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The Making of Mathematics

Heuristic Philosophy of Mathematics

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Chapter 1
Introduction

Abstract Throughout history, philosophy and mathematics have been related and relevant to each other. Nevertheless, many contemporary mathematicians believe that philosophy, and specifically the philosophy of mathematics, is irrelevant to mathematics. This opinion is due to the fact that mainstream philosophy of mathematics, namely the philosophy of mathematics that has prevailed for the past century, does not account for the making of mathematics, in particular discovery, so it cannot provide any real understanding of the nature of mathematics, let alone contribute to its advancement. This, however, does not mean that the philosophy of mathematics is irrelevant to mathematics, but only that so is mainstream philosophy of mathematics. What is needed is an alternative approach to the philosophy of mathematics.

Keywords Relevance of mathematics to philosophy · Relevance of philosophy to mathematics · Irrelevance view · Working philosophy of the mathematician · Front and back of mathematics · Mainstream philosophy of mathematics · Heuristic philosophy of mathematics

1.1 The Relevance of Mathematics to Philosophy

Throughout history, philosophy and mathematics have been related and relevant to each other. It is no coincidence that some of the earliest philosophers, notably Thales, Pythagoras, and Democritus, were also among the earliest mathematicians, and some major philosophers of the early modern period, notably Descartes, Pascal, and Leibniz, were also major mathematicians.

Mathematics has been relevant to philosophy from the very beginning. Indeed, it has played an important role in the birth itself of philosophy as discipline.

Philosophy as discipline was not born with the Presocratics, because they did not sharply distinguish philosophy from the magic-religious tradition. Thus, Thales says that "the mind of the world is the god, and the whole is endowed with soul and also full of daemons; and the divine power, penetrating the elementary moisture, moves

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it" (Thales 11 A 23, ed. Diels-Kranz). Pythagoras says that "number" is "the source of the continuing existence of divine natures, gods, and daemons" (Iamblichus, *De Vita Pythagorica*, XXVIII.147, ed. Deubner). Democritus says that "in the universe, there are images endowed with divinity," and "the gods are the elements of mind in that universe," and "they are certain huge images of such a size as to enclose the whole universe externally" (Cicero, *De natura deorum*, I.120).

Philosophy as discipline was not born with the Sophists either, because they did not use philosophy to advance knowledge, but only to earn money by teaching rich young men rhetorical tricks. Thus, Plato says that the sophist is "a paid hunter of rich young men," a "merchant of knowledge about the soul," a "retail-dealer in these very same wares," a "seller of knowledge of his own production," an "athlete in verbal combat, appropriating to himself the art of eristic" (Plato, *Sophista*, 231 d 3–e 2). Xenophon says that "there is a good and a shameful way to dispose of one's beauty and wisdom. If a man sells his beauty for money to anyone who wants it, they call him a prostitute," and, "in the same way, those who sell their wisdom for money to anyone who wants it, they call them sophists, or, as it were, prostitutes of wisdom" (Xenophon, *Memorabilia*, I.6.13). Aristotle says that "the art of the sophist is apparent but not real wisdom, and the sophist is one who makes money from apparent and not real wisdom" (Aristotle, *Sophistici Elenchi,* 165 a 21–23).

Philosophy as discipline was born only with Plato. As Nightingale says, "the discipline of philosophy emerged" in "Athens in the fourth century BCE, when Plato appropriated the term 'philosophy' for a new and specialized discipline – a discipline that was constructed in opposition to the many varieties of 'sophia' or 'wisdom' recognized by Plato's predecessors and contemporaries" (Nightingale 1995, 14). Plato was aware to have given birth to a new discipline. This is apparent from the fact that "Plato makes no mention of philosophic predecessors in the *Republic*," because he "did not consider" the Presocratics and Sophists "to be 'philosophers' in his sense of the term" (ibid., 18 and footnote).

Nightingale, however, fails to mention that Plato gave birth to philosophy as discipline by modelling the method of philosophy on the method of mathematics. Specifically, Plato modelled the method of philosophy on the method used by Hippocrates of Chios to solve problems in mathematics. What is more, Plato gave the first formulation of that method. The same method was used by Hippocrates of Cos to solve problems in medicine, but neither Hippocrates of Chios nor Hippocrates of Cos gave a formulation of the method, they simply used it, Plato gave the first formulation. Today the method is known as the analytic method or method of analysis (see Chap. 5).

Plato modelled the method of philosophy on the method of mathematics, because he believed that mathematics was "a prelude" to "the song that must be learned" (Plato, *Respublica,* VII 531 d 7). Namely, a prelude to philosophy, which "tries, through argument and without using any of the senses, to find the being itself of each thing" (ibid., VII 532 a 5–7). Thus, arithmetic "strongly leads the soul upward, compelling it to consider the numbers themselves" (ibid., VII 525 d 5–6). And "geometry is knowledge of what always is," so it is apt "to draw the soul toward truth" and is a stimulus "to turn the gaze upward" (ibid., VII 527 b 6–9).

Since Plato believed that mathematics was a prelude to philosophy, he demanded that would-be philosophers should first study mathematics. Then, some of them would be selected to study philosophy, by testing "who is able to release himself from the eyes and the rest of sense and, guided by truth, ascend to the being itself" (ibid., VII 537 d 5–7). The demand that would-be philosophers should first study mathematics was also made by the inscription above the entrance to Plato's Academy: "Let no one ignorant of geometry enter [ageometretos medeis eisito]" (Aelius Aristides, *Opera*, III, 464.13, ed. Dindorf). With obvious exaggeration, Aristotle even complained that, for Plato and his Academy, "mathematics has come to be" all of "philosophy" (Aristotle, *Metaphysica*, A 9, 992 a 32–33).

Plato's demand that would-be philosophers should first study mathematics, was considered a valuable one even many centuries later. Thus, Galileo said: "Was not Plato very right when he wished that his pupils should be first of all grounded in mathematics?" (Galilei 1968, VIII, 175). In fact, "if I were to restart my studies, I would follow the advice of Plato and start with mathematics" (ibid., VIII, 134).

1.2 The Continued Relevance of Mathematics to Philosophy

After the birth of philosophy as discipline, mathematics has continued to be relevant to philosophy in many respects. In particular, several philosophers of the early modern period maintained that the method of philosophy is the same as the method of mathematics, identified either with the analytic method or method of analysis (see Chap. 5), or with the analytic-synthetic method or method of analysis and synthesis (see Chap. 6), or with the synthetic method or axiomatic method (see Chap. 6).

Thus, according to Descartes, the method of philosophy is the analytic method. Indeed, he says: "The old geometers only used" the synthetic method "in their writings," because they thought of the analytic method "so highly that they reserved it to themselves as a valuable secret. But I have followed the analytic method alone" in "my *Meditations*," because the synthetic method "cannot be applied so conveniently to these metaphysical matters" (Descartes 1996, VII, 156).

According to Hobbes, the method of philosophy is the analytic-synthetic method. Indeed, he says: "The method of philosophizing" is the investigation "of causes by their known effects" or "of effects by their known causes" (Hobbes 1839–1845, I, 58). Now, the investigation of causes by their known effects is resolution or analysis, and the investigation of effects by their known causes is composition or synthesis. Therefore, "the method of philosophizing" is "partly analytic, partly synthetic" (ibid., I, 66).

According to Wolff, the method of philosophy is the synthetic method. Indeed, he says: "The philosophical method" is "the same as the scientific method and the synthetic method" (Wolff 1728, 634). For, "in philosophy, no terms must be used, but those explained by an accurate definition" (ibid., 53). And "no proposition must be admitted, but that which is legitimately deduced from sufficiently established principles" (ibid., 54).

1.3 The Relevance of Philosophy to Mathematics

The relation between philosophy and mathematics, however, has not been one-way only. As mathematics has been relevant to philosophy, philosophy has been relevant to mathematics. Here are some examples.

(1) Philosophy has provided analyses of mathematical concepts.

Thus, Zeno gave an analysis of the concept of infinite set by saying that "time is composed of instants" (Zeno 29 A 27, ed. Diels–Kranz). In particular, it is composed of infinitely many instants, because "time is infinite" by "division" (Zeno 29 A 25, ed. Diels–Kranz). Also, "half a time is equal to its double" (Zeno 29 A 28, ed. Diels–Kranz). Zeno illustrated these assertions using lengths viewed as infinite sets of points, so he also implied that length is composed of infinitely many points, and half a length is equal to its double.

Now, to say that half a time is equal to its double, or that half a length is equal to its double, amounts to saying that two infinite sets can be equivalent even when one of them is a proper subset of the other. Galileo gave an example of this by pointing out that "the square numbers are as many as all the numbers, because they are as many as their roots, and all numbers are roots" (Galilei 1968, VIII, 78). Then, Dedekind used the property in question as a definition of infinite set: "A system *S* is said to be infinite when it is similar to a proper part of itself" (Dedekind 1996, 806).

(2) Philosophy has exposed the inadequacy of mathematical concepts.

Thus, Berkeley observed that, in the calculus of infinitesimals of Leibniz and Newton, if one arrives at a correct conclusion, it is only because "two errors being equal and contrary destroy each other; the first error of defect being corrected by a second error of excess" (Berkeley 1992, 182). Indeed, "if we remove the veil and look underneath" the basic concepts of the calculus, we "shall discover much emptiness, darkness, and confusion; nay, if I mistake not, direct impossibilities and contradictions" (ibid., 169). The "introducing of things so inconceivable" is "a reproach to mathematics" (ibid., 213).

Berkeley's attack contributed to the development of the calculus, by pointing out some critical questions that had to be addressed to obtain an adequate formulation. Even Robinson, who thought that Leibniz's ideas about infinitesimals could be fully vindicated, says that "the vigorous attack directed by Berkeley against the foundations of the calculus in the forms then proposed is, in the first place, a brilliant exposure of their logical inconsistencies" (Robinson 1966, 280).

(3) Philosophy has formulated new methods of discovery and justification.

Thus, as already mentioned, Plato gave the first formulation of the analytic method (see Chap. 5). Also, in Plato, there is "the only extant example of proof by" complete induction "in the ancient mathematical corpus" (Acerbi 2000, 58).

Aristotle gave the first formulation of the analytic-synthetic method or method of analysis and synthesis, and, as a byproduct, the first formulation of the synthetic method or axiomatic method (see Chap. 6).

In appendix to the *Discours de la Méthode*, Descartes published *La Géométrie* and two other treatises, calling them "essays of this method," namely of the method presented in the *Discours*, because "the things they contain could not be found without it," and "one can know from them what it is worth" (Descartes 1996, I, 349).

Descartes is a glaring example of the fact that, as mathematics has been relevant to philosophy, philosophy has been relevant to mathematics. As Bos says, for Descartes "mathematics was a source of inspiration and an example for his philosophy, and, conversely, his philosophical concerns strongly influenced his style and program in mathematics" (Bos 2001, 228).

1.4 The Irrelevance View

Several contemporary mathematicians, however, believe that philosophy, in particular the philosophy of mathematics, is irrelevant to mathematics.

Thus, Hersh says that, "in books and articles bearing the label 'philosophy of mathematics'," there are only "arguments disconnected from what mathematicians do and think about" (Hersh 2014, 21). Indeed, "the professional philosopher, with hardly any exception, has little to say to the professional mathematician" because "he has only a remote and inadequate notion of what the professional mathematician is doing" (Hersh 1979, 34). In particular, "some philosophers who write about mathematics seem unacquainted with any mathematics more advanced than arithmetic and elementary geometry" (ibid.).

Gowers says: "Suppose a paper were published tomorrow that gave a new and very compelling argument for some position in the philosophy of mathematics," and that the "argument caused many philosophers" to "embrace a whole new -ism" (Gowers 2006, 198). Then, "what would be the effect on mathematics? I contend that there would be almost none" (ibid.). For, "the questions considered fundamental by philosophers are the strange, external ones that seem to make no difference to the real, internal business of doing mathematics" (ibid.).

Cheng says: "The philosophers come up with theories that don't seem to have any impact on what the mathematicians do or think," and "ask questions that have no impact on mathematical practice" (Cheng 2004, 3). So, "daily mathematical practice seems barely affected by the questions the philosophers are considering" (ibid., 1). Indeed, "mathematical practice seems to carry on oblivious of what philosophical theories mathematicians happen to subscribe to" (ibid., 2).

1.5 The Reason of the Irrelevance View

If several contemporary mathematicians believe that philosophy, in particular the philosophy of mathematics, is irrelevant to mathematics, it is not because they think that philosophical questions concerning the nature of mathematics are of no consequence to mathematics. Not only they do not think so, but some of them even say that it is impossible to do mathematics without a philosophy that tells you what mathematics is.

Other philosophers, however, disagree. They claim that one can very well interest oneself in the philosophy of mathematics, and understand a good deal of the debates on the subject, even with little knowledge of mathematics.

Thus, Dummett says: “If you have little knowledge of mathematics, you do not need to remedy that defect before interesting yourself in the philosophy of mathematics” (Dummett 1998, 124). For, “you can very well understand a good deal of the debates on the subject and a good deal of the theories advanced concerning it without an extensive knowledge of its subject-matter” (ibid.).

But, if one can very well interest oneself in the philosophy of mathematics, and understand a good deal of the debates on the subject, even with little knowledge of mathematics, it is not because little knowledge of mathematics is enough to say what mathematics is. It is rather because such debates are about artificial issues, which have no connection with, and hence shed no light on, the real mathematical process.

On the other hand, however, that the professional philosopher has serious limitations, does not mean that the professional mathematician understands what he is doing, and hence can say what mathematics is. He may not have the necessary skills.

As Hanna and Larvor observe, “the usual reservations about practitioner-testimony apply to mathematics. Adepts in any practice can fail to understand what they are doing, how they are doing it and what conditions make it possible” (Hanna and Larvor 2020, 1137).

Even several mathematicians admit that the professional mathematician may not have the necessary skills.

Thus, Bourbaki says that “the opinions of mathematicians on topics in philosophy, even when these questions are concerned with their field, are most often opinions received at second or third hand, coming from doubtful sources” (Bourbaki 1994, 11).

Byers says that “most practicing mathematicians have no time for anything that is philosophical. They are too busy living within their paradigm, that is, proving theorems” (Byers 2017, 59).

Hersh says that “the art of philosophical discourse is not well developed today among mathematicians, even among the most brilliant,” while “philosophical issues” require “careful argument, fully developed analysis, and due consideration of objections. A bald statement of one’s own opinion is not an argument, even in philosophy” (Hersh 1979, 34–35).

The professional mathematician may not have the necessary skills, because his function is to create new mathematics, not to say what mathematics is.

As Hardy says, “the function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done” (Hardy 1992, 61).

Of course, nothing excludes that the professional mathematician may say what mathematics is. But this requires that he be skilled, not only in creating new mathematics, but also in reflecting on what mathematicians are doing, how they are doing it, and what conditions make it possible.

For this reason, Hardy began a lecture on the subject of mathematical proof saying: “I have chosen” mathematical proof as “a subject for this lecture, after much hesitation,” because the subject is “not from technical mathematics,” so “I shall be setting myself a task for which I have no sufficient qualifications” (Hardy 1929, 1).

1.8 The Front and the Back of Mathematics

In the philosophy of mathematics, an important question is the difference between finished mathematics, namely mathematics presented in finished form in journals, textbooks, or lectures, and the making of mathematics, namely the actual process of mathematical research.

Hersh expresses this difference in terms of Goffman’s “concept of ‘front’ and ‘back’” (Hersh 1997, 35). The “front is where the public is admitted,” and the “back is where it’s excluded” (ibid.). For example, in a restaurant, the front is “the dining area,” and the back is “the kitchen,” in a theater, the front is the “stage,” and the back is the “backstage” (ibid.). But Goffman extends “‘front’ and ‘back’ from restaurants and theaters to all institutions” (ibid.). Then, mathematics too has a front and a back, where “the front is mathematics in finished form,” as it is presented in “lectures, textbooks, journals,” and the “back is mathematics” as it appears “among working mathematicians, told in offices or at cafe tables” (ibid., 36).

This view is opposed by “mainstream philosophy” of mathematics, which “doesn’t know that mathematics has a back. Finished, published mathematics – the front – is taken as a self-subsistent entity” (ibid.). But this is absurd. For mainstream philosophy of mathematics, not to know that mathematics has a back, is like “for a restaurant critic not to know there are kitchens, or a theater critic not to know there is backstage” (ibid., 37). It means ignoring that “the performance in front” was “concocted behind the scenes” (ibid.). Therefore, it is “impossible to understand the front while ignoring the back” (ibid.).

Hersh is quite right in saying that it is impossible to understand the front while ignoring the back. But his position has a weakness. As we have seen, he assumes that the back is mathematics as it appears among working mathematicians, told in offices or at café tables.

This allows Greiffenhagen and Sharrock to criticize Hersh, arguing that his “treatment of the ‘front’ and the ‘back’ as a contrastive pair downplays the continuity of the two” (Greiffenhagen and Sharrock 2011, 841). The continuity is clear from a comparison between mathematical lectures, as one example of mathematics in the ‘front’, and “meetings between a supervisor and his doctoral students,” as “one example of mathematics in the ‘back’” (ibid., 854). The comparison shows that “the difference between the ‘front’ and the ‘back’” is “not between two kinds of proof,” but only “between different stages: of working with an incomplete idea of a possible proof as opposed to presenting a (presumably) complete, thoroughly worked-out proof” (ibid., 858). Thus, “the ‘finished’ product in the ‘front’” is only “a later stage and product of the ‘currently unfinished’ work in the ‘back’” (ibid., 841). Therefore, "it should not be expected that increased familiarity with what goes on 'in the mathematical back' will lead to any significant revision of understanding of what is on show 'out front'" (ibid., 861).

This objection, however, entirely depends on Hersh's assumption that the back is mathematics as it appears among working mathematicians, told in offices or at café tables. It is only because of this assumption that Greiffenhagen and Sharrock may claim that the finished product in the front is only a later stage and product of the currently unfinished work in the back.

But Hersh's assumption is invalid. The back is not mathematics as it appears among working mathematicians, told in offices or at café tables. It is instead the making of mathematics, in particular discovery, and the process of discovery is not reflected virtually to any extent in finished mathematics (see Chap. 3).

If the back is the making of mathematics, in particular discovery, then it is invalid to say, as Greiffenhagen and Sharrock do, that the difference between the 'front' and the 'back' is not between two kinds of proof, but only between different stages: of working with an incomplete idea of a possible proof as opposed to presenting a (presumably) complete, thoroughly worked-out proof. Indeed, the difference between the 'front' and the 'back' of mathematics is precisely the difference between two kinds of demonstration: axiomatic demonstration, namely demonstration based on the axiomatic method, the front, and analytic demonstration, namely demonstration based on the analytic method, the back (see Chap. 10). Axiomatic demonstration is only a means to present, justify, and teach already acquired propositions. But, for the working mathematician, demonstration is primarily a means to discover solutions to problems, and only analytic demonstration is such a means.

It is also invalid to say, as Greiffenhagen and Sharrock do, that it should not be expected that increased familiarity with what goes on 'in the mathematical back' will lead to any significant revision of understanding of what is on show 'out front'. Mathematics presented in finished form has little or nothing to do with the way it was discovered (see Chap. 3). So, what is on show 'out front' gives no understanding of the making of mathematics, in particular discovery. Only familiarity with what goes on 'in the mathematical back', hence with analytic demonstration, gives such an understanding, and this leads to a significant revision of understanding of what is on show 'out front', because analytic demonstration is explanatory (see Chap. 14).

1.9 The Need for an Alternative Approach

Since, as Hersh says, mainstream philosophy of mathematics recognizes only the front, it does not account for the making of mathematics. This justifies the belief of several contemporary mathematicians, that philosophy, and specifically the philosophy of mathematics, is irrelevant to mathematics. A philosophy that does not account for the making of mathematics, namely for the actual process of mathematical research, cannot provide any real understanding of mathematics, let alone contribute to its advancement.

Indeed, as Byers says, “it is not possible to do justice to mathematics” by “separating the content of mathematical theory from the process through which that theory is developed and understood,” so “it is of the utmost importance” to “develop a way of talking about mathematics that contains the entire mathematical experience, not just some formalized version of the results of that experience” (Byers 2007, 5).

Even some philosophers agree that mainstream philosophy of mathematics cannot provide any real understanding of the nature of mathematics, let alone contribute to its advancement.

Thus, Kreisel says that, even those who have “high hopes for (the subject of) philosophy, especially of mathematics,” can have “little trust” in “the work of contemporary professional philosophers,” in particular in “the logic chopping and obviously minor distinctions of which contemporary (Anglo-Saxon) philosophy is full” (Kreisel 1967, 212). Such work “is intended to clarify ideas in the Socratic manner; but it only keeps the outer forms including the banter of Plato, not the substance, namely the serious search for general definitions” (ibid.). Moreover, it focuses on insignificant aspects, and “there is no evidence that careful work on insignificant aspects leads one” to “recognize what is essential” (ibid.).

Putnam says that today in the philosophy of mathematics “nothing works,” indeed “every philosophy seems to fail when it comes to explaining the phenomenon of mathematical knowledge” (Putnam 1994, 499). The “much touted problems in the philosophy of mathematics” are only “problems internal to the thought of various system builders. The systems are” only “intellectual exercises,” and they, “without exception, need not be taken seriously” (Putnam 1975–1983, I, 43).

Kitcher says that, if one asks “what the philosophy of mathematics is” today, “many practicing mathematicians and historians of mathematics will have a brusque reply to” this “question: a subject noted as much for its irrelevance as for its vaunted rigor, carried out with minute attention to a small number of atypical parts of mathematics and with enormous neglect of what most mathematicians spend most of their time doing” (Kitcher 1988, 293).

Corfield says that “by far the larger part of activity in what goes by the name ‘philosophy of mathematics’ is dead to what mathematicians think and have thought, aside from an unbalanced interest in the ‘foundational’ ideas of the 1880–1930 period, yielding too often a distorted picture of that time” (Corfield 2003, 5).

But the inability to provide any real understanding of the nature of mathematics, let alone to contribute to its advancement, is a failure of mainstream philosophy of mathematics, not of philosophy as such. Indeed, as argued above, in the past, philosophy has been relevant to mathematics. What is needed is an alternative approach.

1.10 Aim of the Book

The aim of the book is to highlight the limitations of mainstream philosophy of mathematics, and to offer an alternative approach to the philosophy of mathematics. The alternative approach should satisfy two demands.

First, the alternative approach should account for the making of mathematics, in particular discovery. This contrasts with mainstream philosophy of mathematics, according to which the philosophy of mathematics cannot concern itself with the making of mathematics, but only with finished mathematics.

Second, the alternative approach should possibly contribute to the advancement of mathematics, as philosophy has done in the past. This contrasts with mainstream philosophy of mathematics, according to which the philosophy of mathematics cannot contribute to the advancement of mathematics.

Since the alternative approach should account for the making of mathematics, in particular discovery, it can be called ‘heuristic philosophy of mathematics’. For, ‘heuristic’ comes from the Greek ‘heuriskein’, which means ‘to discover’.

1.11 Organization of the Book

The book is divided into five parts after the present Introduction, which occurs as Chap. 1.

Part I, ‘Heuristic vs. Mainstream’, proposes heuristic philosophy of mathematics as an alternative to mainstream philosophy of mathematics. In particular, Chap. 2 describes characters, origin, and aim of mainstream philosophy of mathematics, and argues that it does not provide an adequate account of mathematics. Chapter 3 describes characters, origin, and aim of heuristic philosophy of mathematics, and argues that it provides an adequate account.

Part II, ‘Discourse on Method’, describes the basic methods that have been devised for mathematics. In particular, Chap. 4 describes the ancient origin of method, the oblivion of method, the role of method in the rise of modern science, the decline and end of method and its negative effects. Chapter 5 describes the analytic method, its origin, characters, and fortune, and points out the differences between the analytic method, on the one hand, abduction and reductio ad absurdum, on the other hand. Chapter 6 describes Aristotle’s analytic-synthetic method, its difference from the analytic method, Pappus’s analytic-synthetic method and its relation to reductio ad absurdum, the material axiomatic method, and the formal axiomatic method. Chapter 7 describes several rules of discovery: several kinds of induction, and several kinds of analogy, metaphor, metonymy, generalization, and specialization. Chapter 8 describes two views of theories and theory building: the axiomatic view and the analytic view.

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Part I
Heuristic vs. Mainstream

Chapter 2
Mainstream Philosophy of Mathematics

Abstract The chapter describes characters, origin, and goal of mainstream philosophy of mathematics, namely the philosophy of mathematics that has prevailed for the past century. According to it, the philosophy of mathematics cannot concern itself with the making of mathematics but only with finished mathematics, namely mathematics presented in finished form, and the method of mathematics is the axiomatic method. The chapter argues that, because of Gödel's incompleteness theorems and for several other reasons, mainstream philosophy of mathematics does not provide an adequate account of mathematics.

Keywords Mainstream philosophy to mathematics · Foundationalist view · Top-down approach · Closed systems · Mathematics as theorem proving · Relevance of incompleteness theorems · Shortcomings of reductionism

2.1 The Fabric of Mainstream Philosophy of Mathematics

In the Introduction, reference has been made to mainstream philosophy of mathematics, namely the philosophy of mathematics that has prevailed for the past century.

Mainstream philosophy of mathematics consists of the three big foundationalist schools, logicism, formalism, and intuitionism, and their direct or indirect descendants.

The direct descendants of the three big foundationalist schools are revised versions of them: neo-logicism, neo-formalism, and neo-intuitionism.

The indirect descendants of the three big foundationalist schools are variations on their themes: platonism, abstractionism, structuralism, fictionalism, phenomenology, and empiricism.

However, the indirect descendants of the three big foundationalist schools also comprises the philosophy of mathematical practice. For, contrary to the widespread view that the latter is alternative to mainstream philosophy of mathematics, as argued in Chap. 3, the philosophy of mathematical practice is continuous with mainstream philosophy of mathematics.

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2.2 The Characters of Mainstream Philosophy of Mathematics

Mainstream philosophy of mathematics has the following characters.

(1) The philosophy of mathematics cannot concern itself with the making of mathematics, in particular discovery, because discovery is a subjective process, so it cannot be accounted for.

Thus, Dieudonné says that the philosophy of mathematics cannot concern itself with the making of mathematics, because it is impossible to explain how mathematicians "arrived at their results" (Dieudonné 1998, 27). For, "what goes on in a creative mind never has a rational 'explanation', in mathematics any more than elsewhere. All that we know is that it" involves "sudden 'illuminations'" (ibid., 27).

Feferman says that the philosophy of mathematics cannot concern itself with the making of mathematics, because the "individual processes of mathematical discovery appear haphazard and illogical" (Feferman 1998, 77). Therefore, "the creative and intuitive aspects of mathematical work evade logical encapsulation" (ibid., 178).

(2) The philosophy of mathematics can concern itself only with finished mathematics, namely mathematics presented in finished form, because only finished mathematics is objective, so it can be completely justified.

Thus, Pólya says that the philosophy of mathematics can concern itself only with "finished mathematics" because only finished mathematics is objective, being "purely demonstrative, consisting of proofs only" (Pólya 1954, I, vi). Finished mathematics can be completely justified, because "we secure our mathematical knowledge by demonstrative reasoning," which "is safe, beyond controversy, and final" (ibid., I, v).

Dummett says that "the philosophy of mathematics" can concern itself only "with the product of mathematical thought," because only the latter is objective and hence can be completely justified, conversely, "the study of the process of production is the concern of psychology, not of philosophy" (Dummett 1991, 305).

(3) Since the philosophy of mathematics cannot concern itself with the making of mathematics, it cannot contribute to the advancement of mathematics.

Thus, Körner says that, "as the philosophy of law does not legislate, or the philosophy of science devise or test scientific hypotheses, so – we must realize from the outset – the philosophy of mathematics does not add to the number of mathematical theorems and theories" (Körner 1986, 9).

Dummett says that, while mathematicians advance knowledge, philosophers of mathematics only cast "light on what we already know from other sources" (Dummett 2010, 7). So, the philosophy of mathematics "does not advance knowledge," it only "clarifies what we already know" (ibid., 21).

(4) The task of the philosophy of mathematics is primarily to give an answer to the question: How do mathematical propositions come to be completely justified? And, subordinately to it, to the question: Do objects exist in virtue of which mathematical propositions are true, and if so what is their nature?

Thus, Lehman says that the task of the philosophy of mathematics is primarily to give an answer to the question of "how mathematical beliefs come to be completely justified," and, subordinately to it, to the question of "whether there are entities in virtue of which the propositions are true," and "if so what their nature is" (Lehman 1979, 1).

Shapiro says that the task of the philosophy of mathematics is primarily to give an answer to the question of what are the true "justifications for mathematical propositions," and, subordinately to it, to the question of whether there are mathematical objects in virtue of which the propositions are true, and if so what is the "underlying nature of mathematical objects" (Shapiro 2004, 37).

(5) The method of mathematics is the axiomatic method. The latter is the method according to which, to demonstrate a proposition, one starts from given principles which are true, in some sense of 'true', and deduces the proposition from them (see Chap. 6).

Thus, Kac and Ulam say that "mathematics owes its unique position to its adherence to the axiomatic method," which "consists in starting with a few statements (axioms) whose truth is taken for granted and then deriving other statements from them by the application of rules of logic alone" (Kac and Ulam 1992, 139).

Rota says that, according to "the accepted description of mathematical truth," a "mathematical statement is held to be true if it is correctly derived from the axioms by application of the rules of inference," because "the truth of all theorems can 'in principle' be 'found' in the axioms" (Rota 1997, 109).

(6) The role of axiomatic demonstration, namely demonstration based on the axiomatic method, is to guarantee the truth of a proposition.

Thus, Bass says that axiomatic demonstration "is the defining source of mathematical truth" (Bass 2015, 129). For, "saying that a mathematical claim is true means, for a mathematician, that there exists" an axiomatic demonstration "of it" (ibid., 132).

Jaffe says that axiomatic demonstration "has the highest degree of certainty possible for man" (Jaffe 1997, 135). While "scientific hypotheses come and go," the truth of a mathematical proposition obtained by an axiomatic demonstration "lasts forever" (ibid., 139).

(7) Since the method of mathematics is the axiomatic method, mathematics is a body of truths, and indeed truths that are certain. Therefore, mathematics is about truth and certainty.

Thus, Feferman says that, since the method of mathematics is the axiomatic method, mathematics is "the paradigm of certain and final knowledge: not fixed, to be sure, but a steadily accumulating coherent body of truths obtained by successive deduction from the most evident truths" (Feferman 1998, 77).

Chihara says that, since the method of mathematics is the axiomatic method, "mathematics is a system of truths and mathematicians are attempting to arrive at truths" (Chihara 1990, 171). For hundreds of years mathematicians "have reasoned and constructed their theories with the tacit belief that" the "principles of

(9) The philosophy of mathematics is a new independent subject, which has been made possible by "a renaissance of logic" (Frege 2013, I, xxvi).

(10) The philosophy of mathematics can be developed independently of experience, because mathematics involves an "a priori mode of cognition" (Frege 1979, 277). So mathematics does not depend on experience, but only on a priori "intuition" in "its proofs" (ibid., 278).

2.4 The Foundationalist View of Mathematics

Mainstream philosophy of mathematics puts forward the foundationalist view of mathematics, which is based on the following assumptions.

(I) There is immediately justified knowledge, namely knowledge which is justified without inference, and all other knowledge is deduced from it, therefore it is justified knowledge.

The foundationalist view owes its name to the architectural metaphor, according to which knowledge is an edifice whose foundation consists of the immediately justified knowledge, and whose body consists of all other knowledge deduced from it, hence knowledge anchored to the foundation via deductive inference.

The immediately justified knowledge are the axioms, all other knowledge deduced from it are the theorems. Therefore, mathematics is theorem proving by the axiomatic method.

(II) The immediately justified knowledge is true and certain because it is based on intuition, and hence so is all knowledge deduced from it. Therefore, mathematics is true and certain because it is based on intuition.

(III) There is a part of mathematics such that all other parts of mathematics can be reduced to it. Specifically, there is a mathematical theory such that all other mathematical theories can be reduced to it. This mathematical theory is The Foundation.

2.5 Original Formulation of the Foundationalist View

Aristotle gave the first formulation of the foundationalist view of mathematics, with the exception of assumption (III).

Aristotle argues for assumption (I) of the foundationalist view as follows. Suppose there is no immediately justified knowledge. Then there will be no primitive premisses, the series of the premisses will be infinite, and one will be "led back in an infinite regress" (Aristotle, *Analytica Posteriora*, A 3, 72 b 8–9). But "it is impossible to traverse an infinite series" (ibid., A 3, 72 b 10–11). So, if the series of the premisses is infinite, then "there is no knowledge" (ibid., A 3, 72 b 5–6). But, as a matter of fact, there is knowledge. Contradiction. Therefore, there is immediately justified knowledge, and all other knowledge is deduced from it, hence it is justified knowledge. Aristotle also introduces the architectural metaphor by saying that, since all other knowledge is deduced from it, the immediately justified knowledge is "that from which a thing first arises," therefore it is like "the foundation of a house" (Aristotle, *Metaphysica*, Δ 1, 1013 a 4–5).

Aristotle argues for assumption (II) of the foundationalist view as follows. Since the immediately justified knowledge is that from which all other knowledge is deduced, the immediately justified knowledge is indemonstrable. Then, "it is intuition, and not discursive thinking, that apprehends the primitive things," namely "it is intuition that apprehends the unchanging and first terms in the order of demonstrations" (Aristotle, *Ethica Nicomachea*, Z 11, 1143 a 36–1143 b 3). Now, intuition is intuition of the essence of things, and "when intuition is of the essence of things, it is true" (Aristotle, *De Anima*, Γ 6, 430 b 28). And, about the essence of things, "it is not possible to be mistaken" (Aristotle, *Metaphysica*, Θ 10, 1051 b 31). Therefore, the immediately justified knowledge is true and certain. Also, all knowledge deduced from the immediately justified knowledge is true and certain. For, "a conclusion from truths is always true" (Aristotle, *Analytica Posteriora*, A 6, 75 a 5–6). And a conclusion from premisses which are certain "results by necessity because these things are so" (Aristotle, *Analytica Priora*, A 1, 24 b 19–20).

On the other hand, while arguing for assumptions (I) and (II) of the foundationalist view, Aristotle does not argue for assumption (III). This depends on the fact that "the diagonal" of the square "is incommensurable with the side" (Aristotle, *Topica*, Θ 13, 163 a 11–12). Therefore, geometry cannot be reduced to arithmetic.

The assumption (III) acquired credibility only in the second half of the nineteenth century, when the basis was laid down for a reduction of arithmetic and geometry to set theory.

2.6 A Remark on the Original Formulation of the Foundationalist View

That Aristotle gave the first formulation of the foundationalist view, with the exception of assumption (III), does not mean, however, that for Aristotle the axiomatic method is the method of the making of mathematics.

For Aristotle, the axiomatic method is only the method of finished mathematics, because the purpose of the axiomatic method is not to acquire new knowledge, but only to present, justify, and teach already acquired propositions (see Chap. 6).

Conversely, the method of the making of mathematics is the analytic-synthetic method or method of analysis and synthesis.

The latter is the method according to which, to solve a problem, one looks for some hypothesis that is a sufficient condition for solving the problem, namely such that a solution to the problem can be deduced from the hypothesis. The hypothesis is obtained from the problem, and possibly other data already available, by some non-deductive rule, and must be plausible, namely such that the arguments for the hypothesis are stronger than the arguments against it, on the basis of experience. If the hypothesis so obtained is not a principle (or a proposition deduced from principles), one looks for another hypothesis that is a sufficient condition for solving the problem posed by the previous hypothesis, it is obtained from the latter, and possibly other data already available, by some non-deductive rule, and must be plausible. And so on, until one arrives at a principle (or a proposition already deduced from principles). The principles must be true. When one arrives at a principle (or a proposition deduced from principles), the process terminates. This is analysis.

At this point, one tries to see whether, inverting the order of the steps followed in analysis, one obtains a deduction of the solution of the problem from the principle (or proposition deduced from principles) arrived at in analysis. This is synthesis (see Chap. 6).

2.7 Logicism and the Foundationalist View

The three big foundationalist schools, logicism, formalism, and intuitionism, make all the assumptions (I) – (III) of the foundationalist view, although they differ in their way of implementing them.

As to logicism, Frege says that, "if we start from a theorem and trace the chains of inference backwards," we "must eventually come to an end by arriving at truths which cannot themselves be inferred in turn from other truths" (Frege 1979, 204). These truths are immediately justified knowledge.

Immediately justified knowledge is knowledge such that one cannot be "in doubt about its truth" (ibid., 205). For, it is based on intuition. Specifically, the immediately justified knowledge of arithmetic is based on intellectual intuition, which is "the logical source of knowledge" (ibid., 267). Then, all other knowledge deduced from it will be knowledge such that one cannot be in doubt about its truth.

The character of the immediately justified knowledge of arithmetic is clear from the fact that "we can count just about everything that can be an object of thought: the ideal as well as the real, concepts as well as objects, temporal as well as spatial entities, events as well as bodies" (Frege 1984, 112). Since "the basic propositions on which arithmetic is based," namely the immediately justified knowledge of arithmetic, "must extend to everything that can be thought," surely "we are justified in ascribing such extremely general propositions to logic" (ibid.). Thus, "there is no such thing as a peculiarly arithmetical mode of inference that cannot be reduced to the general inference-modes of logic" (ibid., 113).

Then, "a rigorous establishment of arithmetical laws reduces them to purely logical laws" (ibid., 145). Specifically, it reduces them to the basic logical laws of Frege's logical system. Therefore, Frege's logical system is The Foundation.

2.8 Formalism and the Foundationalist View

As to formalism, Hilbert says that, in mathematics, there are “a few distinguished propositions” which “suffice by themselves for the construction, in accordance with logical principles, of the entire framework” (Hilbert 1996a, 1108). These basic propositions are immediately justified knowledge.

The basic propositions are true and certain because they are based on intuition, specifically, on Kant’s pure intuition of space and time. However, a distinction must be made between them, because some of them are based on intuition directly, others indirectly.

The basic propositions which are based on intuition directly are the propositions of finitary mathematics, namely the mathematics which can be expressed without using actual infinite sets. These propositions are “real propositions” (Hilbert 1967b, 470). For, they are about certain “concrete objects that are intuitively present as immediate experience prior to all thought” (ibid., 464). Therefore, they can be based on intuition directly.

The basic propositions which are based on intuition indirectly are the propositions of infinitary mathematics, namely mathematics which cannot be expressed without using actual infinite sets. These propositions are “ideal propositions,” because they are about certain abstract objects which are only “ideal objects” (ibid., 470). They are not intuitively present, and hence cannot be based on intuition directly.

Nevertheless, the ideal propositions can be based on intuition indirectly. For, “there is a condition, a single but absolutely necessary one, to which the use” of the ideal propositions “is subject, and that is the proof of consistency” (Hilbert 1967a, 383). Namely, it must be proved that, from the ideal propositions, “it is impossible for us to obtain two logically contradictory propositions, A and $\neg A$” (Hilbert 1967b, 471).

To prove this, the ideal propositions must be formalized. This is necessary because “the ideal propositions, insofar as they do not express finitary assertions, do not mean anything in themselves,” so “the logical operations cannot be applied to them in a contentual way,” hence “it is necessary to formalize the logical operations and also the mathematical proofs themselves” (Hilbert 1967a, 381). Thus, mathematics becomes “manipulation of signs according to rules” (Hilbert 1967b, 467).

Once the ideal propositions have been formalized, their consistency must be proved, and must be proved by a proof based on the “intuitive mode of thought” (Hilbert 1996d, 1150). For, only the intuitive mode of thought can avoid “any dubious or problematic mode of inference” (Hilbert 1996c, 1139). A consistency proof based on the intuitive mode of thought will warrant that the basic propositions which are ideal propositions are “incontestable and ultimate truths” (Hilbert 1996b, 1121). Indeed, if they “do not contradict one another with all their consequences, then they are true” (Hilbert 1980, 39).

Through a consistency proof based on the intuitive mode of thought, the ideal propositions are based on intuition indirectly, because their consistency “rests on a kind of intuitive insight” (Hilbert 1996e, 1161).

Although there are several mathematical theories, Zermelo set theory "encompasses all mathematical theories (like number theory, analysis, geometry), in the sense that the relations which obtain between the objects of one of these mathematical theories are perfectly represented by the relations which obtain within a subdomain of Zermelo set theory" (Hilbert 2013, 356). Thus, "Zermelo axiom system represents the most comprehensive axiomatic system" (ibid.). Therefore, Zermelo set theory is The Foundation.

2.9 Intuitionism and the Foundationalist View

As to intuitionism, Brouwer says that infinitary mathematics is merely a "linguistic science, operating on meaningless words or symbols by means of logical rules" (Brouwer 1975, 522). By so operating, "no more is obtained than a linguistic structure" which "irrevocably remains separated from mathematics" (ibid., 97). Then, Hilbert's attempt to justify infinitary mathematics by a consistency proof based on the intuitive mode of thought "contains a circulus vitiosus since such justification is based on the (contentual) correctness of the assertion that correctness of a proposition follows from its noncontradictority" (Brouwer 1998, 41).

We "can obtain knowledge of mathematics" only if mathematics is "constructed by direct intuition" (Brouwer 1975, 75). And specifically, only if it is constructed by Kant's pure "intuition of time," which is "the basic intuition of mathematics" (ibid., 71). Therefore infinitary mathematics must be replaced with an alternative mathematics, "intuitionistic mathematics," which is built starting from basic propositions based on intuition, and "deducing theorems" from them "exclusively by means of introspective construction" (ibid., 488). Namely, by means of deductive inferences based on intuition. The basic propositions are immediately justified knowledge.

The basic propositions are true and certain because they are based on intuition, and intuition is "the origin of mathematical certainty" (ibid., 508). All propositions deduced from them are also true and certain, because they are deduced exclusively by means of deductive inferences based on intuition. Therefore, mathematics is true and certain being based on intuition.

By deducing theorems from basic propositions, intuitionistic mathematics proceeds by the axiomatic method. It might be thought that, from the intuitionistic point of view, the axiomatic method is unimportant, but it is not so. As Heyting says, of course, "from the intuitionistic point of view," the axiomatic "method cannot be used in its creative function," because "a mathematical object is considered to exist" only "after its construction" by intuition, so "it cannot be brought into existence by a system of axioms" (Heyting 1962, 239). But nothing prevents from using the axiomatic method in its descriptive function, as a description of constructions already made, and "the descriptive function of a system of axioms is as important intuitionistically as it is classically" (ibid.).

Ulam says that “mathematicians start with axioms whose validity they don’t question. You might say it is just a game” which “we play according to certain rules” of deduction, “starting with statements which we cannot analyze further” (Ulam 1986, 13–14).

Gowers says that the mathematician “starts with axioms” and “proceeds to the desired conclusion by means of only the most elementary logical rules” (Gowers 2002, 39).

Sternheimer says that “in mathematics one starts with axioms and uses logical deduction therefrom to obtain results that are absolute truth in that framework” (Sternheimer 2011, 42).

2.13 Inadequacy of the Infinite Regress Argument

Although assumption (I) of the foundationalist view is shared by the majority of mathematicians, nevertheless it is invalid.

First, Aristotle’s argument for assumption (I), that for any part of mathematics there is immediately justified knowledge, is invalid. This can be seen as follows.

Aristotle argues that, if there is no immediately justified knowledge, then there will be no primitive premisses, so the series of the premisses will be infinite. But it is impossible to traverse an infinite series. So, if the series of the premisses is infinite, then there is no knowledge. But, as a matter of fact, there is knowledge. Contradiction. Therefore, there is immediately justified knowledge.

Now, Aristotle is quite right in saying that it is impossible to traverse an infinite series. But this only means that, if the series of the premisses is infinite, then there is no immediately justified knowledge. It does not mean that there is no knowledge.

There would be no knowledge only if the premisses occurring in the infinite series were arbitrary. But they need not be arbitrary, they can be plausible, namely such that the arguments for them are stronger than the arguments against them, on the basis of experience. Now, if the premisses are plausible, then there is knowledge. Admittedly, such knowledge is not absolutely certain, it is provisional, always in need of further consideration. But, as it will be argued below, this is the only kind of knowledge that is possible for us.

At each stage, we may check whether the premisses used until then are plausible. To have knowledge, it is not necessary that we arrive at immediately justified premisses, but only that, at each stage, the premisses used at that stage are plausible. Therefore, Aristotle’s argument is invalid.

2.14 The Foundationalist View and Gödel's Incompleteness Theorems

Not only Aristotle's argument for assumption (I) of the foundationalist view is invalid, but no other argument for it could be valid. For, assumption (I) is refuted by Gödel's incompleteness theorems.

Assumption (I) is refuted by Gödel's first incompleteness theorem. For, by the latter, for any consistent, sufficiently strong, formal system, there are propositions of the system that are true but cannot be deduced from the axioms of the system. This implies that mathematics cannot consist in the deduction of propositions from given axioms because, for any choice of axioms for a given part of mathematics, there will always be true propositions of that part which cannot be deduced from the axioms.

Assumption (I) is also refuted by Gödel's second incompleteness theorem. For, by the latter, for any consistent, sufficiently strong, formal system, it is impossible to demonstrate, by absolutely reliable means, that the axioms of the system are consistent. So there is no guarantee that the propositions deduced from the axioms are justified knowledge.

2.15 Gödel's Attempt to Reaffirm Mathematics as Theorem Proving

Although assumption (I) of the foundationalist view is refuted by Gödel's incompleteness theorems, mathematicians have greatly resisted accepting this conclusion. Gödel himself tries to reaffirm that mathematics is theorem proving by the axiomatic method, by arguing that his incompleteness theorems merely require that, instead of being formalizable in a single formal system, mathematics is formalizable in an infinite sequence of formal systems.

Indeed, Gödel says that, although his first incompleteness theorem implies that it is "impossible to formalize all of mathematics in a single formal system, a fact that intuitionism has asserted all along," nevertheless "everything mathematical is formalizable" (Gödel 1986–2002, I, 389). It is formalizable not in a single formal system, but in "a sequence (continuable into the transfinite) of formal systems" (ibid., I, 237).

However, for this argument to be credible, the transition from a formal system to the next one in the sequence of formal systems should itself be formal. For, if the transition is not formal and requires an appeal to intuition, it will be impossible to say that everything mathematical is formalizable, the appeal to intuition will lead beyond what is formalizable.

But, if the transition from a formal system to the next one in the sequence of formal systems is itself formal, then, as McCarthy argues, for the sequence of formal systems it will be possible to demonstrate a theorem that "is an exact analogue" of "Gödel's first" incompleteness "theorem" (McCarthy 1994, 427). Therefore, not everything mathematical will be formalizable in the continuable sequence of formal systems.

Since not everything mathematical will be formalizable in the continuable sequence of formal systems, then, contrary to Gödel's claim, mathematics cannot consist in the activity of "an idealized mathematician who entertains a sequence of successive" formal systems, and whose choices of formal systems "are effectively determined at each stage" (ibid., 444). Mathematics cannot consist in that, even if one identifies mathematics with the activity of "an idealized mathematician whose epistemic alternatives are effectively determined at each stage, but who may have a choice among these alternatives" (ibid., 446). Therefore, Gödel's argument is invalid.

2.16 Recalcitrant Mathematicians

Already Post stigmatized the great resistance of mathematicians to accept that assumption (I) of the foundationalist view is refuted by Gödel's incompleteness theorems. In 1941 he wrote: "It is to the writer's continuing amazement that ten years after Gödel's remarkable achievement current views on the nature of mathematics are thereby affected only to the point of seeing the need of many formal systems, instead of a universal one" (Post 1965, 345). Instead, "has it seemed to us to be inevitable that these developments will result in a reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries," and that axiomatic "thinking will then remain as but one phase of mathematical thinking" (ibid.).

A fortiori, it is to our continuing amazement that today mathematicians continue to say that "the axiomatic method is 'the' method of mathematics, in fact, it is mathematics" (Naylor and Sell 2000, 6).

This involves denying the relevance of Gödel's first incompleteness theorems to assumption (I) of the foundationalist view. To this purpose, the following arguments have been used.

(1) Gödel's incompleteness theorems do not refute the assumption that mathematics is theorem proving by the axiomatic method. They merely involve that, instead of being formalizable in a single formal system, mathematics is formalizable in a network of formal systems.

Thus, Curry says that "the propositions of mathematics are the propositions" of "some formal system," so "we have not confined mathematics to a single formal system" (Curry 1954, 231). Of course, by Gödel's first incompleteness theorem, "it is hopeless to find a single formal system which will include all of mathematics," so "the essence of mathematics" cannot lie "in any particular kind of formal system" (Curry 1951, 56). But, instead of lying in any particular kind of formal system, "the essence of mathematics lies in the formal method as such" and hence in formal systems, and "in this sense mathematics is the science of formal systems" (Curry 1977, 14). Since, by Gödel's first incompleteness theorem, "the concept of intuitively valid proof cannot be exhausted by any single formalization," it follows that "mathematical proof is precisely that sort of growing thing which the intuitionists have postulated for certain infinite sets" (ibid., 15).

This argument, however, is the same as Gödel's argument that has been discussed above, therefore it is invalid for the reasons stated there. Moreover, the claim that mathematical proof is precisely that sort of growing thing which the intuitionists have postulated for certain infinite sets, is incompatible with the concept of formal proof, because a formal proof cannot be a growing thing. Therefore, mathematical proof transcends the formal method as such.

(2) Gödel's incompleteness theorems do not refute the assumption that mathematics is theorem proving by the axiomatic method, because the propositions that are true but indemonstrable from the axioms are very artificial. They have no connection with real mathematics, and hence are mathematically insignificant.

Thus, Dieudonné says: "Let us suppose that tomorrow, as if by magic, all the works of logic written after 1925," in particular Gödel's incompleteness paper, "disappeared: well, no mathematician, when he proves a theorem, would notice it" (Dieudonné 1981, 22). For, "the undecidable proposition A described by Gödel appears to be very artificial, without any connection with any other part of the current theory of numbers" (Dieudonné 1998, 231). In fact, "among the numerous classical questions which are not resolved within number theory, it has not yet been established, as far as I know, that any of them is undecidable" (ibid.).

This argument, however, is invalid because there are propositions of the theory of numbers of the usual kind, which are true but cannot be demonstrated in first-order Peano arithmetic PA. An example of such propositions is Goodstein's theorem, which can be stated as follows.

The pure base n representation of a natural number m is the result of writing m as a sum of powers of n, then rewriting the various exponents of n themselves as sums of powers of n, and so on until possible (writing n^0 as simply 1).

For example, the pure base 2 representation of 26 is $26 = 2^{2^{2^1}} + 2^{2^1+1} + 2^1$.

For any natural number m and $n \geq 1$, the Goodstein sequence starting from m is the sequence of natural numbers $g(m, n)$ defined as follows:

$$g(m, 1) = m$$

$$g(m, n+1) = \text{the result of taking the pure base } n+1 \text{ representation of } g(m, n), \text{ then replacing each occurrence of the base base } n+1 \text{ with } n+2, \text{ and finally subtracting } 1.$$

For example, for $m = 3$, $g(3, 1) = 3 = 2^1 + 1$, $g(3, 2) = 3^1 + 1 - 1 = 3^1$, $g(3, 3) = 4^1 - 1 = 3$, $g(3, 4) = 3 - 1 = 2$, $g(3, 5) = 2 - 1 = 1$, $g(3, 6) = 1 - 1 = 0$. Thus, for $m = 3$, there is an n, namely $n = 6$, such that $g(m, n) = 0$.

Goodstein's theorem states that this holds generally: For any natural number m there is an n such that $g(m, n) = 0$.

Goodstein's theorem is a proposition of number theory of the usual kind, and can be expressed in first-order Peano arithmetic PA. Now, by a result of Kirby and Paris, we have: Goodstein's theorem $\Rightarrow$ consistency of PA. On the other hand, by Gödel's second incompleteness theorem, the consistency of PA cannot be proved in PA. Therefore, Goodstein's theorem cannot be proved in PA. So, despite Goodstein's theorem being purely number-theoretic in character and "being expressible in first-order" Peano "arithmetic, we cannot give a proof of it in Peano arithmetic" (Kirby and Paris 1982, 286).

In addition to Goodstein's theorem, there are other propositions of the theory of numbers of the usual kind, which are true but cannot be demonstrated in first-order Peano arithmetic PA.

2.17 The Failure of Deductivism

Assumption (I) of the foundationalist view, that mathematics is theorem proving by the axiomatic method, is an expression of deductivism, the view that mathematical reasoning is either deductive or defective. The mathematical reasoning to which deductivism refers includes not only first-order reasoning but also higher-order reasoning. For, as Hilbert acknowledges, mathematics requires reasoning involving "higher types of variables" (Hilbert 1998, 231).

But deductivism conflicts with the strong incompleteness theorem for second-order logic. By the latter, there is no consistent formal system for second-order logic capable of deducing all second-order logical consequences of any given set of propositions.

Indeed, assume that there is a consistent formal system L^2 for second-order logic capable of deducing all second-order logical consequences of any given set of propositions.

Now, by Gödel's first incompleteness theorem for second-order Peano arithmetic PA^2, there is a proposition G which is true in the intended interpretation of PA^2, but cannot be deduced from the axioms of PA^2 by means of the rules of L^2.

By our assumption about L^2, from this it follows that G cannot be a second-order logical consequence of the axioms of PA^2. This means that there must be some full interpretation in which the axioms of PA^2 are true and G is false. (A full interpretation is an interpretation where the domain of second-order variables consists of all subsets of the domain of first-order variables).

But the axioms of PA^2 are categorical, namely any full interpretation in which the axioms of PA^2 are true is isomorphic to the intended interpretation of PA^2. Then, since there must be some full interpretation in which the axioms of PA^2 are true and G is false, from the fact that G is false in such full interpretation, it follows that G must be false in the intended interpretation of PA^2.

But G is true in the intended interpretation of PA^2. Contradiction. Therefore, there is no consistent formal system for second-order logic capable of deducing all second-order logical consequences of any given set of propositions.

The strong incompleteness theorem for second-order logic means that deduction is not strong enough to obtain all second-order logical consequences of any given set of propositions, therefore deductivism does not account for mathematical reasoning, because mathematics requires reasoning involving higher types of variables. This is another reason why assumption (I) of the foundationalist view is invalid.

If the equilateral triangle of Step 0 has area a and perimeter p, then the result of the first application of Step 1 has area $(3/4) \times a$ because we must subtract the area of the removed triangle, and perimeter $(3/2) \times p$ because we must add the perimeter of the removed triangle. The result of the second application of Step 1 has area $(3/4)^2 \times a$ and perimeter $(3/2)^2 \times p$. The result of the third application of Step 1 has area $(3/4)^3 \times a$ and perimeter $(3/2)^3 \times p$. Generally, the result of the n-th application of Step 1 has area $(3/4)^n \times a$ and perimeter $(3/2)^n \times p$.

Thus, at each application of Step 1, the area decreases and the perimeter increases. As we continue the process indefinitely, the area converges to zero and the perimeter diverges to infinity. Therefore, the Sierpiński triangle has zero area and infinite perimeter.

The Sierpiński triangle is only one of several counterexamples to the conclusions of intuition. They show that intuition is unreliable and inadequate as a basis for mathematics.

2.20 Foundationalist Programs and Intuition

That assumption (II) of the foundationalist view is invalid because intuition is unreliable and inadequate as a basis for mathematics, is also clear from the fact that the attempts of the three big foundationalist schools to base mathematics on intuition ended in failure.

(1) The main basic logical law of Frege's logical system is a principle concerning extensions of concepts, the Basic Law V: For any concepts F and G, the extension of F is identical to the extension of G if and only if, for every object a, $F(a)$ if and only if $G(a)$. Frege believed that the truth of the Basic Law V was guaranteed by intuition, indeed he claimed that the Basic Law V is what one thinks when "one speaks of extensions of concepts" (Frege 2013, I, VII). Frege had such a confidence in intuition as to claim: "It is from the outset unlikely that," being based on the Basic Law V, my logical system "could be built on an insecure, defective basis," indeed "I could only acknowledge it as a refutation" if "someone proved to me that my basic principles lead to manifestly false conclusions. But no one will succeed in doing so" (ibid., I, XXVI).

But Frege deluded himself. By showing that the Basic Law V leads to a contradiction, Russell just succeeded in doing so. Indeed, let R be the concept defined by: for any object x, $R(x)$ if and only if there is a concept F such that x is the extension of F and not-$F(x)$. Let r be the extension of R. Now, assume $R(r)$. Then, by the definition of R, there is a concept F such that r is the extension of F and not-$F(r)$. From this, since r is also the extension of R, by the Basic Law V, it follows not-$R(r)$. Thus, if $R(r)$, then not-$R(r)$. Conversely, assume not-$R(r)$. Then there is a concept F (namely R) such that r is the extension of F and not-$F(r)$. Hence, by the definition of R, $R(r)$. Thus, if not-$R(r)$, then $R(r)$. Therefore we conclude that $R(r)$ if and only if not-$R(r)$. This is a contradiction, known as Russell's paradox.

Russell's paradox confirms that intuition is unreliable as a basis for mathematics. As Gödel says, Russell brought "to light the amazing fact that our logical intuitions (i.e. intuitions concerning such notions as: truth, concept, being, class, etc.) are self-contradictory" (Gödel 1986–2002, II, 124).

(2) Hilbert assumed that the consistency of Zermelo set theory could be proved by a proof based on the intuitive mode of thought. In a paper from 1931, he even went so far as to say: "I believe" that "I have fully attained what I desired and promised: The world has thereby been rid, once and for all, of the question of the foundations of mathematics as such" (Hilbert 1996e, 1157).

But Hilbert deluded himself. By Gödel's second incompleteness theorem, it is impossible to prove the consistency of Zermelo set theory by a proof based on the intuitive mode of thought. Therefore, intuition is inadequate as a basis for mathematics. Moreover, even if it were possible to demonstrate the consistency of Zermelo set theory by a demonstration based on the intuitive mode of thought, this would not guarantee that mathematical theorems are incontestable and ultimate truths. For, by the theorem on the false extensions, any consistent sufficiently strong formal system has a consistent extension in which some false proposition is provable.

The theorem on the false extensions is a corollary of Gödel's first incompleteness theorem. Indeed, by the latter, for any consistent, sufficiently strong, formal system S, there is a proposition G of S which is true but unprovable in S. Then, let T be the formal system obtained from S by adding $\neg G$ as a new axiom. Since G is unprovable in S, the system T is consistent. Trivially $\neg G$, being an axiom of T, is provable in T. On the other hand, since G is true, $\neg G$ is false. Therefore, T is a consistent extension of S in which the false proposition $\neg G$ is provable.

The theorem on the false extensions shows that consistency is no guarantee against falsity. Hilbert need not have waited for Gödel to realize that, he could have learned it from Kant. For, Kant had made it clear that it is, "to be sure, a necessary logical condition" that in "a concept no contradiction must be contained," but this is "far from sufficient for the objective reality of the concept, i.e., for the possibility of such an object as is thought through the concept" (Kant 1998, A220/B268). Indeed, it is not enough to assume, as "condition of all our judgments whatsoever," that "they do not contradict themselves," because "for all that a judgment may be free of any internal contradiction, it can still be either false or groundless" (ibid., A150/B190).

(3) Brouwer assumed that intuition is "sufficient to build up all mathematics" (Brouwer 1975, 61). Therefore, "man builds up pure mathematics out of the basic intuition of the intellect" (ibid., 53).

But this is invalid. On the one hand, the assumption that intuition is sufficient to build up all mathematics conflicts with the fact that it is impossible to construct certain mathematical objects that are important for physics by direct intuition. For example, let f be the function defined by: $f(x) = 0$ if $x = 0$, $f(x) = 1$ if $x \neq 0$, for any real number x. Now, it is impossible to construct this function on the basis of intuition (see Cellucci 2007, Section 3.10).

On the other hand, the assumption that intuition is sufficient to build up all mathematics conflicts with Gödel's first incompleteness theorem. By the latter, for any consistent, sufficiently strong, formal system, there are propositions of the system that are true but cannot be deduced from the axioms of the system. Then, such propositions are mathematical truths that are not based on intuition because, as we have seen above, according to Brouwer, a proposition is a mathematical truth only if it is deduced from basic propositions based on intuition, by means of introspective construction. Thus, even when the axioms are based on intuition, there will be mathematical truths not based on intuition. Therefore, intuition is not sufficient to build up all mathematics.

Kreisel says that Brouwer "devoted the ten years after publication" of Gödel's incompleteness paper "to non-scientific activities" because "he had received an intellectual shock" (Kreisel and Newman 1969, 45). For, he had tried to refute the basic assumptions of Hilbert's formalism but had failed, while Gödel had succeeded in refuting them by his "immensely natural proofs" (ibid.). Then, "seeing the incomparable superiority of" Gödel's incompleteness results in refuting the basic assumptions of Hilbert's formalism, "Brouwer had to face the question" to "what extent he had even begun to master his own logical ideas" (ibid.).

But it seems more likely that the intellectual shock Brouwer had received was due to the fact that Gödel's first incompleteness theorem had shown that intuition is not sufficient to build up all mathematics, thus refuting the basic assumption of Brouwer's intuitionism.

2.21 Foundationalist Programs, the World, the Elephant, and the Tortoise

The failure of the attempts of the three big foundationalist schools to base mathematics on intuition brings to mind the story, mentioned by Locke, of the Indian "who, saying that the world was supported by a great elephant, was asked, what the elephant rested on; to which his answer was, a great tortoise: But being again pressed to know what gave support to the broad-back'd tortoise, replied, something, he knew not what" (Locke 1975, 296).

The world, the elephant and the tortoise are, respectively, in the case of Frege, arithmetic, the purely logical laws, and intellectual intuition; in the case of Hilbert, mathematics, the consistency proof, and Kant's pure intuition; and, in the case of Brouwer, mathematics, constructions, and Kant's pure intuition of time.

One cannot base mathematics on intuition any more than one can base the world on a tortoise, the disproportion between, on the one hand, what is to be supported, and, on the other hand, what is supposed to support it, is too large. For this reason, the attempts of the three big foundationalist schools to base mathematics on intuition were doomed to fail.

Russell, who extended logicism from arithmetic, to which Frege had limited it, to all of pure mathematics, eventually admitted the inevitability of failure. Originally, he had believed that all our knowledge, in particular all our mathematical knowledge, "is either intuitive or inferred" from "intuitive knowledge from which it follows logically" (Russell 1998, 81). And is inferred "by the use of self-evident principles of deduction," so all our knowledge ultimately "depends upon our intuitive knowledge" (ibid., 63). On the other hand, only intuitive knowledge gives "an infallible guarantee of truth" (ibid., 68). Therefore, Russell had attempted to base all of mathematics on intuition.

But, "as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure than the elephant" (Russell 1956, 54–55)). So, "after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable" (ibid., 55).

2.22 Mathematics, Truth, and Certainty

Assumption (II) of the foundationalist view, that mathematics is true and certain being based on intuition, is shared by several mathematicians.

For example, Byers claims that "mathematics is about truth," it "is a way of using the mind with the goal of knowing the truth, that is, of obtaining certainty" (Byers 2007, 327). For, "truth" is "knowledge that is certain" (ibid., 330). Indeed, "truth in mathematics and the certainty that arises when that truth is made manifest are not two separate phenomena; they are inseparable from one another – different aspects of the same underlying phenomenon" (ibid., 329). The certainty of mathematics is such that one cannot "have the slightest doubt" about it, "mathematical truth has this kind of certainty, this quality of inexorability. This is its essence" (ibid., 328). Mathematics is true and certain because it is based on intuition, for, "without the flash of insight there is no truth just as there is no understanding, which is, after all, another word for this quality of certainty that we are discussing" (ibid., 346).

But these claims are invalid, because they conflict with Gödel's second incompleteness theorem, by which, for any consistent, sufficiently strong, formal system, it is impossible to demonstrate, by absolutely reliable means, that the axioms of the system are consistent, let alone that they are true and certain. Therefore, mathematics cannot be said to be true and certain.

2.23 The Relevance of Gödel's Second Incompleteness Theorem

Some people have denied the relevance of Gödel's second incompleteness theorem to assumption (II) of the foundationalist view, by raising some objections. But the objections are invalid. Here are the main ones.

(1) If mathematics cannot be said to be true and certain, then Gödel's second incompleteness theorem, being a mathematical result, cannot be said to be true and certain. But the claim that, by Gödel's second incompleteness theorem, mathematics cannot be said to be true and certain, depends on the assumption that Gödel's second incompleteness theorem can be said to be true and certain. Therefore, the claim that, by Gödel's second incompleteness theorem, mathematics cannot be said to be true and certain, is invalid. (This objection was raised in correspondence by Hersh, acting as 'advocatus diaboli', not because he shared it).

The objection is invalid because the claim that, by Gödel's second incompleteness theorem, mathematics cannot be said to be true and certain, does not depend on the assumption that Gödel's second incompleteness theorem can be said to be true and certain. It is a reductio ad absurdum, being of the following kind. Let us suppose, for argument's sake, that mathematics can be said to be true and certain. Then Gödel's second incompleteness theorem, being a mathematical result, can be said to be true and certain. But, by Gödel's second incompleteness theorem, mathematics cannot be said to be true and certain. This contradicts our assumption that mathematics can be said to be true and certain. Therefore, by reductio ad absurdum, we conclude that mathematics cannot be said to be true and certain.

(2) Nothing in Gödel's second incompleteness theorem "in any way contradicts the view that there is no doubt whatever about the consistency of any of the formal systems" *T* "that we use in mathematics" (Franzén 2005, 105). For, either we have no doubts about the consistency of *T*, or we do have doubts about the consistency of *T*. Now, "if we have no doubts about the consistency" of *T*, then "there is nothing in the second incompleteness theorem to give rise to any such doubts. And if we do have doubts about the consistency" of *T*, then "we have no reason to believe that a consistency proof" for *T* "formalizable" in *T* "would do anything to remove those doubts" (ibid., 105–106). For, the consistency of *T* "is precisely what is in question" (ibid., 105).

The objection is invalid because, if we have no doubts about the consistency of *T*, we are rationally justified in having no such doubts only if we can demonstrate by absolutely reliable means that *T* is consistent. But, by Gödel's second incompleteness theorem, this is impossible. On the other hand, if we do have doubts about the consistency of *T*, then the question is not whether a consistency proof for *T* formalizable in *T* would do anything to remove those doubts. It is, instead, whether a consistency demonstration for *T* by absolutely reliable means would do anything to remove them, and the answer is yes, definitely.

But this is unjustified. There are several historical cases of mathematical demonstrations containing errors that have not been almost immediately discovered. Thus, the errors in Kempe's demonstration of the four colour theorem were detected eleven years after its publication. The errors in Jordan's demonstration of the Jordan curve theorem were detected twenty years after its publication. What is more important, there is no evidence that some of the demonstrations which today are considered to be valid do not contain errors.

2.26 Shortcomings of Reductionism in the Main Foundationalist Programs

Assumption (III) of the foundationalist view, that there is a part of mathematics such that all other parts of mathematics can be reduced to it, specifically, there is a mathematical theory such that all other mathematical theories can be reduced to it, is also invalid.

Assumption (III) is invalid as implemented by the main foundational programs. Indeed:

(1) Assumption (III) is invalid as implemented by logicism because, as we have seen above, Frege's logical system leads to a contradiction, Russell's paradox.

(2) Assumption (III) is invalid as implemented by formalism because Zermelo set theory does not permit to establish several mathematical results, for example, that all Borel sets are determined.

(3) Assumption (III) is invalid as implemented by intuitionism because, as said above, the theory of more or less freely proceeding infinite sequences and the theory of species together do not account for many mathematical results that are important to physics.

2.27 Shortcomings of Set Theoretical Reductionism

It may be objected that, while the fact that Zermelo set theory does not permit to establish several mathematical results, for example, that all Borel sets are determined, poses an obstacle to the reduction of all parts of mathematics to set theory, Zermelo-Fraenkel set theory removes this obstacle. For, it permits to establish all these results. Then, Zermelo-Fraenkel set theory ZF is The Foundation.

In the past century, the credibility of assumption (III) of the foundationalist view, has essentially relied on the conviction that all parts of mathematics can be reduced to ZF.

Thus, Maddy says that ZF provides "a framework in which all classical mathematical objects and structures can be defined and all classical mathematical theorems proved" (Maddy 2000, 344). Indeed, "for all mathematical objects and structures, there are set theoretic surrogates and instantiations, and the set theoretic versions of all classical mathematical theorems can be proved from the standard axioms for the theory of sets" (Maddy 1997, 34).

However, assumption (III) with ZF as The Foundation is invalid for the following reasons.

(1) Assumption (III) conflicts with the fact that, as Mac Lane points out, although set theory provides a standard foundation for mathematics, many interesting "questions cannot be settled on the basis of Zermelo-Fraenkel axioms for set theory" (Mac Lane 1986, 385). As regards these questions, an alternative is provided by "category theory" (ibid., 406). However, while "categories and functors are everywhere in topology and in parts of algebra," they "do not as yet relate well to most of analysis" (ibid., 407). So, many interesting questions cannot be settled on the basis of the axioms for category theory. Therefore, while Zermelo-Fraenkel set theory and category theory are each adequate for certain questions, neither of them "is wholly successful" (ibid., 407). So, today there is no mathematical theory such that all other mathematical theories can be reduced to it.

(2) Assumption (III) is conclusively refuted by Gödel's first incompleteness theorem, by which there are arithmetic truths that cannot be deduced from the axioms of ZF. Therefore, even arithmetic cannot be reduced to ZF.

(3) Assumption (III) is purely ideological. Even Maddy admits that "the average algebraist or geometer loses little time over set theory" (Maddy 1990, 4). In fact, "it cannot be denied that mathematicians from various branches of the subject – algebraists, analysts, number theorists, geometers – have different characteristic modes of thought, and that the subject would be crippled if this variety were somehow curtailed" (Maddy 1997, 33).

(4) Assumption (III) does not provide unique set theoretic surrogates for all mathematical objects, starting with the natural numbers. For example, Zermelo identifies natural numbers 0, 1, 2, 3, . . . with the sets $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$, $\{\{\{\varnothing\}\}\}$, . . ., while von Neumann identifies them with the sets $\varnothing$, $\{\varnothing\}$, $\{\varnothing,\{\varnothing\}\}$, $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$, ... But, as even Maddy admits, there is no reason "deep enough to motivate a metaphysical argument that one rather than the other uncovers the true identity of the natural numbers. And the other identifications, of integers, rationals, reals, functions, etc., all share this type of arbitrariness" (Maddy 1997, 24).

(5) Assumption (III) does not take into account that the reduction of mathematical objects to sets may attach properties to mathematical objects that say nothing about their nature. For example, on the one hand, Zermelo's identification of 1 with $\{\varnothing\}$, and of 3 with $\{\{\{\varnothing\}\}\}$, attaches the property $1 \notin 3$ to numbers 1 and 3 because $\{\varnothing\}\notin\{\{\{\varnothing\}\}\}$. On the other hand, von Neumann's identification of 1 with $\{\varnothing\}$, and of 3 with $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$, attaches the property $1 \in 3$ to numbers 1 and 3 because $\{\varnothing\}\in\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$. But this says nothing about the nature of the numbers 1 and 3.

(6) Assumption (III) does not take into account that the reduction of mathematical objects to sets may attach properties to mathematical objects that are inconsistent with each other. For example, as we have just seen, on the one hand, Zermelo's definition attaches the property $1 \notin 3$ to numbers 1 and 3, on the other hand, von Neumann's definition attaches the property $1 \in 3$ to them.

(7) Assumption (III) does not take into account that the most important advances in mathematics do not consist in the reduction of mathematical objects to a single kind of objects, but rather in the introduction of new concepts and new hypotheses to solve problems. Thus, Dedekind says that "every theorem of algebra and higher analysis, no matter how remote, can be expressed as a theorem about natural numbers," but there is "nothing meritorious" in "actually performing this wearisome circumlocution and insisting on the use and recognition of none other than natural numbers" (Dedekind 1996, 792). Indeed, "the greatest and most fruitful advances in mathematics" have "invariably been made by the creation and introduction of new concepts, rendered necessary by the frequent recurrence of complex phenomena which could be mastered by the old notions only with difficulty" (ibid.).

2.28 The Irrelevance of the Existence of Mathematical Objects

That assumptions (I) – (III) of the foundationalist view are all invalid, is not the only shortcoming of mainstream philosophy of mathematics. Another shortcoming arises from the fact that, according to mainstream philosophy of mathematics, a main task of the philosophy of mathematics is to give an answer to the question: Do objects exist in virtue of which mathematical propositions are true, and if so what is their nature?

The question of the existence of mathematical objects has been a central focus of mainstream philosophy of mathematics from the first half of the twentieth century to the present, meaning existence in the metaphysical sense of one of the schools of mainstream philosophy of mathematics. But, in fact, such question is irrelevant to mathematics, because the work of mathematicians on certain mathematical objects does not depend on, and is not affected by, an answer to the question of whether those mathematical objects exist or not, in the metaphysical sense of one of those schools. Mathematicians accept, or reject, mathematical objects not on metaphysical grounds, but because they are, or are not, functional to the advancement of mathematics.

For example, mathematicians eventually accepted imaginary numbers as a legitimate kind of numbers, not because they became convinced that imaginary numbers existed in some metaphysical sense, but because imaginary numbers were functional to the advancement of mathematics. Thus, Gauss said that analysis "would lose immensely in beauty and roundness, and would be forced to add very hampering restrictions to truths which otherwise would hold generally, if these imaginary quantities were to be neglected" (Gauss 1880, 156).

As another example, mathematicians eventually rejected infinitesimals as introduced by Leibniz and Newton, not because they became convinced that infinitesimals did not exist in some metaphysical sense, but because infinitesimals were not functional to the advancement of mathematics. Thus, Abel said that the calculus of infinitesimals of Leibniz and Newton "is so lacking in plan and overall idea, that it is really quite stunning that it can be studied by so many people" (Abel 1902b, 23). In particular, "it is a shame that we dare to base the slightest demonstration on" divergent series, for, "by using them, one may draw any conclusion he pleases, and it is they that have produced so many fallacies and so many paradoxes," therefore, "what is more important in mathematics is without foundation" (Abel 1902a, 16).

That the question of the existence of mathematical objects is irrelevant to mathematics is also clear from the demonstrations by reductio ad absurdum, in which one reasons about what does not exist. As Thomas says, in mathematics "we must be able to reason as dependably about what does not exist – even in a mathematical sense – as about what does, for instance in 'reductio' proofs," so, "whether some things exist or not is not of any practical importance" (Thomas 2014, 248).

A proper approach to the question of the existence of mathematical objects is put forward by Kant, who says: "In mathematical problems the question is not" about "existence as such at all, but about the properties of the objects in themselves, solely insofar as these are" connected "with the concept of them" (Kant 1998, A719/B747).

That the question of the existence of mathematical objects is irrelevant to mathematics, in the sense stated above, has been underlined also by several contemporary mathematicians.

Thus, Nelson says: "Share with me a fantasy: we open our morning newspaper to find a report with a banner headline, 'Numbers Vanish! – Early last night the natural numbers" suddenly "disappeared. Mathematicians have expressed stunned despair. Without numbers, they say, they can no longer prove theorems'" (Nelson 1994, 571). But "this is nonsense," indeed, if natural numbers disappeared, "the newspaper could still put marks 2, 3, etc., on the inside pages," and "we mathematicians could continue to put marks on paper, just as before, and hopefully submit them to editors of mathematical journals" (ibid.).

Rota says: "The existence of mathematical items is a chapter in the philosophy of mathematics that is devoid of consequence" (Rota 1997, 161). If "someone proved beyond any reasonable doubt that mathematical items do not exist," this would not "affect the truth of any mathematical statement" (ibid.). Indeed, "it does not matter whether mathematical items exist," one "can spend a lifetime working on mathematics without ever having any idea whether mathematical items exist, nor does one have to care about such a question" (ibid.).

Gowers says: "There certainly are philosophers who take seriously the question of whether numbers exist," but "this distinguishes them from mathematicians," who "can, and even should, happily ignore this seemingly fundamental question" (Gowers 2002, 17). In fact, "rather than worrying about the existence, or otherwise," of something, mathematicians think "about its properties" (ibid., 70). One might wonder: "How can one consider a set of properties without first establishing that there is something that has those properties? But this is not difficult at all. For example, one can speculate about the character a female president of the United States would be likely to have, even though there is no guarantee that there will ever be one" (ibid., 70–71).

2.29 Other Shortcomings of Mainstream Philosophy of Mathematics

In fact, all the characters of mainstream philosophy of mathematics have shortcomings.

(1) According to mainstream philosophy of mathematics, the philosophy of mathematics cannot concern itself with the making of mathematics, in particular discovery, because discovery is a subjective process, so it cannot be accounted for.

But this is invalid. Already Greek mathematicians invented a method of discovery, the analytic method, they used it as a basis for their mathematical practice, and even reported their processes of discovery by publishing their analyses.

This was possible because discovery is not a subjective process that cannot be accounted for. The hypotheses for solving mathematical problems are obtained by non-deductive rules, and the choice among alternative hypotheses is made by comparing the arguments for and against them on the basis of experience (see Chap. 5). So, discovery can be accounted for.

(2) According to mainstream philosophy of mathematics, the philosophy of mathematics can concern itself only with finished mathematics, because only finished mathematics is objective, so it can be completely justified.

But this is invalid, because justification is not entirely objective, it may involve subjective considerations.

For example, the calculus of infinitesimals of Leibniz and Newton was inconsistent, because infinitesimals were taken to be zero within some demonstrations and non-zero within other demonstrations, and were even taken to be zero at one place and non-zero at another place within the same demonstration. This led to falsities. For example, according to L'Hôpital 2015 presentation of the calculus of infinitesimals of Leibniz described above, by Postulate I two quantities that differ by an infinitely small quantity may be used interchangeably. So $dx + dx = dx$, hence $2dx = dx$, therefore $2 = 1$. Nevertheless, for over 150 years the calculus of infinitesimals of Leibniz and Newton was considered to be justified, because it was very effective in solving problems in science and engineering.

What is more important, by Gödel's second incompleteness theorem, finished mathematics cannot be completely justified.

(3) According to mainstream philosophy of mathematics, since the philosophy of mathematics cannot concern itself with the making of mathematics, it cannot contribute to the advancement of mathematics.

But this is invalid because, as argued in the Introduction, philosophy has contributed to the advancement of mathematics.

(4) According to mainstream philosophy of mathematics, the task of the philosophy of mathematics is primarily to give an answer to the questions: How do mathematical propositions come to be completely justified? And, subordinately to it, to the question: Do objects exist in virtue of which mathematical propositions are true, and if so what is their nature?

Halmos says that “to be a scholar of mathematics” you need genius, because “you must be born with talent, insight, concentration, taste, luck, drive and the ability to visualize and guess” (Halmos 1985, 400).

Byers says that there are “people who find a way to transcend their limitations” and “dare to do what appears to be impossible” (Byers 2007, 16). The “impossible is rendered possible through acts of genius,” and “mathematics boasts genius in abundance” (ibid.). Genius is a matter of insight, and “insight often reveals itself in a flash” (ibid., 329). Thus “mathematics transcends logic” (ibid., 26).

But the myth of mathematical genius is at odds with facts. As it will be argued in the following chapters, the making of mathematics, in particular discovery, is not an irrational process based on leaps of intuition, but a rational process that can be analyzed in terms of rules. And it is not the result of extraordinary thought processes, which are the hallmark of mathematical genius, but of ordinary thought processes that produce an extraordinary outcome.

Rather than mathematical genius, the discoverer must have enough knowledge to go to the edge of the field, while being flexible enough to go over the border, and must be able to undergo long periods of total absorption in the problem. Indeed, when Newton was asked how he could make his discoveries, “he answered: Nocte dieque incubando [By thinking about it day and night]” (Ortega y Gasset 1957, 47). His discoveries were “due to nothing but industry and patient thought” (Newton 2004, 94). And Einstein said: “It’s not that I’m so smart, it’s just that I stay with problems longer” (Einstein 2011, 481).

2.31 Mainstream Philosophy of Mathematics and Mathematical Logic

The second question concerning mainstream philosophy of mathematics that may be considered is mathematical logic.

According to mainstream philosophy of mathematics, mathematical logic is a proper and adequate tool for the philosophy of mathematics.

The three big foundationalist schools make this claim.

Thus, Frege says that, by means of mathematical logic, “every gap in the chain of deductions is eliminated with the greatest care,” so we can “say with certainty upon what primitive truths the proof depends” (Frege 1960, 4).

Hilbert says that “in the logical calculus we possess a sign language that is capable of representing” all “mathematical propositions in formulas and of expressing” all “logical inference through formal processes” (Hilbert 1967a, 381).

Brouwer says that, admittedly, standard mathematical logic is only “a mathematical study of linguistic symbols” (Brouwer 1975, 96). But “intuitionist mathematics has its general introspective theory of mathematical assertions, a theory which with some right may be called ‘intuitionist mathematical logic’” (ibid., 524). By means of it, intuitionism has “built a new structure of mathematics proper with unshakeable certainty” (Brouwer 1998, 42).

The direct and indirect descendants of the three big foundationalist schools also make the claim.

Thus, Kreisel says that the approach to philosophical problems of the philosophical tradition is valid only "at an early stage, when we know too little about the phenomenon involved and about our knowledge of it in order to ask sensible specific questions" (Kreisel 1984, 82). At a mature stage, this approach must be replaced with one based on mathematical logic, which is "a tool in the philosophy of mathematics; just as other mathematics, for example the theory of partial differential equations, is a tool in what used to be called natural philosophy" (Kreisel 1967, 201).

Dummett says that "it is rash to tackle the philosophy of mathematics unless one has" a "reasonable knowledge of mathematical logic," not "so much as part of the object of study as serving as a tool of inquiry" (Dummett 1998, 123–124). Thus, "if you have little knowledge of mathematical logic, you would be strongly advised to acquire some" (ibid., 124).

But the claim that mathematical logic is a proper and adequate tool for the philosophy of mathematics is invalid, because mathematical logic has failed to provide a foundation for mathematics and to give an account of mathematical reasoning.

As Rota says, "mathematical logic has given up all claims of providing a foundation to mathematics," and "very few logicians of our day believe that mathematical logic has anything to do with the way we think" (Rota 1997, 92–93).

For example, Wang admits that, "as we understand the nature of mathematical logic better, we find that the early belief in its philosophical relevance was largely an illusion" (Wang 2016, 28). Gradually, "the inadequacies of mathematical logic as the basic tool for the philosophy of mathematics and for general philosophy have come to be felt" (ibid., 30).

2.32 Mainstream Philosophy of Mathematics and Philosophy

The third question concerning mainstream philosophy of mathematics that may be considered is the attitude towards philosophy. In this respect, there is a difference between the three big foundationalist schools and their direct or indirect descendants.

The attitude of the three big foundationalist schools is anti-philosophical. According to them the justification of mathematics cannot be given by philosophy but only by mathematics itself, so it can only be a self-justification.

For example, Hilbert says that "mathematics is a presuppositionless science" (Hilbert 1967b, 479). The justification of mathematics cannot be given by philosophy, but only by mathematical logic, which is a part of mathematics that enables to "bring mathematical concept-formations and inferences into such a form that they are irrefutable and yet furnish a model of the entire science" (Hilbert 1996d, 1152).

Conversely, the attitude of the direct or indirect descendants of the three big foundationalist schools is not anti-philosophical. However, it consigns philosophy to irrelevance. For, most descendants of the three big foundationalist schools are part of analytic philosophy. And the basic assumption that the philosophy of mathematics cannot concern itself with the making of mathematics and hence cannot contribute to the advancement of mathematics, is part of the assumption of analytic philosophy that, through philosophy, "we do not seek to learn anything new," but only "to understand something that is already in plain view" (Wittgenstein 2009, I, § 89). Now, as Frodeman says, analytic philosophy "has led philosophy, potentially the most relevant of subjects, to become a synonym for irrelevance" (Frodeman 2013, 1918). This is argued for at length in Cellucci (2018, 2019).

Therefore, the attitude of mainstream philosophy of mathematics towards philosophy is either anti-philosophical, or consigns philosophy to irrelevance.

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(4) The task of the philosophy of mathematics is primarily to give an answer to the question: How is mathematics made? And, subordinately to it, to the questions: What is the method of mathematics? What is the nature of mathematical objects, demonstrations, definitions, diagrams, notations? What is the nature of mathematical explanations? What is the nature of mathematical beauty? Why is mathematics applicable to the world? In what sense is mathematics knowledge?

(5) The method of mathematics is the analytic method. The latter is the method according to which, to solve a problem, one looks for some hypothesis that is a sufficient condition for solving the problem, namely such that a solution to the problem can be deduced from the hypothesis. The hypothesis is obtained from the problem, and possibly other data, by some non-deductive rule, and must be plausible, namely such that the arguments for the hypothesis are stronger than the arguments against it, on the basis of experience. But the hypothesis is in turn a problem that must be solved, and is solved in the same way. Namely, one looks for another hypothesis that is a sufficient condition for solving the problem posed by the previous hypothesis, it is obtained from the latter, and possibly other data already available including data acquired from mathematical diagrams, by some non-deductive rule, and must be plausible. And so on. Thus, solving a problem is a potentially infinite process (see Chap. 5).

(6) The role of analytic demonstration, namely demonstration based on the analytic method, is to discover a solution to a problem.

(7) Since the method of mathematics is the analytic method, mathematics is a body of problems and solutions to them that are plausible. Therefore, mathematics is about plausibility.

(8) Since the method of mathematics is the analytic method, mathematical reasoning consists of both deductive reasoning and non-deductive reasoning.

(9) The philosophy of mathematics goes back to the beginning of philosophy, many major philosophers have made substantial contributions to it, and their work remains important even today.

(10) The philosophy of mathematics cannot be developed independently of experience. For, several mathematical problems have an extra-mathematical origin, and the solutions to mathematical problems are only plausible, so their evaluation depends on experience.

3.2 Original Formulation of Heuristic Philosophy of Mathematics

The original formulation of heuristic philosophy of mathematics can be credited to Lakatos's Ph.D. dissertation (Lakatos 1961). Unfortunately, the dissertation is still unpublished as a whole. Pieces of it have been published separately in Lakatos 1963–1964, Lakatos 1976, and Lakatos 1978, II, Chap. 5, but from them it is not easy to get an overall picture.

Moreover, in Lakatos 1976, the editors Worrall and Zahar have added notes in which, as Davis and Hersh say, they “correct Lakatos whenever he calls into question the existence of a final solution of the problem of mathematical rigor,” because they believe that “a modern formal deductive proof is infallible” (Davis and Hersh 1981, 353). But they “are wrong. What is more surprising, their objection is rooted in the very error which Lakatos attacked so vehemently,” namely “the error of identifying mathematics” with its representation in “first-order logic” (ibid.).

Indeed, Lakatos criticizes mainstream philosophy of mathematics because it identifies “mathematics with its formal axiomatic abstraction,” in which “mathematical theories are replaced by formal systems, proofs” by derivations in first-order logic, and “definitions by ‘abbreviatory devices’ which are ‘theoretically dispensable’ but ‘typographically convenient’” (Lakatos 1976, 1). So, mainstream philosophy of mathematics “denies the status of mathematics to most of what has been commonly understood to be mathematics, and can say nothing about its growth,” in particular it can say nothing about “the ‘creative’ periods” and “the ‘critical’ periods of mathematical theories” (ibid., 2). Therefore, in mainstream philosophy of mathematics “there is no proper place for methodology qua logic of discovery” (ibid., 3).

Contrary to mainstream philosophy of mathematics, heuristic philosophy of mathematics is concerned with methodology qua logic of discovery. According to it, although there is no infallibilist logic of discovery, namely “one which would infallibly lead to results,” nevertheless “there is a fallibilist logic of discovery” (ibid., 143–144, footnote 2). The latter consists in “the method of proof and refutations” (ibid., 50). The rules of the method can be found through case studies in the history of mathematics, because there is a strict relation between “the history of mathematics and the logic of mathematical discovery” (ibid., 4). In particular, the rules of the method can be found through the study of the history of Euler’s conjecture for polyhedra: The number of vertices V, edges E, and faces F in a convex polyhedron satisfy the equality $V - E + F = 2$.

That Lakatos is the initiator of an alternative to mainstream philosophy of mathematics is widely acknowledged.

Thus, Hersh says that, “starting with Imre Lakatos’ 1976 *Proofs and Refutations*, some writers have been turning away from the search for a ‘foundation’ for mathematics and instead, seeking to understand and clarify the actual practice of mathematics – what ‘real mathematicians really do’” (Hersh 2014, 241).

Rota says that Lakatos’s views, “published in the book *Proofs and Refutations*, were met with a great deal of anger on the part of the mathematical public who held the axiomatic method to be sacred and inviolable. Lakatos’ book became anathema among philosophers of mathematics of the positivistic school. The truth hurts” (Rota 1997, 50).

Nickles says that Lakatos rejected the “reduction of mathematics to formalized mathematics,” his *Proofs and Refutations* is “a highly original investigation of creative problem solving and the growth of knowledge in mathematics,” which makes Lakatos “the most important philosopher of mathematics” since “the mid-twentieth century” (Nickles 2000, 207).

Admittedly, before Lakatos, Wittgenstein had made some critical remarks against mainstream philosophy of mathematics. Thus, Wittgenstein says that the so-called "foundations are no more the foundations of mathematics than the painted rock is the support of the painted tower" (Wittgenstein 1978, V, § 13). And "logic and mathematics are not based on axioms, any more than a group is based on the elements and operations that define it" (Wittgenstein 2005, 377). But, contrary to Lakatos, Wittgenstein denies the possibility of a methodology qua logic of discovery, because he says that "mathematical discovery is always unmethodical: you have no method for making the discovery" (Wittgenstein 2016, 46).

Therefore, Lakatos has been the first to criticize mainstream philosophy of mathematics for not having a proper place for methodology qua logic of discovery. This justifies the claim that the original formulation of heuristic philosophy of mathematics can be credited to him.

3.3 Shortcomings of the Original Formulation

Despite its merits, however, Lakatos's formulation of heuristic philosophy of mathematics has some serious shortcomings.

Lakatos says that the first step of the method of proof and refutations is the naive conjecture, because discovery "moves from the naive conjecture" (Lakatos 1976, 42). Then, one would expect that the first rule of Lakatos's method of proof and refutations would indicate how to arrive at the naive conjecture. But the rule does nothing of the kind. For, it states: "Rule 1. If you have a conjecture, set out to prove it and to refute it. Inspect the proof carefully to prepare a list of non-trivial lemmas (proof-analysis); find counterexamples both to the conjecture (global counterexamples) and to the suspect lemmas (local counterexamples)" (ibid., 50).

So, the rule assumes that you already have a conjecture. Therefore, Lakatos's method of proof and refutations does not account for how to arrive at the naive conjecture.

Regarding his conjecture $V - E + F = 2$, Euler declares: "From the consideration of many types of solids I have been led to understand that the properties, which I had discerned in them, clearly extended to all solids, even if I was not allowed to show this by rigorous demonstration" (Euler 1758, 141). Thus, Euler states that he arrived at his conjecture by induction from observed cases.

But Lakatos rejects Euler's statement because, following Popper, he claims that "there are no such things as inductive conjectures" (Lakatos 1976, 73). The method of proof and refutations requires "no inductivist starting point at all" (ibid., 72). The "naive conjectures are not inductive conjectures: we arrive at them by trial and error" (ibid., 73).

This, however, conflicts with the fact that, as we will see in Chap. 17, Popper himself says that the success of trials depends very largely on the number and variety of the trials: the more we try, the more likely it is that one of our attempts will be successful. This amounts to admitting that trial and error depends on induction.

Then, it is contradictory to claim, as Lakatos does, that naive conjectures are not inductive conjectures, we arrive at them by trial and error.

In particular, regarding Euler's conjecture, Lakatos says that, "after much trial and error" it was noticed "that for all regular polyhedra $V - E + F = 2$," it was guessed "that this may apply for any polyhedron whatsoever," and this was put forward as a "conjecture" (ibid., 6–7). The "trials and errors" through which the conjecture $V - E + F = 2$ was reached "are beautifully reconstructed by Pólya" (ibid., 73, footnote 3).

But this conflicts with Pólya's own account, because Pólya says: "To begin with, we can scarcely do anything better than examine examples, particular polyhedra" (Pólya 1954, I, 35). Examining them, we observe that $V - E + F = 2$. Moreover, "this relation is verified in all" the polyhedra examined, and "it seems unlikely that such a persistent regularity should be mere coincidence" (ibid., I, 37). So "we are led to the conjecture that, not only in the observed cases, but in any polyhedron the number of faces increased by the number of vertices is equal to the number of edges increased by two" (ibid., I, 38). Therefore, like Euler, Pólya says that we are led to the conjecture $V - E + F = 2$, not by trial and error, but by induction from observed cases.

Lakatos even claims that "it took in this case nearly 2000 years to reach" Euler's "naive conjecture" by "'naive trial and error'. This 'naive' period, the first stage of mathematical discovery, lasted in this particular case from Euclid to Descartes" (Lakatos 1978, II, 96). But this claim seems far-fetched, Euler arrived at his conjecture far more quickly than that, by induction from observed cases. By claiming that it took nearly 2000 years to reach Euler's conjecture by trial and error, Lakatos admits that trial and error is very inefficient, so inefficient that it cannot account for the successes of mathematics. Indeed, the number of trials a mathematician can make is very small with respect to all possible ones, so the probability that he can reach a valuable conjecture by trial and error is very low. This is contradicted by the fact that over 100,000 research papers in mathematics are published every year.

In addition to the shortcomings of Lakatos's account of how Euler's conjecture was reached, Lakatos's assumption that there is a strict relation between the history of mathematics and the logic of mathematical discovery is invalid. For, the history of mathematics is mostly written on the basis of mathematics presented in finished form, and the latter has little or nothing to do with the way it was discovered (see below). Therefore, the history of mathematics does not provide an adequate basis for finding the rules of methodology qua logic of discovery.

Lakatos also claims that the method of proof and refutations is an extension of Pappus's analytic-synthetic method. For, he says that, after we "reach the naive conjecture" by "trial and error," the "naive conjecture is subjected to a sophisticated attempted refutation; analysis and synthesis starts" (Lakatos 1978, II, 96). By 'analysis and synthesis' Lakatos means "Pappusian analysis-synthesis" (ibid., II, 93). Namely, Pappus's analytic-synthetic method. But this does not contribute to the credibility of Lakatos's method of proof and refutations, because Pappus's analytic-synthetic method has serious shortcomings (see Chap. 6).

Therefore, it seems fair to say that Lakatos's method of proof and refutations does not provide a basis for methodology qua logic of discovery.

Lakatos himself ends up admitting it. For, he says that, while "in the seventeenth or even eighteenth century" it "was hoped that methodology would provide scientists" with "rules for solving problems," this hope "has now been given up: modern methodologies or 'logics of discovery' consist merely of a set" of "rules for the appraisal of ready, articulated theories" (ibid., I, 103). In particular, Lakatos's own "'methodology', older connotations of this term notwithstanding," does not presume "to give advice to the scientist" about "how to arrive at good theories," it "only appraises fully articulated theories" (Lakatos 1971, 174). The methodological rules are normative rules, where, however, "'normative' no longer means rules for arriving at solutions, but merely directions for the appraisal of solutions already there" (Lakatos 1978, I, 103, footnote 1). All we can have are rules for the appraisal of solutions already there.

As Nickles observes, it "is astonishing" that "Lakatos's methodology provides ways to appraise" solutions already there, "but stops short of giving advice" (Nickles 1987, 119). For, "the very idea of a method is the idea of something that guides inquiry, however fallibly; and the very idea of methodology is that of something that endorses specific method as preferred directives for future behavior" (ibid., 119–120). So, "the idea of a heuristic methodology which gives no advice is a contradiction in terms. Bluntly stated, Lakatos has no methodology" (ibid., 120).

At least, Lakatos has no methodology qua logic of discovery. This does not invalidate the claim that the original formulation of heuristic philosophy of mathematics can be credited to Lakatos. But it means that, with respect to heuristic philosophy of mathematics, Lakatos is a sort of 'non-playing captain', namely a captain who is not in the field when the game takes place.

3.4 Difference from Practical Heuristics

Heuristic philosophy of mathematics must not be confused with other approaches to mathematics that might seem similar to it.

Thus, heuristic philosophy of mathematics must not be confused with practical heuristics, as formulated by Pólya.

Pólya says that "the greater part of our conscious thinking is concerned with problems" (Pólya 1981, I, 117). And "the most characteristically human activity is solving problems, thinking for a purpose, devising means to some desired end" (ibid., I, 118). This applies also to mathematics, because "mathematics in the making resembles any other human knowledge in the making" (Pólya 1954, I, vi).

But, according to Pólya, there are no general rules for solving problems. Finding "rules applicable to all sorts of problems is an old philosophical dream," rules of that kind "would work magic; but there is no such thing as magic," such rules are like "the philosophers' stone, vainly sought by the alchemists," they are a dream which "will never be more than a dream" (Pólya 2004, 172).

been essential in the development of the foundationalist programs," to be "ineffective in dealing with the questions concerning the dynamics of mathematical discovery" (ibid., 4). Indeed, "many of them work, or have worked, also as mathematical logicians" (ibid.).

The philosophers of mathematical practice are only "calling for an extension" of the foundationalist tradition to "topics that the foundationalist tradition has ignored," namely some "aspects of mathematical practice" (ibid., 18).

This does not mean that the three big foundationalist schools "were removed from such concerns" (ibid., 6–7). Indeed, Frege's development of a formal language "which aimed at capturing formally all valid forms of reasoning occurring in mathematics, required a keen understanding of the reasoning patterns to be found in mathematical practice" (ibid., 7). Hilbert's "distinction between real and ideal elements" also "originates in mathematical practice" (ibid.). Brouwer's intuitionism "takes its origin from the distinction between constructive vs. non-constructive procedures" which was prominent in, "just to name one area, the debates in algebraic number theory in the late nineteenth century (Kronecker vs. Dedekind)" (ibid.). The direct and indirect descendants of the three big foundationalist schools "are also, to various extents, concerned with certain aspects of mathematical practice" (ibid.).

The only difference is that the philosophers of mathematical practice propose to investigate a broader range of aspects of mathematical practice. But this is only a difference in quantity, not in quality.

From this it is clear that, in Mancosu's formulation, there is no conflict between the philosophy of mathematical practice and mainstream philosophy of mathematics. This view is shared by Carter: "I do not intend to claim that there is a necessary tension or conflict between 'philosophy of mathematical practice' and" mainstream "'philosophy of mathematics'" (Carter 2019, 2). Rather, the philosophy of mathematical practice is continuous with mainstream philosophy of mathematics. Therefore, the philosophy of mathematical practice shares the shortcomings of the latter.

Another shortcoming arises from the fact that, according to the philosophers of mathematical practice, "mathematical practice is embodied in the concrete work of mathematicians and that work has taken place in history" (ibid., 13). Therefore, a main concern of the philosophers of mathematical practice is to "cover a broad spectrum" of "case studies arising from mathematical practice" (ibid., 18).

But case studies in the history of mathematics are usually carried out on the basis of finished mathematics, namely mathematics presented in finished form, and the latter has little or nothing to do with the way it was discovered (see below). Therefore, historical case studies can teach us about the sequence of published results and theories, not about discovery. This makes it clear that the aim of the philosophy of mathematical practice is not to account for the making of mathematics, in particular discovery, but only for finished mathematics.