The JOY of MATHEMATICS

Marvels, Novelties, and Neglected Gems That Are Rarely Taught in Math Class

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INTRODUCTION

For many decades the mathematics curriculum has been stocked with lots of essential building blocks to enable the student to navigate properly such disciplines as science, finance, engineering, architecture, and everyday life—just to name a few! With much to cover and the stress of moving ahead at a steady clip, there are many interesting and important mathematics concepts, topics, and applications that rarely ever get mentioned in the classroom.

When students are presented with finance applications in the school curriculum, such as computing the effect of interest on a given principal in the bank, there is a missed chance to enrich them with a mathematical peculiarity such as the "Rule of 72," which enables one to determine how much time is required for a bank account to double the deposited money at a specified interest rate.

There are amazingly simple geometric phenomena that are rarely ever shown in classrooms, simply for lack of time. These might include some of the special yet straightforward characteristics of quadrilaterals inscribed in a circle, such as the incredible relationship between the diagonals of a quadrilateral and its sides, where the product of the diagonals is equal to the sum of the products of the opposite sides. Remember, this is only true when the quadrilateral's four vertices lie on the same circle.

Another opportunity missed is to show how the random placement of a point inside an equilateral triangle shares a common property with any other point in that triangle, namely that the sum of the perpendiculars to each of the three sides is always the same.

There are everyday applications that regularly seem to escape classroom presentations. For example, mechanical algorithms to multiply numbers mentally. Even though there are electronic calculators readily available, the facility of manipulating numbers mentally is clearly an asset that seems to be seen as relatively unimportant in today's technological world. Here we choose to fill in this void.

A clever application of the Fibonacci numbers—perhaps the most ubiquitous numbers in our culture—allows us to convert from miles to kilometers (or the reverse) mentally. This is particularly useful for Americans traveling outside of their country's borders, when they will have to convert distance from kilometer-indicating signs to the more familiar mileage measures.

There are many uses of algebra that can explain many mathematical curiosities, which simply bewonder the uninformed. For example, most teachers will show their students how to look at a number and determine if it is divisible by 3, but they will not take the time to explain why this "trick" works. We believe that knowing *why* this works is almost as important as knowing *how* it works. The same is true for other divisibility rules that we will present in the pages that follow.

Studying conic sections is a standard part of the high school curriculum. Teachers typically show physical models but miss quite a few truly astonishing applications. Take, for example, using the rays of light emanating from a flashlight to show that they can generate conic sections. Shining a flashlight onto the ground or onto a wall at different angles allows different shapes of light to arise. The boundaries of these shapes are conic sections (assuming that the aperture of the flashlight is circular). Depending on the angle, we obtain a circle, an ellipse, a parabola, or a hyperbola. Similarly, a circular arc (being part of a building, for instance) can appear as an elliptic arc, a hyperbola, or a parabola if we look at it from different angles. Furthermore, aspects of the geometry of conic sections can be found in many architectural masterpieces. Mathematics can also explain how we create a visual depth perception and draw a picture. The concept of perspectivity, found in many famous paintings of the Italian Renaissance, has set the stage to study and perfect this concept further. Notable among the artists is Leonardo da Vinci, who also became a model for the famous German artist Albrecht Dürer. These are some of the aspects of our artistic culture that can be explained through mathematics, but, sadly, they are often neglected in the teaching of mathematics.

The topic of probability, which is gaining ever more presence in the standard curriculum today, has some truly astonishing and counterintuitive applications that all too often are not shared with students studying this topic. The famous "birthday problem" is unfortunately not presented to many classes. This "problem" offers some highly counterintuitive results. For example, it determines that the highly unanticipated probability of two people having the same birth date in a room of thirty people is 70 percent; and, perhaps more amazingly, it determines that the probability of two people having the same birth date in a group of fifty-five people is 99 percent. Such omissions clearly weaken the instructional program, so we now take the opportunity to allow the general readership to make up for previously lost chances.

The ever-present drive by teachers, who are being rated by their students' test performance, to "teach to the test" is one reason why so many mathematical wonders are omitted from the instructional program on a nationwide level. *The Joy of Mathematics* is an attempt to fill in the many gaps in the American mathematics education system and at the same time to show the average American citizen that there are lots of entertaining and useful mathematics gems that may have eluded them during their school days. We will do this in short units so as to keep the presentation crisp and intelligible. We will also use photos and diagrams extensively to enhance the attraction and function of our examples. To keep everything accessible, we have used language and content geared toward the general reader, not the math savant. As such, we have kept in mind the idea of the French mathematician Joseph Diaz Gergonne (1771–1859) that "it is not possible to feel satisfied at having said the last word about some theory as long as it cannot be explained in a few words to any passer-by encountered in the street."[1]

We hope to provide you with a better grasp of mathematics, and above all, a greater appreciation for its usefulness, not only in its many applications, but also in exhibiting the power and beauty of the subject in its own right.

CHAPTER 1

ARITHMETIC NOVELTIES

When you think of arithmetic, you typically consider the four basic arithmetic operations. With a little more thought, you tend to tag on the square root operation as well. Unfortunately, most of our school curriculum focuses on ensuring that we have a good mechanical command of the arithmetic operations and know the number facts as best we can to service us efficiently in our everyday life. As a result, most adults are not aware of the many amazing relationships that can be exhibited arithmetically with numbers. Some of these can be extremely useful in our everyday life as well. For example, just by looking at a number and determining if it is divisible by 3, 9, or 11 can be very useful, especially if it can be done at a glance. When it involves determining divisibility by 2, we do this without much thought, by simply inspecting the last digit. We shall extend this discussion to considering divisibility by a prime number, something that clearly is not presented in the school curriculum, with which we hope to motivate the reader to investigate further primes beyond those shown here. We truly expect that the wonders that our number system holds, many of which we will present in this book, will motivate you to search for more of these curiosities along with their justifications. Some of the units in this chapter will also provide you with a deeper understanding for our number system beyond merely arithmetic manipulations. Our introduction to a variety of special numbers will generate a greater appreciation of arithmetic than the typical school courses provide. Let us begin our journey through numbers and their operations.

WHEN IS A NUMBER DIVISIBLE BY 3 OR 9?

Teachers at various grade levels often neglect to mention to students that in order to determine whether a number is divisible by 3 or 9, you just have to apply a simple rule: If the sum of the digits of a number is divisible by 3 (or 9), then the original number is divisible by 3 (or 9).

An example will best firm up your understanding of this rule. Consider the number 296,357. Let's test it for divisibility by 3 (or 9). The sum of the digits is $2 + 9 + 6 + 3 + 5 + 7 = 32$, which is not divisible by 3 or 9. Therefore, the original number, 296, 357, is not divisible by 3 or 9.

Now suppose the number we consider is 457,875. Is it divisible by 3 or 9? The sum of the digits is $4 + 5 + 7 + 8 + 7 + 5 = 36$, which is divisible by 9 (and then, of course, divisible by 3 as well), so the number 457,875 is divisible by 3 and by 9. If by some remote chance it is not immediately clear to you whether the sum of the digits is divisible by 3 or 9, then continue with this process; take the sum of the digits of your original sum and continue adding the digits until you can visually make an immediate determination of divisibility by 3 or 9.

Let's consider another example. Is the number 27,987 divisible by 3 or 9? The sum of the digits is $2 + 7 + 9 + 8 + 7 = 33$, which is divisible by 3 but not by 9; therefore, the number 27,987 is divisible by 3 and not by 9.

If this divisibility rule is mentioned in school settings, what is typically missing from the instruction of this rule is *why* it works. Here is a brief discussion about why this rule works as it does. Consider the decimal number *abcde*, whose value can be expressed in the following way:

$$N = 10^4a + 10^3b + 10^2c + 10d + e = (9 + 1)^4a + (9 + 1)^3b + (9 + 1)^2c + (9 + 1)d + e.$$

After expanding each of the binomials, we can now represent all of the multiples of 9 as $9M$ to simplify this as

$$N = [9M + (1)^4]a + [9M + (1)^3]b + [9M + (1)^2]c + [9 + (1)]d + e.$$

Then, factoring out $9M$, we get $N = 9M[a + b + c + d] + a + b + c + d + e$, which implies that the divisibility of N by 3 or 9 depends on the divisibility of $a + b + c + d + e$ by 3 or 9, which is the sum of the digits.

As you can see, things become so much better understood and appreciated when the reason for a “rule” is presented.

WHEN IS A NUMBER DIVISIBLE BY 11?

When a teacher shows the class something that is not directly specified in the school curriculum, it often generates some enjoyment and can be motivating. Take, for example, a method of determining whether a number is divisible by 11, without actually carrying out the division process. The problem is easily solved if you have a calculator at hand, but that is not always the case. Besides, there is such a clever “rule” for testing for divisibility by 11 that it is worth knowing just for its cleverness.

The rule is quite simple: If the difference of the sums of the alternate digits is divisible by 11, then the original number is also divisible by 11. That sounds a bit complicated, but it really isn't. Finding the sums of the alternate digits means that you begin at one end of the number, and you take the first, third, fifth, etc., digits and add them together. Then you add the remaining (even-placed) digits. Subtract the two sums and inspect for divisibility by 11.

This rule is probably best demonstrated through an example. Suppose we test 918,082 for divisibility by 11. We begin by finding the sums of the alternate digits: 9 + 8 + 8 = 25 and 1 + 0 + 2 = 3. Their difference is 25 – 3 = 22, which is divisible by 11, and so the number 918,082 is divisible by 11. We should point out that if the difference of the sums is equal to zero, then we can conclude that the original number is divisible by 11, since zero is divisible by all numbers. We see this in the following example: testing the number 768,614 for divisibility by 11, we find that the difference of the sums of the alternate digits (7 + 8 + 1 = 16 and 6 + 6 + 4 = 16) is 16 – 16 = 0, which is divisible by 11. Therefore, we can conclude that 768,614 is divisible by 11.

In case you may be wondering why this technique works, we offer the following. Consider the decimal number $N = abcde$, which then can be expressed as

$$N = 10^4a + 10^3b + 10^2c + 10d + e = (11-1)^4a + (11-1)^3b + (11-1)^2c + (11-1)d + e.$$

This can be written as

$$N = [11M + (-1)^4]a + [11M + (-1)^3]b + [11M + (-1)^2]c + [11 + (-1)]d + e,$$

where, after expanding each of the binomials, $11M$ represents the terms which are multiples of 11 written together. Factoring out the $11M$ terms, we get $N = 11M[a + b + c + d] + a - b + c - d + e$, which leaves us with an expression that would be divisible by 11, but only if this last part of the previous expression is divisible by 11, namely, $a - b + c - d + e = (a + c + e) - (b + d)$, which just happens to be the difference of the sums of the alternate digits. This is a handy little “trick” that can also enhance your understanding of arithmetic. By the way, another way of looking at this trick is to say that the number 24,847,291 is divisible by 11 if and only if we obtain a number that is divisible by 11; let's see what we get: 2 – 4 + 8 – 4 + 7 – 2 + 9 – 1 = 15. Therefore, since the difference of the sums was 15, which is not divisible by 11, we know that 24,847,291 is not divisible by 11.

DIVISIBILITY BY PRIME NUMBERS

In today's technological world, arithmetic skills and competencies seem to be relegated to a back burner, since a calculator is so easily available. We can assume that most adults can determine when a number is divisible by 2 or by 5, simply by looking at the last digit (i.e., the units digit) of the number. That is, if the last digit is even (such as 2, 4, 6, 8, 0), then the number itself will be divisible by 2. Furthermore, if the number formed by the last two digits is divisible by 4, then the original number itself is divisible by 4. Also, if the number formed by the last three digits is divisible by 8, then the original number itself is divisible by 8. This rule can be extended to divisibility by higher powers of 2 as well.

Similarly, for the number 5: If the last digit of the number being inspected for divisibility by 5 is either a 0 or 5, then the number itself will be divisible by 5. If the number formed by the last two digits is divisible by 25, then the original number itself is divisible by 25. This is analogous to the rule for powers of 2. Have you guessed what the relationship here is between powers of 2 and 5? Yes, they are the factors of 10, the basis of our decimal number system.

Having completed in the previous discussions, the nifty techniques for determining whether a number is divisible by the primes 3, 9, and 11, the question then is: Are there also rules for divisibility by other prime numbers? Let's consider divisibility rules by prime numbers.

Aside from the potential usefulness of being able to determine whether a number is divisible by a prime number, the investigation of such rules will provide for a better appreciation of mathematics, that is, divisibility rules provide an interesting “window” into the nature of numbers and their properties. Although this is a topic that is typically neglected from the school curriculum, it can prove useful in everyday life.

The smallest prime number that we have not yet discussed in our quest for divisibility rules is the number 7. As you will soon see, some of the divisibility rules for prime numbers are almost as cumbersome as an actual division algorithm, yet they are fun, and, believe it or not, can come in handy. As we begin our quest for divisibility rules for the early prime numbers, we will begin with the following rule for divisibility by 7.

The rule for divisibility by 7: Delete the last digit from the given number, and then subtract twice this deleted digit from the remaining number. If the result is divisible by 7, then the original number is divisible by 7. This process may be repeated until we reach a number that we can visually inspect as one that is divisible by 7.

Let's consider an example to see how this rule works. Suppose we want to test the number 876,547 for divisibility by 7. Begin with 876,547 and delete its units digit, 7, and subtract its double, 14, from the remaining number: 87,654 – 14 = 87,640. Since we cannot yet visually inspect the resulting number for divisibility by 7, we continue the process. We delete the units digit, 0, from the previously resulting number 87,640, and subtract its double (which is still 0) from the remaining number to get 8,764 – 0 = 8,764. It is unlikely that we can visually determine whether this number, 8,764, is divisible by 7, so we continue the process. Again, we delete the last digit, 4, and subtract its double, 8, from the remaining number to get 876 – 8 = 868. Since we still cannot visually inspect the resulting number, 868, for divisibility by 7, we again continue the process.

Continuing with the resulting number, 868, we once again delete its units digit, 8, and subtract its double, 16, from the remaining number to get 86 – 16 = 70, which is divisible by 7. Therefore, the number 876,547 is divisible by 7.

Before continuing with our discussion of divisibility of prime numbers, you might want to practice this rule with a few randomly selected numbers, and then check your results with a calculator.

Now for the beauty of mathematics! Why does this rather strange procedure actually work? To see why things work is the wonderful aspect of mathematics—it enlightens us!

To justify the technique of determining divisibility by 7, consider the various possible terminal digits (that we are "dropping") and the corresponding subtraction that is actually being done after dropping the last digit. In the chart below you will see how in dropping the terminal digit and doubling it, we are essentially subtracting a multiple of 7. That is, we have taken "bundles of 7" away from the original number. Therefore, if the remaining number is divisible by 7, then so is the original number, because you have separated the original number into two parts, each of which is divisible by 7, and therefore, the entire number must be divisible by 7.

There is another way to argue why this method always works, and you may also want to give this some thought: Removing the final digit and then subtracting twice this digit from the remaining number is equivalent to subtracting 21 times the final digit from the number and then dividing the resulting number by 10. (The latter is certainly possible, since the number resulting from the first step must terminate in the digit 0.) Since 21 is divisible by 7, and 10 is not, the resulting number is divisible by 7 if and only if the original number was divisible by 7.

Terminal Digit	Number Subtracted from Original		Terminal Digit	Number Subtracted from Original
1	$20 + 1 = 21 = 3 \cdot 7$		5	$100 + 5 = 105 = 15 \cdot 7$
2	$40 + 2 = 42 = 6 \cdot 7$		6	$120 + 6 = 126 = 18 \cdot 7$
3	$60 + 3 = 63 = 9 \cdot 7$		7	$140 + 7 = 147 = 21 \cdot 7$
4	$80 + 4 = 84 = 12 \cdot 7$		8	$160 + 8 = 168 = 24 \cdot 7$
			9	$180 + 9 = 189 = 27 \cdot 7$

The next prime number that we have not yet considered for divisibility is the number 13.

The rule for divisibility by 13: The procedure here is similar to that used for testing divisibility by 7, except that instead of subtracting *twice* the deleted digit, we subtract *nine* times the deleted digit each time.

Perhaps it is best for us to do an example applying this rule. Let us check for divisibility by 13 for the number 5,616. We begin with our starting number, 5,616, and delete its units digit, 6, and subtract nine times 6, or 54, from the remaining number to get 561 – 54 = 507.

Since we still cannot visually inspect the resulting number for divisibility by 13, we continue the process. With this last resulting number, 507, we delete its units digit, 7, and subtract nine times this digit, 63, from the remaining number, which gives us 50 – 63 = –13, which *is* divisible by 13; therefore, the original number, 5,616, is divisible by 13.

In this rule for divisibility by 13, you might wonder how we determined the "multiplier" to be 9. We sought the smallest multiple of 13 that ends in a

1. That was 91, where the tens digit is 9 times the units digit. Once again consider the various possible terminal digits and the corresponding subtractions in the following table.

Terminal Digit	Number Subtracted from Original	Terminal Digit	Number Subtracted from Original
1	$90 + 1 = 91 = 7 \cdot 13$	5	$450 + 5 = 455 = 35 \cdot 13$
2	$180 + 2 = 182 = 14 \cdot 13$	6	$540 + 6 = 546 = 42 \cdot 13$
3	$270 + 3 = 273 = 21 \cdot 13$	7	$630 + 7 = 637 = 49 \cdot 13$
4	$360 + 4 = 364 = 28 \cdot 13$	8	$720 + 8 = 728 = 56 \cdot 13$
		9	$810 + 9 = 819 = 63 \cdot 13$

In each case, a multiple of 13 is being subtracted one or more times from the original number. Hence, if the remaining number is divisible by 13, then the original number is divisible by 13.

Divisibility by 17: Delete the units digit and subtract *five* times the deleted digit from the remaining number until you reach a number small enough to determine its divisibility by 17.

We justify the rule for divisibility by 17 as we did for the rules for 7 and 13. Each step of the procedure subtracts a “bundle of 17s” from the original number until we reduce the number to a manageable size and can make a visual inspection for divisibility by 17.

The patterns developed in the preceding three divisibility rules (for 7, 13, and 17) should lead you to develop similar rules for testing divisibility by larger primes. The following chart presents the “multipliers” of the deleted terminal digits for various primes.

To Test Divisibility by	7	11	13	17	19	23	29	31	37	41	43	47
Multiplier	2	1	9	5	17	16	26	3	11	4	30	14

You may want to extend this chart. It's fun, and it will increase your perception of mathematics. You may also want to extend your knowledge of divisibility rules to include composite (i.e., nonprime) numbers.

Divisibility by composite numbers: A given number is divisible by a composite number if it is divisible by each of its relatively prime factors. The chart below offers illustrations of this rule. You might want to complete the chart to include composite numbers up to the number 48.

To Be Divisible by	6	10	12	15	18	21	24	26	28
The Number Must Be Divisible by	2, and 3	2, and 5	3, and 4	3, and 5	2, and 9	3, and 7	3, and 8	2, and 13	4, and 7

You now have a rather comprehensive list of rules for testing divisibility, as well as an interesting insight into elementary number theory. An interested reader may want to test these rules (to instill even greater familiarity with numbers) and try to develop rules to test divisibility by other numbers in base ten and to generalize these rules to other bases.

SQUARING NUMBERS QUICKLY

We all learned in school how to multiply two multidigit numbers using pencil and paper. However, if we want to multiply a number by itself (that is, to square the number), there exist shortcuts to get the answer. Moreover, the multiplication of any two numbers can be written as a combination of squares of sums and differences of these numbers. Hence, knowing how to add, subtract, and square numbers is actually enough to compute the product of any two numbers.

Squaring Numbers with a Last Digit of 5

Here is a quick way to square any number with a last digit of 5: We delete the last digit and we are left with some number N. Multiplying N by $N + 1$ and appending the digits 2 and 5 at the end yields the correct result.

For example, to compute 85^2, we delete 5, multiply the remaining digit, 8, by 9, giving 72, and append 25 at the end. The result is 7,225, which is 85^2.

Why does this rule work? If we let N denote the number that remains after we have dropped the last digit, then we can write the square of the number as $(10 \bullet N + 5)^2 = 100 \bullet N^2 + 100 \bullet N + 25 = 100 \bullet N \bullet (N + 1) + 25$. The product $N \bullet (N + 1)$ represents the amount of hundreds in the result. But by writing its numerical value in front of the digits 2 and 5, we assign the place value of a hundred to this number and, according to our little calculation, will end up with the square of the original number.

Squaring the Numbers between 40 and 60

There is also a quick way to square the numbers between 40 and 60. Perhaps you have already figured out the rule by yourself. We merely develop a proof similar to the earlier one. So here is the trick: Any number between 40 and 60 (not including 40 and 60) can be written as $50 \pm N$, where N is a single-digit number (for example, $58 = 50 + 8$ and $43 = 50 - 7$). To do this quick calculation for 57^2, we begin by adding $25 + 7 = 32$ and tagging on $7^2 = 49$ to get 3,249. By the way, the 7 comes from $57 = 50 + 7$. Similarly, to calculate 48^2, we subtract $25 - 2 = 23$ and tag on $2^2 = 4$, which we write as 04, to get 2,304. Again, we get the 2 since $48 = 50 - 2$. The reason that this works is that the square of such a number gives $(50 \pm N)^2 = 2{,}500 \pm 100N + N^2 = 100(25 \pm N) + N^2$, so the leading digits of $(50 \pm N)^2$

are 25 ± N, followed by N^2, written as a two-digit number.

Squaring Arbitrary Numbers

The two tricks we have just discussed settle the case for when the first or the last digit is a 5 as well as for all numbers between 40 and 60. But what about all the other numbers? Although the trick presented above relied on the fact that 2 • 5 = 10, we can use the same kind of reasoning to simplify computing the squares of arbitrary numbers. Let us compute the square of a number in which the last digit is less than 5, such as, for example, 73^2. It is helpful to think of it as $(70 + 3)^2$ = 4,900 + 2 • 210 + 9 = 5,329. On the other hand, if the last digit is greater than 5, such as 29^2, we resort to writing this as $29^2 = (30 – 1)^2$ = 900 – 2 • 30 + 1 = 841.

Summing up, when you want to square a number, you can often simplify this problem by first decomposing this number in a clever way, or making use of the digit-5 tricks presented here. Being very good at squaring numbers can also help you perform arbitrary multiplications, and it gives you a more sophisticated view of arithmetic.

SQUARES AND SUMS

Squares are rather ubiquitous in mathematics. Yet what is not very well known is that every integer number is either a square number or the sum of two, three, or four square numbers. Although conjectured by the Greek mathematician Diophantus (201–285 CE) in his book *Arithmetica*, he was not able to provide a proof to justify his belief. This astonishing fact was first proved by the French mathematician Joseph-Louis Lagrange (1736–1813). The result is known as Lagrange's *four-square theorem*, a concept unfortunately not presented in the school years.

Let's take a look at what this theorem tells us. Consider the number 18, and we will try to represent it as a sum of four or fewer squares: $18 = 3^2 + 3^2 = 4^2 + 1^2 + 1^2 = 3^2 + 2^2 + 2^2 + 1^2$. Here we have represented 18 at the sum of two, three, and four squares.

Here are a few more examples:

$23 = 3^2 + 2^2 + 2^2 + 1^2$
$43 = 5^2 + 3^2 + 3^2$
$97 = 8^2 + 5^2 + 2^2 + 2^2$

An interested reader may want verify this unusual result with other numbers.

USING SQUARES TO MULTIPLY ARBITRARY

NUMBERS

If you want to multiply two numbers whose sum happens to be an even number (that is, two odd numbers or two even numbers), you can use the formula $(a + b)(a - b) = a^2 - b^2$ to reduce the problem to computing two squares and taking their difference. For example, the product 47 • 59 can be written as $(53 - 6)(53 + 6) = 53^2 - 6^2 = 2{,}809 - 36 = 2{,}773$ (and, by the way, we already discussed earlier how to very quickly compute 53^2). Remember, this trick does not work when one number is odd and the other is even. However, the product of any two such numbers of different parity can also be computed as a difference of squares by employing the formula $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

We calculate $(a + b)^2 - (a - b)^2 = a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)$ and obtain

$$a \cdot b = \frac{(a+b)^2 - (a-b)^2}{4},$$

which is a representation of an arbitrary product $a \bullet b$ in terms of the difference of two square numbers.

As a matter of fact, knowing from memory the squares of all numbers from, say, 1 to 20, and being aware of the formulas presented above, you can easily compute the product of any two such numbers. In this sense, remembering multiplication tables for mental arithmetic is not necessary, it suffices to remember all the squares. The Babylonian clay tablets indicate that the Babylonians used tables of squares and multiplied numbers in the way we presented here, that is, by transforming products to differences of squares.[1]

AN ALTERNATIVE METHOD FOR EXTRACTING A SQUARE ROOT

Why would anyone want to find the square root of a number today without using a calculator? Surely, no one would do such a thing. However, you might be curious to know what is actually being done in the process of finding the square root of a number. This would allow you some independence from the calculator. The procedure typically taught in schools many years ago was somewhat rote and had little meaning to the students other than obtaining an answer. We will present a method that was generally not taught in the schools but gives a good insight into the meaning of a square root. The beauty of this method is that it really allows you to understand what is going on, unlike the algorithm that was taught in schools before the advent of calculators. This method was first published in 1690 by the English mathematician Joseph Raphson (1648–1715) in his book, *Analysis alquationum universalis*; Raphson attributed it to Sir Isaac Newton (1643–1727) in his 1671 book *Method of Fluxions*, which was not officially published until 1736. Therefore, the algorithm bears both names, the *Newton-Raphson method*.

It is perhaps best to see the method as it is used in a specific example: Suppose we wish to find $\sqrt{27}$. Obviously, the calculator could be used here. However, you might like to guess at what this value might be. Certainly, it is between $\sqrt{25}$ and $\sqrt{36}$, or between 5 and 6, but closer to 5.

Suppose we guess at 5.2. If this were the correct square root of 27, then if we were to divide 27 by 5.2, we would get 5.2. But this is not the case; since $\frac{27}{5.2} \neq 5.2$, we know that $\sqrt{27} \neq 5.2$.

In order to find a closer approximation, we will calculate $\frac{27}{5.2} = 5.192$. Since 27 ≈ (5.2) • (5.192), one of the factors (in this case, 5.2) must be bigger than $\sqrt{27}$ and the other factor (in this case, 5.192) must be less than $\sqrt{27}$. Hence, $\sqrt{27}$ is sandwiched between the two numbers 5.2 and 5.192; that is, $5.192 < \sqrt{27} < 5.2$. So it is plausible to infer that the average of these two numbers, that is, $\frac{5.2+5.192}{2} = 5.196$, is a better approximation for $\sqrt{27}$ than either 5.2 or 5.192.

This process continues, each time with additional decimal places, so that an allowance is made for a closer approximation. That is, $\frac{5.192+5.196}{2} = 5.194$, then $\frac{27}{5.194} = 5.19831$. Taking this another step to get an even closer approximation of $\sqrt{27}$, we continue this process: $\frac{27}{5.19831} = 5.193996$, then $\frac{5.19831+5.193996}{2} = 5.1961530$.

This continuous process provides insight into the finding of the square root of a number that is not a perfect square. As seemingly cumbersome as the method may be, it surely provides you with a genuine understanding about the value of a square root.

SENSIBLE NUMBER COMPARISONS

Comparing large numbers in today's technological world is something that should not be neglected in the school curriculum. There are numerous techniques for comparing numbers that are not simply written out in their typical decimal form but, rather, in exponential form. We will consider one here, just to give you an opportunity to see the kind of manipulations we can make to answer questions that initially seem impossible to decipher.

The question we could be faced with is which of the two values is greater, 31^{11} or 17^{14}? In order to answer this question, we will change these bases to numbers that can be reduced to a common base. It is clear that $31^{11} < 32^{11} = \left(2^5\right)^{11} = 2^{511} = 2^{55}$. Whereas $17^{14} > 16^{14} = \left(2^4\right)^{14} = 2^{56}$. Now we can clearly see that because $2^{56} > 2^{55}$, we can conclude that $17^{14} > 31^{11}$. Because of the enormous magnitude of each of these two numbers, it would be very difficult to determine which is larger without converting these to common bases.

Another comparison of number magnitudes can be demonstrated by determining the following. Which of the two following expressions is larger, $\sqrt[9]{9!}$ or $\sqrt[10]{10!}$ (where the factorial expression $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot n$)? In this case, we will raise each of the two numbers to be compared to the ninetieth power, since 90 is the common multiple of 9 and 10.

$$\left(\sqrt[9]{9!}\right)^{90} = (9!)^{\frac{1}{9}\cdot 90} = (9!)^{10} = (9!)^{9} \cdot (9!)$$

$$\left(\sqrt[10]{10!}\right)^{90} = (10!)^{\frac{1}{10}\cdot 90} = (10!)^{9} = (9!)^{9} \cdot (10)^{9}$$

If we divide each of the two end results by the same number, in this case $(9!)^9$, we find that the remaining numbers are then 9! and 10^9. Since each of the nine factors of 9! is smaller than the nine factors of 10^9, 9! is clearly less than 10^9; therefore, we can conclude that $\left(\sqrt[9]{9!}\right)^{90} < \sqrt[10]{10!}$. Again you will notice how searching for commonality allows us to make comparisons more easily than actually computing these incredibly large numbers.

EUCLIDEAN ALGORITHM TO FIND THE GCD

What is the *greatest common divisor* (gcd) of 15 and 10? Most people would know intuitively that the answer is 5. This intuition is most likely built up through the study of the multiplication table and through practice with arithmetic. Going further, what is gcd(364, 270)? (Symbolically, this means the greatest common divisor, gcd, of 364 and 270.) At this point, intuition doesn't help as much as it did when considering the more familiar numbers 15 and 10. One option is to calculate the prime decompositions of both numbers and obtain the gcd by looking at the lowest powers of the distinct primes showing up in both the prime decompositions. Another method is to perform the Euclidean algorithm.

Consider two positive integers a and b, where $a > b$. We can always use long division to find the remainder when we divide a by b, that is, $a = qb + r$ where q is the quotient and r is the remainder. If we set $a = 364$ and $b = 270$, and calculate, then we have $364 = 1 \cdot 270 + 94$. The Euclidean algorithm revolves around the fact that gcd(a, b) = gcd(b, r). (Any divisor of both a and b is certainly a divisor of r, and any divisor of both b and r is certainly a divisor of a.) In our example, gcd(364, 270) = gcd(270, 94).

At this point, performing the long division of 270 divided by 94 would yield $270 = 2 \cdot 94 + 82$. If we think of $a = 270$ and $b = 94$, then notice the previous observation about gcds applies once more: gcd(270, 94) = gcd(94, 82).

This process is to be repeated until we get a remainder of 0.

$94 = 1 \cdot 82 + 12$, so we have gcd(94, 82) = gcd(82, 12).
$82 = 6 \cdot 12 + 10$, so gcd(82, 12) = gcd(12, 10).

$12 = 1 \cdot 10 + 2$, so $\gcd(12, 10) = \gcd(10, 2)$.
$10 = 5 \cdot 2 + 0$, so $\gcd(10, 2) = \gcd(2, 0)$.

But the gcd of 2 and 0 is 2 itself, since any integer is a factor of 0, that is, $0 = 0 \cdot n$ for any n. More explicitly, $2 = 1 \cdot 2$ and $0 = 0 \cdot 2$, showing that 2 is a divisor of both 2 and 0. Clearly, 2 is the largest divisor that can go into 2, hence $\gcd(2, 0) = 2$.

Using a string of equalities, we have:

$$\gcd(364, 270) = \gcd(270, 94) = \gcd(94, 82) = \gcd(82, 12) = \gcd(12, 10) = \gcd(10, 2) = \gcd(2, 0) = 2.$$

Let's check this result using the prime decomposition method mentioned earlier. This gives us $364 = 2^2 \cdot 7 \cdot 13$ and $270 = 2 \cdot 3^3 \cdot 5$. The only common prime is 2, and the lowest power of 2 shown in the prime decompositions is 1, hence the $\gcd(364, 270) = 2^1 = 2$.

Why would we use the Euclidean algorithm if the prime decomposition method is available? It seems like the prime decomposition method can be faster if you can quickly compute the prime decompositions of the integers in question. The “quickly” part turns out to be the problem. For very large integers, the prime decomposition can be difficult or inefficient to compute. In fact, much of the security in commerce and the internet today depends on the difficulty of figuring out whether or not a large integer is prime. The Euclidean algorithm avoids this problem if you merely want to the find the gcd of the numbers in question.

The Euclidean algorithm is a very old and efficient algorithm that can compute the greatest common divisor of two integers. While intuition is sufficient in the cases involving relatively small integers, the Euclidean algorithm is able to leverage the knowledge of long division to find the greatest common divisors of as large a pair of integers as we desire to compute. The modest prerequisites combined with the usefulness of the algorithm ensure its lasting place in our arithmetic tool kit.

SUMS OF POSITIVE INTEGERS

You may have heard the often-told childhood story of the famous German mathematician Carl Friedrich Gauss (1777–1855), who performed a remarkable feat when he was just in elementary school. His math teacher had given the class the task of adding all the positive integers from 1 to 100. The teacher expected the task of evaluating $1 + 2 + 3 + \cdots + 100$ to keep the students, including the young Gauss, busy for some time. After all, Gauss was just a little boy! To his teacher's amazement, Gauss did the calculation in just a few seconds, and apparently he was the only one with the right answer.

Young Gauss explained that rather than adding the numbers sequentially, as the rest of his class was doing, he realized that the one hundred terms in this sum can be broken up into pairs: 1 + 100, 2 + 99, 3 + 98, 4 + 97, and so on. There are 50 such pairs, with each pair having a sum of 101. Therefore, the total sum is 50 • 101 = 5,050.

Many will recall that Gauss's technique can be extended to find a formula for the sum $1+2+\cdots+n=\frac{n(n+1)}{2}$, where *n* is an arbitrary positive integer. There are other simple ways to establish this formula, which may not have been shown in school, such as the following visual demonstration.

Consider the diagram shown in figure 1.1, with *n* boxes in the bottom row and right-side column.

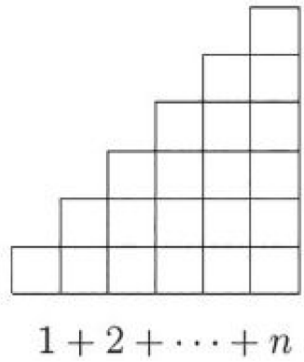

Figure 1.1

The "staircase" in figure 1.1 represents the sum 1 + 2 + 3 + • • • + *n*. To see this, break up the staircase into vertical columns. Looking from left to right, the left-most vertical column has 1 square, the next column over has 2 squares stacked vertically, the third column has 3 squares, and so on. The last column has *n* squares stacked vertically. The area of the staircase is the sum of the areas of the columns, thus the area of the staircase is 1 + 2 + 3 + • • • + *n*.

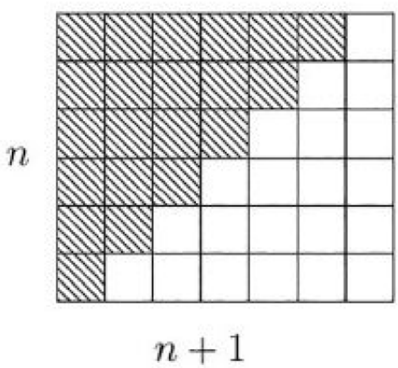

Figure 1.2

Create a shaded copy of the inverted staircase and join it with the original one to form the rectangle shown in the figure 1.2. The rectangle has area $n(n + 1)$. The shaded and unshaded "staircases" have the same area. Thus, dividing the rectangle's area in half, each staircase has an area of $\frac{n(n+1)}{2}$. Recall that the staircase represents the sum of the integers from 1 to n. We can, therefore, conclude that $1+2+\cdots+n=\frac{n(n+1)}{2}$.

This sum formula can be demonstrated in different ways, the most famous of which is probably the one that the young Gauss used. For those who prefer the visual version, the staircase method provides another elegant way of seeing how this formula holds true.

SUMS OF ODD POSITIVE INTEGERS

A few simple calculations can sometimes be enough to reveal marvelous patterns in numbers. Notice that $1 = 1^2$, which is a perfect square, that is, a number that is equal to the product of two equal integers. Notice that $1 + 3 = 4 = 2^2$, also a perfect square. Notice that $1 + 3 + 5 = 9 = 3^2$, is again a perfect square. This pattern of squares continues as expected.

We can use the following squares shown in figure 1.3 to understand this pattern:

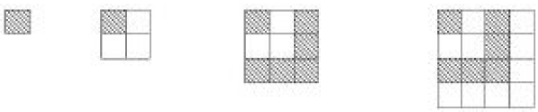

Figure 1.3

Looking from left to right, we are building progressively larger squares. At each stage, we are adding on a new L-shaped piece on the bottom right corner to get to the next larger square. The total area of the original upper left corner square and the additional L-shaped pieces equals the area of the whole square. For example, the square on the right represents $1 + 3 + 5 + 7 = 16 = 4^2$, where the 1 and 5 are areas of the shaded pieces, and the 3 and 7 are areas of the unshaded pieces. The L-shaped pieces (and the original square) all have odd areas, and the sum of these odd areas is equal to the area of the whole square, which justifies our arithmetic statement about the sum of odd integers being equal to squares.

We can also construct this pattern another way. Instead of adding odd positive integers together to get squares, let's consider the following table of squares and notice where the odd positive integers show up:

n	n^2	
0	0	0 + 1 = 1

1	1	1 + 3 = 4
2	4	4 + 5 = 9
3	9	9 + 7 = 16
4	16	16 + 9 = 25

This is a table that helps us see another pattern. To go from one row to the next, we are adding odd positive integers to the perfect squares in order to get the next perfect square. Let's focus on the third column, which represents the changes to the second column to get to the next row. Observe that from the first row to the second row, 0 + 1 = 1. From the second row to the third row, we have 1 + 3 = 4. From the third row to the fourth row, notice 4 + 5 = 9. Similarly, moving along to the next row, we obtain 9 + 7 = 16. In other words, the differences between consecutive perfect squares are the consecutive *odd* positive integers, which we might have expected—given our geometric demonstration above. Notice that by going backward from the last row up, 16 – 9 = 7 and 9 – 4 = 5. Going further back, we also have 4 – 1 = 3 and 1 – 0 = 1. Thus, starting at 0, we can add up these differences to get 16: 1 + 3 + 5 + 7 = 16, which once again leads us to notice that the sum of consecutive odd numbers results in a square number.

Such demonstrations may be helpful if you prefer to learn visually. But do not despair, for those who prefer algebra, this idea can also be demonstrated algebraically. Consider the consecutive squares n^2 and $(n + 1)^2$, for some non-negative integer n. The difference can be computed as follows:

$$(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1.$$

Notice that the difference simplifies to $2n + 1$, which is an odd positive integer when n is non-negative.

The connection between the sum of odd positive integers and perfect squares has very humble prerequisites. The pattern itself can be seen in elementary calculations. Tables of values for the squaring function, some simple geometry, and a little bit of algebra all work together to further demonstrate this marvelous pattern in numbers.

THE REALM OF NONTERMINATING DECIMALS

Nonterminating decimals are numbers that have an infinite sequence of digits after their decimal point. They arise in various situations, some of which you are certainly familiar with. For instance, dividing one integer by another may lead to a non-terminating decimal, as well as taking the square root of some integers. The two most important mathematical constants, e = 2.718281…and π = 3.141592…, are also nonterminating decimals. They have infinite sequences of digits after the decimal point, and human intuition often fails when encountering mathematical notions of infinity. There are many astonishing and remarkable facts about nonterminating decimals, some of which we believe you may not have been aware of.

Repeating Decimals

Dividing two integers using pencil and paper is a basic topic in elementary arithmetic with which we are all familiar. The result can be a terminating decimal (for example, $\frac{21}{7}$ = 3, and $\frac{7}{4}$ = 1.75, etc.) or a decimal with a fractional part consisting of a repeating sequence of digits (for example, $\frac{7}{3} = 2.\overline{3} = 2.333333\ldots$). Notice that the bar over the 3 indicates that it repeats endlessly, or infinitely. The inverse problem is sometimes less emphasized, that is, given a decimal number with a fractional part, how can we construct the fraction that this number represents? This is a very easy task if it is a terminating decimal, but what if the number is something such as 1.428574285742857...? At first sight it is not so obvious as to how to convert this number into a fraction. Unfortunately, using a pocket calculator may not be of much help either. Nevertheless, it can be done without much effort by applying a little trick. First, we need to know the "length" of the repeating part, measured in digits. For example, the repeating part of $x = 1.\overline{42857}$ is 5 digits long. Now multiply *x* by a corresponding power of ten and subtract *x* from the result. This leads directly to the desired representation of *x* as a fraction. In our example we would have $x = 1.\overline{42857}$, then, since the repetition is 5 digits, we multiply by 10^5 to get: $100{,}000x = 142857.\overline{42857}$·

Subtracting the first equation from the second yields 100,000*x* – *x* = 142,856. So, we obtain 99,999*x* = 142,856 and, thus, $x = \frac{142856}{99999}$, which cannot be reduced any further. Applying this conversion procedure to the repeating decimal $0.\overline{9} = 0.999999\ldots$ reveals that the decimal representation of a number is not always unique: For $x = 0.\overline{9}$, we would get 10*x* = $9.\overline{9}$, and by subtracting we get 9*x* = 9, or *x* = 1. Thus implying that $0.\overline{9} = 1$. This means that $0.\overline{9}$ is really just another representation of 1. Intuitively you might be tempted to believe that $0.\overline{9}$ should be a tiny little bit smaller than 1, but it isn't (as we just proved). We cannot always trust our intuition when dealing with infinite sequences.

Irrational Numbers

If a nonterminating decimal does not have a repeating pattern of numbers, then it cannot be written as a fraction. Thus, it is not a rational number and therefore, such numbers are then called irrational. Euler's number *e* and π are both examples of irrational numbers. The square root of a natural number is irrational whenever the number is not a square number; for example, $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}$, and $\sqrt{11}$ are all irrational numbers. They have infinite nonrepeating sequences of digits after the decimal point. Clearly, it is not possible to know all decimal values of , *e*, or $\sqrt{2}$, since they go on infinitely far and without any obvious pattern.

Mathematicians are always in search of patterns among the digits of the decimal approximation of π, where sometimes patterns can be established. For example, the British mathematician John Conway (1937–) has indicated that if you separate the decimal value of π into groups of ten places, the probability of each of the ten digits appearing in any of these blocks is about 1 in 40,000. Yet he noted that it does occur in the seventh such group of ten places, as you can see from the grouping below:

π = 3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 **5923078164** 0628620899 8628034825 3421170679 8214808651 3282306647 0938446095 5058223172 5359408128....

However, accompanied by the advances in computational power, the record precision of numerical approximations to is steadily increasing. Recently, the first 22,459,157,718,361 (about 22.5 trillion) decimal places of have been computed, but most likely at your reading, this record will already have been broken. Irrational numbers are a challenge not only for computers but also for people who like to memorize numbers. Believe it or not, there are even separate world-ranking lists for memorizing digits of , *e*, and $\sqrt{2}$. For example, the current record (as of March 2017) for is held by Suresh Kumar, who was able to recite its first 70,030 digits, and likely this will have been superseded at this reading.[2] On the other hand, it is often overlooked that not all irrational numbers have decimal representations that are hard to memorize. For example, consider the number 0.12345678910111213141516171819202122232425262728293031323334....

Can you see how this sequence continues? This number is called Champernowne's constant, named after the English mathematician D. G. Champernowne (1912–2000), who published it as an undergraduate student in 1933. Its fractional part is obtained by concatenating all positive integers in order—a sequence everyone can write down immediately. Observe that any finite sequence of numbers will occur somewhere in the infinite sequence of digits of this number. In fact, any finite sequence of digits will even occur infinitely often in Champernowne's number. For example, if we would use a sequence of numbers to represent the code imprinted in the sequence of nucleobases in a DNA molecule, then the exact sequence of numbers representing your own DNA molecule would occur somewhere in the endless digits of Champernowne's number. Of course, the same would be true for the DNA molecule of any other organism that lives, or has ever lived, on Earth. This may be hard to believe, but it is just a simple consequence of the concept of infinity and the definition of Champernowne's constant (and it does not imply that this number has any special meaning).

These strange features of the seemingly all-encompassing Champernowne's constant once again illustrate that infinite sequences (and mathematical notions of infinity in general) are completely different from everything we experience in real life. If you are not used to dealing with such concepts, then you may be puzzled by some of the counterintuitive facts accompanying them.

ATOMS IN THE UNIVERSE OF NUMBERS

In chemistry class, we are introduced to the periodic table of elements, which contains all known extant chemical elements. Some of them have only been produced in laboratories and do not occur in nature. As far as we know, all visible matter in the universe is made up of 94 different natural chemical elements, from the lightest element, hydrogen, to the heaviest, plutonium. Although 24 even heavier elements have been produced artificially, they have extremely short half-lives and could not be observed in nature. The 94 natural chemical elements, representing 94 different sorts of atoms, can be regarded as the elementary building blocks of our world. Every piece of matter can be decomposed into a finite number of atoms, each belonging to one of the different elements. For instance, a droplet of water is made up of a vast number of water molecules, each of which is formed by two hydrogen atoms and one oxygen atom. Thus, the droplet contains some number of oxygen atoms and twice that number of hydrogen atoms. Similarly, we can decompose every isolated conglomerate of matter into individual atoms and sort these atoms according to the chemical elements they represent. The word *atom* was created by ancient Greek philosophers and meant the “indivisible,” that is, the smallest unit of matter. In ancient Greece, philosophy, physics, and mathematics were not separate disciplines; they all belonged together as “natural philosophy,” meaning the philosophical study of nature and the physical universe. Ancient Greek philosophers also noticed that smallest, indivisible units exist in the world of numbers as well. They are now called prime numbers—from Latin *numerus primus* (meaning “first numbers”). A prime number is a natural number that has exactly two natural numbers as divisors (the number itself and 1). Bear in mind that 1 is not a prime number by this definition, since it has no other divisor than itself.

Just as a piece of matter can be decomposed into individual atoms, each representing a certain chemical element, every integer greater than 1 can be decomposed into indivisible factors, each of them representing a certain prime number. But while there are only 94 different natural chemical elements in nature, there are infinitely many prime numbers. Notwithstanding the infinite number of primes, the decomposition of an integer into prime factors is unique, just like the decomposition of matter into atoms. This important statement is called the *fundamental theorem of arithmetic*. Euclid of Alexandria (fl. 300 BCE) gave a proof of this theorem in his famous book Στοιχεῖα (Greek: *Stoicheia*), which is now known as Euclid's *Elements*. Although the proof is elementary from a mathematical viewpoint, we will not present it here, since it requires some special notation with which not all readers would be familiar. We will instead try to motivate and explain the result on a heuristic level.

The Fundamental Theorem of Arithmetic

Suppose we are given an arbitrary integer greater than 1. Then either this number is a prime number, meaning it has no divisors other than 1 and itself, or it is not a prime number. If it is prime, the number itself represents its unique prime factor decomposition. However, if the number is not prime, then we can always break it up into prime factors, thereby obtaining a product of prime numbers representing the given number. Such a nonprime number is called a *composite number*. The following table shows the prime factor decompositions of the forty smallest integers greater than 1, which are composite numbers. The prime numbers, which are "missing" from the list are the numbers 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, and 53, and they are identical to their prime factorizations.

$4 = 2^2$	$20 = 2^2 \cdot 5$	$33 = 3 \cdot 11$	$46 = 2 \cdot 23$
$6 = 2 \cdot 3$	$21 = 3 \cdot 7$	$34 = 2 \cdot 17$	$48 = 2^4 \cdot 3$
$8 = 2^3$	$22 = 2 \cdot 11$	$35 = 5 \cdot 7$	$49 = 7^2$
$9 = 3^2$	$24 = 2^3 \cdot 3$	$36 = 2^2 \cdot 3^2$	$50 = 2 \cdot 5^2$
$10 = 2 \cdot 5$	$25 = 5^2$	$38 = 2 \cdot 19$	$51 = 3 \cdot 17$
$12 = 2^2 \cdot 3$	$26 = 2 \cdot 13$	$39 = 3 \cdot 13$	$52 = 2^2 \cdot 13$
$14 = 2 \cdot 7$	$27 = 3^3$	$40 = 2^3 \cdot 5$	$54 = 2 \cdot 3^3$
$15 = 3 \cdot 5$	$28 = 2^2 \cdot 7$	$42 = 2 \cdot 3 \cdot 7$	$55 = 5 \cdot 11$
$16 = 2^4$	$30 = 2 \cdot 3 \cdot 5$	$44 = 2^2 \cdot 11$	$56 = 2^3 \cdot 7$
$18 = 2 \cdot 3^2$	$32 = 2^5$	$45 = 3^2 \cdot 5$	$57 = 3 \cdot 19$

The fact that every integer greater than 1 can be broken up into the product of prime numbers is not so surprising, given the definition of a prime number. A number that is not prime must have integer divisors other than 1 and itself and can therefore be factored, that is, split up into factors (e.g., $12 = 4 \cdot 3$). If any of these factors is not prime, it can also be split up into smaller factors, and so forth. Evidently, the procedure of factoring will stop when all obtained factors cannot be divided any further, that is, when they are all prime numbers (e.g., $12 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3$). So it is actually quite obvious that integers can be represented as products of prime numbers. However, the fundamental theorem of arithmetic also states that this decomposition is unique (the order of the factors is not important; e.g., $12 = 2 \cdot 2 \cdot 3 = 2 \cdot 3 \cdot 2 = 3 \cdot 2 \cdot 2$). For example, $2016 = 2^5 \cdot 3^2 \cdot 7$, and there is no other way to represent 2016 as the product of prime numbers. Thus, independent of the way we do the factorization (by using a certain algorithm or simply by using a trial-and-error strategy), we will end up with five 2s, two 3s, and one 7.

Is Prime Factorization Really Special?

Another way of looking at the fundamental theorem of arithmetic is to view it as a statement on the "composition" of integers rather than on the "decomposition": All integers greater than 1 can be constructed or "composed" by multiplying prime numbers, and for each integer there is only one special composition of primes representing this integer. Hence, prime numbers can truly be regarded as the basic building blocks (or the "atoms") of integers.

One could argue that every integer can as well be constructed by adding a unique number of 1s; for example, 12 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1, so 1 could then be called the building block of all integers. Yet there is a decisive difference to prime factorization: If we want a sum of 1s to become 12, we also need twelve 1s. More generally, if we want to represent an integer *N* as a sum of 1s, we need *N* of them. So we actually need *N* to "construct" *N* as a sum of 1s. As a matter of fact, representing *N* as a sum of 1s does not provide any additional information about *N*; it is essentially just another way of writing this number (such as writing a number using Roman numerals). In contrast, when multiplying primes, the prime numbers themselves "construct" the number *N*. For instance, to obtain 12, the prime factorization $2^2 \cdot 3$ already contains all the information —nothing more is needed.

The Atoms of Numbers

As we pointed out earlier, all molecules (and, more generally, all pieces of matter) consist of specific numbers of atoms from different chemical elements. Analogously, every integer greater than 1 consists of specific numbers of different primes. We can represent molecules by chemical formulas. For example, H_2O for the water molecule (H_2O stands for 2 hydrogen atoms and 1 oxygen atom). Furthermore, every integer greater than 1 can be represented by a unique product of primes, the prime factorization of this number. For instance, the expression $2^5 \cdot 3^2 \cdot 7$ plays the same role for the number 2016 as the chemical formula H_2O does for the water molecule.

Applications of Prime Factorization

For more than 2,000 years, prime numbers and the fundamental theorem of arithmetic seemed to be of little practical value. This changed with the advent of computer technology. The fundamental theorem of arithmetic gives no information about how to obtain an integer's prime factorization; it only guarantees its existence. While there exist systematic methods to decompose an integer into prime factors, the number of operations required in such procedures increases very rapidly with the number of digits of the given integer. Factoring many-digit integers is only possible with the help of computers. However, if the number to be factored is sufficiently large (say, several hundred digits), prime factorization is virtually impossible, even for the most powerful supercomputers. It would simply take too much time. Many public-key cryptosystems for secure data transmission are based on this fact. In public-key cryptography, each user has a pair of cryptographic keys—a public encryption key and a private decryption key. The public encryption key may be widely distributed, while the private decryption key is known only to its owner. A typical application of public-key cryptography are digital signatures used in financial transactions to demonstrate the authenticity of a digital message. There is a mathematical relation between the encryption key and the decryption key, but calculating the private key from the public key is unfeasible, since it would involve finding the prime factors of a very large number. Thus, the safety of such cryptosystems directly depends on the mathematical difficulty of factoring large numbers. Interestingly, a hypothetical quantum computer (that is, a computation system making direct use of quantum-mechanical phenomena) could factor even large integers quickly. This was shown by the American mathematician Peter Williston Shor (1959–), who developed an algorithm for quantum computers that runs exponentially faster than the best currently known algorithm running on a classical computer. However, it has not yet been proved that there does not exist an efficient prime factorization algorithm for classical computers. It's just that nobody has yet found one.

FUN WITH NUMBER RELATIONSHIPS

Today's the school curriculum seems very heavily focused on testing students. This has many teachers gearing their instruction toward passing these examinations. It would be refreshing to encourage teachers to entertain students with number peculiarities. Frankly, the one advantage of taking time to show these number relationships is to demonstrate the beauty that lies well hidden in our number system, which should motivate students toward embracing mathematics. These unexpected relationships are boundless in their manifestations. We will present some of these here as a form of entertainment with the hope that you will then be motivated to seek other such clever patterns.

Let's begin by considering numbers where we will raise each of the digits of the number to the third power and show that their sum is equal to the original number:

$407 = 4^3 + 0^3 + 7^3$
$153 = 1^3 + 5^3 + 3^3$
$371 = 3^3 + 7^3 + 1^3$

A similar situation can be shown for fourth and fifth powers, as in the following examples:

$1,634 = 1^4 + 6^4 + 3^4 + 4^4$
$4,150 = 4^5 + 1^5 + 5^5 + 0^5$.

There are many other numbers that can be expressed as the sum of its digits each taken to the same power. We invite you to begin your search. But first we will start you off with a clue to one such number: 8,208, which can be expressed as the sum of its digits taken to a power. We leave it to you to determine to which power these digits need to be raised.

We can do this again, but this time we have two numbers related to each other in a similar fashion as above, that is, each number can be expressed as the sum of the digits of the other number, each taken to the same power. In our example that follows, the number 136 and the number 244 have this relationship:

$136 = 2^3 + 4^3 + 4^3$; now we take these bases to form the number $244 = 1^3 + 3^3 + 6^3$, whose bases determine the original number.

Another unusual arrangement of powers can be seen from the value of 204^2, which can be shown to be equal to three consecutive numbers taken to third power: $204^2 = 23^3 + 24^3 + 25^3$.

Taking this a step further, we consider the number $8{,}000 = 20^3$, which can also be expressed as a sum of consecutive cubes—this time four cubes—as follows: $20^3 = 11^3 + 12^3 + 13^3 + 14^3$.

There are other numbers that can also be expressed as the sum of consecutive numbers taken to the same power. Before you search for others, we will further entice you with one more example:

$4{,}900 = 70^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \bullet\bullet\bullet + 20^2 + 21^2 + 22^2 + 23^2 + 24^2$.

Now to consider consecutive exponents. There are also numbers that are equal to the sum of the digits raised to consecutive powers, such as the following:

$135 = 1^1 + 3^2 + 5^3$
$175 = 1^1 + 7^2 + 5^3$
$518 = 5^1 + 1^2 + 8^3$
$598 = 5^1 + 9^2 + 8^3$

Expressing numbers as the sum of powers also provides us some further entertainment. Some of these are quite ingenious. For example, in 1772 the famous Swiss mathematician Leonhard Euler (1707–1783) discovered that $159^4 + 158^4 = 635{,}318{,}657 = 133^4 + 134^4$. This can be extended to considering a number, 6,578, which can be expressed as a sum of three fourth-powers in two distinct ways: $6{,}578 = 1^4 + 2^4 + 9^4 = 3^4 + 7^4 + 8^4$. This happens to be the smallest number for which this is possible.

We can also express certain numbers as a sum of equal powers—less than the fourth power:

$65 = 8^2 + 1^2 = 7^2 + 4^2$
$125 = 10^2 + 5^2 = 11^2 + 2^2 = 5^3$
$250 = 5^3 + 5^3 = 13^2 + 9^2 = 15^2 + 5^2$
$251 = 1^3 + 5^3 + 5^3 = 2^3 + 3^3 + 6^3$

Another unusual sum of powers, where each of the numbers is taken to the same power as the original number, is shown here: $102^7 = 12^7 + 35^7 + 53^7 + 58^7 + 64^7 + 83^7 + 85^7 + 90^7$.

Perhaps you will be amused to find a number that is equal to the sum of all of the two-digit numbers that can be formed by the digits of the original number. Our example will be with the number 132, which also happens be the smallest number for which this is possible, namely, 132 = 12 + 13 + 21 + 23 + 31 + 32.

There are boundless curious patterns that exist in our number system. Unfortunately, there rarely seems to be a really good opportunity for them to be presented to students or the general public. However, the fun of discovering these unusual relationships adds to an aspect of mathematics that can be entertaining and also enlightening as you search for more such patterns.

FRIENDLY NUMBERS

As we have indicated, unfortunately, there is hardly enough time in the course of learning mathematics in school to show some of the unusual properties that numbers have, and which have given mathematicians throughout the centuries material to further investigate. We are all aware that certain numbers have properties in common. For example, even numbers are all divisible by 2. We know that odd numbers are not divisible by 2. These are common relationships among numbers. There are, however, relationships between numbers that are quite unusual. One such relationship has been termed numbers that are "friendly" to each other. What could possibly make two numbers friendly? Mathematicians have decided that two numbers are to be considered friendly—or as is sometimes used in the more sophisticated literature, "amicable"—if the sum of the proper divisors[3] (or factors) of one number equals the second number *and* the sum of the proper divisors of the second number equals the first number as well. Sounds complicated? It really isn't. Just take a look at the smallest pair of friendly numbers: 220 and 284. The proper divisors (or factors) of **220** are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110. Their sum is 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = **284**. The proper divisors of **284** are 1, 2, 4, 71, and 142, and their sum is 1 + 2 + 4 + 71 + 142 = **220**. This shows that the two numbers can be considered a pair of *friendly numbers*.

The second pair of friendly numbers, which were discovered by the famous French mathematician Pierre de Fermat (1601–1665) is 17,296 and 18,416. In order for us to establish their friendliness relationship, we need to find all of the prime factors of each of the numbers: $17{,}296 = 2^4 \cdot 23 \cdot 47$, and $18{,}416 = 2^4 \cdot 1{,}151$. Then we need to create all the numbers from these prime factors as follows. The sum of the factors of 17,296 is

1 + 2 + 4 + 8 + 16 + 23 + 46 + 47 + 92 + 94 + 184 + 188 + 368 + 376 + 752 +1081 + 2162 + 4324 + 8648 = 18,416.

The sum of the factors of 18,416 is

$1 + 2 + 4 + 8 + 16 + 1151 + 2302 + 4604 + 9208 = 17,296.$

Once again, we notice that the sum of the factors of 17,296, is equal to 18,416, and, conversely, the sum of the factors of 18,416, is equal to 17,296. This qualifies these two numbers to be considered a pair of friendly numbers.

There are many more such pairs of friendly numbers. The following are a few of such pairs of friendly numbers for your consideration:

1,184 and 1,210
2,620 and 2,924
5,020 and 5,564
6,232 and 6,368
10,744 and 10,856
9,363,584 and 9,437,056
111,448,537,712 and 118,853,793,424

If you are feeling ambitious, you may want to verify the above pairs' friendliness!

For the experts, the following is one method for finding pairs of friendly numbers:

Let $a = 3 \bullet 2^n$, $b = 3 \bullet 2^{n-1} - 1$, and $c = 3^2 \bullet 2^{n-1} - 1$, where n is an integer greater than or equal to 2, and a, b, and c are all prime numbers. It then follows that $2^n ab$ and $2^n c$ are friendly numbers.

We can always look for fascinating relationships between numbers. We now know what is meant by pairs of friendly numbers. With some creativity, we can establish another form of "friendliness" between numbers. Some of them can be truly mind-boggling! Take for example the pair of numbers 6,205 and 3,869.

At first glance, there seems to be no apparent relationship between these two numbers. But with some luck and imagination, we can get some fantastic results:

$6,205 = 38^2 + 69^2$, and $3,869 = 62^2 + 05^2$.

We can even find another pair of numbers with a similar relationship. Consider these:

$5,965 = 77^2 + 06^2$, and $7,706 = 59^2 + 65^2$.

Beyond the enjoyment of seeing this wonderful pattern, there isn't much mathematics in these examples. However, the relationship is truly amazing and worth noting. Again, mathematics has its hidden treasures, many of which have passed by the average math student without proper fanfare!

PALINDROMIC NUMBERS

The average school curriculum is rather limited with regard to the types of numbers that are presented to students throughout their mathematics instruction. Surely, students know about odd numbers, even numbers, prime numbers, and even perfect numbers, which we will discuss later in this chapter. However, there are other kinds of numbers that have an unusual property and are often neglected, such as numbers that read the same in both directions. These numbers are called *palindromic numbers*; they read the same left to right as they do from right to left. First, recall that a palindrome can also be a word, phrase, or sentence that reads the same in both directions. Figure 1.4 shows a few amusing palindromes.

A
EVE
RADAR
REVIVER
ROTATOR
LEPERS REPEL
MADAM I'M ADAM
STEP NOT ON PETS
DO GEESE SEE GOD
PULL UP IF I PULL UP
NO LEMONS, NO MELON
DENNIS AND EDNA SINNED
ABLE WAS I ERE I SAW ELBA
A MAN, A PLAN, A CANAL, PANAMA
A SANTA LIVED AS A DEVIL AT NASA
SUMS ARE NOT SET AS A TEST ON ERASMUS
ON A CLOVER, IF ALIVE, ERUPTS A VAST, PURE EVIL; A FIRE VOLCANO

Figure 1.4

A palindrome in mathematics would be a number, such as 666 or 123321, that reads the same in either direction. For example, the first four powers of 11 are palindromic numbers:

$11^0 = 1$
$11^1 = 11$
$11^2 = 121$
$11^3 = 1331$
$11^4 = 14641$

It is interesting to see how a palindromic number can be generated from randomly selected numbers. All you need to do is to continually add a number to its reversal (that is, the number written in the reverse order of digits) until you arrive at a palindrome. For example, a palindrome can be reached with a single addition when the starting number is 23: 23 + 32 = 55, which is a palindrome. Or it might take two steps, such as when the starting number is 75: 75 + 57 = 132 and 132 + 231 = 363, which has led us to a