(b) If **6 + 8 = 10**, then **1 + 1 = 2**.
(c) If **2 + 1 = 4**, then **2 + 3 = 6**.
(d) If **3 + 7 = 10**, then **4 + 7 = 14**.

(a)	2 + 3 = 5	T
	6 + 9 = 15	T
(b)	6 + 8 = 10	F
	1 + 1 = 2	T
(c)	2 + 1 = 4	F
	2 + 3 = 6	F
(d)	3 + 7 = 10	T
	4 + 7 = 14	F

Solution

We observe that (d) is the only conditional statement having the hypothesis "**3 + 7 = 10**" true and the conclusion "**4 + 7 = 14**" false. Thus, (d) is a false statement. Statements (a), (b), and (c) are all true. Note that even though the hypothesis in both (b) and (c) is false, the resulting conditional statement is true because of Property 5.

(a) **F→T** is **T**
(b) **T→T** is **T**
(c) **F→F** is **T**
(d) **T→F** is **F**

TABLE 2.5 Truth Table for the conditional statement $p \rightarrow q$

Column	1	2	3
	p	q	$p \rightarrow q$
	T	T	T
	T	F	F
	F	T	T
	F	F	T

Common: Conditional Statement

There are four common ways in which the **conditional statement** $p \rightarrow q$ can be expressed.

1. ***p implies q*** (Direct statement)
2. ***p only if q***
3. ***p*** is ***sufficient*** for ***q***
4. ***q*** is ***necessary*** for ***p***

If science and ***logic*** *chatters as fine and as fast as he can; though I am no judge of such matters, I'm sure he's a talented man*

W. M. Praed
Write of "The Talented Man" (1830)

The following example depicts the usage of the preceding expressions. However, it should be noted that a frequent usage of "p implies q" is the meaning that q is true whenever p is true. The conditional "$p \to q$" is a new statement compounded from two given statements, while the implication "p implies q" is a relation between two statements. The connection is the following: "p implies q" means the conditional statement $p \to q$ is always true.

Necessary

Being essential, indispensable, or requisite

Existing by necessity

Sufficient

Adequate for the purpose; enough

Of a condition

Example 2.2.9 Necessary and Sufficient

Obtain the truth value of each of the following conditional statements:

(a) $3 + 7 = 11$ is a sufficient condition for $4 + 8 = 12$.

(b) $4 + 11 = 15$ is a necessary condition that $3 \times 3 = 9$.

(c) Sugar is sour implies vinegar is sweet.

(d) $4 + 10 = 14$ only if $5 \times 4 = 23$.

Solution

Statement (a) can be expressed as "If $3 + 7 = 11$, then $4 + 8 = 12$," which is true because the hypothesis "$3 + 7 = 11$" is false and the conclusion "$4 + 8 = 12$" is true. Statement (b) can be expressed as "If $3 \times 3 = 9$, then $4 + 11 = 15$," which is true. Statement (c) can be restated in the form "If sugar is sour, then vinegar is sweet." We observe that since both hypotheses and the conclusion are false, the given conditional statement is true. Statement (d) can be restated as "If $4 + 10 = 14$, then $5 \times 4 = 23$," which is false because the hypothesis "$4 + 10 = 14$" is true but the conclusion "$5 \times 4 = 23$" is false.

$p \leftrightarrow q$

A bi-conditional statement

Bi

*From the Latin "**bis**" meaning twice*

$p \leftrightarrow q$

Definition 2.2.9 Bi-conditional Statement: $p \leftrightarrow q$

A statement of the form "p if and only if q" is called a **bi-conditional statement**, written in shorthand, you will often write "p iff q," symbolically, $p \leftrightarrow q$.

Its truth value is obtained from the property 2.2.6. If p and q are both true or if p and q are both false, then $p \leftrightarrow q$ is true; if p and q have opposite truth values, then $p \leftrightarrow q$ is false. This is due to the fact that the statement $p \leftrightarrow q$ is equivalent to the conjunction of conditional statements, $(p \to q) \wedge (q \to p)$, hence if both p and q are both true then both $p \to q$ and $q \to p$ are true and if both p and q are both false then both $p \to q$ and $q \to p$ are vacuously true; thus $(p \to q) \wedge (q \to p)$ is true.

A convenient way to analyze logically compound statements formed by the connectives $\wedge$, $\vee$, $\sim$, $\to$ and $\leftrightarrow$ is by structuring their truth tables. Truth tables will be introduced in Section 2.4.

Example 2.2.10 Bi-Conditional Statement

Let p represent the statement "Maria is happy" and q "Maria looks beautiful"; then $p \leftrightarrow q$ denotes the bi-conditional statement "Maria is happy if and only if she looks beautiful."

The following property is used to determine the truth value of the bi-conditional statement:

TABLE 2.6 **Truth Table for the bi-conditional statement $p \leftrightarrow q$**

Column	1	2	3
	p	q	$p \leftrightarrow q$
	T	T	T
	T	F	F
	F	T	F
	F	F	T

Property 2.2.6 Bi-Conditional Statement: $p \leftrightarrow q$

If p and q are either both statements are true or both statements are false, then $p \leftrightarrow q$ is true; if p and q have opposite truth values, then $p \leftrightarrow q$ is false.

Row 1: *When both* p *and* q *are true, the statement "*p *iff* q*" is true.*
Row 2: *When* p *is true, but* q *is not true, the statement "*p *iff* q*" is false.*
Row 3: *When* p *is false, and only* q *is true, the statement "*p *iff* q*" is false.*
Row 4: *However, when both* p *and* q *are false, the statement "*p *iff* q*" is true.*

The truth value of the biconditional statement $p \leftrightarrow q$ is given in Table 2.6.

In Table 2.6, Column 3 tells us that the bi-conditional statement $p \leftrightarrow q$ is true only if p and q have the same truth value, as in the first and fourth lines of the table.

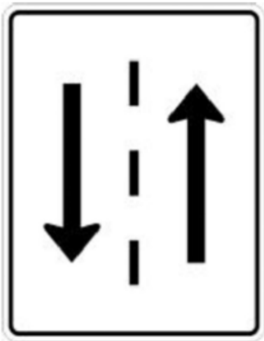
Example 2.2.11 Bi-Conditional Statement Truth Value

Determine the truth value of each of the following bi-conditional statements:

(a) $\mathbf{4+8=12}$ if and only if $\mathbf{3+7=10}$.
(b) London is in France if and only if "Sharks" live on the moon.
(c) $\mathbf{14+5=20}$ if and only if $\mathbf{3+8=15}$.
(d) $\mathbf{5+3=8}$ if and only if **6** is greater than **8**.

Solution

We observe that only in Statements (a), (b), and (c) have p and q have the same truth value, both being true in (a) and both false in (b) and (c). Thus, we can conclude that Statements (a), (b), and (c) are true, whereas Statement (d) is false.



(a) $\mathbf{T \leftrightarrow T}$ is **T**
(b) $\mathbf{F \leftrightarrow F}$ is **T**
(c) $\mathbf{F \leftrightarrow F}$ is **T**
(d) $\mathbf{T \leftrightarrow F}$ is **F**

We should also mention here that a bi-conditional statement, $p \leftrightarrow q$, is often stated in the form "p is necessary and sufficient for q."

Example 2.2.12 Bi-Conditional Statement Truth Value

Obtain the truth value of each of the following bi-conditional statements:

(a) $\mathbf{4+5=9}$ is a necessary and sufficient condition for $\mathbf{8+7=15}$.
(b) Honey is sweet is necessary and sufficient for $\mathbf{8+10=25}$.

Solution

We can rewrite Statement (a) as "$\mathbf{4 + 5 = 9}$ if and only if $\mathbf{8 + 7 = 15}$," which is true. Statement (b) can be restated in the form "Honey is sweet if and only if $\mathbf{8 + 10 = 25}$," which is a false bi-conditional statement since p and q have opposite truth values.

If it was so, it might be; and if it were so, it would be: but as it isn't, it ain't. That's logic

Lewis Carroll
Logician in **Through the Looking Glass** (1872)

EXERCISES

Critical Thinking

Indicate which phrases are acceptable statements; and for those which are, indicate whether they are true or false.

2.2.1. Sparta is in Greece.
2.2.2. $\mathbf{11 + 1 = 10}$.
2.2.3. Who is coming?
2.2.4. Put the fish in the water.
2.2.5. Florida has a cold climate.
2.2.6. Tennis is fun.
2.2.7. $\mathbf{-3+5=2}$.
2.2.8. Goodbye Columbus.
2.2.9. Put your shoes on.
2.2.10. Casablanca is in North Africa.
2.2.11. Tampa is the capital of Florida.
2.2.12. Answer the phone.
2.2.13. This statement is false.

Analyze the given statements and indicate the appropriate connectives and the simple statements.

2.2.14. $\mathbf{4 + 1 = 5}$ and Maria is pretty.
2.2.15. If you are swimming in the Gulf, then Washington is in England or $\mathbf{4+7=10}$.
2.2.16. If Columbus was from Italy or Columbus was from Portugal, then 1492 was an Italian year.
2.2.17. If Jonathan is thirsty, then Maria needs water, but 1977 is a dry year.
2.2.18. If $\mathbf{4-11=7}$, then $\mathbf{-4+11=10}$ and $\mathbf{6-7=-1}$.
2.2.19. Deb is athletic if Maria and Jonathan are, but Mathew is a basketball player.
2.2.20. It is not the case that it is raining and the sun is shining.

Basic Problems

Write each statement in symbolic form using the indicated letters to represent the corresponding simple statement.

2.2.21. Roses are red ($\boldsymbol{r}$), Violets are blue ($\boldsymbol{b}$)
2.2.22. I will travel by train ($\boldsymbol{t}$), plane ($\boldsymbol{p}$) or automobile ($\boldsymbol{a}$).
2.2.23. I will study art ($\boldsymbol{a}$) and music ($\boldsymbol{m}$) as a minor.
2.2.24. You may go to the movies ($\boldsymbol{m}$) if and only if you clean your room ($\boldsymbol{r}$).
2.2.25. If I win the lottery ($\boldsymbol{l}$), then I will buy you dinner ($\boldsymbol{d}$).

2.2.26. Give a verbal translation of each compound statement given $\boldsymbol{p}$ represents "I love Lucy" and $\boldsymbol{q}$ represents "Lucy is a research scientist."

a. $\boldsymbol{p \wedge q}$
b. $\boldsymbol{p \vee q}$
c. $\boldsymbol{\sim p \wedge q}$
d. $\boldsymbol{\sim (p \wedge q)}$
e. $\boldsymbol{\sim p \vee \sim q}$
f. $\boldsymbol{p \wedge \sim q}$
g. $\boldsymbol{\sim p \vee q}$

Write the negative of each statement.

2.2.27. Our coffee shop showed profit this year.
2.2.28. My cats name is Snowflake.
2.2.29. My cat is not a Siamese.
2.2.30. Peggy loves chocolate cake.
2.2.31. All positive integers are even
2.2.32. All rhombi are rectangles.
2.2.33. Some democrats are politicians.
2.2.34. The number 5 is a whole number.

2.2.35. Everyone likes ice cream.
2.2.36. There are no absolutes; that is nothing is certain.
2.2.37. All circles are round.

Determine the truth value of the given statements or give the conditions that will make the statement true.

2.2.38. $\mathbf{4+16=19}$ and Florida is in Canada.
2.2.39. Peter is a swimmer and Alex a boxer.
2.2.40. $\mathbf{4-6=0}$ and $\mathbf{5+5=10}$.
2.2.41. Sopoto is in Greece and Chris is from Sopoto.
2.2.42. $\mathbf{3-3=0}$ or $\mathbf{9+1=8}$.
2.2.43. Miami is in Mississippi or Newport is in Rhode Island.
2.2.44. The earth is round or the moon is square.
2.2.45. If Houston is in Texas, then Boston is in New York.
2.2.46. If $\mathbf{6+3=8}$ then $\mathbf{6-3=8}$.
2.2.47. If $\mathbf{5+3=8}$, then $\mathbf{6+2=8}$.
2.2.48. If the sun is shining, then it is cold.
2.2.49. If $\mathbf{6-11=-5}$, then Chris is Greek.
2.2.50. $\mathbf{6+6=12}$ if and only if $\mathbf{3+6=9}$.
2.2.51. Chicago is in Michigan if Tampa is in Florida.
2.2.52. Patras is in the Corinthian Gulf if and only if Peter is from Sopoto.
2.2.53. $\mathbf{11+4=15}$ if and only if $\mathbf{6-11=-5}$.
2.2.54. $\mathbf{6+11=17}$ is a sufficient condition for $\mathbf{6+8=12}$.
2.2.55. Honey is sweet implies Jan is sour.
2.2.56. $\mathbf{6+14=20}$ is a necessary condition for $\mathbf{3+8=11}$.
2.2.57. $\mathbf{9+22=31}$ only if $\mathbf{17+6=21}$.
2.2.58. If $\mathbf{16+22=38}$, then Alex is pretty.
2.2.59. San Francisco is in California.

2.3 TRUTH TABLES

$\wedge$ *And*

$\vee$ *Or*

$\sim$ *Not*

$\rightarrow$ *Implies, if …, then …*

$\leftrightarrow$ *… if and only if … "iff"*

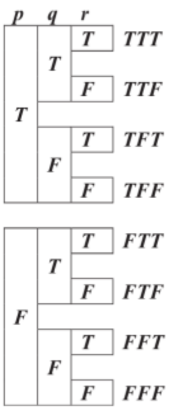
Often, in logical reasoning, complex compound statements are formed by the logical connectives such as conjunctive, disjunctive, negation, implication and bi-implications written symbolically:

$$\wedge, \vee, \sim, \rightarrow \text{ and} \leftrightarrow$$

along with simple statements $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}\ldots$ Our aim in this section is to learn how to determine the truth value of such a compound statement when the truth values of its components $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}\ldots$ are known. An effective way of doing this is by using truth tables.

Given $\boldsymbol{n}$ simple statements, then there are $\mathbf{2^n}$ different possible combinations of true values. This can be seen in previous examples; when there is a single statement as in Table 2.4 when the negation is considered, there are only $\mathbf{2^1=2}$ possibilities, either $\boldsymbol{p}$ is true or $\boldsymbol{p}$ is false. When there are two simple statements as in Table 2.1 when the conjunction of two simple statements is considered, there are $\mathbf{2^2=4}$ possibilities: **TT, TF, FT,** and **FF**. Notice, half the first are **T** and half are **F**; this is also true for the second, however they are "half"; that is, half the **T**'s for the first statement are **T** for the second statement and half for the second statement are **F** as well as half the **F**'s for the first statement are **T** for the second statement and half for the second statement are **F**.

Extending this "halving" technique, for three simple statements, there are $\mathbf{2^3=8}$ different possible combinations; for the first statement, four are **T** and four are **F**; for the four **T**'s statements, two are **T** and two are **F** which is



If I do not have to work today, then I will go to the store.

Not true is false and not false is true:

(a) **∼T≡F**
(b) **∼T≡F**
(c) **∼F≡T**
(d) **∼F≡T**

Fails only when premise is true and the conclusion does not follow:

(a) **∼T→T** is **T**
(b) **∼T→F** is **T**
(c) **∼F→T** is **T**
(d) **∼F→F** is **F**

TABLE 2.7 **Truth values for three simple statements**

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

the same for the **F**'s. Then, for each of the given possibilities, the last statement is either true or false (Table 2.7).

Example 2.3.1

Determine the truth table for $\sim p \to q$.

Solution

First we observe that this statement is composed of the simple statements $\sim p$ and q along with the connectives $\sim$ and $\to$. Since there are just two simple statements, p and q, involved, each one being either true or false, the truth table will have 4 rows, giving all possible truth-value combinations of p and q. We present this procedure in five steps, as illustrated below:

STEP 1: Construct the truth table for the given number of simple statements

p	q
T	T
T	F
F	T
F	F

STEP 2: We determine the truth values for $\sim p$ by using Table 2.4

p	q	$\sim p$
T	T	F
T	F	F
F	T	T
F	F	T

STEP 3: Lastly, we determine the truth values for $\sim p \to q$ using statement builder tables.

p	q	$\sim p$	$\sim p \to q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

Solution—cont'd

The last column was obtained by using the truth values of $\sim p$ and q along with the truth table in Property 2.2.5, defining the truth value of a conditional statement.

An alternative way of constructing the truth table is to reproduce this information as in the chart below allowing a column for each connective.

STEP 4: Reproduce information need in final statement.

$\sim p$	$\rightarrow$	q
F		T
F		F
T		T
T		F

STEP 5: Then use this information to construct the conditional statement $\sim p \rightarrow q$

$\sim p$	$\rightarrow$	q
F	T	T
F	T	F
T	T	T
T	F	F

The entire problem presented in a single table as follows:

1	2	3	4	6	5
p	q	$\sim p$	$\sim p$	$\rightarrow$	q
T	T	F	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Example 2.3.2

Construct a truth table for the $(p \vee q) \rightarrow p$

Solution

p	q	$p \vee q$	$(p \vee q) \rightarrow p$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Or, alternatively

1	2	3	5	4
p	q	$p \vee q$	$\rightarrow$	p
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

We note from the answer column indicates that $(p \vee q) \rightarrow p$ is false only when p is false and q is true; otherwise $(p \vee q) \rightarrow p$ is true.

If you have pride or money, then you have pride.

Hence, when you ***have pride*** *and* ***have money****, then you do have* ***pride****, this is true.*

Then ***when you don't have pride*** *but* ***you do have money****, then stating this implies* ***pride*** *is fallacious.*

When you ***have pride****, but* ***do not have money****, then you can conclude you have* ***pride****.*

However, when you ***don't have pride*** *and you* ***don't have money****, then stating "****If you have pride or money, then you have pride****" is not false, and is therefore true.*

If we may believe our ***logicians****, man is distinguished from all others creatures by the faculty of laughter*

The Spectator (1712)

If you know Math, English and Science, then you know Math or English.

Table 2.1

1	2	3
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2.2

1	2	3
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2.5

1	2	3
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Regardless of the validity of each simple statement, the statement is always true; hence, a tautology.

Example 2.3.3

Construct a truth table for the statement $[p \to (p \wedge q)] \vee \sim p$

Solution

p	q	$p \wedge q$	$p \to (p \wedge q)$	$\sim p$	$[p \to (p \wedge q)] \vee \sim p$
T	T	T	T	F	T
T	F	F	F	F	F
F	T	F	T	T	T
F	F	F	T	T	T

Or, alternatively

1	2	3	5	4	7	6
p	q	$[p$	$\to$	$p \wedge q]$	$\vee$	$\sim p$
T	T	T	T	T	T	T
T	F	T	F	F	F	F
F	T	F	T	F	T	T
F	F	F	T	F	T	T

➤ ***Step 5** is obtained from Steps 3 and 4, giving the truth value of* $p \to (p \wedge q)$*. We see that* $p \to (p \wedge q)$ *is false only when* p *is true and* $(p \wedge q)$ *is false. Table 2.4 is used to complete the column.*

➤ ***Step 7** is obtained by using the truth values in Steps 5 and 6 along with Table 2.2 for the disjunction. Note that the disjunction* $[p \to (p \wedge q)] \vee \sim p$ *is false only in the second line, where both* $p \to (p \wedge q)$ *and* $\sim p$ *are false.*

Example 2.3.4

Determine the truth table for $[(p \wedge q) \wedge r] \to (p \vee q)$

Solution

Since there are three simple statements forming this compound statement; namely, p, q, and r, the truth table will have $2^3 = 8$ rows.

Following the indicated steps carefully, using Table 2.1 for the conjunction $\wedge$, Table 2.2 for the disjunction $\vee$ and Table 2.5 for the conditional $\to$, we have

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$p \vee q$	$[(p \wedge q) \wedge r] \to (p \vee q)$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	F	F	T	T
F	T	F	F	F	T	T
F	F	T	F	F	F	T
F	F	F	F	F	F	T

Alternatively,

1	2	3	4	6	5	8	7
p	q	r	$[(p \wedge q)$	$\wedge$	$r]$	$\rightarrow$	$(p \vee q)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	F	F	T	T	T
T	F	F	F	F	F	T	T
F	T	T	F	F	T	T	T
F	T	F	F	F	F	T	T
F	F	T	F	F	F	T	F
F	F	F	F	F	F	T	F

An analysis of Step 8 reveals that the compound statement $[(p \wedge q) \wedge r] \rightarrow (p \vee q)$ is true, regardless of the truth values of p, q, and r. Step 8 was computed from Steps 6 and 7 by use of Table 2.5, which defines the conditional.

Example 2.3.5

Construct a truth table for $(p \wedge \sim q) \leftrightarrow r$.

Solution

Following the indicated steps carefully, we can write

p	q	r	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \leftrightarrow r$ OR $r \leftrightarrow (p \wedge \sim q)$
T	T	T	F	F	F
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	F
F	T	F	F	F	T
F	F	T	T	F	F
F	F	F	T	F	T

Alternatively,

1	2	3	4	6	8	7
p	q	r	$\sim q$	$p \wedge \sim q$	$\leftrightarrow$	r
T	T	T	F	F	F	T
T	T	F	F	F	T	F
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	F	T
F	T	F	F	F	T	F
F	F	T	T	F	F	T
F	F	F	T	F	T	F

Note that **Step 7** was obtained from **Steps 5** and **6**, making use of **Table 2.6** for the bi-conditional statement.

Example 2.3.6

Construct a truth table for the statement $(p \underline{\vee} q) \rightarrow (p \wedge \sim q)$

There is rain and not sunshine if and only if there are clouds in the sky.

Let ***p*** *represent "****there is rain,****"* ***q*** *represent "****the sun is shining****" and* ***r*** *represent "****there are clouds****."*

This statement is not true when:

There is ***sunshine*** *with the* ***rain,*** *regardless of the* ***clouds****; or, there is* ***no rain*** *when there are* ***clouds.***

Table 2.6

1	2	3
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

"You can have ice cream and not cake if and only if you finish your dinner"

This statement is false when: I have ***cake*** *or* ***ice cream without finishing my dinner****; this includes* ***cake and ice cream****; or I* ***don't have cake*** *and I* ***don't have ice cream*** *when I do* ***finish my dinner****.*

*"****If you want to drink either chocolate milk or soda, then you may have chocolate milk, but not a soda.****" This statement is false when: I have the* ***soda*** *and* ***not chocolate milk.***

If there is either a cheaper guitar or a payment plan r, then I will buy a cheaper guitar and not take the payment plan. *This statement is false when: There is* ***not a cheaper guitar*** *and I do* ***take the payment plan.***

Solution

Following the indicated steps we have

p	q	$p\veebar q$	$\sim q$	$p\wedge\sim q$	$(p\veebar q)\rightarrow(p\wedge\sim q)$
T	T	F	F	F	T
T	F	T	T	T	T
F	T	T	F	F	F
F	F	F	T	F	T

Or, alternatively

1	2	3	7	4	6	5
p	q	$(p\veebar q)$	$\rightarrow$	$(p$	$\wedge$	$\sim q)$
T	T	F	T	T	F	F
T	F	T	T	T	T	T
F	T	T	F	F	F	F
F	F	F	T	F	F	T

$p\Delta q \equiv \sim(p\rightarrow q)$

Let ***p*** *represent "****John is smart****,"* ***q*** *represent "****Leroy is tall****" and* ***r*** *represent "****Sherrie is bashful****."*

It is the logic of our times, no subject for immortal verse – that we who lived by honest dreams. Defend the bad against the worse

C. Day-Lewis
Anglo-Irish Poet (1943)

Example 2.3.7

Let the connective **Δ** be defined by the truth table,

p	q	$p\Delta q$
T	T	F
T	F	F
F	T	T
F	F	F

Find the truth value for $p\rightarrow[\sim p\Delta(\sim q\wedge p)]$

p	q	$\sim p$	$\sim q$	$\sim q\wedge p$	$\sim p\Delta(\sim q\wedge p)$	$p\rightarrow[\sim p\Delta(\sim q\wedge p)]$
T	T	F	F	F	F	F
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	T	F	F	T

Alternatively,

1	2	3	9	4	8	5	7	6
p	q	p	$\rightarrow$	$[\sim p$	Δ	$(\sim q$	$\wedge$	$p)]$
T	T	T	T	F	F	F	F	T
T	F	T	T	F	T	T	T	T
F	T	F	F	T	F	F	F	F
F	F	F	T	T	F	T	F	F

Step 7 giving the truth values of $\sim q\wedge p$ was obtained from **Steps 5 and 6**, making use of Table 2.1 for $\wedge$. The truth values for $\sim p\Delta(\sim q\wedge p)$ were obtained from **Steps 4 and 7** and the truth table for the connective $\wedge$ given at the beginning of this problem. Finally, the answer in **Step 9** was derived from **Steps 3 and 8**, making use of Table 2.5 for the conditional statement.

Example 2.3.8

Determine the truth value of the following statement: "**John** is smart or **Leroy** is tall if and only if **Sherrie** is bashful implies **John** is not smart." Given that **John** is smart, **Leroy** is not tall and Sherrie is bashful.

Solution

First, we note that "**John** is smart or **Leroy** is tall" can be written symbolically as

$$p \vee q$$

Secondly, we can write the statement "**Sherrie** is bashful implies **John** is not smart" as

$$r \rightarrow \sim p.$$

Thirdly, joining the two statements with the bi-conditional "if and only if," we can represent the entire statement as

$$(p \vee q) \leftrightarrow (r \rightarrow \sim p)$$

Now, since $(p \vee q)$ is true when p is true and q is false and $(r \rightarrow \sim p)$ is false when r is true and p is true, we can write

$$(T \vee F) \leftrightarrow (T \rightarrow \sim T)$$
$$(T) \leftrightarrow (T \rightarrow F)$$

This leads us to $T \leftrightarrow F$, which means that the given statement is *false*.

EXERCISES

Critical Thinking

2.3.1. How many rows are required in a truth table for a compound statement that contains four simple statements, $p, q, r,$ and s?

2.3.2. How many rows are required in a truth table for a compound statement that contains five simple statements, p, q, r, s and t?

Basic Problems

Construct the truth table for the following statements:

2.3.3. $\sim q \wedge p$

2.3.4. $\sim (p \wedge q)$

2.3.5. $\sim (\sim p \vee \sim q)$

2.3.6. $p \wedge (q \vee p)$

2.3.7. $(p \vee q) \rightarrow p$

2.3.8. $(p \wedge q) \rightarrow p$

2.3.9. $(p \vee q) \rightarrow (p \wedge q)$

2.3.10. $[(p \rightarrow q) \vee p]$

2.3.11. $[(p \wedge q) \vee q] \rightarrow p$

2.3.12. $[(p \rightarrow q) \vee \sim p] \rightarrow q$

2.3.13. $p \vee (q \wedge r)$

2.3.14. $(p \vee q) \wedge (p \vee r)$

2.3.15. $p \Delta \sim q$

2.3.16. $\sim p \Delta q$

Determine the truth value of the following statements given p is true, q is false and r is true.

2.3.17. $p \rightarrow (q \vee r)$

2.3.18. $p \wedge (q \vee r)$

2.3.19. $p \vee \sim p$

2.3.20. $p \wedge \sim (q \rightarrow r)$

2.4 PROPERTIES OF LOGIC

In certain cases of logical reasoning we are concerned with statements that are always true; for example, $p \vee \sim p$ is always *true*. If p is true then we have $T \vee F$, which is true; however, if p is false then we have $F \vee T$, which is true. More specifically, consider the following definition:

Definition 2.4.1 Tautology

A compound statement, τ, is said to be a **tautology** or **logically true** if it is true for all possible truth values of its components.

In other words a **tautology,** τ, is a compound statement which has only true, **T,** in the last column of its truth table. Note that in **Example 2.16** the $[(p \wedge q) \wedge r] \rightarrow (p \vee q)$ is a **tautology**.

Example 2.4.1 Tautology

Show that the statement $\sim(p \wedge q) \vee p$ is a tautology.

Tautology

From the Greek word **ταυτολογία** *meaning a formula which is* ***unconditional true***, ***tautology*** *was first applied by* ***Ludwig Wittgenstein*** *to redundancies of propositional logic in 1921.*

A tune is a kind of tautology … complete in itself
Wittgenstein

If you do ***not*** *have both* ***p and q***, *then you are missing at least one; hence, you either* ***don't*** *have* ***p*** *(or* ***not q***), *or you do have* ***p***—*this is always true. With anything, you either you have it, or your do not, this is an unconditional truth.*

If you don't know, then you know.
You must clean both your room and the kitchen, but you don't have to do at least one of these.

Solution

Constructing the truth table for the statement $\sim(p \wedge q) \vee p$, we have

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim(p \wedge q) \vee p$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Since the truth table for the statement $\sim(p \wedge q) \vee p$ is true, **T**, for all possible truth value combinations of p and q, the statement is a **tautology**.

Analogous to **tautologies**, there are statements that are logically false; for example, $p \wedge \sim p$ is always *false*. If p is true then we have $T \wedge F$, which is false; however, if p is false then we have $F \wedge T$, which is false.

That is,

Definition 2.4.2 Contradiction

A compound statement, ϕ, is said to be a **self-contradiction** or **logically false** if it is false for all possible truth values of its components.

This means that a self-contradiction, φ, is a compound statement that has only false, **F**, in the last column of its truth table. Consider the following example:

Example 2.4.2 Self-Contradiction

Show that the statement $(p \wedge q) \wedge \sim(p \vee q)$ is a self-contradiction.

1	2	3	4	5	6
p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Observe that Step 6 was obtained from Steps 3 and 5, making use of Table 2.1 for the conjunction. Since only **F** appears in Step 6, we conclude that the statement $(p \wedge q) \wedge \sim(p \vee q)$ is a self-contradiction.

Definition 2.4.3 Paradox

A **paradox** is an apparently true statement or group of statements that leads to a contradiction.

For example, the liars' **paradox**: "this statement is false"; when this statement is true, this implies it is false or vice versa.

In logical reasoning we often encounter statements that are the same or equivalent.

That is,

Definition 2.4.4 Equivalent

Two statements r and s are said to be **logically equivalent** or simply **equivalent** if they have identical truth tables; that is, if $r \leftrightarrow s$ is a tautology. To symbolize two equivalent statements r and s, we write $r \equiv s$ or $r \Leftrightarrow s$.

Similarly, we write $p \Rightarrow q$ when the statement $p \rightarrow q$ is a **tautology**.

Example 2.4.3 Equivalent Statements

Show that the statement $\sim p \vee q$ is equivalent to the statement $p \rightarrow q$.

Solution

We begin by constructing the truth table for $(\sim p \vee q) \leftrightarrow (p \rightarrow q)$.

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow q$	$(\sim p \vee q) \leftrightarrow (p \rightarrow q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since $(\sim p \vee q) \leftrightarrow (p \rightarrow q)$ is a tautology, we see that $(\sim p \vee q) \equiv (p \rightarrow q)$.

*When you do both, then you did not listen when I stated you **don't have to do at least one**, and when you don't do both, you did not listen to the first statement which stated do **both your room and the kitchen**.*

Archival Note

*The **Epimenides paradox** (circa 600 BCE), is a liar's paradox; what do you think form a man from **Crete** when he states "Cretans are always liars."*

Syntactically

$r \equiv s$

Semantically

$r \Leftrightarrow s$

I will not win the lottery ticket or I will take you to dinner** is equivalent to stating **if I win the lottery, then I will take you to dinner.

Lottery

*From the Italian **lotteria** meaning arrangement for a distribution of prizes by chance*

If the sun is shining, then I will go swimming is equivalent to stating if I don't go swimming, then the sun was not shining.

If you are seventeen, then you are a minor is equivalent to stating if you are not a minor, then you are not seventeen.

If you have an A in this course, then you understand logic is equivalent to stating if you do not know logic, then you will not have an A in this course.

***If you are seventeen, then you are a minor** is NOT equivalent to stating **if you are a minor, then you are seventeen.** There are many ways to be a minor and not be seventeen.*

Minor

A person under the age of full legal responsibility

Example 2.4.4 Contrapositive: $\sim q \rightarrow \sim p$

Show that $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$.

Solution

It suffices to show that the statement $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is a tautology. We can show this by structuring the appropriate truth table.

That is,

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Example 2.4.5 Converse: $q \rightarrow p$

Show that the statement $p \rightarrow q$ is not logically equivalent to the statement $q \rightarrow p$.

If the figure is a square, then the rectangle *is NOT equivalent to stating* ***if the figure is a rectangle, then the figure is a square.*** *A rectangle has equal angles whereas a square has both equal angles and equal sides.*

Given it is not the case that if you are an adult, then you do drive*, this is equivalent to stating* ***there are adults out there that do not drive.***

Solution

We begin by constructing the truth table for the statement $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$.

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

It is clear from the results of Step 5 that the statement $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is not a tautology; thus

$$(p \rightarrow q) \not\equiv (\sim q \rightarrow \sim p),$$

where $\not\equiv$ is read "not equivalent."

Example 2.4.6 Not Implied: $\sim (p \rightarrow q)$

Is the statement $\sim (p \rightarrow q)$ is equivalent to the statement $p \wedge \sim q$?

Solution

We begin by constructing the truth table for the statement $\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q)$.

1	2	3	4	5	6	7
p	q	$\sim q$	$p \rightarrow q$	$\sim (p \rightarrow q)$	$p \wedge \sim q$	$\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q)$
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	F	T	F	F	T
F	F	T	T	F	F	T

We observe that the truth table for the statement $\sim(p \rightarrow q)$ in **Step 5** is the same as the truth table for $p \wedge \sim q$ in Step 6. Thus, we conclude that $\sim(p \rightarrow q) \equiv (p \wedge \sim q)$. This means that the negation of the conditional statement, $p \rightarrow q$, is equivalent to $p \wedge \sim q$. Moreover, using this equivalence in conjunction with the equivalence in Example 2.4.3, $\sim(\sim p \vee q) \equiv (p \wedge \sim q)$; this distributive property called De Morgan's Laws.

Example 2.4.7 Equivalent Statements

Show that $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

Solution

The equivalence relation of these two statements can be seen by constructing the following truth table:

1	2	3	4	5	6	7	8
p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$
T	T	T	T	F	T	F	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	T	T
F	T	T	F	F	T	F	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	F	F	F	F	F	F	T

Since the truth table for $(p \wedge q) \wedge r$ in Step 5 is identical to that $p \wedge (q \wedge r)$ in **Step 7**, we conclude that, indeed, the two statements are equivalent. This is further seen in **Step 8** which shows that $(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$ is a **tautology**.

The statements that we have studied under the equivalence relation, $\equiv$, satisfy some very basic laws of algebra listed below. We shall state these laws and illustrate their usefulness with some examples. In order to prove any of these laws of algebra it is sufficient to construct a truth table to show that the given equivalence statement is indeed true.

Let p, q, and r represent given statements.

Rule 2.4.1 Idempotent

$$p \vee p \equiv p \text{ and } p \wedge p \equiv p$$

Idempotent describes the property of operations, as in mathematics and computer science, which yield the same result after the operation is applied multiple times.

Given it is not the case that if you try to get pregnant that you will get pregnant *is equivalent to stating* ***there are those out there who try to get pregnant and do not get pregnant.***

Given it is not the case that having money implies happiness *is equivalent to stating* ***there are those who have money, but are not happy.***

Commutative

From Latin ***commutate*** *to change*

Involving the quality that quantities connected by operators give the same result when commuted

$$a \times b = b \times a$$

Tautology

From the Greek ***tautologia****—the same saying*

A statement that is true by necessity or by virtue of its logical form

Idempotence

Applied a multiple number of times without change

Example:

$$f(x) = x$$
$$f(f(x)) = f(x) = x$$
$$\vdots$$

Comparable to Association in Addition and Multiplication

$$(a+b)+c=a+(b+c)$$
$$(a\times b)\times c=a\times(b\times c)$$

Comparable to Commutative in Addition and Multiplication

$$a+b=b+a$$

$$a\times b=b\times a$$

Not directly comparable to Distributive in Addition and Multiplications *multiplication is distributive over addition, but not the reverse.*

Whereas in Logic, conjunction distributions over disjunction and disjunction distributions over conjunction.

Not exactly comparable to the Identity in Addition and Multiplication *as the last statement would not follow in addition*

$a+0=a$ ✓
$a\times 1=a$ ✓
$a\times 0=0$ ✓
$a+1=a$ ✗

Directly comparable to maximum of 0/1 versus minimum 0/1; that is,
$a=0$ (false) and $a=1$ (true):

$max\{a, 0\}=a$ ✓
$min\{a, 1\}=a$ ✓
$min\{a, 0\}=0$ ✓
$max\{a, 1\}=1$ ✓

or ≡ *addition*
and ≡ *multiply*

Rule 2.4.2 Associative

$$(p\vee q)\vee r\equiv p\vee(q\vee r) \text{ and } (p\wedge q)\wedge r\equiv p\wedge(q\wedge r)$$

Associative describes the property of operations that enables statements to re-associate while yielding the same result. The associative law in logic is comparable to the associative law of addition and multiplication in algebra; that is, $(a+b)+c=a+(b+c)$ and $(a\times b)\times c=a\times(b\times c)$.

Rule 2.4.3 Commutative

$$p\vee q\equiv q\vee p \text{ and } p\wedge q\equiv q\wedge p$$

Commutative describes the property of operations that enables statements to move or commute while yielding the same result. The commutative law in logic is comparable to the commutative law of addition and multiplication in algebra; that is, $a+b=b+a$ and $a\times b=b\times a$.

Rule 2.4.4 Distributive

$$p\vee(q\wedge r)\equiv(p\vee q)\wedge(p\vee r) \text{ and } p\wedge(q\vee r)\equiv(p\wedge q)\vee(p\wedge r)$$

Distributive describes the property of one operator to be expanded in a particular way which yield the same result; that is, an equivalent expression. The distributive law in logic is more extensive than the distributive property in algebra. In logic, the operator for "and" is distributive over the operator for "or" and vise verse, whereas in algebra, this relation only works for "multiplication" over "addition" but not "addition over multiplication." However, the idea of distribution of one operator over a second operator is comparable; for multiplication over addition we have $a(b+c)=ab+ac$ and for power over multiplication we have $(a\times b)^n=a^n\times b^n$.

Rule 2.4.5 Identity

If τ is a **tautology** and φ is any **contradiction**, then

$$p\vee\phi\equiv p,\ p\wedge\tau\equiv p,\ p\wedge\phi\equiv\phi \text{ and } p\vee\tau\equiv\tau$$

Identity is the state or fact of remaining the same one or ones, as under varying aspects or conditions, to identify. Hence, a contradiction in a disjunction identifies the truth value of the statement p, but in conjunction identifies the contradiction. A tautology in disjunction identifies the tautology, whereas in conjunction identifies the true value of the statement p. The first two are comparable to the identity property of addition and multiplication: $a+0=a$ and $a\times 1=a$, where 0 is compared to the contradiction and 1 is compared to the tautology; therefore, "or" is comparable to "addition" and "and" is comparable to "multiplication." The third is comparable to zero property, $a\times 0=0$; however, the four as will the law of distribution is not exactly comparable to any given algebraic property or law. These comparisons will come back into play in Chapter 3, which introduces the fundamental principle of counting; basically "and" means "multiply" (when there are more than one independent events) and "or" means "add" (minus overlap).

Rule 2.4.6 Complementary

If τ is a **tautology** and ϕ is any **contradiction**, then

$$p \vee \sim p \equiv \tau,\ p \wedge \sim p \equiv \phi,\ \sim(\sim p) \equiv p,\ \sim \tau \equiv \phi \text{ and } \sim \phi \equiv \tau$$

Directly comparable to maximum of 0/1 versus minimum 0/1; that is, $a=0$ (false) and $a=1$ (true); if p is a, then $\sim p$ is $1-a$:

$$\max\{a, 1-a\} = 1$$
$$\min\{a, 1-a\} = 0$$
$$1-1=0$$
$$1-0=1$$

A **complement** is the part needed to make complete or perfect; in **logic**, this is the relationship between ***true*** and ***false***, an event and not the event, etc. In **logic**, the statement is true or the statement is false and thus $p \vee \sim p$ is a **tautology**, τ; however, a statement cannot be both true and false; hence, $p \wedge \sim p$ is a contradiction, ϕ. Furthermore, as in English, a double negative is the positive statement; not "not p" then p. Finally, logically speaking, the negation of a **tautology** is a **contradiction** and the negation of a **contradiction** is a **tautology**. They complement each other, without tautologies, contradictions would not exist.

Rule 2.4.7 De Morgan's Rule

$$\sim(p \vee q) \equiv \sim p \wedge \sim q \text{ and } \sim(p \wedge q) \equiv \sim p \vee \sim q$$

De Morgan's Rule illustrates the fact that "or" is the complement of "and" and vice versa. In logic, "or" means at least one and "and" means both; hence, when ***you do not have at least one***, $\sim(p \vee q)$, this is equivalent to you do not have either, or you are missing both, $\sim p \wedge \sim q$; ***you do not have the first and you do not have the second***. Similarly, when ***you do not have both***, $\sim(p \wedge q)$, this is equivalent to you are missing at least one, $\sim p \vee \sim q$; ***you do not have the first or you do not have the second***. The idea of "negation" will be discuss further in Section 2.6.

It is left to the student as an exercise to construct the truth tables to show the stated equivalence of the preceding laws of algebra. However, we use these laws to simplify compound statements.

De Morgan was a British mathematician and logician who formulated De Morgan's laws and introduced rigor to mathematical induction.

This will be similar in sets or events:
When you do not have at least one, then you do not have the first and you do not have the second.

When you do have both, you are missing at least one; either the first one or the second one (or you are missing both), you just do not have both.

Example 2.4.8 Properties of Logic

Simplify each of the following statements by using the laws of statements and using t for **tautologies** and ϕ for **contradictions**:

(a) $(p \vee q) \wedge \sim p$ **(b)** $p \vee (p \wedge q)$
(c) $[\sim(p \vee q)] \vee (\sim p \wedge q)$ **(d)** $[\sim(p \wedge q)] \vee (p \wedge \sim q)$

Let's use the following laws:
- *Commutative*
- *Distribution*
- *Complement*
- *Identity*
- *De Morgan's Law*

Solution

	Statement (a)	Reason
1.	$(p \vee q) \wedge \sim p \equiv \sim p \wedge (p \vee q)$	**Commutative Law**
2.	$\equiv (\sim p \wedge p) \vee (\sim p \wedge q)$	**Distributive Law**
3.	$\equiv \phi \vee (\sim p \wedge q)$	**Complement Law**
4.	$\equiv (\sim p \wedge q)$	**Identity Law**

I will do the dishes or wash the car, but I will not do the dishes *is equivalent to saying* ***I will not do the dishes and I will wash the car***.

I have ice cream, or I have ice cream and cake *is equivalent to* ***I have ice cream, and I have ice cream or cake***.

Either I don't have at least one (p or q) or I am missing the first (p) and have the second (q) *is equivalent to* ***I don't have the first (p)***. *As for the second, I either have it or I do not—it is irrelevant.*

Either I do not have both (p and q) or I have the first (p) and not the second (q) *is equivalent to stating* ***I do not have both the first and second.***

I lost my pen and paper, or I have the pen and not the paper *is equivalent to stating* ***I lost either the pen or I lost the paper***; *that is,* ***I do not have both the pen and paper.***

	Statement (b)	Reason
1.	$p \vee (p \wedge q) \equiv (p \vee p) \wedge (p \vee q)$	**Distributive Law**
2.	$\equiv p \wedge (p \vee q)$	**Identity Law**

	Statement (c)	Reason
1.	$[\sim(p \vee q)] \vee (\sim p \wedge q)$ $\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$	**De Morgan's Law**
2.	$\equiv \sim p \wedge (\sim q \vee q)$	**Distributive Law**
3.	$\equiv \sim p \wedge \tau$	**Complement Law**
4.	$\equiv \sim p$	**Identity Law**

	Statement (d)	Reason
1.	$[\sim(p \wedge q)] \vee (p \wedge \sim q)$ $\equiv (\sim p \vee \sim q) \vee (p \wedge \sim q)$	De Morgan's Law
2.	$\equiv [(\sim p \vee \sim q) \vee p] \wedge [(\sim p \vee \sim q) \vee \sim q]$	Distributive Law
3.	$\equiv [p \vee (\sim p \vee \sim q)] \wedge [(\sim p \vee \sim q) \vee \sim q]$	Commutative Law
4.	$\equiv [(p \vee \sim p) \vee q] \wedge [\sim p \vee (\sim q \vee \sim q)]$	Associative Law
5.	$\equiv [\tau \vee \sim q] \wedge [\sim p \vee (\sim q \vee \sim q)]$	Complement Law
6.	$\equiv [\tau \vee \sim q] \wedge [\sim p \vee \sim q]$	Idempotent Law
7.	$\equiv \tau \wedge [\sim p \vee \sim q]$	Identity Law
8.	$\equiv \sim p \vee \sim q$	Identity Law
9.	$\equiv \sim(p \wedge q)$	De Morgan's Law

EXERCISES

Summary

➲ Idempotent Law

$$p \wedge p = p \qquad p \vee p = p$$

➲ Associative Law

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r p \wedge p = p \qquad p \vee (q \vee r) = (p \vee q) \vee r p \vee p = p$$

➲ Commutative Law

$$p \wedge q = q \wedge p \qquad p \vee q = q \vee p$$

➲ Distributive Laws

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee q) \qquad p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

➲ Identity Law

$$p \vee \phi = p, \quad p \wedge \tau = p, \quad p \wedge \phi = \phi, \quad p \vee \tau = \tau$$

➲ Complement Law

$$p \vee \sim p = \tau, \quad p \wedge \sim p = \phi, \quad \sim(\sim p) = p, \quad \sim \tau = \phi, \quad \sim \phi = \tau$$

➲ De Morgan's Laws

$$\sim(p \vee q) = \sim p \wedge \sim q, \quad \sim(p \wedge q) = \sim p \vee \sim q$$

Critical Thinking

2.4.1. Is the statement $\sim(p \vee q) \wedge p$ a tautology?
2.4.2. Show that $p \vee \sim(p \wedge q)$ is a tautology.
2.4.3. Determine whether or not the statement $(\sim p \wedge \sim q) \vee (p \vee q)$ is a tautology.
2.4.4. Is the statement $(p \vee q) \vee \sim(p \wedge q)$ a tautology?
2.4.5. Determine whether or not the statement $(p \vee q) \rightarrow \sim(p \wedge q)$ is a self-contradiction.
2.4.6. What can you say about the two statements $q \rightarrow p$ and $\sim p \rightarrow \sim q$

2.4.7. Show that the statement $(p \wedge q) \vee \sim (p \vee q)$ is a self-contradiction.
2.4.8. Is the statement $p \rightarrow q$ logically equivalent to the statement $q \rightarrow \sim p$?
2.4.9. Prove that $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
2.4.10. Show that $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$.
2.4.11. Is the statement $p \vee q$ equivalent to $\sim(\sim p \wedge \sim q)$?
2.4.12. Verify the associative law $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ by constructing the appropriate truth table.
2.4.13. Prove the distributive law $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

Basic Problems

2.4.14. Show that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.
2.4.15. What can you say about the statements $\sim(\sim p)$ and p?
2.4.16. Simplify the statement $(p \wedge q) \wedge \sim p$.
2.4.17. Using the laws of algebra simplify the statement $p \wedge (p \wedge q)$.
2.4.18. Simplify $\sim(p \wedge q) \wedge (\sim p \wedge q)$.
2.4.19. Simplify $\sim(\sim p \wedge q)$.

Determine if the following statement is a **tautology**, **self-contradiction** and **paradox**.

2.4.20. There is an exception to every rule; except this rule.
2.4.21. To be or not to be.
2.4.22. You travel back in time and kill your grandfather before he meets your grandmother. Hence, you are never born and, therefore, you couldn't go back in time and kill your grandfather.
2.4.23. Yes and no, and I don't mean maybe.

2.5 VARIATIONS OF THE CONDITIONAL STATEMENT

We have seen that equivalent statements have identical truth tables and may be thought of as different forms of the same statement. In this section we shall be concerned with some of the different forms by which the conditional statement $p \rightarrow q$ can be expressed. That is, given a conditional statement $p \rightarrow q$, we shall study three variations formed from statements p, q, and the logical connectives, $\rightarrow$ and $\sim$.

Statement	Name
$p \rightarrow q$	Conditional Statement
$q \rightarrow p$	**Converse** of $p \rightarrow q$
$\sim p \rightarrow \sim q$	**Inverse** of $p \rightarrow q$
$\sim q \rightarrow \sim p$	**Contra-positive** of $p \rightarrow q$

A comparison of the truth tables for these statements is given by Table 2.8.

TABLE 2.8 Conditional variations

1	2	3	4	5	6	7	8
p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Conditional

Subject to one or more requirements

Sentences that discuss factual implication

Example:

If you are seventeen then you are a minor.

If you are a minor, then you are seventeen.

If you are not seventeen, then you are not a minor.

If you are not a minor, then you are not seventeen.

Given the conditional statement is true, the inverse and the converse need not follow.

If the converse and the inverse do hold, this would be a bi-conditional.

Common: Equivalent Statements

Analyzing Table 2.8 on the previous page we observe that

1. $p \rightarrow q$ is not equivalent to $q \rightarrow p$ because Columns 5 and 6 are **not identical**.
2. $p \rightarrow q$ is not equivalent to $\sim p \rightarrow \sim q$ because Columns 5 and 7 are **not identical**.
3. $p \rightarrow q$ is equivalent to $\sim q \rightarrow \sim p$ because Columns 5 and 8 are **identical**.
4. $q \rightarrow p$ is equivalent to $\sim p \rightarrow \sim q$ because Columns 6 and 7 are **identical**.

Therefore, there are two equivalences: $p \rightarrow q \equiv \sim q \rightarrow \sim p$ and similarly $q \rightarrow p \equiv \sim p \rightarrow \sim q$. Hence, a conditional statement is equivalent to the counter of the positive statement, that is, the contra-positive.

Example 2.5.1 Contra-positive

Given the conditional statement "If you are seventeen, then you are a minor," find the ***converse***, ***inverse***, and the ***contra-positive*** of the statement.

Solution

Converse: If you are a minor, then you are seventeen.
Inverse: If you are not seventeen, then you are not minor.
Contra-positive: If you not a minor, then you are not seventeen.

By definition of a minor, both the original conditional statement and the contra-positive statement both true (that is, equivalent statements) whereas, the converse and inverse are not true. If you are a minor, you might be twelve and not seventeen; and if you are not seventeen, then you might be sixteen which is still a minor. Hence, in this case the conditional statement and the contra-positive statement are true and the converse and the inverse are false; however, this need not be the case. If all four statements are true, then the conditional statement is actually a bi-conditional statement. However, in general, $\rightarrow$ is not commutative; that is, $p \rightarrow q \not\equiv q \rightarrow p$.

Example 2.5.2 Conditional Statements

Suppose that p is true and q is false. What is the truth value of the following statements?

(a) $p \rightarrow q$
(b) The inverse of $\sim q \rightarrow p$
(c) The contra-positive of $\sim q \rightarrow \sim p$
(d) q is sufficient for $\sim p$
(e) $\sim p$ is necessary for $\sim q$
(f) The converse of $\sim q$ only if p

Solution

(a) For $p \rightarrow q$ we have $T \rightarrow F$, which is false by **Table 2.8**.
(b) The **inverse** of $\sim q \rightarrow p$ is $\sim(\sim q) \rightarrow \sim p$ or **equivalently** $q \rightarrow \sim p$. Thus, we have $F \rightarrow \sim T$ or $F \rightarrow F$, which is vacuously true.
(c) The **contra-positive** of $\sim q \rightarrow \sim p$ is $\sim(\sim q) \rightarrow \sim(\sim p)$ or equivalently $q \rightarrow p$. Hence, $F \rightarrow T$, which is vacuously true.
(d) The statement "q is sufficient for $\sim p$" is another way of saying "if q then $\sim p$" or, symbolically $q \rightarrow \sim p$. Thus, we have $F \rightarrow \sim T$, which is equivalent to $F \rightarrow F$, which is vacuously true.
(e) The statement "$\sim p$ is necessary for $\sim q$" is another way of saying "if you have $\sim q$, then it was necessary that you had $\sim p$"; in other words, "if $\sim q$ then $\sim p$" or written symbolically $\sim q \rightarrow \sim p$. Hence, $\sim F \rightarrow \sim T$, which gives us $T \rightarrow F$, which is false. Alternatively, $\sim q \rightarrow \sim p$ is the contra-positive of $p \rightarrow q$ which, by (a) is false.
(f) "$\sim q$ only if p" can be written as $\sim q \rightarrow p$ and therefore the converse is $p \rightarrow \sim q$. Since p is true and q is false, we have $T \rightarrow \sim F$, that is, $T \rightarrow T$, which is true.

If I win the lottery, then I will buy you dinner.

If I buy you dinner, then I will win the lottery? *This is not equivalent.*

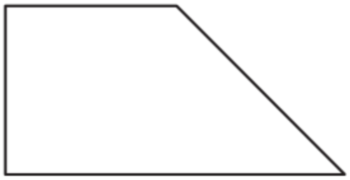
If I do not win the lottery, then I cannot buy you dinner? *This does not follow.*

If I do not take you to dinner, then I have not won the lottery. *This is true.*

Among the three different forms statements, the first two are equivalent to each other.

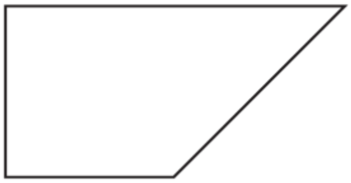
Conditional Statement
$p \rightarrow q$
Converse
$q \rightarrow p$
Inverse
$\sim p \rightarrow \sim q$
Contrapositive
$\sim q \rightarrow \sim p$

Original Statement:
A well supported figure.



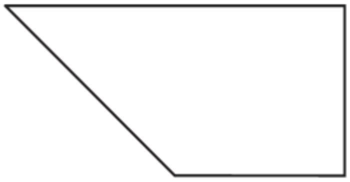
Inverse Statement:

Not the same support



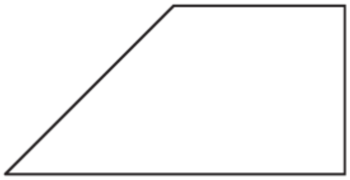
Converse Statement:

Not the same support as the original, but the same as the inverse.



Contrapositive Statement:

Same support as the original figure.



Example 2.5.3 Associative

Show that the associative rule does not hold for $\rightarrow$; that is, $(p\rightarrow q)\rightarrow r\not\equiv p\rightarrow(q\rightarrow r)$.

Solution

Here, to prove that $(p\rightarrow q)\rightarrow r$ is not equivalent to $p\rightarrow(q\rightarrow r)$.

This is shown by structuring the following truth table:

1	2	3	4	5	6	7	8
p	q	r	$p\rightarrow q$	$q\rightarrow r$	$(p\rightarrow q)\rightarrow r$	$p\rightarrow(q\rightarrow r)$	$(p\rightarrow q)\rightarrow r \leftrightarrow p\rightarrow(q\rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	F

Since **Steps 6** and **7** do not yield identical truth tables for $(p\rightarrow q)\rightarrow r$ and $p\rightarrow(q\rightarrow r)$, we observe that these two statements are not equivalent; that is, $(p\rightarrow q)\rightarrow r\leftrightarrow p\rightarrow(q\rightarrow r)$ is not a tautology. Thus $\rightarrow$ is not associative.

$$p\rightarrow q\equiv q\vee\sim p$$
$$\&$$
$$q\rightarrow r\equiv r\vee\sim q$$

are equivalent to

$$(p\rightarrow q)\equiv(q\vee\sim p)\rightarrow r$$
$$\&$$
$$p\rightarrow(q\rightarrow r)\equiv$$
$$p\rightarrow(r\vee\sim q)$$

therefore

$$LHS=r\vee\sim(q\vee\sim p)$$
$$=r\vee\sim q\wedge p$$

and

$$RHS=(r\vee\sim q)\vee\sim p$$
$$=r\vee\sim q\vee\sim p$$

If you have significant debt than you should not by more cloths is equivalent to If you have money to buy new cloths, then you are not in significant debt.

If you have a National Merit Scholarship, then you can afford to go to university in the State of Florida or out of State is equivalent to If you cannot afford to go to university out of state and you cannot afford to go to university in the State of Florida, then you must not have a National Merit Scholarship.

Example 2.5.4 Equivalency

Show that the following statements are equivalent without the use of truth tables:

(a) $p\rightarrow\sim q$ and $\sim q\rightarrow p$

(b) $p\rightarrow(q\vee p)$ and $(\sim q\wedge\sim r)\rightarrow\sim p$

Solution

(a) The contra-positive of $p\rightarrow\sim q$ is $\sim q\rightarrow\sim(\sim p)$, which we know is equivalent to $\sim q\rightarrow p$

(b) The contra-positive of $p\rightarrow(q\vee p)$ is $\sim(q\vee r)\rightarrow\sim p$ and since by De Morgan's Law, $\sim(q\vee r)\equiv\sim q\wedge\sim r$; hence, $p\rightarrow(q\vee p)$ is equivalent to $(\sim q\wedge\sim r)\rightarrow\sim p$.

In logical reasoning it is often necessary to prove a statement of the form "p if and only if q"; that is, $p\leftrightarrow q$. A truth table may be used to show that $p\leftrightarrow q$ is equivalent to $(p\rightarrow q)\wedge(q\rightarrow p)$; that is, $p\leftrightarrow q\equiv(p\rightarrow q)\wedge(q\rightarrow p)$.

EXERCISES

Critical Thinking

Given the logical statement, give the **converse**, **inverse**, and **contra-positive**.

2.5.1. "If John is blonde then Sandra is a brunette"
2.5.2. Given the logical statement "If Maria leaves for the moon then Liz will be going to Albuquerque"

Let p and q represent two statements where p is **false** and q is **true**. Determine the truth value of the following statements:

2.5.3. $p \rightarrow q$,
2.5.4. The inverse of $\sim p \rightarrow q$,
2.5.5. The converse of $\sim p$ only if q.

Let p and q represent two statements where p is **false** and q is **true**. Determine the truth value of the following statements:

2.5.6. $p \rightarrow q$
2.5.7. The inverse of $\sim p \rightarrow q$,
2.5.8. The converse of $\sim p$ only if q.

Show that the logical statements are **equivalent**.

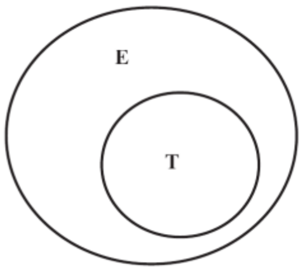
2.5.9. $\sim(p \vee q) \vee (p \wedge \sim q)$ and $\sim q$
2.5.10. $p \veebar q$ and $(p \vee q) \wedge \sim(p \wedge q)$

Archival Note
*Categorical propositions are discussed in Aristotle's **Prior Analytics**. These types of propositions occur in categorical syllogisms.*

Syllogism
From the Greek **συλλογισμός** *meaning to conclude or infer*

A discourse in which, certain things having been supposed, something different from the things supposed results of necessity because these things are so.
Aristotle

*Leonhard Euler was a Swiss mathematician and physicist working in such fields as mechanics, fluid dynamics, optics and astronomy; whereas in set theory, he introduce **Euler circles**, he also defined the mathematical constant **e**.*



2.6 QUANTIFIED STATEMENT

There are four other commonly used forms of English phrases with which one should become familiar. These are the quantifying statements:

I. All p are q — ***Universal Affirmative***
II. No p are q — ***Universal Negative***
III. Some p are not q — ***Particular Affirmative***
IV. Some p are q — ***Particular Negative***

In order to see how to negate these forms, we must first consider exactly what they mean.

FORM I: Universal Affirmation

Form I: The ***universal affirmative***; this extreme "all p are q" is actually a conditional statement; for example in the statement "all teachers (are people who) give exams," if the **antecedent** is that "you are a teacher," then **consequence** is that "you give exams." In addition, as with any conditional statement, this implication only goes one way. That is, if you are known to give exams, this does not necessarily imply that you are in fact a teacher: for example, you may be a doctor giving a physical exam. Hence, the statement "all teachers give exams" is equivalent to "if you are a teacher, then you give exams." Thus, abbreviate "you are a teacher" as t and "people who give exams" as e, then mathematically, the statement "all teachers give exams" written as $t \rightarrow e$.

Another way to think of "all t are e" is in the context of containment (or as subsets). The group of teachers is contained in the large group of people who give exams. This idea can be illustrated using circles: a Swiss mathematician name Euler used them in the 1700s and for this reason these circles are sometimes called Euler circles. Whatever, if we let one circle represent the group of "teachers" (abbreviated **T**) and another circle represent the group of people who give "exams" (abbreviated **E**), then the relationship between these two circles or sets of people can be illustrated as shown to the left.

Example 2.6.1 "All p are q" to "If p, then q"

"All mothers are teachers" written in "if … then …" form is "if you are a mother, then you are a teacher."

Hence, "all $\boldsymbol{p}$ are $\boldsymbol{q}$" is logically equivalent to $p \rightarrow q$.

- ***A*** *proposition* is a universal affirmative: ***All S is P***
- ***E*** *proposition is a universal negative*: ***No S is P***
- ***I*** *proposition* is a particular affirmative: ***Some S is P***
- ***O*** *proposition* is a particular negative: ***Some S is not P***

Universal Negatives:

➢ ***No man is mortal.***
➢ ***No publicity is bad***
➢ ***No man is an island***

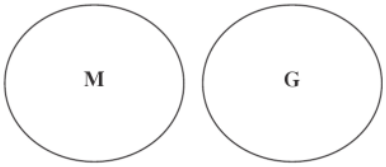
The above statements are equivalent to:

➢ ***If you are a man, then you are not mortal***
➢ ***If it gets you publicity, then it is not bad***
➢ ***If you are a man, then you are not an island***

FORM II: Universal Negative

Form II: The ***universal negative*** is the extreme "none (no) $\boldsymbol{p}$ are $\boldsymbol{q}$" is also a conditional; for example consider the statement "there are no good men" or "no man is good." Opinions aside, if the antecedent is that "you are a man," the consequence is that "you are not good." In addition, as with any conditional statement, this implication only goes one way. That is, if you are known to be bad (not good), this does not necessarily imply that you are in fact a man: because if you are bad, then you could have been a woman. Hence, the statement "there are no good men" is equivalent to "if you are a man, then your not good." Note: for all those who think this statement is false will have the opportunity to contradict me later, but for now this statement is assumed to be TRUE. Thus, if we let "you are a man" be abbreviate as $\boldsymbol{m}$ and "you are good" be abbreviated as $\boldsymbol{g}$, then the statement "there are no good men" can be written mathematically as $m \rightarrow \sim g$.

Another way to think of "none (no) $\boldsymbol{m}$ are $\boldsymbol{g}$" is in the context of non-containment (mutually exclusive). The group of men has nothing to do with the group of people who are good and the group of good people has nothing to do with the group of men. If we let one circle represent the group of "men" (abbreviated **M**) and another circle represent the group of people are "good" (abbreviated **G**), then the relationship between these two circles or sets of people can be illustrated as follows.



Example 2.6.2 "No p are q" to "If …, then …"

"No vegan eats eggs" written in "if … then…" form is "If you are a vegan, then you do not eat eggs" or "if you eat eggs, then you are not a vegan." Hence, "no $\boldsymbol{p}$ are $\boldsymbol{q}$" is logically equivalent to $p \rightarrow \sim q$ or $q \rightarrow \sim p$.

FORM III: Negation of "All p are q"

Form III: This particular level of logic is not an extreme; in fact, "some $\boldsymbol{p}$ are not $\boldsymbol{q}$" is the exact opposite of the extreme "all $\boldsymbol{p}$ are $\boldsymbol{q}$"; for example, in the statement "some teachers do not give exams" is equivalent to the statement "Not all teachers give exams." Common sense aside, this statement leads to several consequences; that is, for you to be the one who make this statement true, then you must both be a teacher and not give exams; maybe you teach kindergarten? However, if you are not a teacher, you still may or may not give exams. Hence, the statement "Some teachers do not give exams" is not as easy to write any other way. Therefore, the best way to consider of "some are not" is in the context of partial-containment (overlapping sets). The group of teachers

*Using **Euler circles**, we can illustrate the universal negative as mutually exclusive events.*

Vegan

*Coined by **Donald Watson** to distinguish those who abstain from all animal products including eggs and cheese and not just those who merely refuse to eat the meat from an animal.*

This second logic, then, I mean the worse one, the teach to talk unjustly, and prevail

Aristophanes
The Clouds (423 BCE)

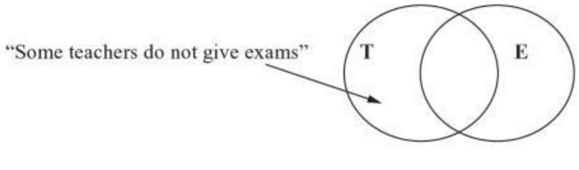
Particular Affirmative:

- *Some men are boys*
- *Sometimes too much is bad thing*
- *Some horses are white*

Is there any relationship among them?

$\forall$ is read as "for all"

is only partially contained in the group of people who give exams. If we let one circle represent the group of "teachers" (abbreviated **T**) and another circle represent the group of people who give "exams" (abbreviated **E**), then the relationship between these two circles or sets of people can be illustrated as follows.

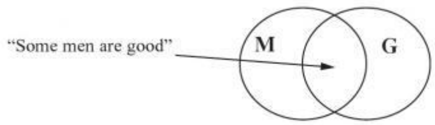


Example 2.6.3 "Some *p* are not *q*"

"Some automobiles are not cars" translates to "It is not the case that all automobiles are cars" or "it is not the case that if a vehicle is an automobile then it is a car." Hence, "some ***p*** are not ***q***" is logically equivalent to $\sim(p \rightarrow q)$.

FORM IV: Negation of "No *p* are *q*"

Form IV: This particular level of logic is not an extreme; in fact "some ***p*** are ***q***" is the exact opposite of the extreme "none (no) ***p*** are ***q***"; for example the statement "some men are good" is equivalent to the statement "it is not that there no good men." Proper English aside, this statement can again lead to several consequences; that is if you are the one who make this statement true, then you must both be a man and you must be good; there are a few of you? However, if you are not a man, you still may or may not be good. Hence, the statement "some men are good" is not as easy to write any other way. Therefore, the best way to consider "some … are …" is again in the context of partial-containment (overlapping sets). The group of men is only partial contained in the group of people who are good. If we let one circle represent the group of "men" (abbreviated **M**) and another circle represent the group of people who are "good" (abbreviated **G**), then the relationship between these two circles or sets of people can be illustrated as follows.



Example 2.6.4 "Some *p* are *q*"

"Some children are well behaved" translates to "It is not the case that there are no well-behaved children" or "It is not the case that if you are a child that you are not well behaved."

Hence, "some ***p*** are ***q***" is logically equivalent to $\sim(p \rightarrow \sim q)$ or $\sim(q \rightarrow \sim p)$.

Let $p(x)$ be an ***open statement*** or ***predicate***; for example, let $p(x) =$ "if x is odd, then $x-2 \geq 7$." Consider the truth value of this statement for $U=\{1, 2, 3, \dots\}$, then the open sentence $p(x)$ represents many statements, one for each $x \in U$.

$p(1) =$ "if 1 is odd, then $1-2 \geq 7$" is false, whereas

$p(2) =$ "if 2 is odd, then $2-2 \geq 7$" is vacuously true.

Definition 2.6.1 Universal Quantifier: $\forall$

The **universal quantifier** is "for all", denote by an upside-down A, $\forall$. The statement $\forall x \in U[p(x)]$ is true if and only if $p(x)$ is true for all $x \in U$.

For example, let the universal set be the set of integers, then "for all natural numbers n, n is greater than zero" can be translated as

$$\forall x(x>0)$$

Example 2.6.5 "For all"

Given $U=\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, determine the truth value for $\forall x\left(x^2<81\right)$.

Solution

$\forall x\left(x^2<81\right)$ translates to "for all digits, the digit squared is less than 81." However, the set of digits squared is $\{0,1,4,9,16,25,36,49,64,81\}$ and hence the truth value of this is false; there exists a digit such that the digit square is not less than 81, but equal to 81.

Definition 2.6.2 Existential Quantifier: $\exists$

The **existential quantifier** is "**there exist**," denote by a backwards E, $\exists$. The statement $\exists x \in U[p(x)]$ is true if and only if there exist at least one $x \in U$ for which $p(x)$ is true.

For example, "there exists a integer such that this integer squared is less than 5," can be translated as

$$\exists x\left(x^2<5\right).$$

Example 2.6.6 "There exist"

Given $U=\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, determine the truth value for $\exists x(x+3\geq 10)$

Solution

$\exists x(x+3\geq 10)$ translates to "there exists a digit such the value three more than the digit is greater than or equal to 10." Since the subset $A=\{7, 8, 9\}$ has three elements such that $x \in A \rightarrow x+3\geq 10$, there exists at least one value of x such that $x+3\geq 10$ is true, and thus $\exists x(x+3\geq 10)$ is true.

Example 2.6.7 "For all" and "There exist"

Let the universal set be the set of all college students and

$p(x,y)=x$ is a friend of y

$q(x,y)=x$ takes a class with y

Write the English sentence from the symbolic statement:

(a) $\forall x \forall y[p(x,y)\rightarrow q(x,y)]$

(b) $\forall x \exists y[p(x,y)]$

$\forall$

vs.

$\exists$

1. ***The statement $P(x,y)$ holds for all x and for all y if and only if the statement $P(x,y)$ holds for all y and for all x.***
2. ***Given there exists an x and there exists a y such that the statement holds true if and only if there exists a y and there exists an x such that the statement holds true.***
3. ***Given there exists an x such that for all y the statement $P(x,y)$ holds true, if and only if for all y, there exists an x such that the statement $P(x,y)$ holds true.***

Solution

(a) For all college students, x and y, if x is a friend of y, then x takes a class with student y.

(b) For all college students, x, there exist a student y such that x is a friend of y.

Let $p(x,y)$ be an open statement regarding two variables x and y, the following are equivalent

(1) $\forall x \forall y\, p(x,y) \Leftrightarrow \forall y \forall x\, p(x,y)$

(2) $\exists x \exists y\, p(x,y) \Leftrightarrow \exists y \exists x\, p(x,y)$

(3) $\exists x \forall y\, p(x,y) \Leftrightarrow \forall y \exists x\, p(x,y)$

However, it should be noted that any other exchanges of $\forall$ and $\exists$ needs to be handled very carefully as they are unlikely to give equivalent statements for all cases.

EXERCISES

Critical Thinking

Write the given statements in symbolic form and illustrate with Euler circles.

2.6.1. All elephants are pink.
2.6.2. Some cars are Hondas
2.6.3. Some politicians are dishonest
2.6.4. Some people are not Democrats
2.6.5. No children are allow to drive

The following statements are from the writings of Lewis Carroll. Write each statement in "if … then …" form.

2.6.6. All my poultry are ducks.
2.6.7. All my sons are slim.
2.6.8. Opium-eaters have no self-control.
2.6.9. Donkeys do have not horns.
2.6.10. Some apples are not ripe.
2.6.11. No porcupines are talkative.
2.6.12. Some chickens are cats.

Translate the following in to complete English statements assuming the universal discourse is the set of all real numbers.

2.6.13. $\forall x, x^2 \geq 0$
2.6.14. $\forall x \exists y (y = 2x)$
2.6.15. $\forall x \forall y \left[(x = y) \to (x^2 = y^2)\right]$
2.6.16. $\exists x \exists y (xy = 2)$

2.7 NEGATING STATEMENTS

Archival Note
Symbolic logic *was discussed in* ***Aristotle's Prior Analytics*** *as part of deductive reasoning*

Symbolic logic may be usefully employed to solve an interesting problem; namely, that of forming an accurate and concise negation of an English statement. In order to correctly negate a statement, one must first translate it into symbolic form. To accomplish this task, we can use some of the algebra of statements discussed in previous sections. We should then be able to translate the symbolic negation into smooth and correct English. The principal rules that are involved in our task have negating English has been previously discussed. They are as follows:

Common: Negations

$\sim(\sim p) \equiv p$	Double negative, is the positive statement $\sim(p \vee q) \equiv \sim p \wedge \sim q$ Not at least one is equivalent to not having either; that is, not the first and not the second (De Morgan's Law)
$\sim(p \wedge q) \equiv \sim p \vee \sim q$	"Not both" is equivalent to not having at least one; that is, "not the first or not the second." (De Morgan's Law)
$\sim(p \rightarrow q) \equiv p \wedge \sim q$	When the first statement does not imply the second statement, the first statement can be true when (and) the second statement fails (that is, is false)

Note that **De Morgan's laws** state clearly the correct way to negate both the *disjunction* and the *conjunction* of two statements p and q. That is, De Morgan's laws indicate the method to use in negating English statements that involve the words "or" or "and" as the primary connective. Example 2.7.1 is concerned with the negation of conditional statements. With these rules one can accurately negate even very complex English statements. We shall illustrate their use in the following examples:

Example 2.7.1 ACC Championship

Consider the statement "**The ACC championship was won by UNC or Wake Forest**." What is its negation?

Solution

Suppose that we wish to negate this statement. We could simply say that "it is not the case that the ACC championship was won by UNC or Wake Forest."

However, this style is stilted and it is still not clear exactly what is meant. To clarify matters let us first translate the original statement into symbolic form by letting

u: UNC won the ACC championship
w: Wake Forest won the ACC championship.

The original statement is $u \vee w$. By De Morgan's law, we have $\sim(u \vee w) \equiv \sim u \wedge \sim w$. Translating this into English we see that the negation of our original statement is

The ACC championship was not won by UNC and it was not won by Wake Forest.

Another way to express this is to say

The ACC championship was won by neither UNC nor Wake Forest.

This statement is concise and cannot be easily misunderstood. Also, note that u represents **U**NC and w represents **W**ake instead of the conventional p and q. Hence, statements can be represented by the standard lower case letters, p, q and r; or statements can be represented by distinct lower case letters that better represent the statement itself.

First symbolism in logic:
a = belongs to every
e = belongs to no
i = belongs to some
o = does not belong to some

Hence, categorical sentences may then be abbreviated as follows:
$AaB = A$ belongs to every B (Every B is A)
$AeB = A$ belongs to no B (No B is A)
$AiB = A$ belongs to some B (Some B is A)
$AoB = A$ does not belong to some B (Some B is not A)

Disjunction
vs.
Conjunctions

In Football, AAC stands for the Atlantic Coast Conference
UNC stands for the University of North Carolina
Wake Forest University is located in Winston-Salem, North Carolina

Example 2.7.2 Voter Approved

Let us now negate the statement "**The voters approved amendments one and three to the constitution.**"

Solution

This statement is the conjunction of the statements by letting

o: The voters approved amendment one.

t: The voters approved amendment three.

That is, the original statement symbolically is

$$o \wedge t$$

By **De Morgan's law** we have

$$\sim(o \wedge t) \equiv \sim o \vee \sim t$$

That is, the negation of the original statement is the English statement

The voters did not approve amendment one or they did not approve amendment three.

Another way to express this is to say

The voters failed to approve at least one of the two amendments offered.

Example 2.7.3 High School Pranks

Consider the conditional statement

If the fountain is turned on, then the students will put Jell-O in it.

Solution

Letting

f: The fountain is turned on,

j: The students put Jell-O in the fountain,

the original statement symbolically becomes $f \rightarrow j$.

If the cold weather does not break, then gas will become scarce and schools will close.

Example 2.7.4 Weather causes schools to close

Negate the statement "If the cold weather does not break, then gas will become scarce and school will be closed."

Solution

Letting

c: The cold weather breaks,

g: Gas will become scarce,

s: School will close,

we have $\sim c \rightarrow (g \wedge s)$. Negating this symbolically we get

$$\sim[\sim c \rightarrow (g \wedge s)] \equiv \sim c \wedge \sim(g \wedge s) \equiv \sim c \wedge (\sim g \vee \sim s).$$

Solution—cont'd

Thus, the English negation of the original statement reads as follows:
"The cold will not break but gas will not become scarce or school will not close."

The negation of this statement (that is, when this statement is false) is when ***the cold weather breaks and gas is not scarce and schools do not close***.

Common: Quantifying Statements

There are four other commonly used forms of English phrases with which one should become familiar. These are the **quantifying statements**:

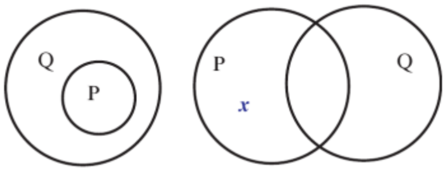
I.	All ***p*** are ***q***	Not "All ***p*** are ***q***" is equivalent to "Some ***p*** are not ***q***"
II.	No ***p*** are ***q***	Not "No ***p*** are ***q***" is equivalent to "Some ***p*** are ***q***"
III.	Some ***p*** are ***q***	Not "Some ***p*** are ***q***" is equivalent to "No ***p*** are ***q***"
IV.	Some ***p*** are not ***q***	Not "Some ***p*** are not ***q***" is equivalent to "All ***p*** are ***q***"

In order to see how to negate these forms, we must first consider exactly what they mean.

Consider first **Form I**. This extreme "all ***p*** are ***q***" is actually a conditional statement $p \to q$. The negation of Form I is, therefore,

$$\sim(p \to q) \equiv p \wedge \sim q.$$

That is, we may negate a statement of the form "All ***p*** are ***q***" with a statement of the form "Some ***p*** are not ***q***."



"All ***p*** are ***q***" Not "all ***p*** are ***q***" "Some ***p*** are not ***q***"

Example 2.7.5 "All elections are honest"

Negate the statement "**All elections are honest**."

Solution

Letting

e: The occurrence is an election,
h: Occurrence is honest,
the original statement can be written as $e \to h$. Hence,

$$\sim(e \to h) \equiv e \wedge \sim h$$

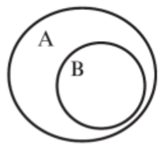
That is, "**The occurrence is an election which is not honest**." Putting this thought into smoother English, we see that the negation of the statement "**All elections are honest**" is the statement "**Some elections are dishonest**."

Form II can be handled similarly. To say that "no ***p*** are ***q***" is to say that "if one is a ***p*** then he is not ***q***." Hence, Form II can be expressed symbolically as

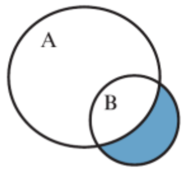
$$p \to \sim q$$

and negated symbolically by $\sim(p \to \sim q) \equiv p \wedge q$.

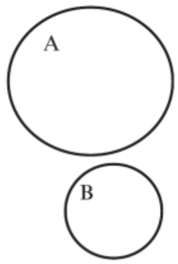
"All B are A"



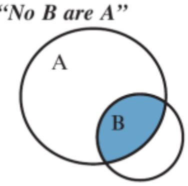
Negative of "All B are A"



"No B are A"

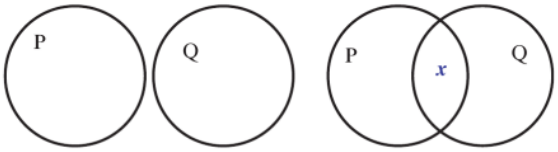


Negative of "No B are A"



All P are Q $\equiv p \to q$
Not all P are Q $\equiv p \wedge \sim q$
No P are Q $\equiv p \to \sim q$
Not the case that no P are Q $\equiv p \wedge q$

That is, we may negate a statement of the form "no ***p*** are ***q***" with a statement of the form "Some ***p*** are ***q***."



"No ***p*** are ***q***" Not "No ***p*** are ***q***" "Some ***p*** are ***q***"

When it is ***not*** *the case that "****no fights are fixed****" then we can say that "****some fights are fixed.****"*

When is ***not*** *the case that "****some people are democrats****" then we can say that* ***nobody is a democrat****.*

Example 2.7.6 "No fights are fixed"

The negation of the statement "No fights are fixed" is the statement "Some fights are fixed." Here we see that to negate a **Form II** statement we use **Form III**. Hence, in order to negate **Form III**, we should use **Form II.** That is, the negation of a statement of the type "Some ***p*** are ***q***" is a statement of the type "No ***p*** are ***q***."

Negation: "Some fights are fixed"

Example 2.7.7 "Some people are Democrats"

Negate the statement "Some people are Democrats." The desired negation is literally "No people are Democrats" or, in better English, "Nobody is a Democrat."

Example 2.7.8 Translations

In each of the following cases, express the given statement into symbolic form; negate the statement symbolically; and translate the negation into smooth, concise English:

(a) You will lose weight only if you stop stuffing yourself.

(b) A necessary condition for being a successful student is that you own a pair of grubby jeans.

(c) No mathematicians are sneaky.

(d) Cafeteria food is cheap and nourishing.

Solution

(a) Let

l:	You lose weight,
s:	You stop stuffing yourself.
Original statement:	$\sim s \rightarrow \sim l \equiv l \rightarrow s$
Negation:	$l \wedge \sim s$
Translation:	You can lose weight and continue to stuff yourself.

(b) Let

s:	You are a successful student,
g:	You own a pair of grubby jeans.
Original statement:	$\sim g \rightarrow \sim s \rightarrow s \rightarrow g$
Negation:	$s \wedge \sim g$
Translation:	You can be a successful student but not own a pair of grubby jeans.

Solution—cont'd

(c) Let

m:	You are a mathematician,
s:	You are sneaky.
Original statement:	$m \rightarrow \sim s$
Negation:	$m \wedge s$
Translation:	Some mathematicians are sneaky.

(d) Let

c:	Cafeteria food is cheap,
n:	Cafeteria food is nourishing
Original statement:	$c \wedge n$
Negation:	$\sim c \vee \sim n$
Translation:	Cafeteria food is not cheap or else it is not nourishing.

The following exercise is intended to extend the ideas presented in this section as well as to give the student practice in translating statements from English into symbols and vice versa. The ability to make such translations quickly is essential to the work to follow concerning the testing of arguments for validity.

Common: Universal/Existential Qualifiers
In addition, there are the **universal quantifier** and the **existential qualifier**, ∀ and ∃, respectively,

1. $\sim[\forall x \forall y\, p(x,y)] \Leftrightarrow \exists x \exists y[\sim p(x,y)]$
2. $\sim[\forall x \exists y\, p(x,y)] \Leftrightarrow \exists x \forall y[\sim p(x,y)]$
3. $\sim[\exists x \forall y\, p(x,y)] \Leftrightarrow \forall x \exists y[\sim p(x,y)]$
4. $\sim[\exists x \exists y\, p(x,y)] \Leftrightarrow \forall x \forall y[\sim p(x,y)]$

SYMBOLIC EQUVALENCES
$\sim \equiv$ "*not*"
$\forall \equiv$ "*for all*"
$\exists \equiv$ "*there exist*"
$\Leftrightarrow \equiv$ "*logical equvialent to*"

The above four equivalence, the first of which translates to "when it is not true that for all x and y the open statement $p(x,y)$ holds" is equivalent to "there exist an x and y for which $p(x,y)$ does not hold." The second statement translates to "when it is not true that for all x, there exist a y such that the open statement $p(x,y)$ holds" is equivalent to "there exist a x such that for all y, $p(x,y)$ does not hold."

EXERCISES

Critical Thinking

2.7.1. Consider the bi-conditional statement $p \leftrightarrow q$. Note that we have shown that $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$. Use this information to symbolically negate $p \leftrightarrow q$.

2.7.2. Consider the exclusive disjunction:
a. Show that $p \underline{\vee} q \equiv (p \vee q) \wedge \sim (p \wedge q)$.
b. Use the above equivalence to symbolically negate $p \underline{\vee} q$.
c. Verbally negate the statement "You can take calculus or finite mathematics but not both."

Basic Problem

2.7.3. Let

d:	He drinks Singapore slings.
p:	He sees pink elephants.
g:	He has a good time at parties.

Using this notation put each of the following into symbolic form, negate each symbolically, and translate the negation into smooth English:

a. If he drinks Singapore slings then he will see pink elephants.
b. Drinking Singapore slings is necessary for having a good time at parties.
c. He sees pink elephants only if he drinks Singapore slings.
d. All people who see pink elephants have a good time at parties.
e. Drinking Singapore slings is necessary and sufficient for having a good time at parties.
f. In order not to see pink elephants, it is sufficient that one not drink Singapore slings.

2.7.4. Negate verbally each of the following:
a. All politicians are devious.
b. Some college presidents are devious.
c. Some college presidents are politicians.
d. All Southerners like grits.
e. No Republicans voted for a Democrat.
f. Some people did not vote.
g. No Italians do not like spaghetti.

2.7.5. Let r: You can register early.
a: You are an athlete.
t: You are tall.
s: You are in style.

Using this notation put the following into symbolic form, negate each symbolically, and translate the negation into concise English:

a. If you are an athlete or tall, then you can register early.
b. If you are an athlete and tall, then you are in style.
c. You are an athlete or tall.
d. You are short but in style.
e. You are neither tall nor in style but you can register early.
f. For each integer, x, there exists an integer, y, such that $y=\sqrt{c}$

2.8 TESTING THE VALIDITY OF AN ARGUMENT

One of the most important applications of logic is to determine whether an argument is valid or fallacious (false). We begin our study of this topic with the following definitions:

Definition 2.8.1 Deductive Reasoning

Reasoning or **deductive reasoning** is a cognitive process using arguments to move from given statements or premises, which are true by assumption, to conclusions. The conclusions must be true when the premises are true.

Archival Note
Deductive reasoning *was largely advanced by the French philosopher and Mathematician* ***Rene Descartes***.

Example 2.8.1 "All men are mortal"

Given "all men are mortal" and "Aristotle is a man," therefore we can deduce "Aristotle is mortal."

Deductive reasoning is often contrasted with inductive reasoning in that inductive reasoning is the process of reasoning in which the premises are an argument are believed to support the conclusion, how do not entail it; that is, they do not ensure it but is a generalization.

Example 2.8.2 "That which goes up, must come down"

Given the proposition, "**this object fell when dropped**," therefore one can infer "**all objects fall when dropped**."

Definition 2.8.2 Argument

An **argument** is an assertion that a given collection of statements $p_1, p_2, \ldots, p_n$ called **premises** yields another statement r called the **conclusion**.

We symbolize an argument as

$$\left.\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \end{array}\right\} \textit{premises}$$
$$\therefore r$$

Where the symbol ∴ is read "**therefore**" and the ***p's*** represent the statement of the argument; the horizontal line simply separates these premises from the conclusion.

Definition 2.8.3 Valid/Fallacy

An argument is **valid** if the conclusion r is true whenever the conjunction of the premises $p_1, p_2, \ldots, p_n$ is true; that is, $p_1 \wedge p_2 \wedge \ldots \wedge p_n \to r$ is a tautology. Otherwise the argument is said to be a **fallacy**. In other words, an argument is valid whenever *all* the premises are true, the conclusion is true.

The validity of an argument can be checked by constructing a truth table. This procedure is illustrated by several examples.

Example 2.8.3 Law of Detachment

Test the validity of the following argument:

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Solution

It suffices to show that $[(p \to q) \wedge p] \to q$ is a tautology. This means that if both premises, $p \to q$ and p, are true, then the conclusion q is true.

The truth table is

1	2	3	4	5
p	q	$p \to q$	$(p \to q) \wedge p$	$[(p \to q) \wedge p] \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Hence, $[(p \to q) \wedge p] \to q$ is a **tautology**; and therefore if $p \to q$ is true and p is true, then the conclusion q is true. The above argument

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Continued

Archival Note

The word ***argument*** *comes from the French meaning* ***a statement and reasoning in support of a proposition****. The word* ***premise*** *is the grounds or basis of the argument, that which comes before. The word* ***conclusion*** *is the deduction reached by reasoning.*

∴
is read
"therefore"

Valid
vs.
Fallacy

If you have a dime, you have ten cents.
You have a dime.
Therefore, you have ten cents.

This is rather redundant except the first two are premises and the third is a conclusion. The second statement has been detached from the condition set forth in the first statement.

Solution—cont'd

is called the **law of detachment.** In this form, the law of detachment is called **modus ponens**. Similarly, the argument

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

using the contrapositive statement, the above argument is one form of the law of detachment called **modus tollens**.

Law of Detachment
&
Modus Ponents
&
Modus Tollens

Example 2.8.4 Law of Syllogism

Show that the following argument is valid:

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Comparable to transitivity in equality:
If $a=b$ and $b=c$, then $a=c$.

Archival Note
*The word **syllogism** comes from old French **silogisme** meaning **inference, conclusion, computation or calculation**.*

Solution

It suffices to show that $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a **tautology**. This can be shown by constructing its truth table.

1	2	3	4	5	6	7	8
p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The above argument is called the **law of syllogism** (hypothetical syllogism) or the transitive property of implication.

Example 2.8.5 Fallacies in Arguments

Show that the following argument is fallacious:

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

Solution

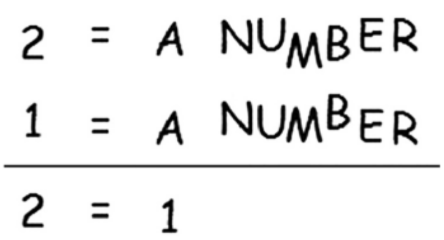
If suffices to show that the statement $[(p \to q) \wedge q] \to p$ is not a tautology.

The truth table is

1	2	3	4	5
p	q	$p \to q$	$(p \to q) \wedge q$	$[(p \to q) \wedge p] \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Solution—cont'd

The third line tells us that the conclusion p is false when both the premise $p \to q$ and q is true. Hence, the argument is a fallacy or fallacious.



Example 2.8.6 Test the Validity

Test the validity of the argument
If it snows, then Maria will ski.
I did not snow.
Therefore, Maria will not ski.

Solution

We first translate the argument into symbolic form. Let p represent "it snows" and q represent "Maria will ski." Thus, the argument takes the form

$$\begin{array}{l} p \to q \\ \sim p \\ \hline \therefore \sim q \end{array}$$

The truth table of the argument is given by

1	2	3	4	5	6	7
p	q	$p \to q$	$\sim p$	$(p \to q) \wedge \sim p$	$\sim q$	$[(p \to q) \wedge \sim p] \to \sim q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

If I win the lottery, then I will take you to dinner.
I took you to dinner.
...I win the lottery???

In the third line of **Step 7** we observe that both the premise $p \to q$ and $\sim p$ are true, but the conclusion $\sim q$ is false. Thus, the argument is a fallacy.

Example 2.8.7 Test the Validity

Test the validity of the following argument:
If I study, then I will not fail statistics.
If I do not play tennis, then I will study.
I failed statistics.
Therefore, I played tennis.

If I study, then I will not fail statistics.
If I do not play tennis, then I can study.
I failed statistics.
... I played tennis

Solution

Let p represent "**I study**," q represent "**I failed statistics**," and r represent "**I play tennis**." Thus, we can translate the argument into symbolic form:

$$\begin{array}{l} p \to \sim q \\ \sim r \to p \\ q \\ \hline \therefore r \end{array}$$

To test the validity of this argument, we must show that whenever the premises $p \to \sim q$, $\sim r \to p$ and q are true, that the conclusion r must also be true.

	1	2	3	4	5	6	7
CASE	p	q	r	$\sim q$	$\sim r$	$p \to \sim q$	$\sim r \to q$
1	T	T	**T**	F	F	F	T
2	T	T	F	F	T	F	T
3	T	F	T	T	F	T	T
4	T	F	F	T	T	T	T
5	**F**	T	**T**	**F**	**F**	T	T
6	F	T	F	F	T	T	F
7	F	F	T	T	F	T	T
8	F	F	F	T	T	T	F

We observe, by crossing out all those that don't hold true, that the premises $p \to \sim q$, $\sim r \to p$ and q are true only in **Case 5**, and in that case the conclusion r is also true. Thus, the above argument is valid. Otherwise, we could have extended the table to include the conjunction of the premises and then finally the conditional statement $(p \to \sim q) \wedge (\sim r \to p) \wedge q \to r$; however, this becomes extremely tedious. Alternatively, we could have used equivalence and the law of detachment and law of syllogism.

Since $p \to \sim q \equiv q \to \sim p$ and $\sim r \to p \equiv \sim p \to r$, the contrapositive, the argument becomes

If I fail statistics, then I did not study.
If I did not study, then I will play tennis.
I failed statistics.
*∴ **I played tennis***

$$\begin{array}{l} q \to \sim p \\ \sim p \to r \\ \underline{q} \\ \therefore r \end{array}$$

Hence, by the law of syllogism,

If I fail statistics, then I did not study and if I don't study, then I will play tennis.
∴ If I fail statistics, then I played tennis.

$$\begin{array}{l} q \to \sim p \\ \underline{\sim p \to r} \\ \therefore q \to r \end{array}$$

yielding

If I fail statistics, then I play tennis.
I failed statistics.
*∴ **I played tennis***

$$\begin{array}{l} q \to r \\ \underline{q} \\ \therefore r \end{array}$$

this is valid by the law of detachment or modus ponens. This type of argument will be extended to proof patterns later in this section.

Example 2.8.8 Test the Validity

Test the validity of the argument
If Fred loves Maria, then Bill will leave town.
Either Bill leaves town or Maria is divorced.
Therefore, if Maria is divorced, then Fred does not love Maria.

Solution

Let p represent "Fred loves Maria," q represent "Bill will leave town," and r represent "Maria is divorced." Thus, the argument takes the form

$$\begin{array}{l} p \to q \\ \underline{q \vee r} \\ \therefore r \to \sim p \end{array}$$

Solution—cont'd

We now construct the truth table of the statements $p \to q$, $q \vee r$ and $r \to \sim p$

	1	2	3	4	5	6	7
CASE	p	q	r	$\sim p$	$p \to q$	$q \vee r$	$r \to \sim p$
1	T	T	T	F	T	T	F
2	T	T	F	F	T	T	T
3	T	F	T	F	F	T	F
4	T	F	F	F	F	F	T
5	F	T	T	T	T	T	T
6	F	T	F	T	T	T	T
7	F	F	T	T	T	T	T
8	F	F	F	T	T	F	T

Recall that an argument is valid if the conclusion is true whenever the premises are true. However, in Case 1 of the preceding truth table, the premises $p \to q$ and $q \vee r$ are both true, but the conclusion $r \to \sim p$ is false. Thus, the argument in Example 1.7.6 is a fallacy.

If an argument has two or three premises, we can always use the concept of a truth table to check its validity. However, when there are more than three premises, a truth-table analysis is quite awkward. An easier approach to check validity in such instances is by use of proof patterns. We have already seen two such patterns; namely, in Example 1.8.1: law of detachment

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

and Example 1.8.4: law of syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

The law of syllogism may be extended to more than two premises, all of which are conditionals. For example, we can write

$$\begin{array}{l} p \to q \\ q \to r \\ r \to s \\ s \to t \\ t \to u \\ \hline \therefore p \to u \end{array}$$

We observe that in the preceding proof pattern, one just follows the arrows. The validity of most arguments can be tested using only the laws of detachment and syllogism. However, when only these proof patterns are used, it is often necessary to replace one or more of the premises with an equivalent statement; for example,

Statement	Equivalnce	Reason
$p \to q$	$\sim q \to \sim p$	Contrapositive
$\sim(p \wedge q)$	$\sim p \vee \sim q$	De Morgan's Law
$\sim(p \vee q)$	$\sim p \wedge \sim q$	De Morgan's Law
$\sim p \vee q$	$p \to q$	Disjunctive Syllogism

The last statement $\sim p \vee q$ is true when the implication is true since, $\sim p \vee p$ is a **tautology**; hence, if $p \to q$ is true, then

Detachment

The action of detaching

The condition of being detached

Syllogism

Bring together, the premise and conclusion

Deductive Reasoning

Contrapositive
vs.
Disjunction
vs.
De Morgan's Laws

$$\sim p \vee \underset{\displaystyle q}{\underset{\downarrow}{p}} \equiv \sim p \vee q.$$

Alternatively, $p \vee q$ is equivalent to $\sim p \rightarrow q$ since by double negatives, $p \vee q \equiv \sim(\sim p) \vee q$, which is equivalent to $\sim p \rightarrow q$ by implication.

We shall now give some examples on the use of proof patterns to prove the validity of certain arguments.

Example 2.8.9 Test the Validity

Prove the validity of the following argument using a proof pattern:
It is raining.
If it is cold, then it is not raining.
If it is not cold, then I cannot go skating.
Therefore, I cannot go skating.

Solution

Let p, q, and r represent the given statements:
p: It is raining
q: It is cold
r: I can go skating

In symbolic form the argument translates to

$$\begin{array}{l} p \\ q \rightarrow \sim p \\ \underline{\sim q \rightarrow \sim r} \\ \quad \therefore \sim r \end{array}$$

	Statement	Reason
1.	p	Premise
2.	$q \rightarrow \sim p$	Premise
3.	$\sim q \rightarrow \sim r$	Premise
4.	$p \rightarrow \sim q$	Contrapositive of (2.)
5.	$p \rightarrow \sim r$	Syllogism using (3.) and (4.)
6.	$\sim r$	Detachment using (1.) and (5.)

Thus, the argument in Example 1.6.7 is valid because, when the premises p, $q \rightarrow \sim p$ and $\sim q \rightarrow \sim r$ are true, then the conclusion $\sim r$ is true.

Example 2.8.10 Test the Validity

Prove the validity of the following argument using a proof pattern:
If Jacob graduates, then he will go to Greece.
If he goes to Greece, then he will visit Athens.
If he does not visit Sparta, then he will not visit Athens.
Jacob did graduate.
Therefore, Jacob will visit Sparta.

Solution

Let p, q, r, and s represent the following statements:
p: Jacob graduates,

Solution—cont'd

q: He will go to Greece,
r: He will visit Athens,
s: He will visit Sparta.

Then, the above argument translates into

$$\begin{array}{l} p \to q \\ q \to r \\ \sim s \to \sim r \\ \underline{p \quad\quad} \\ \therefore s \end{array}$$

A proof pattern of the argument is constructed as follows:

Statement		Reason
1.	$p \to q$	Premise
2.	$q \to r$	Premise
3.	$\sim s \to \sim r$	Premise
4.	p	Premise
5.	$p \to r$	Syllogism using (1.) and (2.)
6.	$r \to s$	Contrapositive of (3.)
7.	$p \to s$	Syllogism using (5.) and (6.)
8.	s	Detachment using (4.) and (7.)

Premise
vs.
Syllogism
vs.
Contrapositive

Thus, we conclude that the argument in Example1.7.8 is valid because the conclusion s is true whenever the premises $p \to q$, $q \to r$, $\sim s \to \sim r$ and p are all true.

Example 2.8.11 Test the Validity

Is the following argument valid?
Maria is a good dancer or Matthew is intelligent.
If Deb is a beautiful girl, then Maria is not a good dancer.
Matthew is not intelligent.
Therefore, Deb is a beautiful girl.

Solution

Let p, q, and r represent the given statements:

p: Maria is a good dancer
q: Matthew is intelligent
r: Deb is a beautiful girl

Then the argument translates to

$$\begin{array}{l} p \vee q \\ r \to \sim p \\ \underline{\sim q \quad\quad} \\ \therefore r \end{array}$$

Statement		Reason
1.	$p \vee q$	Premise
2.	$r \to \sim p$	Premise
3.	$\sim q$	Premise
4.	$\sim p \to q$	Disjunctive Syllogism using (1.) and (3.)
5.	$r \to q$	Syllogism using (2.) and (4.)
6.	$\sim q \to \sim r$	Contrapositive of (5.)
7.	$\sim r$	Detachment using (3.) and (7.)

Hence, r is false by definition of negation and therefore we see that the above argument is *not* valid; that is, this argument is invalid because the conclusion r is false when all the premises are true.

Example 2.8.12 Test the Validity

Is there a valid conclusion to the following argument?

All men eat cake.
I am a man.

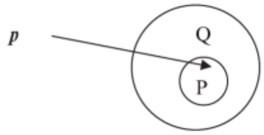
Solution

Let p represent "people who are men" and q represent "people who eat cake."

Recall, this extreme in logic can be rewritten in terms of a conditional; for example, "all p are q" is equivalent to $p \rightarrow q$. Therefore, you could rewrite this extreme as a conditional and use the logic discussed previously.

Alternatively, you can use Euler circles. Just as above, a conclusion can be drawn in two situations. Let **P** be the set satisfying p and **Q** be the set satisfying q.

Then, the Example 2.44, the conditional argument is: "all p are q" and p; this situation can be illustrated as follows.



From this illustration, you can logically deduce q; that is, modus ponens. The premises can be written as

$$\begin{array}{l} p \rightarrow q \\ \underline{p} \\ \therefore ? \end{array}$$

which, by modus ponens, has q is the valid conclusion.

Archival Note
***Euler circles** or diagrams are related to **Venn diagrams** and where used by **Leonhard Euler** to represents sets and their relationship.*

Example 2.8.13 Valid Argument

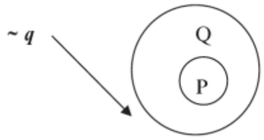
Is the following argument valid?

All boys like bugs.
Alexis does not like bugs.
Therefore, Alexis is not a boy.

Solution

Let p represent "people who are boys" and q represent "people who like bugs."

The given argument is: "all p are q" and $\sim q$; this situation can be illustrated as follows.



From this illustration, you can logically deduce $\sim p$.

Hence, given the two premises, $p \rightarrow q$ and $\sim q$, the logical conclusion is $\sim p$. Note, the premise can be written in argument form are

$$\begin{array}{l} p \rightarrow q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

which, by modus tollens, has $\sim p$ is the valid conclusion and therefore, this is a valid argument.

Example 2.8.14 Valid Conclusion

Is there a valid conclusion?

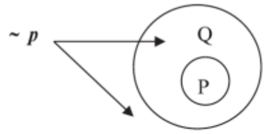
All elephants are pink.

Brownie is a not an elephant.

Solution

Let ***p*** represent "animals that are elephants" and ***q*** represent "animals that are pink."

Consider the argument "all ***p*** are ***q***" and **∼*p***; this situation can be illustrated as follows.



From this illustration, you can see that there are two ways this can be situation can conclude, therefore a single conclusion cannot be logically deduced. Hence, there is no valid conclusion to this argument; any conclusion drawn would be invalid.

Example 2.8.15 Test the Validity

Is the following argument valid?

All gadgets are thingamajigs.

A wiper snap is a thingamajig.

Therefore, all wiper snaps are gadgets.

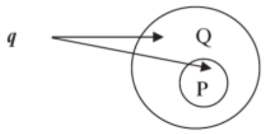
Thingamajig

Something that is hard to classify or whose name is forgotten or unknown

Solution

Let ***p*** represent "things that are gadgets" and ***q*** represent "things that are thingamajigs."

Given the argument "all ***p*** are ***q***" and ***q***; this situation can be illustrated as follows.



From this illustration, you can see that there are still two ways this can be situation can be concluded, therefore a single conclusion cannot be logically deduced; any conclusion drawn under such premises would be invalid. A wiper snap may or may not be a gadget; therefore, this is an invalid argument.

Widgets

A doodad or gadget

An unnamed article considered in a hypothetical example

Jams

Blocked or wedged

Example 2.8.16 Test the Validity

Is there a valid conclusion?

No widgets are wedges.

The doohickey is a widget.

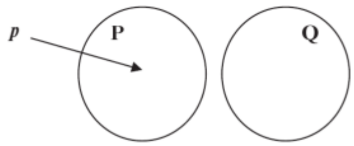
Solution

Let $\boldsymbol{p}$ represent "widgets" and $\boldsymbol{q}$ represent "wedges."

The statement "no $\boldsymbol{p}$ are $\boldsymbol{q}$" is equivalent to $\boldsymbol{p} \to \sim\boldsymbol{q}$. Therefore, you could rewrite this extreme as a conditional and use the logic discussed previously.

Alternatively, you can use **Euler** circles; just as before, a conclusion can be drawn in two situations. Let P be the set satisfying $\boldsymbol{p}$ and **Q** be the set satisfying $\boldsymbol{q}$.

Then, the case given in Example 2.45, "no $\boldsymbol{p}$ are $\boldsymbol{q}$" and $\boldsymbol{p}$ can be illustrated as follows.



From this illustration, you can logically deduce $\sim\boldsymbol{q}$. Therefore, a valid conclusion is that the doohickey is not a wedge.

Example 2.8.17 Valid Conclusion

Is there a valid conclusion?

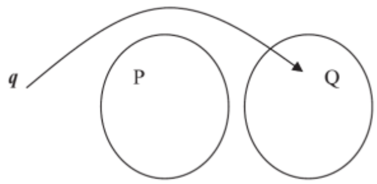
No widgets are wedges.

The jam is a wedge.

Solution

Let $\boldsymbol{p}$ represent "widgets" and $\boldsymbol{q}$ represent "wedges."

Therefore, the argument "no $\boldsymbol{p}$ are $\boldsymbol{q}$" and $\boldsymbol{q}$ can be illustrated as follows.



From this illustration, you can logically deduce $\sim\boldsymbol{p}$. Therefore, the jam is not a wedge.

Example 2.8.18 Valid Argument

Is the following argument valid?

No real man wears hoop skirts.

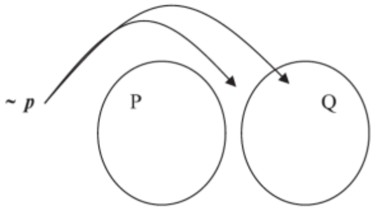
Dana does not wear hoop skirts.

Therefore, Dana is a realman.

Solution

Let p represent "people who are real men" and q represent "people who were hoop skirts."

Consider the argument "no p are q" and $\sim p$; this argument can be illustrated as follows.



From this illustration, you can see that there are two ways this situation can be concluded, therefore a single conclusion cannot be logically deduced. Therefore, the conclusion "Dana is a real man" is an invalid argument. Not because Dana is not a real man, but because there is insufficient evident to prove that Dana is or is not a real man. Dana is a real man does not follow from the argument made.

Example 2.8.19 Valid Conclusion

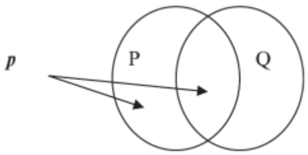
Is there a valid conclusion?

Some women are strong.
Sam is a woman.

Solution

Let p represent "people who are women" and q represent "people who are strong."

Consider the argument: "some p are q" and p, which can be illustrated as follows.



From this illustration, you can see that there are still two ways this can be situation can concluded, therefore a single conclusion cannot be logically deduced. That is, there is no valid conclusion. Sam may or may not be strong.

EXERCISES

Critical Thinking

2.8.1. Test the validity of the argument $\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$

2.8.2. Show that the following argument is valid: $\begin{array}{l} \sim p \to q \\ q \to \sim r \\ \hline \therefore \sim p \to \sim r \end{array}$

2.8.3. Is the following argument valid? $\begin{array}{l} \sim p \to \sim q \\ p \\ \hline \therefore q \end{array}$

2.8.4. Test the validity of the following argument:

If it stops raining, then Chris will play tennis.
It did not stop raining.
Therefore, Chris did not play tennis.

2.8.5. Determine the validity of the argument

If Diane invites Dennis, then John will attend her party.
Diane did not invite John.
___
Therefore, John attended theparty.

2.8.6. Test the validity of the argument

$$\begin{array}{l} \sim p \rightarrow q \\ r \rightarrow \sim p \\ \underline{q} \\ \therefore r \end{array}$$

2.8.7. Test the validity of the following argument:

If Linda does not study, then she will fail her course.
If Linda played tennis, then she did not study.
Linda passed her course.
___
Therefore, Linda did not play tennis.

2.8.8. Is the following argument valid?

If Chris marries Deb, then John joins the Navy.
Either Deb divorces Chris or John joins the Navy.
Therefore, if Deb is divorced, then John joins the Navy.

2.8.9. Using proof patterns prove or disprove the validity of the following argument:

It is hot.
If it is hot, then it is not raining.
If it is not raining, then Maria can go swimming.
Therefore, Maria did not go swimming.

2.8.10. Prove or disprove the validity of the argument

If Sue graduates in the top 10% of her class, then she will go to medical school.
If Sue goes to medical school, then she will specialize in heart disease.
If Sue did not specialize in heart disease, then she did not go to medical school.
Sue graduated in the top 10% of her class.
___
Therefore, Sue specialized in heart disease.

2.8.11. Show that the following argument is false:

If you like finite mathematics, then you will study.

Either you study or you will fail.
___
Therefore, if you failed, then you do not like finite mathematics.

2.8.12. Is the argument following argument valid?

$$\begin{array}{l} p \rightarrow q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

2.8.13. Illustrate Problem 2.8.8 using Euler circles.
2.8.14. Illustrate Problem 2.8.9 using Euler circles.
2.8.15. Illustrate Problem 2.8.10 using Euler circles.

2.9 APPLICATIONS OF LOGIC

Proof by Induction

The logic associated with conditional statements can be used to prove a property holds for an infinitely large set. For example, $2^n \geq n+1$ for all natural numbers, let $P(n)$ be the statement, the property hold for the natural number n. Then if we can show that $P(n) \rightarrow P(n+1)$ and $P(1)$ is true, then $P(n)$ holds for all natural numbers, by induction: $P(1)$ and $P(1) \rightarrow P(2)$ true, implies, by modus ponens, $P(2)$ is true, $P(2)$ and $P(2) \rightarrow P(3)$ true, implies, by modus ponens, $P(3)$ is true, so forth and so on for all n an element of the natural numbers. In general, proof by induction can be bounded below by $n = c$. That is, if the conditional statement in regards to sequentially defined equation "if it is true for $n = k$, then is it is true for $n = c$ greater than or equal to c."

It should be noted that Mathematical induction (proof by induction) is not a form of inductive reasoning, but rather an extended form of deductive reasoning. Proof by induction is a three-step procedure; two of the steps are the proving the conditional statement and one step to show for true for $n = 1$. "Show for true for $n = 1$" can be done first or last, clearly label each step.

Archival Note

Recall, proof by inductions was introduced with great rigor by Aristotle. The basic idea is to get into an infinite loop.

First, show that the statement holds for $n = 1$.

Second, assume that the statement is true for $n = k$ *in general.*

Third, prove using the assumption stated in the second part that the statement holds true for $n = k + 1$.

STEPS: Proof by Induction

Step 1. Show for true for $n = 1$ or for $n = c$, where c is the first n.
Step 2. Assume the antecedent is true; that is assume true for $n = k$.
Step 3. Show true for $n = k + 1$; since the only conditions under which the conditional statement fails it when the antecedent is true and the conclusion is false.

If we can show that true for $n = k$, implies the equation it is true for $n = k + 1$; then we have shown that the conditional statement is always true.

Be sure to clearly state what you "need to show" (***NTS***); clearly define the left hand side (***LHS***) and the right hand side (***RHS***).

NTS≡"Need to show"
LHS≡"Left hand side"
RHS≡"Right hand side"

Natural Number

The ordinary counting numbers: 1, 2, 3, …

Example 2.9.1 Sum of Whole Numbers

Prove $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all natural numbers: $n \in N$, where $\in$ is read "an element of."

Proof

Step 1: Show true for $n = 1$

$$1 = \frac{1(1+1)}{2}$$

$$1 = 1$$

True

Step 2: Assume true for $n = k$; that is,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

Step 3: Show true for $n = k + 1$; that is, we NTS

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}$$

or equivalently,

$$1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Hence, $LHS = 1 + 2 + 3 \cdots + k + (k+1)$, which by assumption becomes

p	q	$p \vee q$	$p \wedge q$
1	1	**Max (1,1) = 1**	**Min (1,1) = 1**
1	0	**Max (1,0) = 1**	**Min (1,0) = 0**
0	1	**Max (0,1) = 1**	**Min (0,1) = 0**
0	0	**Max (0,0) = 0**	**Min (0,0) = 0**

Furthermore, if we consider the negation to be the (truth-value + 1) mod 2; or in layman's terms, the opposite truth-value, we have the following.

p	$\sim p$
1	**0**
0	**1**

Now for conditionals, it is easier to consider its disjunctive equivalent; but if you insist one just an algebraic rule, a conditional is the maximum of the opposite of the antecedent's truth-value and the consequence's truth-value; this is in essence the disjunctive equivalent.

p	q	$\sim p$	$p \rightarrow q \equiv \sim p \vee q$
1	1	0	**Max (0, 1) = 1**
1	0	0	**Max (0, 0) = 0**
0	1	1	**Max (1, 0) = 1**
0	0	1	**Max (1, 0) = 1**

Hence, if we interpret the **1'***s* as true and the **0'***s* as false, these truth-values create the same truth tables introduced previously.

Switching Circuits

The logic of compound statements is often utilized in the design of switching networks in electrical circuit theory. In this section we shall introduce some of the basic theory necessary for the construction of switching networks. We begin by defining what is meant by a switching network.

Definition 2.8.4 Switching Network

A **switching network** is a collection of wires and switches connecting two terminals, A and B. A switch may be either open, O, or closed, C. An open switch will not permit the current to flow while a closed switch will permit current to flow, (Figure 2.1).

We shall now proceed to develop the relationship between logic and the

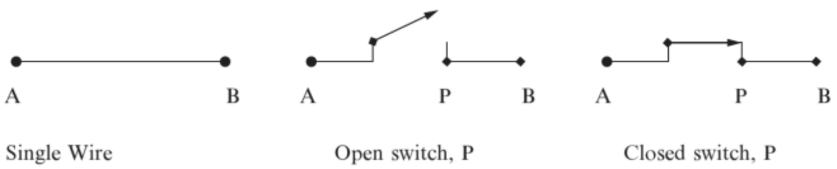


FIGURE 2.1 Various wires.

design of simple switching networks. Given a switch, P, let p be a statement associated with this switch having the property that

p is true if and only if P is closed

or

Archival Note
Boolean logic *is the logical calculus of truth values developed in the 1840s*

Calculus
From Latin **calculus** *"reckoning, account"*

Warning: **+ and ×**
When using **Boolean algebra**, remember we changed the meaning of + and ×; these new definitions lead to similar but very different properties, some of which look alike and some of which look vastly different.

p is false if and only if P is closed.

Two switches, P and Q, may be connected in two fundamental ways: in series or parallel.

I SERIES

Figures 2.2 and 2.3 illustrate how the switches P and Q are connected in series.

Here, we observe that the current will flow from A to B if and only if both

FIGURE 2.2 Closed switches.

FIGURE 2.3 Open switches.

switches P and Q are closed; that is, if and only if $p \wedge q$ is true. Also, current will not flow from A to B if one of the switches is open as shown by Figures 2.3 and 2.4.

That is, $p \wedge q$ is false. The behavior of P and Q when they are connected in

FIGURE 2.4 P closed, Q open.

series is summarized in the following table.

P	Q	Series Circuit		p	q	$p \wedge q$
C	C	C		T	T	T
C	O	O	OR	T	F	F
O	C	O		F	T	F
O	O	O		F	F	F

Thus, it is clear that $p \wedge q$ is true only when both P and Q are closed.

II Parallel

When the switches P and Q are connected in *parallel* they appear as shown by Figures 2.5 and 2.6. These two figures illustrate two of the four possible switch positions.

From these figures it is clear that current will flow from A to B if and only if either P or Q is closed. That is, in logical terms, current will flow from A to B if

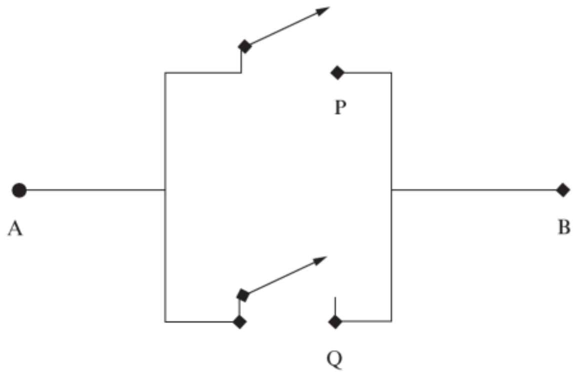


FIGURE 2.5 P and Q open.

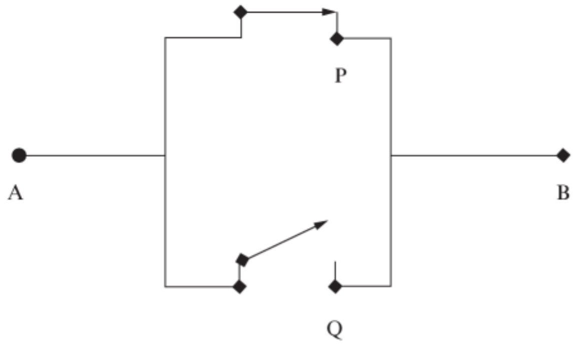


FIGURE 2.6 P closed, Q open.

and only if $p \vee q$ is true. Note also that in Figure 2.5 current will not flow from A to B since P and Q are open. That is, $p \vee q$ is false.

P	Q	*Parallel Circuit*
C	C	C
C	O	C
O	C	C
O	O	O

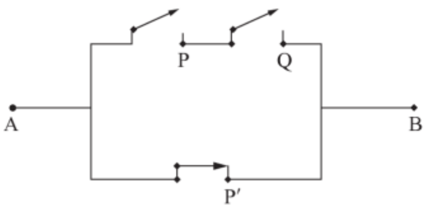
OR

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Hence, current will flow from A to B in three out of four possible combinations; namely, P open and Q open, P closed and Q open, P open and Q closed, or P closed and Q closed.

Example 2.9.3 Current flow from A to B?

Given the following network, when does current flow from A to B?



Definition 2.8.6 Equivalent Electrical Networks

Two electrical networks are said to be **equivalent** if they have the same electrical properties concerning the flow and non-flow of current.

This definition simply states that their corresponding statements are logically equivalent. We shall illustrate this concept in the following examples:

Example 2.9.4 Equivalent Electrical Network

Find a network equivalent to the one given in Figure 2.7.

Note that the dashed line indicates the path of the electrical flow. The current in this network will flow from A to B whenever the logical statement $(p \vee \sim p) \wedge (q \vee \sim q)$ *circuits is true.*

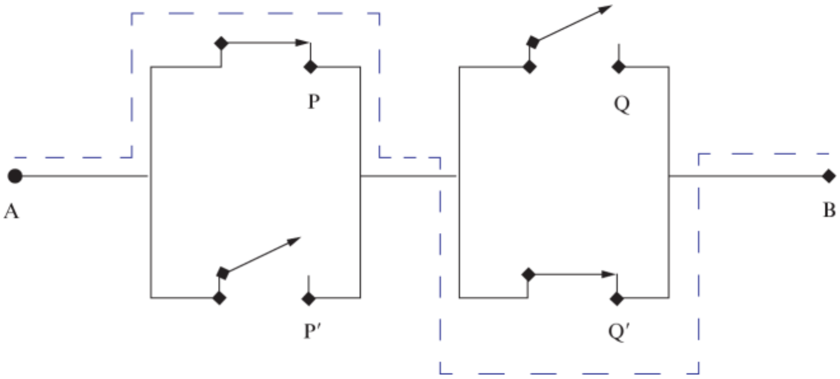


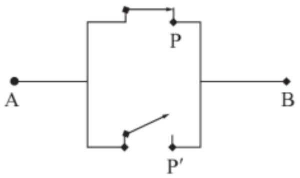
FIGURE 2.7 Network diagram of the electrical circuit $(p \vee \sim p) \wedge (q \vee \sim q)$.

Solution

A network equivalent to Figure 2.7 can be obtained as follows:

	Statement	Reason
1.	$(p \vee \sim p) \wedge (q \vee \sim q)$	Statement describing the circuit
2.	$\equiv \tau \wedge \tau$	Complement Law
3.	$\equiv \tau$	Identity Law

Thus, the above network can be designed equivalently by any tautology such as $p \vee \sim p$. That is,



Example 2.9.5 Equivalent Electrical Network

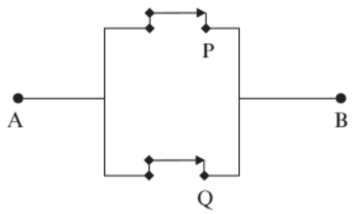
Obtain a network equivalent to the network given in Figure 2.8

Solution—cont'd

Observe that the dashed path indicates the path of the electrical flow from A to B. Now, we proceed to find a statement equivalent to the logical statement already given. This statement is simply $p \vee q$ That is,

	Statement	Reason
1.	$[p \vee (\sim p \wedge q)] \vee (p \wedge \sim q)$	Statement describing the circuit
2.	$\equiv [(\sim p \vee p) \wedge (p \vee q)] \vee (p \wedge \sim q)$	Distributive Law (from the left)
3.	$\equiv [\tau \wedge (p \vee q)] \vee (p \wedge \sim q)$	Complement Law
4.	$\equiv (p \vee q) \vee (p \wedge \sim q)$	Identity Law
5.	$\equiv (p \wedge \sim q) \vee (p \vee q)$	Commutative Law
6.	$\equiv [(p \wedge \sim q) \vee p] \vee q$	Associative Law
7.	$\equiv [(p \wedge \sim q) \vee (p \wedge p)] \vee q$	Distribution Law (from the right)
8.	$\equiv [p \wedge (\sim q \vee p)] \vee q$	Distribution Law (in reverse)
9.	$\equiv (p \vee q) \wedge [(\sim q \vee p) \vee q]$	Distribution Law (from the right)
10.	$\equiv (p \vee q) \wedge [(p \vee \sim q) \vee q]$	Commutative Law
11.	$\equiv (p \vee q) \wedge [p \vee (\sim q \vee q)]$	Associative Law
12.	$\equiv (p \vee q) \wedge (p \vee \tau)$	Complement Law
13.	$\equiv (p \vee q) \wedge \tau$	Identity Law
14.	$\equiv p \vee q$	Identity Law

Thus, the complicated network shown by Figure 2.8 is equivalent to a network made up of two switches P and Q, connected in parallel; that is,



Example 2.9.6 Equivalent Electrical Networks

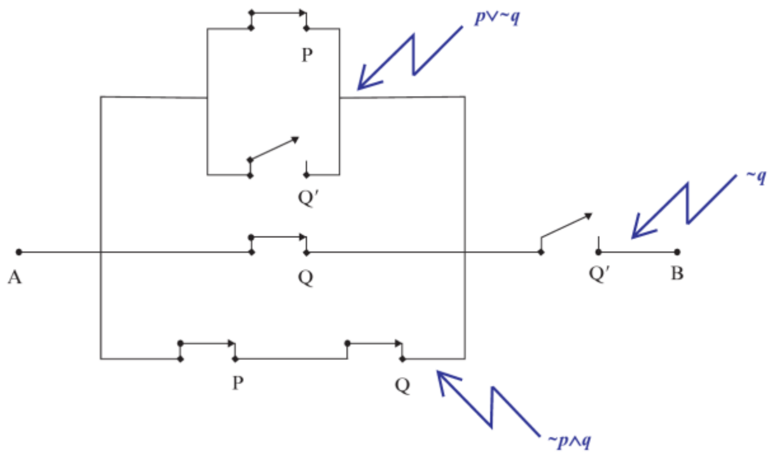
Draw a switching network corresponding to the compound statement

$$[(p \vee \sim q) \vee q \vee (\sim p \wedge q)] \wedge \sim q.$$

Also find an equivalent network simpler in design.

Solution

The switching network described by the preceding logical statement is



Continued

2.4.1. A compound statement, τ, is said to be a **tautology** or **logically true** if it is true for all possible truth values of its components.
2.4.2. A compound statement, ϕ, is said to be a **self-contradiction** or **logically false** if it is false for all possible truth values of its components.
2.4.3. A **paradox** is an apparently true statement or group of statements that leads to a contradiction.
2.4.4. Two statements r and s are said to be **logically equivalent** or simply **equivalent** if they have identical truth tables; that is, if $r \leftrightarrow s$ is a tautology. To symbolize two equivalent statements r and s, we write $r \equiv s$ or $r \Leftrightarrow s$.

Qualifiers are terms that indicate to what extent a property holds: ***all are***, ***none are***, ***some are not*** and ***some are*****.** The universal qualifier is ***for all***, $\forall$, and the existential qualifier is **there exist**, $\exists$.

2.6.1. The **universal quantifier** is "for all," denote by an upside-down A, $\forall$. The statement $\forall x \in U[p(x)]$ is true if and only if $p(x)$ is true for all $x \in U$.
2.6.2. The **existential quantifier** is "**there exist**," denote by a backwards E, $\exists$. The statement $\exists x \in U[p(x)]$ is true if and only if there exist at least one $x \in U$ for which $p(x)$ is true.

2.8.1. **Reasoning** or **deductive reasoning** is a cognitive process using arguments to move from given statements or premises, which are true by assumption, to conclusions. The conclusions must be true when the premises are true.
2.8.2. An **argument** is an assertion that a given collection of statements $p_1, p_2, \ldots, p_n$ called **premises** yields another statement r called the **conclusion**.
2.8.3. An argument is **valid** if the conclusion r is true whenever the conjunction of the premises $p_1, p_2, \ldots, p_n$ is true; that is, $p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow r$ is a tautology. Otherwise the argument is said to be a **fallacy**. In other words, an argument is valid whenever *all* the premises are true, the conclusion is true.
2.8.4. A **switching network** is a collection of wires and switches connecting two terminals, **A** and **B**. A switch may be either open, **O**, or closed, **C**. switch will not permit the current to flow while a closed switch will permit current to flow.
2.8.5. Two switches are said to be **complementary** if one switch is open and the other is closed, and vice versa. Thus, if one switch is **P**, the complementary switch will be labeled **P′** (**P** prime).
2.8.6. Two electrical networks are said to be **equivalent** if they have the same electrical properties concerning the flow and non-flow of current.

We have defined seven basic laws of algebra: ***idempotent***, ***associative***, ***commutative***, ***distributive***, ***identity***, ***complement***, and ***De Morgan's*** for statements under the equivalence relation. We have also considered various forms of the conditional statement.

Properties:
2.2.1. **Conjunction**: If p is true and q is true, then $p \wedge q$ is true; otherwise $p \wedge q$ is false.
2.2.2. **Disjunction (inclusive):** When p is true or q is true or if both p and q are true, then $p \vee q$ is true; otherwise $p \vee q$ is false. Thus, the disjunction $p \vee q$ of the two statements p and q is false only when both p and q are false.
2.2.3. **Disjunction (inclusive):** When p is true and q is false, or when p is false and q is true, then $p \veebar q$ is true; otherwise $p \veebar q$ is false. Thus, this alternative disjunction $p \veebar q$ of the two statements p and q is false only when both p and q are false and when both p and q are true.
2.2.4. **Negation:** When p is true, then $\sim p$ is false; if p is false, then $\sim p$ is true.
2.2.5. **Conditional:** The **conditional statement** $p \rightarrow q$ is false only when p is true and q is false; otherwise $p \rightarrow q$ is true.
2.2.6. **Bi-conditional:** If p and q are either both statements are true or both statements are false, then $p \leftrightarrow q$ is true; if p and q have opposite truth values, then $p \leftrightarrow q$ is false.

Rules:
2.4.1. **Idempotent:** $p \vee p \equiv p$ and $p \wedge p \equiv p$
2.4.2. **Associative:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$ and $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
2.4.3. **Commutative:** $p \vee q \equiv q \vee p$ and $p \wedge q \equiv q \wedge p$
2.4.4. **Distributive:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ and $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
2.4.5. **Identity:** If τ is a **tautology** and φ is any **contradiction**, then $p \vee \varphi \equiv p$, $p \wedge \tau \equiv p$, $p \wedge \varphi \equiv \varphi$ and $p \vee \tau \equiv \tau$
2.4.6. **Complement:** If τ is a **tautology** and φ is any **contradiction**, then $p \vee \sim p \equiv \tau$, $p \wedge \sim p \equiv \varphi$, $\sim(\sim p) \equiv p$, $\sim \tau \equiv \varphi$ and $\sim \varphi \equiv \tau$
2.4.7. **DeMorgan's Rule:** $\sim(p \vee q) \equiv \sim p \wedge \sim q$ and $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Symbol (Abbr.)	Name/Origin	Meaning/ Read as	Description	Example/Description
$\rightarrow$	Implication/Conditional	"If …, then…"	If the antecedent is true that implies that the consequence must be true	"If I win the lottery, then I will buy you dinner"
$\leftrightarrow$	Bi-conditional	"… if and only if …"	The conditional holds in both directions	"I am a male if and only if I have a Y chromosome"
iff		"if and only if"	Abbreviation for "if and only if"	
$\forall$	Universal Qualifier	"for all …"	An upside down A	For all real numbers $x, x^2 \geq 0$
$\exists$	Existential Qualifier	"there exist …"	A backwards E	There exist $x \in Z$ such that $\sqrt{x}=2$
$\equiv$	Equivalent	"Is the same as saying"	Equality between compound statement	Related to the word: Equal
$\therefore$	Hebrew	"Therefore"	The abbreviation for "therefore"	Used when drawing conclusions
ϕ or c	Greek letter "Phi" or an abbreviation of **c**ontradiction	A contradiction	Statement that is never true	"This statement if false"
τ or t	Greek letter "Tau" or an abbreviation of **t**autology	A tautology	Statement that is always true	"A rose is a rose"
$\in$	Greek letter "Epsilon"	"An element of" or "is contained in"	Symbol illustrating containment	Let p represent a statement: $p \in S$
n	Abbreviation of **n**umber	"Number of simple statements"	Used when counting total number of possible situations	Given p, q, r: $n=3$

REVIEW TEST

Multiple-Choice:

1. Which of the following is/are not statement(s)?
   - **i.** Come here
   - **ii.** That is a cute kitty
   - **iii.** $2+2=5$
   - **A.** i only
   - **B.** i and ii only
   - **C.** iii only
   - **D.** all of them are statements
   - **E.** none of them are statements
2. Which of the following is/are not statement(s)?
   - **i.** Watch your step
   - **ii.** Susan will call you tonight.
   - **iii.** $2+5=7$
   - **iv.** $2\times 5=7$
   - **A.** i only
   - **B.** ii and iii only
   - **C.** iii only
   - **D.** all of them are statements
   - **E.** none of them are statements

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Solution

Let x represent a letter in the word *football*. Thus, the desired set in set-builder notation is written

$$A = \{x | x \text{ a letter in the word football}\}$$

The elements in the set can be listed in roster form:

$$A = \{f, o, t, b, a, l\}.$$

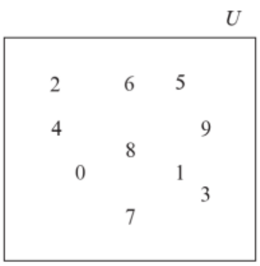
Note that although the word *football* contains two *o*'s and two *l*'s, we write them only once in structuring the set. In this context, the universal discourse is understood to be the letters of the alphabet; however, universal discourse is logistical jargon, in set theory, we shall say the universal set.

Definition 3.2.2 Universal Set

The **universal set** U is the largest set in a given context; that is, the **universal set** is the totality of the elements under consideration. Denoted by the capital letter, U normally written with the upper bars, U, or a tail, U, as to be it distinguishable from the union symbol ∪.

This is the set from which all other sets will be taken, but it is important that you understand the given context. For example, if the set is defined in the context of the natural numbers, then the understood universe is $U = \{1, 2, 3, \ldots\}$ which the set of all counting numbers including large numbers like 100, 1000, and 1,000,000. However, if the set is defined in the context of the digits, then the understood universe is the limited set: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ which does include the element zero by no integer larger than 9.

To help us better understand certain aspects of set theory we shall use circles and rectangles to denote sets. The diagram approach originated with an English logician named John Venn (1834-1923), and we refer to them as Venn diagrams. Similar to Euler circles in Logic, we can draw sets using ***Venn diagrams***. The main difference begin, in Logic there are not always physical boundaries other than those represent by the statements themselves (the sets) and an implied universal discourse; whereas in Set Theory, there is the boundary of the contextual universal set. This universal set in a Venn diagram is emphasized in that it is represented by a rectangle which contains the primary set; for example, the universal set of digits can be illustrated as



Definition 3.2.3 Empty Set

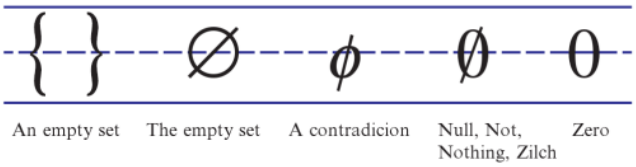
This **empty set**, sometimes call the **null set** is the smallest set in any context; this set contains no elements. If we were to write it in "proper" set notation, it would be obvious that it has no elements, **{ }** or the single symbol $\varnothing$.

$U=\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$U=\{x|x \text{ is a digit}\}$

$A \subseteq U$

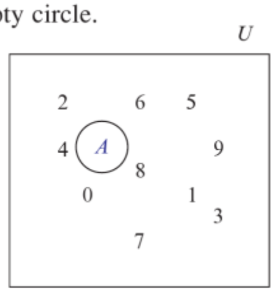
$A=\varnothing$



Hence, it is possible for a set to have no elements, for example, the set

$$A=\{y|y \text{ a unit digit}, y<-1\}$$

has no elements because there is no digit that is less than or equal to -1. In a Venn diagram, letting a set be described by a circle, then the empty set would be represented by an empty circle.



Example 3.2.6 Bounded Sets

List the elements of the set given by

$$B=\{z|z \text{ a positive integer}, z^2=25 \text{ and } 5<z\leq 25\}.$$

Solution

Here, B is empty because there is no value of z that will satisfy both the condition (a) z is a positive integer, (b) $z^2=25$, and (c) that is greater than five and less than or equal to 25. The only positive number that when squared is 25 is 5; which is not greater than five.

There are many situations in which the elements of a given set are also the elements of another set. For example, let us consider the set F which consists of all female students of a finite mathematics course and the set U which consists of all students in our university. Here, all the members of the set F are also members of the set U. That is, all females students are university students; this idea of containment. Before we state the definition of a subset, let us consider another example. Define the sets of even digits E and the set of all digits D as

$$E=\{0, 2, 4, 6, 8\} \text{ and } D=\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

We note here that every element of the set E is also an element of the set D. We symbolize this situation by writing

$$E\subset D$$

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Solution

Putting these observations in a set form, we have

$$C=\{86, 88, 89, 90, 92\}.$$

Recall that all members of a set must be distinct. Therefore, although 88 and 89 were recorded twice, when we put them in set form we include the observations only once. Also, the order in which we list the elements of the set is irrelevant.

Consider temperatures recorded for the same period in Sparkling City:

90, 92, 89, 89, 88, 88, 86

degrees Fahrenheit. Putting these observations in a set form, we have

$$S=\{86, 88, 89, 90, 92\}$$

Since both sets C and S contain the same recordings, $C=S$.

PROBLEMS

Critical Thinking

3.2.1. Indicate which of the following verbal descriptions a well-defined set:
- (a) The players of the Tampa Bay Buccaneers football team,
- (b) The collection of all good United States senators,
- (c) The states of the United States of America,
- (d) The collection of all secretaries who can type at least 75 words per minute,
- (e) The golf players who have won the United States Open Golf Tournament.

3.2.2. Let $B=\{-1,a,2,b,c,d\}$. Indicate which of the following statements are correct:
(a) $\varnothing \in B$, (b) $a \notin B$, (c) $d \in B$, (d) $-1 \in B$

3.2.3. Consider the set $A=\{x|x$ a positive integer $1<x\leq 8\}$ Which of the following statements are true?
(a) $A = \{2, 3, 4, 5, 6, 7\}$, (b) $A = \{1, 2, 3, 4, 5, 6\}$
(c) $4 \in A$, (d) $1 \notin A$
(e) $a \in A$, (f) $7 \notin A$

3.2.4. Let $C=\{z|z$ is an even integer, $1\leq x<13\}$. Indicate which of the following statements are true.
(a) $C=\{1, 2, 4, 6, 8, 10, 12\}$
(b) $C=\{2, 4, 6, 8, 10, 12\}$
(c) $7 \notin C$, (d) $1 \in C$, (e) $8 \in C$

3.2.5. Consider the set of French philosophers during the Age of Reason, an intellectual movement of the 1700s.

$$P = \{\text{de Condorcet, Diderot, Helvetius, Rousseau, Voltaire}\}.$$

Indicate which of the following are true or false:
(a) de Condorcet $\in P$, (b) Diderot $\in P$
(c) Helvetius $\notin P$, (d) Voltaire $\in P$

3.2.6. Consider the set of drugs useful in the treatment of certain cancers.

$$D=\{5-\text{fluorouracil, methotrexate, cytoxin, Vincristine}\}.$$

Indicate which of the following statements are true or false:
(a) 7-fluorouracil $\in D$
(b) cytoxin $\in D$
(c) 5-fluorouracil $\notin D$
(d) Vincristine $\in D$

3.2.7. Write the set which consists of all positive integers less than 15.

3.2.8. Write the set of all integers greater than -3 and less than or equal to 4.

3.2.9. Construct the set of all even integers greater than or equal to zero and less than 16.

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