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Mohan Ganesalingam

The Language of Mathematics

A Linguistic and Philosophical Investigation

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Author

Mohan Ganesalingam
Trinity College
Cambridge CB2 1TQ, UK
E-mail: mg262@cam.ac.uk

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Contents

1 Introduction

1.1 Challenges

The aim of this book is to give a formal, objective and above all precise analysis of the language used by mathematicians in textbooks and papers. Our analysis will closely parallel the analyses of human languages by syntacticians and semanticians in the generative tradition. In particular, it will let us take mathematical sentences, determine their syntactic structure, and extract their underlying meaning in an appropriate logic.

We face a number of central challenges in this task. Some relate to the scope of the analysis and to our methodology, and some to the nature of mathematics itself. In this section, we will outline these main challenges.

First and foremost, we aim to give an analysis that can potentially encompass all of pure mathematics. This is considerably harder than developing an analysis of the language used in a single, isolated area of mathematics, such as group theory or linear algebra. In order to describe so much mathematics with a compact theory, we develop ways for our theory to *adapt* by extracting mathematical terms, notations and concepts, and all properties thereof, from their explicit *definitions* in mathematical texts. So, for example, our theory will need to be able to extract all syntactic and semantic properties of the word 'group' from the definition of a group, as found in any textbook on group theory.

Second, we will require that *every* mathematical term, notation or concept be extracted from mathematical text rather than being an intrinsic part of our analysis. Even foundational material must be extracted from text; the theory must be compatible with the standard foundational accounts using the ZF(C) axioms, but must not be tied to them. This strategy of *full adaptivity* is considerably harder than allowing some mathematics to be encoded directly into our theory. However, it consistently pays dividends when we reach sufficiently advanced mathematics. For example, it is substantially harder to adaptively extract set theory from mathematical texts than to describe it directly. But if we confront and overcome this problem, we find that the

M. Ganesalingam (Ed.): The Language of Mathematics, LNCS 7805, pp. 1–15, 2013.

same methods we used to extract set theory from texts can be used to extract category theory from texts. Conversely, if we had directly encoded set theory into our analysis, we would have encountered real difficulties when facing category theory.

Third, mathematics is written in a mixture of words and symbols. These are very different in character. The words resemble words in natural language texts, but have many differences. The symbols superficially resemble symbols in artificial languages but, as we shall show, they behave in a way that is much more complex than in any artificial language. And the interaction between words and symbols is unlike anything found in any other kind of language, natural or artificial; although the two are entirely dissimilar, they are remarkably interdependent. Thus we will need to develop a new kind of theory of language, unlike theories of both natural languages and artificial languages, to give a unified account of mathematics. This requirement will pervade our analyses of individual phenomena.

Fourth, because the theory must describe such a wide range of mathematics, ambiguity becomes a major problem. As we will see, word sense ambiguity, attachment ambiguity, coordination ambiguity and other kinds of ambiguity from natural language recur in mathematics. But moreover, and more problematically, we will show that the syntactic structure of a fixed expression in symbolic mathematics can depend on what kinds of mathematical objects occur in it, i.e. on the *types* of the objects in it. Thus the syntax of symbolic mathematics is type-dependent in a way that has no parallel in any other kind of language, and that requires novel disambiguation techniques. Additionally, we will demonstrate that ambiguity in words and ambiguity in symbols are very different in character but are inextricably intertwined; neither can be resolved without resolving the other. We will also show that existing methods from linguistics and computer science are unable to remove ambiguity in mathematics. Eventually we will remove the ambiguity by developing a novel method which tracks the flow of type information inside mathematical sentences, treating words and symbols in a unified way.

Fifth, we will show that there is a considerable gap between what mathematicians claim is true and what they believe, and this mismatch causes a number of serious linguistic problems. For example, mathematicians claim that all numbers are really sets, but their use of language consistently reflects a belief that this is not the case. Our attempts to understand what is happening here will lead us deep into the foundations of mathematics, and will show us that our linguistic problems are connected to philosophical problems. Resolving them will prove to be a major undertaking.

Last, our focus on adaptivity and our close examination of mathematical material will lead us to the discovery that a notion of time plays a key part in the language of mathematics in a way that has not previously been realised. For example, we will exhibit instances of a heretofore undescribed phenomenon whereby the meaning of a fixed piece of mathematical language can *change* as time passes and one learns more mathematics. Equally, we will derive constraints on our theory based on the fact that mathematics is learned in order, and that certain parts of mathematics may or must be learnt before other parts. All of our descriptions of phenomena in the language of mathematics will need to be compatible with our novel notion of time. Ultimately, this notion will come to play a key part in our analysis of the foundations; it will allow us to find deficiencies in all of the standard accounts of the foundations of mathematics, and eventually to construct an alternative account which resolves these problems.

To sum up, the main challenges that we will face are as follows:

Breadth. The theory must be able to describe all of pure mathematics.

Full Adaptivity. All mathematical content must be extracted from mathematical text.

Words and Symbols. We will need to analyse all phenomena in mathematics by giving a unified description of their relationships to both the words and symbols in mathematics, despite the fact that these are highly dissimilar.

Ambiguity. We will find that ambiguity is utterly pervasive in mathematics, and that it crosses the line between words and symbols in an unprecedented way. Resolving this will require novel techniques.

Belief and Behaviour. We will need to resolve disparities between the claims mathematicians make about certain mathematical objects and the linguistic behaviour of those objects.

Time. We will discover a novel notion of time underlying mathematics, and all accounts of the language of mathematics and the foundations of mathematics will need to be compatible with this.

We will return to discuss these in the conclusion (Chapter 9).

1.2 Concepts

1.2.1 Linguistics and Mathematics

Our analysis of the language of mathematics is related to generative linguistics in two ways. First, our aim is to give a completely formal and precise description of the language of mathematics. Similarly, the central aim of generative linguistics is to give a completely formal and precise description of natural languages, such as English. Thus, in a broad sense, all of the work in this book may be regarded as 'the application of linguistics to mathematics'. In this respect, linguistics gives us general ideas about how we can formally analyse language: it provides us with a mindset which we can use throughout the entire book.

Second, if we restrict ourselves to the part of mathematics that consists only of words, there are clear parallels with natural languages. We need to be careful not to say that these parts of mathematics are written in natural language; as we shall see in §2.3, many of the conventions are different, so that the same sentence might mean different things in mathematical language and in general natural language. Nevertheless, much of the machinery of linguistics may be adapted to describe textual mathematics, before being combined with novel machinery for describing symbolic mathematics. This is, surprisingly, almost untrodden ground; as the only linguist to analyse mathematics, Aarne Ranta, remarks:

> Linguistically, the study of mathematical language rather than everyday language is rewarding because it offers examples that have complicated grammatical structure but are free from ambiguities. We always know exactly what a sentence means, and there is a determinate structure to be revealed. The informal language of mathematics thus provides a kind of grammatical laboratory. Amazingly little use has been made of this laboratory in linguistics; even the material presented below is just an application of results obtained within the standard linguistic fragment containing donkey sentences etc. It is to be expected that a closer study of mathematical language itself will give experience that is useful in general linguistics as well.
>
> (Ranta, 1994)

We should emphasise that Ranta primarily studies the words in mathematics, rather than the symbols (see §1.5.1 for details). We will be concerned with both, and will therefore have to stray further from the linguistic canon than Ranta does.

Linguistics is divided into many branches. Those that will be most useful to us are generative syntax (Chomsky, 1957) and formal semantics in the Montagovian tradition (Montague, 1970a,b, 1973). We will make particular use of the semantic theory called Discourse Representation Theory (Kamp and Reyle, 1993).

There are some significant ways in which mathematics differs substantially from natural languages, which will affect our general approach. Most of these will be described in Chapter 2, which gives a description of the language of mathematics, but one is worth emphasising here. Mathematics has associated with it a clear, external, notion of meaning. Since the publication of Principia Mathematica (Whitehead and Russell, 1910), it has been accepted that the language of mathematics can be given a complete semantic representation in the form of an appropriate logic. Unlike current semantic representations for natural language, these logical representations for the language of mathematics completely capture what mathematical objects 'are' and how they behave.

Further, when mathematicians are writing modern mathematics in a formal register (see §1.3 for details of registers), they intend to formulate meaningful statements in some underlying logic. If it was pointed out that a particular sentence had no translation into such a logic, a mathematician would genuinely feel that they had been insufficiently precise. (The actual translation into logic is never performed, because it is exceptionally laborious; but the possibility of the translation is held to be crucial.) Thus mathematics has a normative notion of what its content should look like; there is no analogue in natural languages.

1.2.2 Time

As noted in §1.1, a novel notion of time will play a major part in this book. In this section, we will briefly sketch the role which this notion plays and then list the parts of the book in which it is elaborated on.

The importance of time in our theory arises from adaptivity, i.e. the way in which our theory derives mathematical terms, concepts and notation from their definitions in texts. We can only refer to a term, concept or notation *after* the appropriate definition has been encountered; correspondingly, we need to pay considerable attention to the *temporal order* in which definitions and other material are encountered.

The notion of time will lie in the background when we discuss adaptivity and definitions (§2.2), but it will first come to prominence in our discussion in §2.6 of a (previously unnoted) phenomenon which we call reanalysis, whereby the meaning of a fixed mathematical expression can change over time, i.e. as an individual mathematician encounters more mathematics. The notion of time will then remain in the background throughout most of Chapters 3, 4, 5 and 6, implicit whenever we discuss operations like definition, where there is an essential distinction between our picture of mathematics 'before' and 'after' the operation. It will, however, occasionally surface more explicitly. Most notably, we will use it to criticise certain disambiguation mechanisms in §4.1.4, to argue that certain pieces of mathematics must not be able to affect each other in §5.3.3, and to argue that certain categories must be able to grow over time in §5.3.5.

It is only when we come to Chapter 7, discussing the foundations of mathematics, that the concept of time will come into its own. In that chapter, we will contrast the notion of time with another notion of time from the philosophical literature (more specifically, from Lakatos (1976)), introduce terminology to reflect this and then use the notion of time as our central tool throughout the remainder of the chapter. It is not too much of a stretch to say that the significance of the chapter lies in its demonstration of the utility of our novel notion of time.

1.2.3 Full Adaptivity

Adopting a linear temporal perspective allows us to state a remarkable property of mathematical language. It is very nearly the case that the language of mathematics is completely *self-contained*: that every lexical, syntactic or semantic property of a mathematical object can be adaptively extracted from the discourse prior to the first reference to that object. It is this property that allowed us to even formulate our goal of describing mathematics using a fully adaptive theory.

Thus this sentence is in the formal mode. The formal mode also includes the expression of relationships between mathematical facts, as in:

$$\text{By } \underbrace{\text{Theorem 11}}_{\text{mathematical fact}}, \underbrace{\text{every bounded monotonic sequence converges}}_{\text{mathematical fact}}.$$

From a linguistic perspective, the formal mode is more novel and interesting because it is restricted enough to describe completely, both in terms of syntax and semantics. By contrast, the informal mode seems as hard to describe as general natural language. We will therefore look only at mathematics in the formal mode.

Finally, every mathematical text is written in a mixture of some natural language and symbols. We will only concern ourselves with the case where the natural language in question is English. The theory of mathematics in, say, French is unlikely to be that different to the theory of mathematics in English. But by restricting ourselves to English we gain the ability to abstract away from certain concerns of general language which are orthogonal to our primary concerns. In particular, English-language mathematics can be very effectively captured by an unaugmented phrase structure grammar, and exhibits little in the way of agreement.

1.4 Structure

Chapter 2 presents a descriptive overview of mathematics from a linguistic perspective. §2.6 is of particular importance as it highlights a previously undiscussed way in which the meaning of fixed mathematical expressions may change as more mathematics is encountered; this section may be of interest to philosophers of mathematics.

Chapter 3 sets up the basic linguistic framework which is used in the remainder of the book. Syntax is handled using a context-free grammar, and semantics using a variant of Discourse Representation Theory (Kamp and Reyle, 1993) adapted to better fit mathematical phenomena. The theory constructed in this chapter is comparable to that constructed by Ranta (cf. §1.5.1 below), but covers a much wider range of phenomena and overcomes certain deficiencies noted by Ranta in his own work.

We should note that Chapter 3 concludes the analysis of material for which there is a precedent in the literature; subsequent chapters discuss more advanced topics. Consequently, we make a point of noting particular difficulties that any analysis must face at the beginnings of Chapter 4, Chapter 5 and Chapter 6, and also in §4.4. Chapter 7, discussing the foundations of mathematics, contains a comparable discussion of major difficulties in §7.3.

Chapter 4 discusses ambiguity in detail. It begins by demonstrating that ambiguity is pervasive in both words and symbols in mathematics and then

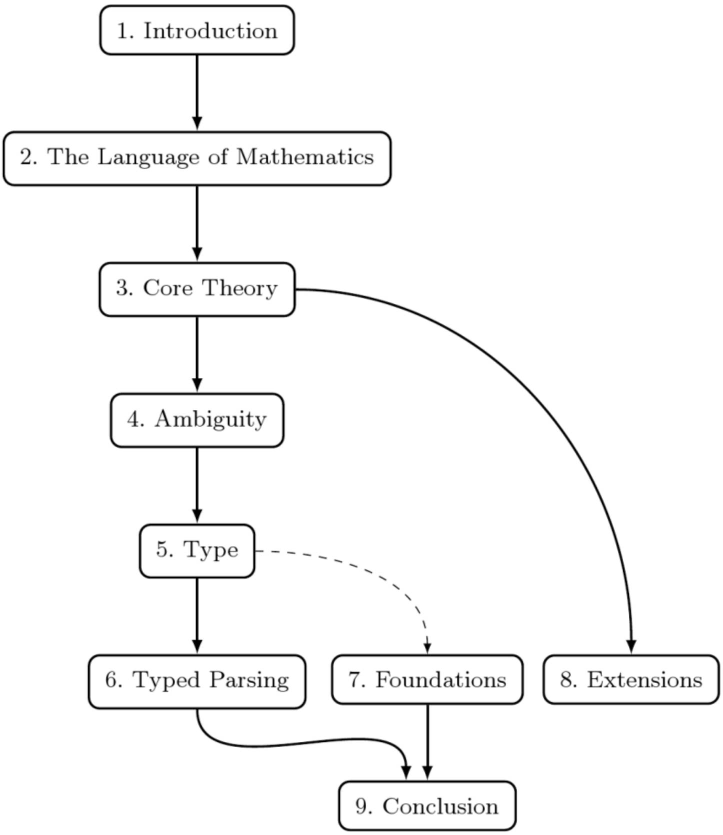

Fig. 1.1 Dependencies between chapters

gives a sequence of examples that show that standard mechanisms for disambiguating formal languages are not adequate for resolving ambiguity in symbolic mathematics, and that the textual ambiguity and symbolic ambiguity in mathematics are too intertwined to be handled separately. Chapter 4 concludes by arguing that a notion of *type* is needed to resolve ambiguity in mathematical language, and mathematics is unique in that the syntax of an expression can depend on the types of the elements it contains.

Chapter 5 begins by showing that showing that the standard notion of 'type' conflates two separate notions, and separates these. It then gives a sequence of examples of the use of mathematical language which narrow down exactly which type distinctions need to be in place to model mathematics. In particular, it demonstrates that mathematics contains *non-extensional types*, which have no parallel in any other kind of language; it then uses the two separated notions of type to characterise and interpret this concept. Finally, it presents the actual formal type system needed to model mathematics. One important point arising in Chapter 5 is that everyday usages of mathematical language cannot be reconciled with the standard account of the foundations of mathematics.

Chapter 6 shows how the notion of type introduced in Chapter 5 can be combined with syntax, and gives a unified model for the process of interpretation of mathematics, including the removal of ambiguity in both words and symbols.

Chapter 7 treats the foundations of mathematics. It picks up the criticisms of the standard foundational accounts which arose in Chapter 5, and connects them to a famous criticism arising in the philosophy of mathematics (Benacerraf, 1965). It then discusses every standard approach to the construction of numbers and other simple objects, and shows that all of these are flawed on a variety of linguistic and philosophical grounds. Its primary tool in this task is a novel notion of *time* in mathematics. Finally, it presents a new approach to the foundations which overcomes all the linguistic and philosophical problems, and which interacts correctly with the aforementioned notion of time. It then illustrates this approach using a range of examples. This chapter may be of interest to philosophers of mathematics and to mathematicians with a particular interest in the foundations.

Chapter 8 briefly discusses miscellaneous minor phenomena in mathematics which are worthy of note, and one major topic which is relatively independent of the remainder of the book.

1.5 Previous Analyses

As we noted in §1.2.1, the language of mathematics has barely been studied from a linguistic perspective. To repeat the relevant quote:

> Amazingly little use has been made of [the language of mathematics] in linguistics; even the material presented below is just an application of the results obtained within the standard linguistic fragment containing donkey sentences etc..
>
> (Ranta, 1994, p. 3)

1.5.1 Ranta

The only substantive work on the language of mathematics appears in papers by Ranta (Ranta, 1994, 1995, 1996, 1997a,b) which operate within the framework of Constructive Type Theory (Martin-Löf, 1984). These papers place a great deal of emphasis on the problem of *sugaring*, that is, the conversion of logical representations into sentences in the language of mathematics. We are not concerned with this problem in this book; our approach follows a more classical linguistic tradition, and is concerned purely with analysis rather than synthesis. Equally, we do not base our analysis on Constructive

Type Theory but on a modified form of Discourse Representation Theory (Kamp and Reyle, 1993) which, as we will show in Chapter 3, is well-suited to describing a range of mathematical phenomena. Nevertheless, there is a substantial amount of material in Ranta's work to which (the earlier parts of) our theory may be compared, and on which our theory improves. In particular, we will solve various problems that Ranta notes in his analyses of plurals (Ranta, 1994, p. 11–12; cf. §3.3.6) and of variables and quantifiers (Ranta, 1994, p. 11–13; cf. §3.5).

Our own work is more ambitious than Ranta's in three respects, which have been introduced in §1.1. First, Ranta exploits Montague's method of fragments (Montague, 1970a,b, 1973) to study small domains in mathematics, such as plane geometry. By contrast, we aim to produce a theory that describes all of mathematics, subject to the restrictions given in §1.3. Thus we are forced to accord much greater weight to ambiguities and notational collisions that arise when the theory simultaneously describes many areas of mathematics.

Second, we extensively analyse symbolic mathematical notation, which is only discussed briefly in Ranta's work. (See §2.4.1 for details of Ranta's analysis). That is, our linguistic theory analyses expressions such as

$$\left|\delta \int_2^\infty \frac{E(t)}{t^{1+\delta}}\,dt\right| < A\delta \int_2^T \frac{dt}{t} + \frac{A\delta}{\log T} \int_T^\infty \frac{dt}{t^{1+\delta}}$$

(Hardy and Wright, 1960, p. 352)

We will find that behaviour of expressions of this kind is complex in a way that has no analogue in natural languages, computer languages, or any other theoretically analysed systems that we are aware of. In addition, there is considerable interaction between such symbolic material and the parts of the language of mathematics that resembles natural language. As noted in §1.1, some of the most challenging problems solved in this book are motivated by the need to find a unified treatment of these two dissimilar parts of mathematical language.

Third, Ranta manually specifies a lexicon for the fragment under consideration. Thus, for example, the theory describing plane geometry is associated with a hand-crafted lexicon containing entries for 'a line', 'a point' and 'the intersection of X and Y'. As noted in §1.1 and §1.2, we do not follow this approach but instead extract all mathematical terms, concepts and notation from definitions.

In addition to these, there are various minor respects in which we extend Ranta's work. For example, Ranta makes little reference to rhetorical structure. Given that mathematics exhibits discourse-level phenomena which do not exist in general language, we need to handle these carefully and explicitly. (Cf. §2.5 and §3.5.) We will highlight such improvements as we encounter them.

Most of the material that directly corresponds to or improves on Ranta's work will be presented in Chapter 3, which presents the basic theoretical framework of the book. (Several minor and self-contained phenomena are relegated to Chapter 8, which outlines miscellaneous extensions to the main theory.) In particular, Chapter 3 will explain how our framework can handle the problems noted by Ranta in his own work. The remainder of the book will introduce more severe problems that arise due to our more ambitious aims, and present solutions to these.

1.5.2 de Bruijn

Parts of this book may be related to de Bruijn's work on a *Mathematical Vernacular* (De Bruijn, 1987), a formal language intermediate between ordinary mathematical practice and formalised computer languages for mathematics (themselves discussed in §1.5.4 below). The Mathematical Vernacular is adaptive (in the sense introduced in §1.1): its lexicon and grammar are not fixed, but expand when definitions in mathematical texts are encountered. Further, de Bruijn's delimitation of the scope of the mathematical vernacular (De Bruijn, 1987) may be compared to our notion of the formal mode (§1.3), with the caveat that the Mathematical Vernacular excludes assertions about relationships between mathematical facts.

Nevertheless, there are significant differences between the our work and de Bruijn's. The Mathematical Vernacular seems to be intended as a practical tool, designed to overcome some of the weaknesses in de Bruijn's computer theorem-proving language Automath. By contrast, we are analysing actual mathematical language from a linguistic perspective. Some consequences of this difference in emphasis are that de Bruijn often takes a prescriptive stance towards mathematics, that he avoids discussing ambiguity by simply inserting extra brackets into mathematical texts (De Bruijn, 1987, p. 935), and that his theory massively overgenerates (see particularly (De Bruijn, 1987, p. 868, §1.10)). Another is that divergences between mathematical practice and the Mathematical Vernacular are often presented without discussion.

In the course of the development of his Mathematical Vernacular, de Bruijn occasionally discusses specific phenomena in the language of mathematics; many of these phenomena will also be considered in this book. As de Bruijn does not base his analysis on any linguistic theory, he tends to organise material according to its surface character, rather than the underlying processes. For example, he makes a detailed taxonomic study of the distinct usages of the indefinite article in Dutch mathematics (De Bruijn, 1982), covering a number of phenomena with analogues in English mathematics. This includes phenomena related to pragmatics (§2.3), variable introduction (§2.4.4, §3.3.4), the treatment of material inside definition blocks (§2.5.1), quantifier scope ambiguity (§4.2.2) and prepositional phrases and genericity (§8.1). In the remainder of the book, we will highlight de Bruijn's observations where they are directly relevant to the phenomena we are analysing.

Naproche's authors remark that it is at a prototypical stage, and that the language is currently 'inelegant' (Koepke, 2009, p. 5, p. 10). It seems likely that it will come closer to the actual language of mathematics, but as it stands, NaProChe is sufficiently far removed from that of the language of mathematics that it of limited relevance to the present work. In particular, we will not refer to the syntactic aspects of NaProChe at all. Its main area of relevance is semantic: NaProChe adapts Discourse Representation Theory (or DRT) (Kamp and Reyle, 1993) to provide a semantic representation, on the grounds that it is one of the longest established theories, and that it can analyse a wide range of phenomena (Kolev, 2008, p. 21). We will also adopt a variant of DRT, albeit for very specific reasons.[6] We will provide a comparison of the two DRT-based approaches in §3.5.4.

1.5.4 Other Work

We should note that there are a number of sources which give a linguistic analysis of mathematics in some capacity, but are not comparable to the work in this book. For example, Wolska and Kruijff-Korbayová (2004) describes an experiment that deals with simulated mathematical dialogs, but focuses on subjects with 'little to fair mathematical knowledge' (Wolska and Kruijff-Korbayová, 2004, p. 2). The material produced by these users is not related to the formal dialect of mathematics studied in this book (cf. §1.3). (The paper also treats material in German, whereas we focus exclusively on English.)

[6] DRT was developed to handle Geach's 'donkey sentences' (Geach, 1980) and, as we will show in §3.3.1, there are similar sentences in real mathematical texts. We will also be able to adapt DRT to overcome deficiencies noted by Ranta in his own work; see §3.3.2 for details.

2

The Language of Mathematics

We will now give an informal description of the language of mathematics, and highlight some of the major issues that arise to confront a theory of mathematical language. No systematic survey of this kind exists in the literature, and we will therefore for the most part construct our description *ab initio*. Exceptions will be drawn in certain areas where Ranta has discussed similar phenomena, especially in §2.4.

We will start by introducing a basic division of the language of mathematics into 'textual' and 'symbolic' halves (§2.1) and introducing the most important way in which the language of mathematics differs from natural languages (§2.2). We will then examine each of textual and symbolic mathematics in greater detail (§2.3 and §2.4), and finally turn to the macroscopic discourse structure of mathematical language (§2.5).

2.1 Text and Symbol

At first sight, the most striking feature of mathematical language is the way in which it mixes material that looks as if it is drawn from a natural language with material built up out of idiosyncratically mathematical symbols. The distinction is illustrated in Figure 2.1.

There are respects in which the 'natural language' part of mathematics differs from genuine natural languages; for example, as we will discuss below, many pragmatic phenomena are blocked in 'mathematical natural language'. Also, the term 'natural language' carries particular connotations; for example, it suggests that one is dealing with a language that has native speakers. We will therefore use the neutral term 'textual' to refer to the parts of mathematics that resemble natural language. The remaining material will be referred to as 'symbolic', and specific pieces of symbolic mathematics will occasionally be referred to as mathematical 'notation'.

The primary function of symbolic mathematics is to abbreviate material that would be too cumbersome to state with text alone. Thus a sentence

M. Ganesalingam (Ed.): The Language of Mathematics, LNCS 7805, pp. 17–38, 2013.

> If $K \leq G$ and there are inclusions $gKg^{-1} \leq K$ for every $g \in G$, then $K \lhd G$: replacing g by g^{-1}, we have the inclusion $g^{-1}Kg \leq K$, and this gives the reverse inclusion $K \leq gKg^{-1}$.
>
> The kernel K of a homomorphism $f : G \to H$ is a normal subgroup: if $a \in K$, then $f(a) = 1$; if $g \in G$, then $f(gag^{-1}) = f(g)f(a)f(g^{-1}) = f(g)f(g^{-1}) = 1$, and so $gag^{-1} \in K$. Hence, $gKg^{-1} \leq K$ for all $g \in G$, and so $K \lhd G$. Conversely, we shall see later that every normal subgroup is the kernel of some homomorphism.

Fig. 2.1 Excerpt from Rotman (1995). Symbolic material highlighted.

> The square root of 2 is irrational.

might be expressed more concisely as

> $\sqrt{2}$ is irrational.

or even as

> $\sqrt{2} \notin \mathbb{Q}$.

Without the capacity of symbolic material to abbreviate text in a remarkably compact way, modern mathematics would quickly become unreadable. For example, the symbolic formula

$$f(gag^{-1}) = f(g)f(a)f(g^{-1})$$

would have to be written as

> The value of f at the product of g and a and the inverse of g is equal to the product of the value of f at g and the value of f at a and the value of f at the inverse of a.

(This last sentence actually contains some residual symbolic material in the form of the variables f, a, and g. Eliminating these would require rewriting the context surrounding the above remark, and the result would be even more unwieldy.)

Because symbolic material functions primarily in an abbreviative capacity, symbolic mathematics tends to occur inside textual mathematics rather than vice versa. Thus mathematical texts are largely composed out of textual sentences, with symbolic material embedded like 'islands' inside text. Most often symbolic terms are embedded in textual contexts that would accept noun phrases, and symbolic formulae are embedded in textual contexts that would accept sentences (i.e. constituents of the category 'S'). For example, in the first sentence of Figure 2.1, the term 'g^{-1}' and the formula '$K \leq G$' appear in contexts that would accept a noun phrase and a sentence respectively. Less frequently, one also find substitutions of symbolic material inside the mathematical equivalents of individual words; for example one might refer to a '$\mathbb{Z}$-module', a '$\mathbb{F}_p$-module', or a '$(\mathbb{Z} \times \mathbb{Z})$-module'.

Ranta states that symbolic material may occur inside textual material, but not vice versa (Ranta, 1997b, pp. 10–11). This remark holds in the fragment he is analysing, which describes only arithmetic and some trigonometry, but is not true of mathematics in general. A counterexample is given by the symbolic term

$$\{(x, y) \in \mathbb{N}^2 \mid x \text{ and } y \text{ are coprime}\}$$

All counterexamples involve symbolic formulae being used in contexts that accept textual sentences. (In fact, formulae and sentences appear to have the same distribution in the register of mathematics we are considering; we know of no context that admits a sentence but not a formula, or vice versa.) When one looks at terms and noun phrases, Ranta's assertion extends to mathematics as a whole; for example, one may not write:

$$\sqrt{\text{the smallest element of } \mathbb{N}}.$$

We will provide a theoretical explanation of this asymmetry of substitutability between terms and noun phrases at the end of §3.3.7.

After discussing another major feature of mathematical language, we will look at each of textual mathematics and symbolic mathematics in more detail (§2.3 and §2.4).

2.2 Adaptivity

When one starts to delve into the language of mathematics, one encounters a phenomenon that is much more remarkable than the use of symbols. Mathematical language *expands* as more mathematics is encountered. The kind of expansion to which we are referring occurs as an individual mathematician reads more and more mathematical texts, and is entirely distinct from the slow change of language over time. All references to 'change' or to 'expansion' in this book will refer to this distinctively mathematical notion, and not to conventional language change. We will call this phenomenon, by which the grammar of an individual mathematician changes as definitions are encountered, *adaptivity*.

Adaptivity occurs when certain mathematical statements, known as definitions, are read. Definitions can change the language of mathematics in two ways. First, they can expand the lexicon of the textual part of mathematics. Second, they can expand the *syntax* of the symbolic part of mathematics. Any given definitions can perform either or both of these functions. For example, consider:

> If $\mathfrak{p}$ is a minimal prime of a graded S-module M, we define the *multiplicity* of M at $\mathfrak{p}$, denoted $\mu_{\mathfrak{p}}(M)$, to be the length of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$.
>
> (Hartshorne, 1977, p. 51)