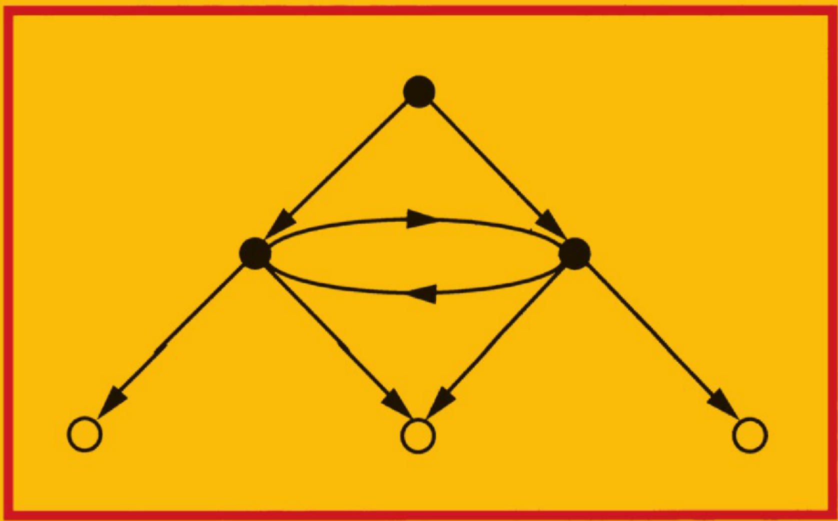
Undergraduate Texts in Mathematics

Keith Devlin

The Joy of Sets

Fundamentals of Contemporary Set Theory

Second Edition



Springer

Undergraduate Texts in Mathematics

Springer Science+Business Media, LLC

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The Joy of Sets

Fundamentals of Contemporary Set Theory

Second Edition

With 11 illustrations

Springer

Keith Devlin
School of Science
Saint Mary's College of California
Moraga, CA 94575
USA

Mathematics Subject Classification (2000): 03-01, 03E30, 03E47

Library of Congress Cataloging-in-Publication Data
Devlin, Keith J.
The joy of sets : fundamentals of contemporary set theory / Keith Devlin. -- 2nd ed., completely re-written.
p. cm. -- (Undergraduate texts in mathematics)
Rev. ed. of: Fundamentals of Contemporary set theory / Keith J. Devlin.
June 1992.
Includes bibliographical references and index.
ISBN 978-1-4612-6941-0 ISBN 978-1-4612-0903-4 (eBook)
DOI 10.1007/978-1-4612-0903-4
1. Set theory. I. Devlin, Keith J. Fundamentals of contemporary set theory. II. Title. III. Series.
QA248.038 1993 93-4692
511,3'22--dc20

Printed on acid-free paper.

Production managed by Karen Phillips, manufacturing supervised by Vincent Scelta.
Photocomposed pages prepared from the author's LATEX file.

9 8 7 6 5

ISBN 978-1-4612-6941-0 SPIN 10842284

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Naive Set Theory

Zermelo–Fraenkel set theory, which forms the main topic of the book, is a rigorous theory, based on a precise set of axioms. However, it is possible to develop the theory of sets considerably without any knowledge of those axioms. Indeed, the axioms can only be fully understood after the theory *has* been investigated to some extent. This state of affairs is to be expected. The concept of a 'set of objects' is a very intuitive one, and, with care, considerable, sound progress may be made on the basis of this intuition alone. Then, by analyzing the nature of the 'set' concept on the basis of that initial progress, the axioms may be 'discovered' in a perfectly natural manner.

Following standard practice, I refer to the initial, intuitive development as 'naive set theory'. A more descriptive, though less concise, title would be 'set theory from the naive viewpoint'. Once the axioms have been introduced, this account of 'naive set theory' can be re-read, without any changes being necessary, as the elementary development of *axiomatic* set theory.

1.1 What is a Set?

In naive set theory we assume the existence of some given domain of 'objects', out of which we may build sets. Just what these objects are is of no interest to us. Our only concern is the behavior of the 'set' concept. This is, of course, a very common situation in mathematics. For example, in algebra, when we discuss a group, we are (usually) not interested in what the elements of the group are, but rather in the way the group operation acts upon those elements. When we come to develop our set theory axiomatically we shall, in fact, remove this assumption of an initial domain, since *everything* will then be a set; but that comes much later.

In set theory, there is really only one fundamental notion:

The ability to regard any collection of objects as a single entity (i.e. as a set).

It is by asking ourselves what may and what may not determine 'a collection' that we shall arrive at the axioms of set theory. For the present, we regard the two words 'set' and 'collection (of objects)' as synonymous and understood.

If a is an object and x is a set, we write

$$a \in x$$

to mean that a is an *element* of (or member of) x , and

$$a \notin x$$

to mean that a is not an element of x.

In set theory, perhaps more than in any other branch of mathematics, it is vital to set up a collection of symbolic abbreviations for various logical concepts. Because the basic assumptions of set theory are absolutely minimal, all but the most trivial assertions about sets tend to be logically complex, and a good system of abbreviations helps to make otherwise complex statements readable. For instance, the symbol $\in$ has already been introduced to abbreviate the phrase *'is an element of'*. I also make considerable use of the following (standard) logical symbols:

$\rightarrow$	abbreviates	'implies'
$\leftrightarrow$	abbreviates	'if and only if'
$\neg$	abbreviates	'not'
$\wedge$	abbreviates	'and'
$\vee$	abbreviates	'or'
$\forall$	abbreviates	'for all'
$\exists$	abbreviates	'there exists'.

Note that in the case of 'or' we adopt the usual, mathematical interpretation, whereby $\phi \vee \psi$ means that either ϕ is true or ψ is true, or else both ϕ and ψ are true, where ϕ, ψ denote any assertions in any language.

The above logical notions are not totally independent, of course. For instance, for any statements, we have

$\phi \leftrightarrow \psi$	is the same as	$(\phi \to \psi) \wedge (\psi \to \phi)$
$\phi \to \psi$	is the same as	$(\neg\phi) \vee \psi$
$\phi \vee \psi$	is the same as	$\neg((\neg\phi) \wedge (\neg\psi))$
$\exists x\phi$	is the same as	$\neg((\forall x)(\neg\phi))$

where the phrase 'is the same as' means that the two expressions are logically equivalent.

Exercise 1.1.1. *Let $\phi\dot{\vee}\psi$ mean that exactly one of ϕ, ψ is true. Express $\phi\dot{\vee}\psi$ in terms of the symbols introduced above.*

Let us return now to the notion of a set. Since a set is the same as a collection of objects, a set will be uniquely determined once we know what its elements are. In symbols, this fact can be expressed as follows:

$$x = y \leftrightarrow \forall a[(a \in x) \leftrightarrow (a \in y)].$$

This principle will, in fact, form one of our axioms of set theory: the *Axiom of Extensionality.*

If x, y are sets, we say x is a subset of y if and only if every element of x is an element of y, and write

$$x \subseteq y$$

in this case. In symbols, this definition reads[1]

$$(x \subseteq y) \leftrightarrow \forall a[(a \in x) \to (a \in y)].$$

We write

$$x \subset y$$

in case x is a subset of y and x is not equal to y; thus:

$$(x \subset y) \leftrightarrow (x \subseteq y) \wedge (x \neq y)$$

where, as usual, we write $x \neq y$ instead of $\neg(x = y)$, just as we did with $\in$. Clearly we have

$$(x = y) \leftrightarrow [(x \subseteq y) \wedge (y \subseteq x)].$$

Exercise 1.1.2. *Check the above assertion by replacing the subset symbol by its definition given above, and reducing the resulting formula logically to the axiom of extensionality. Is the above statement an equivalent formulation of the axiom of extensionality?*

[1]The reader should attain the facility of 'reading' symbolic expressions such as this as soon as possible. In more complex situations the symbolic form can be by far the most intelligible one.

1.2 Operations on Sets

There are a number of simple operations that can be performed on sets, forming new sets from given sets. I consider below the most common of these.

If x and y are sets, the *union* of x and y is the set consisting of the members of x together with the members of y, and is denoted by

$$x \cup y.$$

Thus, in symbols, we have

$$(z = x \cup y) \leftrightarrow \forall a[(a \in z) \leftrightarrow (a \in x \vee a \in y)].$$

In the above, in order to avoid proliferation of brackets, I have adopted the convention that the symbol $\in$ predominates over logical symbols. This convention, and a similar one for $=$, will be adhered to throughout. An alternative way of denoting the above definition is

$$(a \in x \cup y) \leftrightarrow (a \in x \vee a \in y).$$

Using this last formulation, it is easy to show that the union operation on sets is both commutative and associative; thus

$$x \cup y = y \cup x,$$

$$x \cup (y \cup z) = (x \cup y) \cup z.$$

The beginner should check these and any similar assertions made in this chapter.

The *intersection* of sets x and y is the set consisting of those objects that are members of both x and y, and is denoted by

$$x \cap y.$$

Thus

$$(a \in x \cap y) \leftrightarrow (a \in x \wedge a \in y).$$

The intersection operation is also commutative and associative.

The (*set-theoretic*) *difference* of sets x and y is the set consisting of those elements of x that are not elements of y, and is denoted by

$$x - y.$$

Thus

$$(a \in x - y) \leftrightarrow (a \in x \wedge a \notin y).$$

Care should be exercised with the difference operation at first. Notice that $x - y$ is always defined and is always a subset of x, regardless of whether y is a subset of x or not.

Exercise 1.2.1. *Prove the following assertions directly from the definitions. The drawing of 'Venn diagrams' is forbidden; this is an exercise in the manipulation of logical formalisms.*

(i) $x \cup x = x \quad ; \quad x \cap x = x$;

(ii) $x \subseteq x \cup y \quad ; \quad x \cap y \subseteq x$;

(iii) $[(x \subseteq z) \wedge (y \subseteq z)] \rightarrow [x \cup y \subseteq z]$;

(iv) $[(z \subseteq x) \wedge (z \subseteq y)] \rightarrow [z \subseteq x \cap y]$;

(v) $x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$;

(vi) $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$;

(vii) $(x \subseteq y) \leftrightarrow (x \cap y = x) \leftrightarrow (x \cup y = y)$.

Exercise 1.2.2. *Let x, y be subsets of a set z. Prove the following assertions:*

(i) $z - (z - x) = x$;

(ii) $(x \subseteq y) \leftrightarrow [(z - y) \subseteq (z - x)]$;

(iii) $x \cup (z - x) = z$;

(iv) $z - (x \cup y) = (z - x) \cap (z - y)$;

(v) $z - (x \cap y) = (z - x) \cup (z - y)$.

Exercise 1.2.3. *Prove that for any sets x, y,*

$$x - y = x - (x \cap y).$$

In set theory, it is convenient to regard the collection of no objects as a set, the *empty* (or *null*) *set*. This set is usually denoted by the symbol $\emptyset$, a derivation from a Scandinavian letter.

Exercise 1.2.4. *Prove, from the axiom of extensionality, that there is only one empty set. (This requires a sound mastery of the elementary logical concepts introduced earlier.)*

Two sets x and y are said to be *disjoint* if they have no members in common; in symbols,

$$x \cap y = \emptyset.$$

Exercise 1.2.5. *Prove the following*:

(i) $x - \emptyset = x$;

(ii) $x - x = \emptyset$;

(iii) $x \cap (y - x) = \emptyset$;

(iv) $\emptyset \subseteq x$.

1.3 Notation for Sets

Suppose we wish to provide an accurate description of a set x. How can we do this? Well, if the set concerned is finite, we can enumerate its members: if x consists of the objects $a_1, \ldots, a_n$, we can denote x by

$$\{a_1, \ldots, a_n\}.$$

Thus, the statement

$$x = \{a_1, \ldots, a_n\}$$

should be read as 'x is the set whose elements are $a_1, \ldots, a_n$'. For example, the *singleton* of a is the set

$$\{a\}$$

and the *doubleton* of a, b is the set

$$\{a, b\}.$$

In the case of infinite sets, we sometimes write

$$\{a_1, a_2, a_3, \ldots\}$$

to denote the set whose elements are precisely

$$a_1, a_2, a_3, \ldots .$$

An alternative notation is possible in the case where the set concerned is defined by some *property* P: if x is the set of all those a for which $P(a)$ holds, we may write

$$x = \{a \mid P(a)\}.$$

Thus, for example, the set of all real numbers may be denoted by

$$\{a \mid a \text{ is a real number}\}.$$

Exercise 1.3.1. *Prove the following equalities*:

(i) $x \cup y = \{a \mid a \in x \vee a \in y\}$;

(ii) $x \cap y = \{a \mid a \in x \wedge a \in y\}$;

(iii) $x - y = \{a \mid a \in x \wedge a \notin y\}$.

1.4 Sets of Sets

So far, I have been tacitly distinguishing between sets and objects. Admittedly, I did not restrict in any way the choice of initial objects — they could themselves be sets; but I did distinguish these initial objects from the sets of those objects that we could form. However, as I said at the beginning, the main idea in set theory is that any collection of objects can be regarded as a single entity (i.e. a set). Thus we are entitled to build sets out of entities that are themselves sets. Commencing with some given domain of objects then, we can first build sets of those objects, then sets of sets of objects, then sets of sets of sets of objects, and so on. Indeed, we can make more complicated sets, some of whose elements are basic objects, and some of which are sets of basic objects, etc.

For example, we can *define* the ordered pair of two objects a, b by

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

According to this definition, (a, b) is a set: it is a set of sets of objects.

Exercise 1.4.1. *Show that the above definition does define an ordered-pair operation; i.e. prove that for any* a, b, a', b'

$$(a, b) = (a', b') \leftrightarrow (a = a' \wedge b = b').$$

(*Don't forget the case* $a = b$.)

The inverse operations $(-)_0, (-)_1$ to the ordered pair are defined thus: if $x = (a, b)$, then $(x)_0 = a$ and $(x)_1 = b$. If x is not an ordered pair, $(x)_0$ and $(x)_1$ are undefined.

The *n-tuple* $(a_1, \ldots, a_n)$ may now be defined iteratively, thus

$$(a_1, \ldots, a_n) = ((a_1, \ldots, a_{n-1}), a_n).$$

It is clear that

$$(a_1, \ldots, a_n) = (a'_1, \ldots, a'_n) \quad \text{if and only if} \quad a_1 = a'_1 \wedge \ \ldots \ \wedge a_n = a'_n.$$

The inverse operations to the n-tuple are defined in the obvious way, so that if $x = (a_0, \ldots, a_{n-1})$, then $(x)^n_0 = a_0, \ldots, (x)^n_{n-1} = a_{n-1}$.

Of course, it is not important how an ordered-pair operation is defined. What counts is its behavior. Thus, the property described in Exercise 1.4.1 is the only requirement we have of an ordered pair. In naive set theory, we could just take (a, b) as a basic, undefined operation from pairs of objects to objects. But when we come to axiomatic set theory a definition of the ordered pair operation in terms of sets, such as the one above, will be necessary. Though there are other definitions, the one given is the most common, and it is the one I shall use throughout this book.

If x is any set, the collection of *all* subsets of x is a well-defined collection of objects and, hence, may itself be regarded as an entity (i.e. set). It is called the *power set* of x, denoted by $\mathcal{P}(x)$. Thus

$$\mathcal{P}(x) = \{y \mid y \subseteq x\}.$$

Suppose now that x is a set of sets of objects. The *union* of x is the set of all elements of all elements of x, and is denoted by $\bigcup x$. Thus

$$\bigcup x = \{a \mid \exists y(y \in x \wedge a \in y)\}.$$

Extending our logical notation by writing

$$(\exists y \in x)$$

to mean 'there exists a y in x such that', this may be re-written as

$$\bigcup x = \{a \mid (\exists y \in x)(a \in y)\}.$$

The *intersection* of x is the set of all objects that are elements of all elements of x, and is denoted by $\bigcap x$. Thus

$$\bigcap x = \{a \mid \forall y(y \in x \rightarrow a \in y)\}.$$

Or, more succinctly,

$$\bigcap x = \{a \mid (\forall y \in x)(a \in y)\}$$

where $(\forall y \in x)$ means 'for all y in x'.

If $x = \{y_i \mid i \in I\}$ (so I is some indexing set for the elements of x), we often write

$$\textstyle\bigcup_{i \in I} y_i$$

for $\bigcup x$ and

$$\bigcap_{i\in I} y_i$$

for $\bigcap x$. This ties in with our earlier notation to some extent, since we clearly have, for any sets x, y,

$$x \cup y = \bigcup\{x, y\}\,, \quad x \cap y = \bigcap\{x, y\}.$$

Exercise 1.4.2.

(i) *What are* $\bigcup\{x\}$ *and* $\bigcap\{x\}$?

(ii) *What are* $\bigcup\emptyset$ *and* $\bigcap\emptyset$?

Verify your answers.

Exercise 1.4.3. *Prove that if* $\{x_i \mid i \in I\}$ *is a family of sets, then*

(i) $\bigcup_{i\in I} x_i = \{a \mid (\exists i \in I)(a \in x_i)\}$;

(ii) $\bigcap_{i\in I} x_i = \{a \mid (\forall i \in I)(a \in x_i)\}$.

Exercise 1.4.4. *Prove the following*:

(i) $(\forall i \in I)(x_i \subseteq y) \to (\bigcup_{i\in I} x_i \subseteq y)$;

(ii) $(\forall i \in I)(y \subseteq x_i) \to (y \subseteq \bigcap_{i\in I} x_i)$;

(iii) $\bigcup_{i\in I}(x_i \cup y_i) = (\bigcup_{i\in I} x_i) \cup (\bigcup_{i\in I} y_i)$;

(iv) $\bigcap_{i\in I}(x_i \cap y_i) = (\bigcap_{i\in I} x_i) \cap (\bigcap_{i\in I} y_i)$;

(v) $\bigcup_{i\in I}(x_i \cap y) = (\bigcup_{i\in I} x_i) \cap y$;

(vi) $\bigcap_{i\in I}(x_i \cup y) = (\bigcap_{i\in I} x_i) \cup y$.

Exercise 1.4.5. *Let* $\{x_i \mid i \in I\}$ *be a family of subsets of* z. *Prove*:

(i) $z - \bigcup_{i\in I} x_i = \bigcap_{i\in I}(z - x_i)$;

(ii) $z - \bigcap_{i\in I} x_i = \bigcup_{i\in I}(z - x_i)$.

1.5 Relations

If x, y are sets, the *cartesian product* of x and y is defined to be the set

$$x \times y = \{(a, b) \mid a \in x \wedge b \in y\}.$$

More generally, if $x_1, \ldots, x_n$ are sets, we define their cartesian product by

$$x_1 \times \ldots \times x_n = \{(a_1, \ldots, a_n) \mid a_1 \in x_1 \wedge \ \ldots \ \wedge a_n \in x_n\}.$$

A *unary relation* on a set x is defined to be a subset of x. An n-ary relation on x, for $n > 1$, is a subset of the n-fold cartesian product $x \times \ldots \times x$.

Notice that an n-ary relation on x is a unary relation on the n-fold product $x \times \ldots \times x$.

These formal definitions provide a concrete realization within set theory of the intuitive concept of a relation.

However, as is often the case in set theory, having seen how a concept may be defined set-theoretically, we revert at once to the more familiar notation. For example, if P is some property that applies to pairs of elements of a set x, we often speak of 'the binary relation P on x', though strictly speaking, the relation concerned is the *set*

$$\{(a, b) \mid a \in x \wedge b \in x \wedge P(x, y)\}.$$

Also common is the tacit identification of such a property P with the relation it defines, so that $P(a, b)$ and $(a, b) \in P$ mean the same.

Similarly, going in the opposite direction, if R is some binary relation on a set x, I often write $R(a, b)$ instead of $(a, b) \in R$. Indeed, in the specific case of binary relations, I sometimes go even further, writing aRb instead of $R(a, b)$. In the case of ordering relations, this notation is, of course, very common: we rarely write $< (a, b)$ or $(a, b) \in <$, though from a set-theoretic point of view, both could be said to be more accurate than the more common notation $a < b$.

Binary relations play a particularly important role in set theory and, indeed, in mathematics as a whole. The rest of this section is devoted to a rapid review of binary relations.

There are several properties that apply to binary relations. Let R denote

The *range* of R is defined to be the set

$$\text{ran}(R) = \{b \mid \exists a[(a, b) \in R]\}.$$

If $n = 1$, so that R is a binary relation, then it is clear what is meant by these definitions: elements of R are ordered pairs, $\text{dom}(R)$ is the set of first components of members of R, and $\text{ran}(R)$ the set of second components. But what if $n > 1$? In this case, any member of R will be an $(n+1)$-tuple. But what is an $(n+1)$-tuple? Well, by definition, an $(n+1)$-tuple, c, has the form (a, b) where a is an n-tuple and b is an object in x. Thus, even if $n > 1$, the elements of R will still be *ordered pairs*, only now the domain of R will consist not of elements of x but elements of the n-fold product $x \times \ldots \times x$. So in all cases, $\text{dom}(R)$ is the set of first components of members of R and $\text{ran}(R)$ is the set of second components.

Although the notions of domain and range for an arbitrary relation are quite common in more advanced parts of set theory, chances are that the reader is not used to these concepts. But when we define the notion of a function as a special sort of relation, as we do below, you will see at once that the above definitions coincide with what one usually means by the 'domain' and 'range' of a function.

An n-ary *function* on a set x is an $(n+1)$-ary relation, R, on x such that for every $a \in \text{dom}(R)$ there is exactly one $b \in \text{ran}(R)$ such that $(a, b) \in R$.

As usual, if R is an n-ary function on x and $a_1, \ldots, a_n, b \in x$, we write

$$R(a_1, \ldots, a_n) = b$$

instead of

$$(a_1, \ldots, a_n, b) \in R.$$

Exercise 1.6.1. *Comment on the assertion that a set-theorist is a person for whom all functions are unary. (This is a serious exercise, and concerns a subtle point which often causes problems for the beginner.)*

I write

$$f : x \to y$$

to denote that f is a function such that $\text{dom}(f) = x$ and $\text{ran}(f) \subseteq y$.

Notice that if $f : x \to y$, then $f \subseteq x \times y$.

A *constant function* from a set x to a set y is a function of the form

$$f = \{(a, k) \mid a \in \text{dom}(f)\}$$

where k is a fixed member of y.

The *identity function* on x is the unary function defined by

$$\mathrm{id}_x = \{(a, a) \mid a \in x\}.$$

If $f : x \to y$ and $g : y \to z$, we define $g \circ f : x \to z$ by

$$g \circ f(a) = g(f(a))$$

for all $a \in x$.

Exercise 1.6.2. *Express $g \circ f$ as a set of ordered pairs.*

Let $f : x \to y$. If $u \subseteq x$, we define the *image* of u under f to be the set

$$f[u] = \{f(a) \mid a \in u\};$$

and if $v \subseteq y$, we define the *preimage* of v under f to be the set

$$f^{-1}[v] = \{a \in x \mid f(a) \in v\}.$$

Exercise 1.6.3. *Let $f : x \to y$, and let $v_i \in y$, for $i \in I$. Prove that:*

(i) $f^{-1}\bigcup_{i\in I}[v_i] = \bigcup_{i\in I} f^{-1}[v_i]$;

(ii) $f^{-1}\bigcap_{i\in I}[v_i] = \bigcap_{i\in I} f^{-1}[v_i]$;

(iii) $f^{-1}[v_i - v_j] = f^{-1}[v_i] - f^{-1}[v_j]$.

If $f : x \to y$ and $u \subseteq x$, we define the *restriction* of f to u by

$$f \restriction u = \{(a, f(a)) \mid a \in u\}.$$

Notice that $f \restriction u$ is a function, with domain u.

Exercise 1.6.4. *Prove that if $f : x \to y$ and $u \subseteq x$, then*

(i) $f[u] = \mathrm{ran}(f \restriction u)$;

(ii) $f \restriction u = f \cap (u \times ran(f))$.

Let $f : x \to y$. We say f is *injective* (or *one-one*) if and only if

$$a \neq b \to f(a) \neq f(b).$$

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