The MαTH βOOK

Clifford A. Pickover

From Pythagoras to the 57th Dimension, 250 Milestones in the History of Mathematics

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For Martin Gardner

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Contents

Introduction

The Beauty and Utility of Mathematics

> *"An intelligent observer seeing mathematicians at work might conclude that they are devotees of exotic sects, pursuers of esoteric keys to the universe."*
>
> —Philip Davis and Reuben Hersh, *The Mathematical Experience*

Mathematics has permeated every field of scientific endeavor and plays an invaluable role in biology, physics, chemistry, economics, sociology, and engineering. Mathematics can be used to help explain the colors of a sunset or the architecture of our brains. Mathematics helps us build supersonic aircraft and roller coasters, simulate the flow of Earth's natural resources, explore subatomic quantum realities, and image faraway galaxies. Mathematics has changed the way we look at the cosmos.

In this book, I hope to give readers a taste for mathematics using few formulas, while stretching and exercising the imagination. However, the topics in this book are not mere curiosities with little value to the average reader. In fact, reports from the U.S. Department of Education suggest that successfully completing a mathematics class in high school results in better performance at college *whatever major* the student chooses to pursue.

The *usefulness* of mathematics allows us to build spaceships and investigate the geometry of our universe. Numbers may be our first means of communication with intelligent alien races. Some physicists have even speculated that an understanding of higher dimensions and of *topology*—the study of shapes and their interrelationships—may someday allow us to escape our universe, when it ends in either great heat or cold, and then we could call all of space-time our home.

Simultaneous discovery has often occurred in the history of mathematics. As I mention in my book *The Möbius Strip*, in 1858 the German mathematician August Möbius (1790–1868) simultaneously and independently discovered the Möbius strip (a wonderful twisted object with just one side) along with a contemporary scholar, the German mathematician Johann Benedict Listing (1808–1882). This simultaneous discovery of the Möbius band by Möbius and Listing, just like that of calculus by English polymath Isaac Newton (1643–1727) and German mathematician Gottfried Wilhelm Leibniz (1646–1716), makes me wonder why so many discoveries in science were made at the same time by people working independently. For another example, British naturalists Charles Darwin (1809–1882) and Alfred Wallace (1823–1913) both developed the theory of evolution independently and simultaneously. Similarly, Hungarian mathematician János Bolyai (1802–1860) and Russian mathematician Nikolai Lobachevsky (1793–1856) seemed to have developed hyperbolic geometry independently, and at the same time.

Most likely, such simultaneous discoveries have occurred because the time was ripe for such discoveries, given humanity's accumulated knowledge at the time the discoveries were made. Sometimes, two scientists are stimulated by reading the same preliminary research of one of their contemporaries. On the other hand, mystics have suggested that a deeper meaning exists to such coincidences. Austrian biologist Paul Kammerer (1880–1926) wrote, "We thus arrive at the image of a world-mosaic or cosmic kaleidoscope, which, in spite of constant shuffling and rearrangements, also takes care of bringing like and like together." He compared events in our world to the tops of ocean waves that seem isolated and unrelated. According to his controversial theory, we notice the tops of the waves, but beneath the surface some kind of synchronistic mechanism may exist that mysteriously connects events in our world and causes them to cluster.

Georges Ifrah in *The Universal History of Numbers* discusses simultaneity when writing about Mayan mathematics:

> We therefore see yet again how people who have been widely separated in time or space have... been led to very similar if not identical results.... In some cases, the explanation for this may be found in contacts and influences between different groups of people.... The true explanation lies in what we have previously referred to as the profound unity of culture: the intelligence of *Homo sapiens* is universal and its potential is remarkably uniform in all parts of the world.

Ancient people, like the Greeks, had a deep fascination with numbers. Could it be that in difficult times numbers were the only constant thing in an ever-shifting world? To the Pythagoreans, an ancient Greek sect, numbers were tangible, immutable, comfortable, eternal—more reliable than friends, less threatening than Apollo and Zeus.

Many entries in this book deal with whole numbers, or integers. The brilliant mathematician Paul Erdös (1913–1996) was fascinated by number theory—the study of integers—and he had no trouble posing problems, using integers, that were often simple to state but notoriously difficult to solve. Erdös believed that if one can state a problem in mathematics that is unsolved for more than a century, then it is a problem in number theory.

Many aspects of the universe can be expressed by whole numbers. Numerical patterns describe the arrangement of florets in a daisy, the reproduction of rabbits, the orbit of the planets, the harmonies of music, and the relationships between elements in the periodic table. Leopold Kronecker (1823–1891), a German algebraist and number theorist, once said, "The integers came from God and all else was man-made." His implication was that the primary source of all mathematics is the integers.

Since the time of Pythagoras, the role of integer ratios in musical scales has been widely appreciated. More important, integers have been crucial in the evolution of humanity's scientific understanding. For example, French chemist Antoine Lavoisier (1743–1794) discovered that chemical compounds are composed of fixed proportions of elements corresponding to the ratios of small integers. This was very strong evidence for the existence of atoms. In 1925, certain integer relations between the wavelengths of spectral lines emitted by excited atoms gave early clues to the structure of atoms. The near-integer ratios of atomic weights were evidence that the atomic nucleus is made up of an integer number of similar nucleons (protons and neutrons). The deviations from integer ratios led to the discovery of elemental isotopes (variants with nearly identical chemical behavior but with different numbers of neutrons).

Small divergences in the atomic masses of pure isotopes from exact integers confirmed Einstein's famous equation $E = mc^2$ and also the possibility of atomic bombs. Integers are everywhere in atomic physics. Integer relations are fundamental strands in the mathematical weave—or as German mathematician Carl Friedrich Gauss (1777–1855) said, "Mathematics is the queen of sciences—and number theory is the queen of mathematics."

Our mathematical description of the universe grows forever, but our brains and language skills remain entrenched. New kinds of mathematics are being discovered or created all the time, but we need fresh ways to think and to understand. For example, in the last few years, mathematical proofs have been offered for famous problems in the history of mathematics, but the arguments have been far too long and complicated for experts to be certain they are correct. Mathematician Thomas Hales had to wait *five years* before expert reviewers of his geometry paper—submitted to the journal *Annals of Mathematics*—finally decided that they could find no errors and that the journal should publish Hales's proof, but only with the disclaimer saying they were not certain it was right! Moreover, mathematicians like Keith Devlin have admitted in the *New York Times* that "the story of mathematics has reached a stage of such abstraction that many of its frontier problems cannot even be understood by the experts." If experts have such trouble, one can easily see the challenge of conveying this kind of information to a general audience. We do the best we can. Mathematicians can construct theories and perform computations, but they may not be sufficiently able to fully comprehend, explain, or communicate these ideas.

A physics analogy is relevant here. When Werner Heisenberg worried that human beings might never truly understand atoms, Niels Bohr was a bit more optimistic. He replied in the early 1920s, "I think we may yet be able to do so, but in the process we may have to learn what the word *understanding* really means." Today, we use computers to help us reason beyond the limitations of our own intuition. In fact, experiments with computers are leading mathematicians to discoveries and insights never dreamed of before the ubiquity of these devices. Computers and computer graphics allow mathematicians to discover results long before they can prove them formally and open entirely new fields of mathematics. Even simple computer tools like spreadsheets give modern mathematicians power that Gauss, Leonhard Euler, and Newton would have lusted after. As just one example, in the late 1990s, computer programs designed by David Bailey and Helaman Ferguson helped produce new formulas that related pi to log 5 and two other constants. As Erica Klarreich reports in *Science News*, once the computer had produced the formula, proving that it was correct was extremely easy. Often, simply *knowing* the answer is the largest hurdle to overcome when formulating a proof.

Mathematical theories have sometimes been used to predict phenomena that were not confirmed until years later. For example, Maxwell's equations, named after physicist James Clerk Maxwell, predicted radio waves. Einstein's field equations suggested that gravity would bend light and that the universe is expanding. Physicist Paul Dirac once noted that the abstract mathematics we study now gives us a glimpse of physics in the future. In fact, his equations predicted the existence of antimatter, which was subsequently discovered. Similarly, mathematician Nikolai Lobachevsky said that "there is no branch of mathematics, however abstract, which may not someday be applied to the phenomena of the real world."

In this book, you will encounter various interesting geometries that have been thought to hold the keys to the universe. Galileo Galilei (1564–1642) suggested that "Nature's great book is written in mathematical symbols." Johannes Kepler (1571–1630) modeled the solar system with Platonic solids such as the dodecahedron. In the 1960s, physicist Eugene Wigner (1902–1995) was impressed with the "unreasonable effectiveness of mathematics in the natural sciences." Large Lie groups, like E_8—which is discussed in the entry "The Quest for Lie Group E_8 (2007)"—may someday help us create a unified theory of physics. In 2007, Swedish American cosmologist Max Tegmark published both scientific and popular articles on the mathematical universe hypothesis, which states that our physical reality is a mathematical structure—in other words, our universe is not just *described* by mathematics—it *is* mathematics.

Book Organization and Purpose

> *"At every major step, physics has required, and frequently stimulated, the introduction of new mathematical tools and concepts. Our present understanding of the laws of physics, with their extreme precision and universality, is only possible in mathematical terms."*
>
> —Sir Michael Atiyah, "Pulling the Strings," *Nature*

One common characteristic of mathematicians is a passion for completeness—an urge to return to first principles to explain their works. As a result, readers of mathematical texts must often wade through pages of background before getting to the essential findings. To avoid this problem, each entry in this book is short, at most only a few paragraphs in length. This format allows readers to jump right in to ponder a subject, without having to sort through a lot of verbiage. Want to know about infinity? Turn to the entries "Cantor's Transfinite Numbers" (1874) or "Hilbert's Grand Hotel" (1925), and you'll have a quick mental workout. Interested in the first commercially successful portable mechanical calculator, developed by a prisoner in a Nazi concentration camp? Turn to "Curta Calculator" (1948) for a brief introduction.

Wonder how an amusing-sounding theorem may one day help scientists form nanowires for electronics devices? Then browse through the book and read the "Hairy Ball Theorem" (1912) entry. Why did the Nazis compel the president of the Polish Mathematical Society to feed his own blood to lice? Why was the first female mathematician murdered? Is it really possible to turn a sphere inside out? Who was the "Number Pope"? When did humans tie their first knots? Why don't we use Roman numerals anymore? Who was the earliest named individual in the history of mathematics? Can a surface have only one side? We'll tackle these and other thought-provoking questions in the pages that follow.

Of course, my approach has some disadvantages. In just a few paragraphs, I can't go into any depth on a subject. However, I provide suggestions for further reading in the "Notes and Further Reading" section. While I sometimes list primary sources, I have often explicitly listed excellent secondary references that readers can frequently obtain more easily than older primary sources. Readers interested in pursuing any subject can use the references as a useful starting point.

My goal in writing *The Math Book* is to provide a wide audience with a brief guide to important mathematical ideas and thinkers, with entries short enough to digest in a few minutes. Most entries are ones that interested me personally. Alas, not all of the great mathematical milestones are included in this book in order to prevent the book from growing too large. Thus, in celebrating the wonders of mathematics in this short volume, I have been forced to omit many important mathematical marvels. Nevertheless, I believe that I have included a majority of those with historical significance and that have had a strong influence on mathematics, society, or human thought. Some entries are eminently practical, involving topics that range from slide rules and other calculating devices to geodesic domes and the invention of zero. Occasionally, I include several lighter moments, which were nonetheless significant, such as the rise of the Rubik's Cube puzzle or the solving of the Bed Sheet Problem. Sometimes, snippets of information are repeated so that each entry can be read on its own. Occasional text in boldface type points the reader to related entries. Additionally, a small "See also" section at the bottom of each entry helps weave entries together in a web of interconnectedness and may help the reader traverse the book in a playful quest for discovery.

The Math Book reflects my own intellectual shortcomings, and while I try to study as many areas of science and mathematics as I can, it is difficult to become fluent in all aspects, and this book clearly indicates my own personal interests, strengths, and weaknesses. I am responsible for the choice of pivotal entries included in this book and, of course, for any errors and infelicities. This is not a comprehensive or scholarly dissertation, but rather it is intended as recreational reading for students of science and mathematics and interested laypeople. I welcome feedback and suggestions for improvement from readers, as I consider this an ongoing project and a labor of love.

This book is organized chronologically, according to the year of a mathematical milestone or finding. In some cases, the literature may report slightly different dates for the milestone because some sources give the publication date as the discovery date of a finding, while other sources give the actual date that a mathematical principle was discovered, regardless of the fact that the publication date is sometimes a year or more later. If I was uncertain of a precise earlier date of discovery, I often used the publication date.

Dating of entries can also be a question of judgment when more than one individual made a contribution. Often, I have used the earliest date where appropriate, but sometimes I have surveyed colleagues and decided to use the date when a concept gained particular prominence. For example, consider the Gray code, which is used to facilitate error correction in digital communications, such as in TV signal transmission, and to make transmission systems less susceptible to noise. This code was named after Frank Gray, a physicist at Bell Telephone Laboratories in the 1950s and 1960s. During this time, these kinds of codes gained particular prominence, partly due to his patent filed in 1947 and the rise of modern communications. The Gray code entry is thus dated as 1947, although it might also have been dated much earlier, because the roots of the idea go back to Émile Baudot (1845–1903), the French pioneer of the telegraph. In any case, I have attempted to give readers a feel for the span of possible dates in each entry or in the "Notes and Further Reading" section.

Scholars sometimes have disputes with respect to the person to whom a discovery is traditionally attributed. For example, author Heinrich Dörrie cites four scholars who do not believe that a particular version of Archimedes' cattle problem is due to Archimedes, but he also cites four authors who believe that the problem *should* be attributed to Archimedes. Scholars also dispute the authorship of Aristotle's wheel paradox. Where possible, I mention such disputes either in the main text or the "Notes and Further Reading" section.

You will notice that a significant number of milestones have been achieved in just the last few decades. As just one example, in 2007, researchers finally "solved" the game of checkers, proving that if an opponent plays perfectly, the game ends in draw. As already mentioned, part of the rapid recent progress in mathematics is due to the use of the computer as a tool for mathematical experiments. For the checkers solution, the analysis actually began in 1989 and required dozens of computers for the complete solution. The game has roughly 500 billion billion possible positions.

Sometimes, science reporters or famous researchers are quoted in the main entries, but purely for brevity I don't list the source of the quote or the author's full credentials in the entry. I apologize in advance for this occasional compact approach; however, references in the back of the book should help to make the author's identity clearer.

Even the naming of a theorem can be a tricky business. For example, mathematician Keith Devlin writes in his 2005 column for the Mathematical Association of America:

> Most mathematicians prove many theorems in their lives, and the process whereby their name gets attached to one of them is very haphazard. For instance, Euler, Gauss, and Fermat each proved hundreds of theorems, many of them important ones, and yet their names are attached to just a few of them. Sometimes theorems acquire names that are incorrect. Most famously, perhaps, Fermat almost certainly did not prove "Fermat's Last Theorem"; rather, that name was attached by someone else, after his death, to a conjecture the French mathematician had scribbled in the margin of a textbook. And Pythagoras's theorem was known long before Pythagoras came onto the scene.

In closing, let us note that mathematical discoveries provide a framework in which to explore the nature of reality, and mathematical tools allow scientists to make predictions about the universe; thus, the discoveries in this book are among humanity's greatest achievements.

At first glance, this book may seem like a long catalogue of isolated concepts and people with little connection between them. But as you read, I think you'll begin to see many linkages. Obviously, the final goal of scientists and mathematicians is not simply the accumulation of facts and lists of formulas, but rather they seek to understand the patterns, organizing principles, and relationships between these facts to form theorems and entirely new branches of human thought. For me, mathematics cultivates a perpetual state of wonder about the nature of mind, the limits of thoughts, and our place in this vast cosmos.

Our brains, which evolved to make us run from lions on the African savanna, may not be constructed to penetrate the infinite veil of reality. We may need mathematics, science, computers, brain augmentation, and even literature, art, and poetry to help us tear away the veils. For those of you who are about to embark on reading the *The Math Book* from cover to cover, look for the connections, gaze in awe at the evolution of ideas, and sail on the shoreless sea of imagination.

Acknowledgments

I thank Teja Krašek, Dennis Gordon, Nick Hobson, Pete Barnes, and Mark Nandor for their comments and suggestions. I would also like to especially acknowledge Meredith Hale, my editor for this book, as well as Jos Leys, Teja Krašek, and Paul Nylander for allowing me to include their mathematically inspired artworks.

While researching the milestones and pivotal moments presented in this book, I studied a wide array of wonderful reference works and Web sites, many of which are listed in the "Notes and Further Reading" section toward the end of the book. These references include "The MacTutor History of Mathematics Archive" (*www-history.mcs.st-and.ac.uk*), "Wikipedia: The Free Encyclopedia" (*en.wikipedia.org*), "MathWorld" (*mathworld.wolfram.com*), Jan Gullberg's *Mathematics: From the Birth of Numbers*, David Darling's *The Universal Book of Mathematics*, Ivars Peterson's "Math Trek Archives" (*www.maa.org/mathland/mathland_archives.html*), Martin Gardner's *Mathematical Games* (a CD-ROM made available from The Mathematical Association of America), and some of my own books such as *A Passion for Mathematics*.

Ant Odometer

Ants are social insects that evolved from vespoid wasps in the mid-Cretaceous period, about 150 million years ago. After the rise of flowering plants, about 100 million years ago, ants diversified into numerous species.

The Saharan desert ant, *Cataglyphis fortis*, travels immense distances over sandy terrain, often completely devoid of landmarks, as it searches for food. These creatures are able to return to their nest using a direct route rather than by retracing their outbound path. Not only do they judge directions, using light from the sky for orientation, but they also appear to have a built-in "computer" that functions like a pedometer that counts their steps and allows them to measure exact distances. An ant may travel as far as 160 feet (about 50 meters) until it encounters a dead insect, whereupon it tears a piece to carry directly back to its nest, accessed via a hole often less than a millimeter in diameter.

By manipulating the leg lengths of ants to give them longer and shorter strides, a research team of German and Swiss scientists discovered that the ants "count" steps to judge distance. For example, after ants had reached their destination, the legs were lengthened by adding stilts or shortened by partial amputation. The researchers then returned the ants so that the ants could start on their journey back to the nest. Ants with the stilts traveled too far and passed the nest entrance, while those with the amputated legs did not reach it. However, if the ants *started* their journey from their nest with the modified legs, they were able to compute the appropriate distances. This suggests that stride length is the crucial factor. Moreover, the highly sophisticated computer in the ant's brain enables the ant to compute a quantity related to the horizontal projection of its path so that it does not become lost even if the sandy landscape develops hills and valleys during its journey.

SEE ALSO Primates Count (c. 30 Million B.C.) and Cicada-Generated Prime Numbers (c. 1 Million B.C.).

Saharan desert ants may have built-in "pedometers" that count steps and allow the ants to measure exact distances. Ants with stilts glued to their legs (shown in red) travel too far and pass their nest entrance, suggesting that stride length is important for distance determination.

Primates Count

Around 60 million years ago, small, lemur-like primates had evolved in many areas of the world, and 30 million years ago, primates with monkeylike characteristics existed. Could such creatures count? The meaning of *counting* by animals is a highly contentious issue among animal behavior experts. However, many scholars suggest that animals have some sense of number. H. Kalmus writes in his *Nature* article "Animals as Mathematicians":

> There is now little doubt that some animals such as squirrels or parrots can be trained to count.... Counting faculties have been reported in squirrels, rats, and for pollinating insects. Some of these animals and others can distinguish numbers in otherwise similar visual patterns, while others can be trained to recognize and even to reproduce sequences of acoustic signals. A few can even be trained to tap out the numbers of elements (dots) in a visual pattern.... The lack of the spoken numeral and the written symbol makes many people reluctant to accept animals as mathematicians.

Rats have been shown to "count" by performing an activity the correct number of times in exchange for a reward. Chimpanzees can press numbers on a computer that match numbers of bananas in a box. Testsuro Matsuzawa of the Primate Research Institute at Kyoto University in Japan taught a chimpanzee to identify numbers from 1 to 6 by pressing the appropriate computer key when she was shown a certain number of objects on the computer screen.

Michael Beran, a research scientist at Georgia State University in Atlanta, Georgia, trained chimps to use a computer screen and joystick. The screen flashed a numeral and then a series of dots, and the chimps had to match the two. One chimp learned numerals 1 to 7, while another managed to count to 6. When the chimps were tested again after a gap of three years, both chimps were able to match numbers, but with double the error rate.

SEE ALSO Ant Odometer (c. 150 Million B.C.) and Ishango Bone (c. 18,000 B.C.).

Primates appear to have some sense of number, and the higher primates can be taught to identify numbers from 1 to 6 by pressing the appropriate computer key when shown a certain number of objects.

Cicada-Generated Prime Numbers

Cicadas are winged insects that evolved around 1.8 million years ago during the Pleistocene epoch, when glaciers advanced and retreated across North America. Cicadas of the genus *Magicicada* spend most of their lives below the ground, feeding on the juices of plant roots, and then emerge, mate, and die quickly. These creatures display a startling behavior: Their emergence is synchronized with periods of years that are usually the prime numbers 13 and 17. (A prime number is an integer such as 11, 13, and 17 that has only two integer divisors: 1 and itself.) During the spring of their 13th or 17th year, these periodical cicadas construct an exit tunnel. Sometimes more than 1.5 million individuals emerge in a single acre; this abundance of bodies may have survival value as they overwhelm predators such as birds that cannot possibly eat them all at once.

Some researchers have speculated that the evolution of prime-number life cycles occurred so that the creatures increased their chances of evading shorter-lived predators and parasites. For example, if these cicadas had 12-year life cycles, all predators with life cycles of 2, 3, 4, or 6 years might more easily find the insects. Mario Markus of the Max Planck Institute for Molecular Physiology in Dortmund, Germany, and his coworkers discovered that these kinds of prime-number cycles arise naturally from evolutionary mathematical models of interactions between predator and prey. In order to experiment, they first assigned random life-cycle durations to their computer-simulated populations. After some time, a sequence of mutations always locked the synthetic cicadas into a stable prime-number cycle.

Of course, this research is still in its infancy and many questions remain. What is special about 13 and 17? What predators or parasites have actually existed to drive the cicadas to these periods? Also, a mystery remains as to why, of the 1,500 cicada species worldwide, only a small number of the genus *Magicicada* are known to be periodical.

SEE ALSO Ant Odometer (c. 150 Million B.C.), Ishango Bone (c. 18,000 B.C.), Sieve of Eratosthenes (240 B.C.), Goldbach Conjecture (1742), Constructing a Regular Heptadecagon (1796), Gauss's *Disquisitiones Arithmeticae* (1801), Proof of the Prime Number Theorem (1896), Brun's Constant (1919), Gilbreath's Conjecture (1958), Sierpiński Numbers (1960), Ulam Spiral (1963), Erdös and Extreme Collaboration (1971), and Andrica's Conjecture (1985).

Certain cicadas display a startling behavior: Their emergence from the soil is synchronized with periods that are usually the prime numbers 13 and 17. Sometimes more than 1.5 million individuals emerge in a single acre within a short interval of time.

Knots

The use of knots may predate modern humans (*Homo sapiens*). For example, seashells colored with ocher, pierced with holes, and dated to 82,000 years ago have been discovered in a Moroccan cave. Other archeological evidence suggests much older bead use in humans. The piercing implies the use of cords and the use of a knot to hold the objects to a loop, such as a necklace.

The quintessence of ornamental knots is exemplified by *The Book of Kells*, an ornately illustrated Gospel Bible, produced by Celtic monks in about A.D. 800. In modern times, the study of knots, such as the trefoil knot with three crossings, is part of a vast branch of mathematics dealing with closed twisted loops. In 1914, German mathematician Max Dehn (1878–1952) showed that the trefoil knot's mirror images are not equivalent.

For centuries, mathematicians have tried to develop ways to distinguish tangles that *look* like knots (called *unknots*) from true knots and to distinguish true knots from one another. Over the years, mathematicians have created seemingly endless tables of distinct knots. So far, more than 1.7 million nonequivalent knots with pictures containing 16 or fewer crossings have been identified.

Entire conferences are devoted to knots today. Scientists study knots in fields ranging from molecular genetics—to help us understand how to unravel a loop of DNA—to particle physics, in an attempt to represent the fundamental nature of elementary particles.

Knots have been crucial to the development of civilization, where they have been used to tie clothing, to secure weapons to the body, to create shelters, and to permit the sailing of ships and world exploration. Today, knot theory in mathematics has become so advanced that mere mortals find it challenging to understand its most profound applications. In a few millennia, humans have transformed knots from simple necklace ties to models of the very fabric of reality.

SEE ALSO Quipu (c. 3000 B.C.), Borromean Rings (834), Perko Knots (1974), Jones Polynomial (1984), and Murphy's Law and Knots (1988).

The quintessence of ornamental knots is exemplified by The Book of Kells, *an ornately illustrated Gospel Bible, produced by Celtic monks in about A.D. 800. Various knot-like forms can be seen in the details of this illustration.*

c. 18,000 B.C.

Ishango Bone

In 1960, Belgian geologist and explorer Jean de Heinzelin de Braucourt (1920–1998) discovered a baboon bone with markings in what is today the Democratic Republic of the Congo. The Ishango bone, with its sequence of notches, was first thought to be a simple tally stick used by a Stone Age African. However, according to some scientists, the marks suggest a mathematical prowess that goes beyond counting of objects.

The bone was found in Ishango, near the headwaters of the Nile River, the home of a large population of upper Paleolithic people prior to a volcanic eruption that buried the area. One column of marks on the bone begins with three notches that double to six notches. Four notches double to eight. Ten notches halve to five. This may suggest a simple understanding of doubling or halving. Even more striking is the fact that numbers in other columns are all odd (9, 11, 13, 17, 19, and 21). One column contains the prime numbers between 10 and 20, and the numbers in each column sum to 60 or 48, both multiples of 12.

A number of tally sticks have been discovered that predate the Ishango bone. For example, the Swaziland Lebombo bone is a 37,000-year-old baboon fibula with 29 notches. A 32,000-year-old wolf tibia with 57 notches, grouped in fives, was found in Czechoslovakia. Although quite speculative, some have hypothesized that the markings on the Ishango bone form a kind of lunar calendar for a Stone Age woman who kept track of her menstrual cycles, giving rise to the slogan "menstruation created mathematics." Even if the Ishango was a simple bookkeeping device, these tallies seem to set us apart from the animals and represent the first steps to symbolic mathematics. The full mystery of the Ishango bone can't be solved until other similar bones are discovered.

SEE ALSO Primates Count (c. 30 Million B.C.), Cicada-Generated Prime Numbers (c. 1 Million B.C.), and Sieve of Eratosthenes (240 B.C.).

The Ishango baboon bone, with its sequence of notches, was first thought to be a simple tally stick used by a Stone Age African. However, some scientists believe that the marks suggest a mathematical prowess that goes beyond counting of objects.

Quipu

The ancient Incas used *quipus* (pronounced "key-poos"), memory banks made of strings and knots, for storing numbers. Until recently, the oldest-known quipus dated from about A.D. 650. However, in 2005, a quipu from the Peruvian coastal city of Caral was dated to about 5,000 years ago.

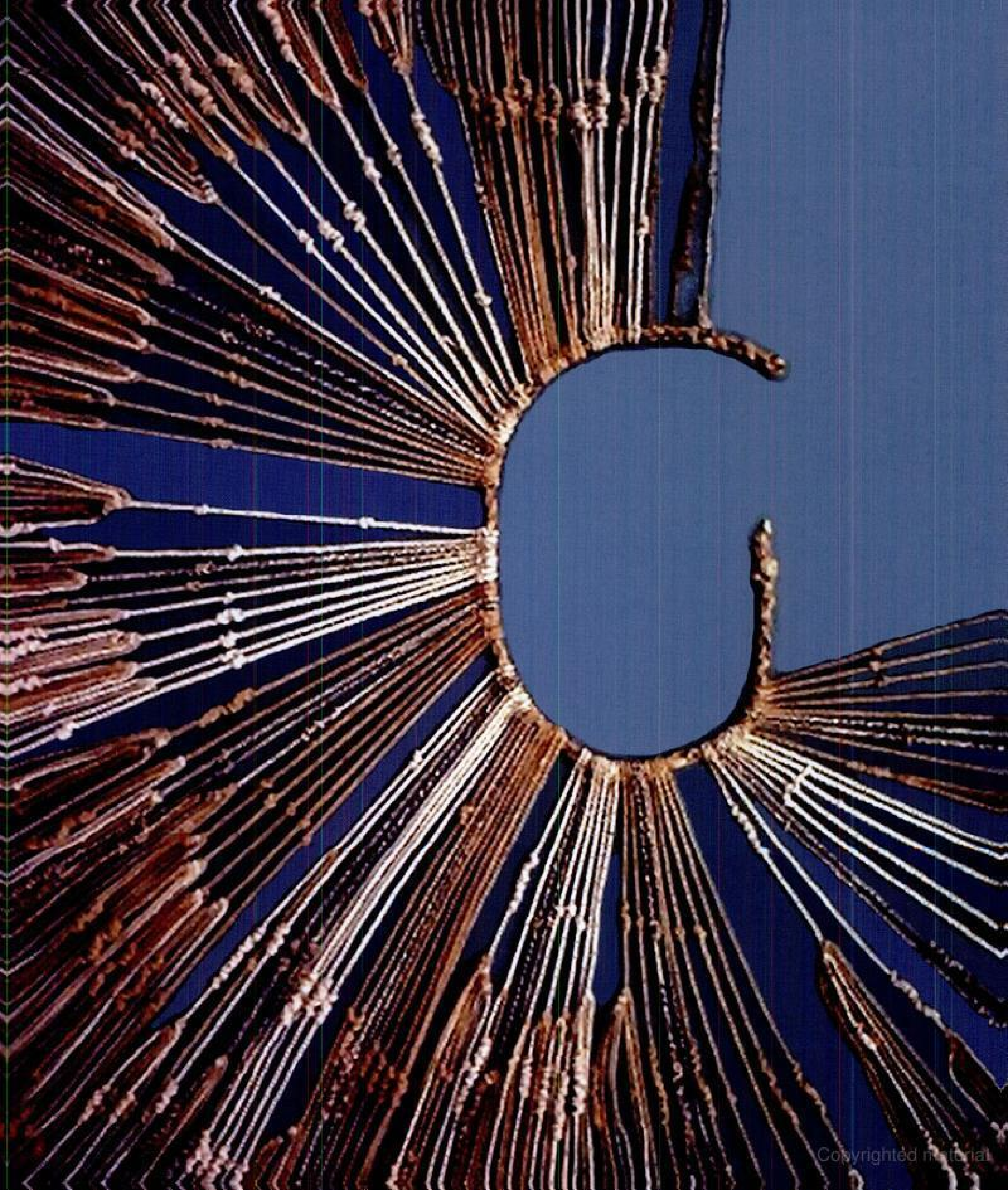
The Incas of South America had a complex civilization with a common state religion and a common language. Although they did not have writing, they kept extensive records encoded by a logical-numerical system on the quipus, which varied in complexity from three to around a thousand cords. Unfortunately, when the Spanish came to South America, they saw the strange quipus and thought they were the works of the Devil. The Spanish destroyed thousands of them in the name of God, and today only about 600 quipus remain.

Knot types and positions, cord directions, cord levels, and color and spacing represent numbers mapped to real-world objects. Different knot groups were used for different powers of 10. The knots were probably used to record human and material resources and calendar information. The quipus may have contained more information such as construction plans, dance patterns, and even aspects of Inca history. The quipu is significant because it dispels the notion that mathematics flourishes only after a civilization has developed writing; however, societies can reach advanced states without ever having developed written records. Interestingly, today there are computer systems whose file managers are called quipus, in honor of this very useful ancient device.

One sinister application of the quipu by the Incas was as a death calculator. Yearly quotas of adults and children were ritually slaughtered, and this enterprise was planned using a quipu. Some quipus represented the empire, and the cords referred to roads and the knots to sacrificial victims.

SEE ALSO Knots (c. 100,000 B.C.) and Abacus (c. 1200).

The ancient Incas used quipus made of knotted strings to store numbers. Knot types and positions, cord directions, cord levels, and colors often represented dates and counts of people and objects.



Dice

Imagine a world without random numbers. In the 1940s, the generation of statistically random numbers was important to physicists simulating thermonuclear explosions, and today, many computer networks employ random numbers to help route Internet traffic to avoid congestion. Political poll-takers use random numbers to select unbiased samples of potential voters.

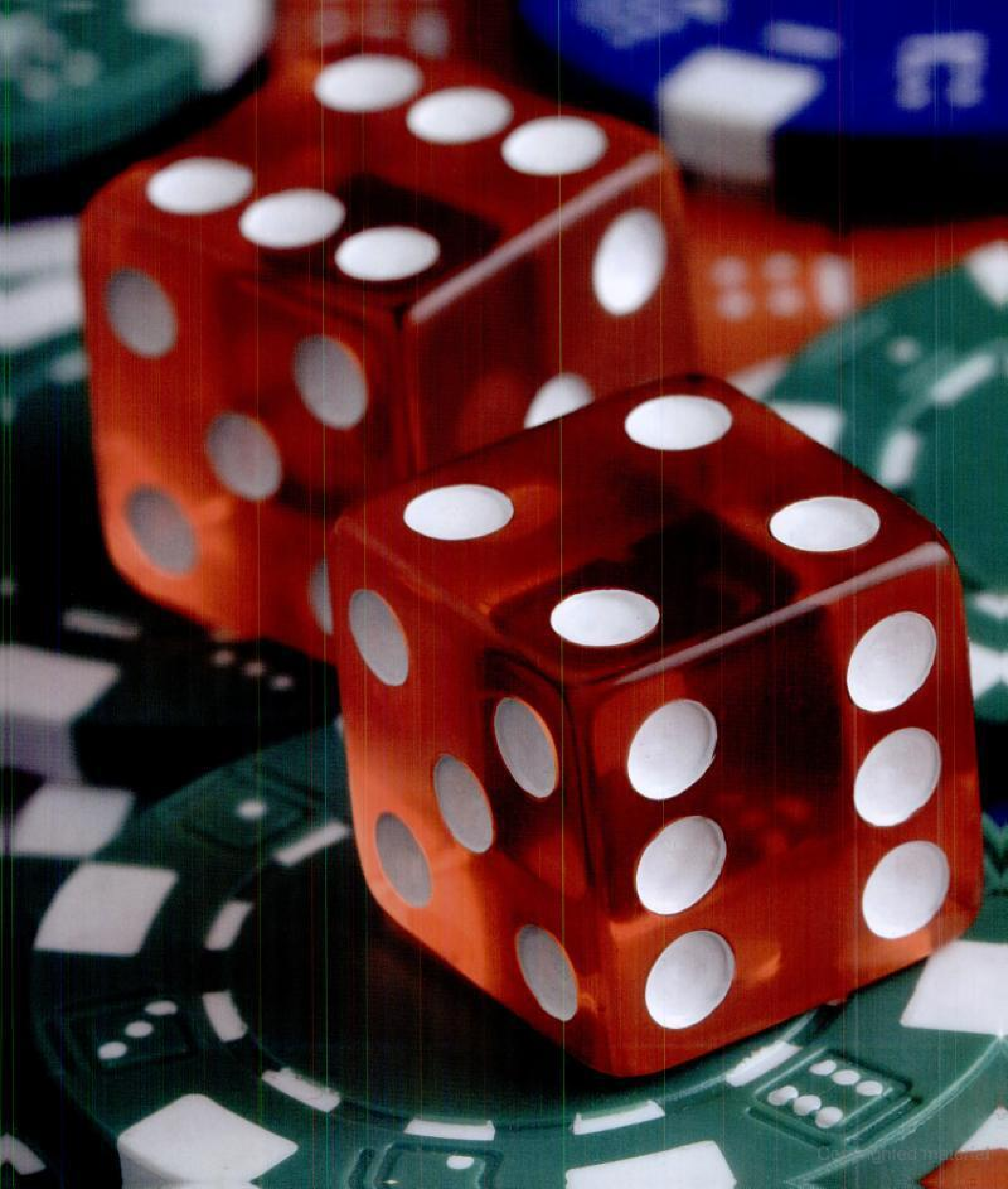
Dice, originally made from the anklebones of hoofed animals, were one of the earliest means for producing random numbers. In ancient civilizations, the gods were believed to control the outcome of dice tosses; thus, dice were relied upon to make crucial decisions, ranging from the selection of rulers to the division of property in an inheritance. Even today, the metaphor of God controlling dice is common, as evidenced by astrophysicist Stephen Hawking's quote, "Not only does God play dice, but He sometimes confuses us by throwing them where they can't be seen."

The oldest-known dice were excavated together with a 5,000-year-old backgammon set from the legendary Burnt City in southeastern Iran. The city represents four stages of civilization that were destroyed by fires before being abandoned in 2100 B.C. At this same site, archeologists also discovered the earliest-known artificial eye, which once stared out hypnotically from the face of an ancient female priestess or soothsayer.

For centuries, dice rolls have been used to teach probability. For a single roll of an n-sided die with a different number on each face, the probability of rolling any value is $1/n$. The probability of rolling a particular sequence of i numbers is $1/n^i$. For example, the chance of rolling a 1 followed by a 4 on a traditional die is $1/6^2 = 1/36$. Using two traditional dice, the probability of throwing any given sum is the number of ways to throw that sum divided by the total number of combinations, which is why a sum of 7 is much more likely than a sum of 2.

SEE ALSO Law of Large Numbers (1713), Buffon's Needle (1777), Least Squares (1795), Laplace's *Théorie Analytique des Probabilités* (1812), Chi-Square (1900), Lost in Hyperspace (1921), The Rise of Randomizing Machines (1938), Pig Game Strategy (1945), and Von Neumann's Middle-Square Randomizer (1946).

Dice were originally made from the anklebones of animals and were among the earliest means for producing random numbers. In ancient civilizations, people used dice to predict the future, believing that the gods influenced dice outcomes.



c. 2200 B.C.

Magic Squares

Bernard Frénicle de Bessy (1602–1675)

Legends suggest that magic squares originated in China and were first mentioned in a manuscript from the time of Emperor Yu, around 2200 B.C. A *magic square* consists of N^2 boxes, called *cells*, filled with integers that are all different. The sums of the numbers in the horizontal rows, vertical columns, and main diagonals are all equal.

If the integers in a magic square are the consecutive numbers from 1 to N^2, the square is said to be of the Nth order, and the *magic number*, or sum of each row, is a constant equal to $N(N^2 + 1)/2$. Renaissance artist Albrecht Dürer created this wonderful 4×4 magic square below in 1514.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Note the two central numbers in the bottom row read "1514," the year of its construction. The rows, columns, and main diagonals sum to 34. In addition, 34 is the sum of the numbers of the corner squares (16 + 13 + 4 + 1) and of the central 2×2 square (10 + 11 + 6 + 7).

As far back as 1693, the 880 different fourth-order magic squares were published posthumously in *Des quassez ou tables magiques* by Bernard Frénicle de Bessy, an eminent amateur French mathematician and one of the leading magic square researchers of all time.

We've come a long way from the simplest 3×3 magic squares venerated by civilizations of almost every period and continent, from the Mayan Indians to the Hasua people of Africa. Today, mathematicians study these magic objects in high dimensions—for example, in the form of four-dimensional hypercubes that have magic sums within all appropriate directions.

SEE ALSO Franklin Magic Square (1769) and Perfect Magic Tesseract (1999).

The Sagrada Família church in Barcelona, Spain, features a 4 × 4 magic square with a magic constant of 33, the age at which Jesus died according to many biblical interpretations. Note that this is not a traditional magic square because some numbers are repeated.


1 14 14 4
11 7 6 9
8 10 10 5
13 2 3 15

c.1800 B.C.

Plimpton 322

George Arthur Plimpton (1855–1936)

Plimpton 322 refers to a mysterious Babylonian clay tablet featuring numbers in cuneiform script in a table of 4 columns and 15 rows. Eleanor Robson, a historian of science, refers to it as "one of the world's most famous mathematical artifacts." Written around 1800 B.C., the table lists Pythagorean triples—that is, whole numbers that specify the side lengths of right triangles that are solutions to the Pythagorean theorem $a^2 + b^2 = c^2$. For example, the numbers 3, 4, and 5 are a Pythagorean triple. The fourth column in the table simply contains the row number. Interpretations vary as to the precise meaning of the numbers in the table, with some scholars suggesting that the numbers were solutions for students studying algebra or trigonometry-like problems.

Plimpton 322 is named after New York publisher George Plimpton who, in 1922, bought the tablet for $10 from a dealer and then donated the tablet to Columbia University. The tablet can be traced to the Old Babylonian civilization that flourished in Mesopotamia, the fertile valley of the Tigris and Euphrates rivers, which is now located in Iraq. To put the era into perspective, the unknown scribe who generated Plimpton 322 lived within about a century of King Hammurabi, famous for his set of laws that included "an eye for an eye, a tooth for a tooth." According to biblical history, Abraham, who is said to have led his people west from the city of Ur on the bank of the Euphrates into Canaan, would have been another near contemporary of the scribe.

The Babylonians wrote on wet clay by pressing a stylus or wedge into the clay. In the Babylonian number system, the number 1 was written with a single stroke and the numbers 2 through 9 were written by combining multiples of a single stroke.

SEE ALSO Pythagorean Theorem and Triangles (c. 600 B.C.).

Plimpton 322 (here shown turned on its side) refers to a Babylonian clay tablet featuring numbers in cuneiform script. These whole numbers specify the side lengths of right triangles that are solutions to the Pythagorean theorem $a^2 + b^2 = c^2$.

Rhind Papyrus

Ahmes (c. 1680 B.C.–c. 1620 B.C.), **Alexander Henry Rhind** (1833–1863)

The Rhind Papyrus is considered to be the most important known source of information concerning ancient Egyptian mathematics. This scroll, about a foot (30 centimeters) high and 18 feet (5.5 meters) long, was found in a tomb in Thebes on the east bank of the river Nile. Ahmes, the scribe, wrote it in hieratic, a script related to the hieroglyphic system. Given that the writing occurred in around 1650 B.C., this makes Ahmes the earliest-named individual in the history of mathematics! The scroll also contains the earliest-known symbols for mathematical operations—*plus* is denoted by a pair of legs walking toward the number to be added.

In 1858, Scottish lawyer and Egyptologist Alexander Henry Rhind had been visiting Egypt for health reasons when he bought the scroll in a market in Luxor. The British Museum in London acquired the scroll in 1864.

Ahmes wrote that the scroll gives an "accurate reckoning for inquiring into things, and the knowledge of all things, mysteries…all secrets." The content of the scroll concerns mathematical problems involving fractions, arithmetic progressions, algebra, and pyramid geometry, as well as practical mathematics useful for surveying, building, and accounting. The problem that intrigues me the most is Problem 79, the interpretation of which was initially baffling.

Today, many interpret Problem 79 as a puzzle, which may be translated as "Seven houses contain seven cats. Each cat kills seven mice. Each mouse had eaten seven ears of grain. Each ear of grain would have produced seven hekats (measures) of wheat. What is the total of all of these?" Interestingly, this indestructible puzzle meme, involving the number 7 and animals, seems to have persisted through thousands of years! We observe something quite similar in Fibonacci's *Liber Abaci* (*Book of Calculation*), published in 1202, and later in the St. Ives puzzle, an Old English children's rhyme involving 7 cats.

SEE ALSO *Ganita Sara Samgraha* (850), Fibonacci's *Liber Abaci* (1202), and *Treviso Arithmetic* (1478).

The Rhind Papyrus is the most important source of information concerning ancient Egyptian mathematics. The scroll, a portion of which is shown here, includes mathematical problems involving fractions, arithmetic progressions, algebra, geometry, and accounting.

Tic Tac Toe

The game of Tic Tac Toe (TTT) is a among humanity's best-known and most ancient games. Although the precise date of TTT with its modern rules may be relatively recent, archeologists can trace what appear to be "three-in-a-row games" to ancient Egypt around 1300 B.C., and I suspect that similar kinds of games originated at the very dawn of human societies. For TTT, two players, O and X, take turns marking their symbols in the spaces of a 3×3 grid. The player who first places three of his own marks in a horizontal, vertical, or diagonal row wins. A draw can always be obtained for the 3×3 board.

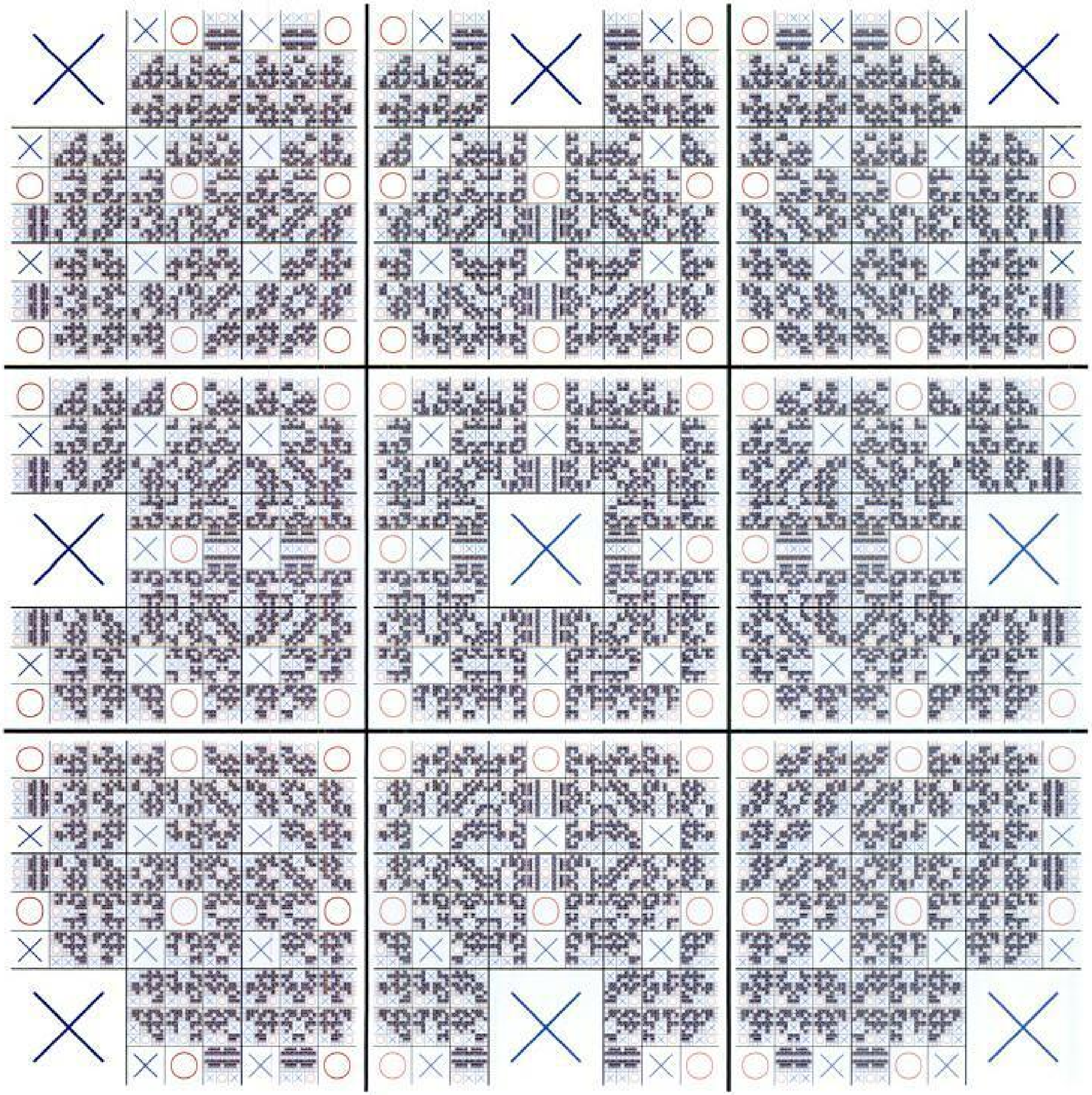
In ancient Egypt, during the time of the great pharaohs, board games played an important role in everyday life, and TTT-like games are known to have been played during these ancient days. TTT may be considered an "atom" upon which the molecules of more advanced games of position were built through the centuries. With the slightest of variations and extensions, the simple game of TTT becomes a fantastic challenge requiring significant time to master.

Mathematicians and puzzle aficionados have extended TTT to larger boards, higher dimensions, and strange playing surfaces such as rectangular or square boards that are connected at their edges to form a torus (doughnut shape) or Klein bottle (a surface with just one side).

Consider some TTT curiosities. Players can place their Xs and Os on the TTT board in 9! = 362,880 ways. There are 255,168 possible games in TTT when considering all possible games that end in 5, 6, 7, 8, and 9 moves. In the early 1980s, computer geniuses Danny Hillis, Brian Silverman, and friends built a Tinkertoy® computer that played TTT. The device was made from 10,000 Tinkertoy parts. In 1998, researchers and students at the University of Toronto created a robot to play three-dimensional ($4 \times 4 \times 4$) TTT with a human.

SEE ALSO Go (548 B.C.), Icosian Game (1857), Solving the Game of Awari (2002), and Checkers Is Solved (2007).

Philosophers Patrick Grim and Paul St. Denis offer an analytic presentation of all possible Tic-Tac-Toe games. Each cell in the Tic-Tac-Toe board is divided into smaller boards to show various possible choices.



Pythagorean Theorem and Triangles

Baudhayana (c. 800 B.C.), **Pythagoras of Samos** (c. 580 B.C.–c. 500 B.C.)

Today, young children sometimes first hear of the famous Pythagorean theorem from the mouth of the Scarecrow, when he finally gets a brain in MGM's 1939 film version of *The Wizard of Oz*. Alas, the Scarecrow's recitation of the famous theorem is completely wrong!

The Pythagorean theorem states that for any right triangle, the square of the hypotenuse length c is equal to the sum of the squares on the two (shorter) "leg" lengths a and b—which is written as $a^2 + b^2 = c^2$. The theorem has more published proofs than any other, and Elisha Scott Loomis's book *Pythagorean Proposition* contains 367 proofs.

Pythagorean triangles (PTs) are right triangles with integer sides. The "3-4-5" PT—with legs of lengths 3 and 4, and a hypotenuse of length 5—is the only PT with three sides as consecutive numbers and the only triangle with integer sides, the sum of whose sides (12) is equal to double its area (6). After the 3-4-5 PT, the next triangle with consecutive leg lengths is 21-20-29. The tenth such triangle is much larger: 27304197-27304196-38613965.

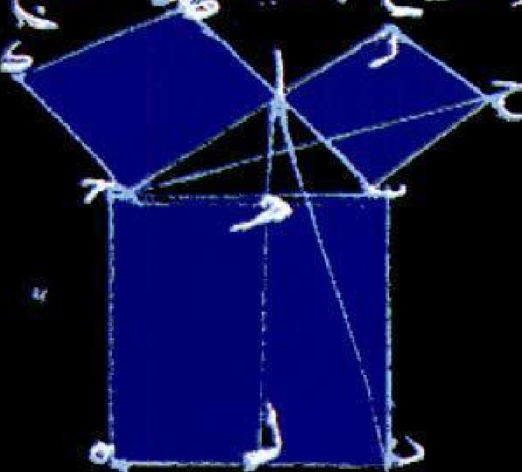
In 1643, French mathematician Pierre de Fermat (1601–1665) asked for a PT, such that both the hypotenuse c and the sum $(a + b)$ had values that were square numbers. It was startling to find that the *smallest* three numbers satisfying these conditions are 4,565,486,027,761, 1,061,652,293,520, and 4,687,298,610,289. It turns out that the second such triangle would be so "large" that if its numbers were represented as feet, the triangle's legs would project from Earth to beyond the sun!

Although Pythagoras is often credited with the formulation of the Pythagorean theorem, evidence suggests that the theorem was developed by the Hindu mathematician Baudhayana centuries earlier around 800 B.C. in his book *Baudhayana Sulba Sutra*. Pythagorean triangles were probably known even earlier to the Babylonians.

SEE ALSO Plimpton 322 (c. 1800 B.C.), Pythagoras Founds Mathematical Brotherhood (c. 530 B.C.), Quadrature of the Lune (c. 440 B.C.), Law of Cosines (c. 1427), and Viviani's Theorem (1659).

Persian mathematician Nasr al-Din al-Tusi (1201–1274) presented a version of Euclid's proof of the Pythagorean theorem. Al-Tusi was a prolific mathematician, astronomer, biologist, chemist, philosopher, physician, and theologian.

ب ز ا ط ك د فنصل ز ا ح خطاً واحداً لكون زاويتي
ب ا ح و ب ا ز قائمتين وكذلك ب ا ط وبخرج من ا ك موازياً
لـ ب د فيقع داخل المثلث لأن زاوية د ب ا أكبر من قائمة فيكون
زاوية ب ا ك أصغر من زاوية ب ا ح القائمة ويقطع لا محالة ب ح
على م وينقسم به مربع ب ه السطحين ب ل ل ح ونصل
ب ح ا د فلأن في مثلثي ح ب ز ب ا د ضلعي ز ب ب ح
وزاوية ب ح مساوية لضلعي ا ب ب د وزاوية ا ب د
يكون المثلثان متساويين ومثلث ح ب ز يساوي نصف مربع



ز ب لكونهما على قاعدة
ح ب بين متوازيي ز ح
ب ه وكذلك مثلث ب ا د
يساوي نصف سطح ب ل
لكونهما على قاعدة ب د
بين متوازيي ب د ا ك
مربع ز ب يساوي
سطح ب ل لتساوي نصفيهما ومثل ذلك تبيّن أن مربع ط ح يساوي
سطح ح ل … مربع ح ب يساوي مربعي ا ح … وذلك ما أردناه.

548 B.C.

Go

Go is a two-player board game that originated in China around 2000 B.C. The earliest written references to the game are from the earliest Chinese work of narrative history, *Zuo Zhuan* (*Chronicle of Zuo*), which describes a man in 548 B.C. who played the game. The game spread to Japan, where it became popular in the thirteenth century. Two players alternately place black and white stones on intersections of a 19 × 19 playing board. A stone or a group of stones is captured and removed if it is tightly surrounded by stones of the opposing color. The objective is to control a larger territory than one's opponent.

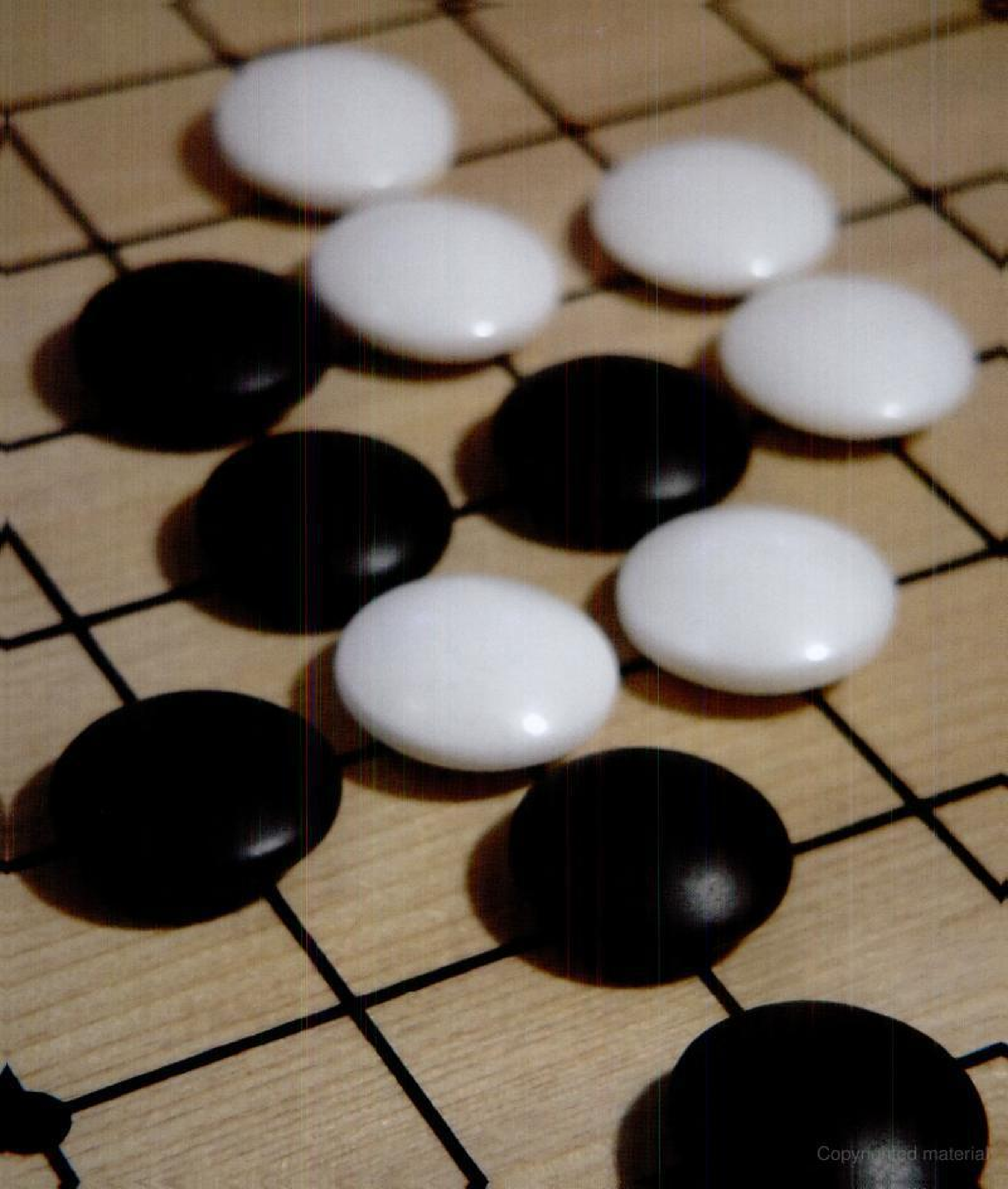
Go is complex for many reasons, including its large game board, multifaceted strategies, and huge numbers of variations in possible games. Simply having more stones than an opponent does not ensure victory. After taking symmetry into account, there are 32,940 opening moves, of which 992 are considered to be strong ones. The number of possible board configurations is usually estimated to be on the order of 10^{172}, with about 10^{768} possible games. Typical games between talented players consist of about 150 moves, with an average of about 250 choices per move. While powerful chess software is capable of defeating top chess players, the best Go programs often lose to skillful children.

Go-playing computers find it difficult to "look ahead" in the game to judge outcomes because many more reasonable moves must be considered in Go than in chess. The process of evaluating the favorability of a position is also quite difficult because a difference of a single unoccupied grid point can affect large groups of stones.

In 2006, two Hungarian researchers reported that an algorithm called UCT (for Upper Confidence bounds applied to Trees) could compete with professional Go players, but only on 9 × 9 boards. UCT helps the computer focus its search on the most promising moves.

SEE ALSO Tic Tac Toe (c. 1300 B.C.), Solving the Game of Awari (2002), and Checkers Is Solved (2007).

The game of Go is complex, due in part to the large game board, complicated strategies, and huge numbers of variations in possible games. While powerful chess software is capable of defeating top chess players, the best Go programs often lose to skillful children.



c. 530 B.C.

Pythagoras Founds Mathematical Brotherhood

Pythagoras of Samos (c. 580 B.C.–c. 500 B.C.)

Around 530 B.C., the Greek mathematician Pythagoras moved to Croton, Italy, to teach mathematics, music, and reincarnation. Although many of Pythagoras's accomplishments may actually have been due to his disciples, the ideas of his brotherhood influenced both numerology and mathematics for centuries. Pythagoras is usually credited with discovering mathematical relationships relevant to musical harmonies. For example, he observed that vibrating strings produce harmonious sounds when the ratios of the lengths of the strings are whole numbers. He also studied triangular numbers (based on patterns of dots in a triangular shape) and perfect numbers (integers that are the sum of their proper positive divisors). Although the famous theorem that bears his name, $a^2 + b^2 = c^2$ for a right triangle with legs a and b and hypotenuse c, may have been known to the Indians and Babylonians much earlier, some scholars have suggested that Pythagoras or his students were among the first Greeks to prove it.

To Pythagoras and his followers, numbers were like gods, pure and free from material change. Worship of the numbers 1 through 10 was a kind of polytheism for the Pythagoreans. They believed that numbers were alive, with a telepathic form of consciousness. Humans could relinquish their three-dimensional lives and telepathize with these number beings by using various forms of meditation.

Some of these seemingly odd ideas are not foreign to modern mathematicians who often debate whether mathematics is a creation of the human mind or if it is simply a part of the universe, independent of human thought. To the Pythagoreans, mathematics was an ecstatic revelation. Mathematical and theological blending flourished under the Pythagoreans and eventually affected much of the religious philosophy in Greece, played a role in religion of the Middle Ages, and extended to philosopher Immanuel Kant in modern times. Bertrand Russell mused that if it were not for Pythagoras, theologians would not have so frequently sought logical proofs of God and immortality.

SEE ALSO Plimpton 322 (c. 1800 B.C.) and Pythagorean Theorem and Triangles (c. 600 B.C.).

Pythagoras (the bearded man at bottom left with a book) is teaching music to a youth in The School of Athens *by Raphael (1483–1520), the famous Renaissance Italian painter and architect.*

c. 445 B.C.

Zeno's Paradoxes

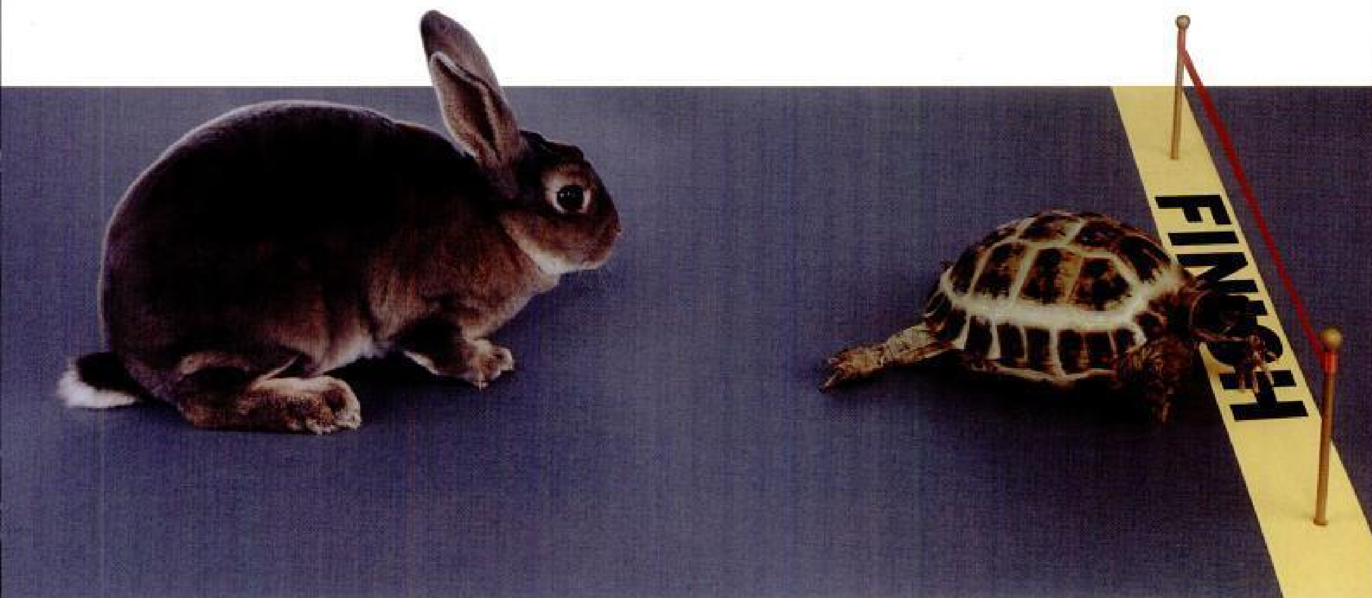
Zeno of Elea (c. 490 B.C.–c. 430 B.C.)

For more than a thousand years, philosophers and mathematicians have tried to understand Zeno's paradoxes, a set of riddles that suggest that motion should be impossible or that it is an illusion. Zeno was a pre-Socratic Greek philosopher from southern Italy. His most famous paradox involves the Greek hero Achilles and a slow tortoise that Achilles can never overtake during a race once the tortoise is given a head start. In fact, the paradox seems to imply that you can never leave the room you are in. In order to reach the door, you must first travel half the distance there. You'll also need to continue to half the remaining distance, and half again, and so on. You won't reach the door in a finite number of jumps! Mathematically one can represent this limit of an infinite sequence of actions as the sum of the series ($1/2 + 1/4 + 1/8 + \ldots$). One modern tendency is to attempt to resolve Zeno's paradox by insisting that the sum of this infinite series $1/2 + 1/4 + 1/8$ is *equal* to 1. If each step is done in half as much time, the actual time to complete the infinite series is no different than the real time required to leave the room.

However, this approach may not provide a satisfying resolution because it does not explain how one is able to *finish* going through an *infinite* number of points, one after the other. Today, mathematicians make use of infinitesimals (unimaginably tiny quantities that are almost but not quite zero) to provide a microscopic analysis of the paradox. Coupled with a branch of mathematics called nonstandard analysis and, in particular, internal set theory, we may have resolved the paradox, but debate continues. Some have also argued that if space and time are *discrete*, the total number of jumps in going from one point to another *must* be finite.

SEE ALSO Aristotle's Wheel Paradox (c. 320 B.C.), Harmonic Series Diverges (c. 1350), Discovery of Series Formula for π (c. 1500), Discovery of Calculus (c. 1665), St. Petersburg Paradox (1738), Barber Paradox (1901), Banach-Tarski Paradox (1924), Hilbert's Grand Hotel (1925), Birthday Paradox (1939), Coastline Paradox (c. 1950), Newcomb's Paradox (1960), and Parrondo's Paradox (1999).

According to Zeno's most famous paradox, the rabbit can never overtake the tortoise once the tortoise is given a head start. In fact, the paradox seems to imply that neither can ever cross the finish line.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c. 440 B.C.

Quadrature of the Lune

Hippocrates of Chios (c. 470 B.C.–c. 400 B.C.)

Ancient Greek mathematicians were enchanted by the beauty, symmetry, and order of geometry. Succumbing to this passion, Greek mathematician Hippocrates of Chios demonstrated how to construct a square with an area equal to a particular lune. A lune is a crescent-shaped area, bounded by two concave circular arcs, and this Quadrature of the Lune is one of the earliest-known proofs in mathematics. In other words, Hippocrates demonstrated that the area of these lunes could be expressed exactly as a rectilinear area, or "quadrature." In the example depicted here, two yellow lunes associated with the sides of a right triangle have a combined area equal to that of the triangle.

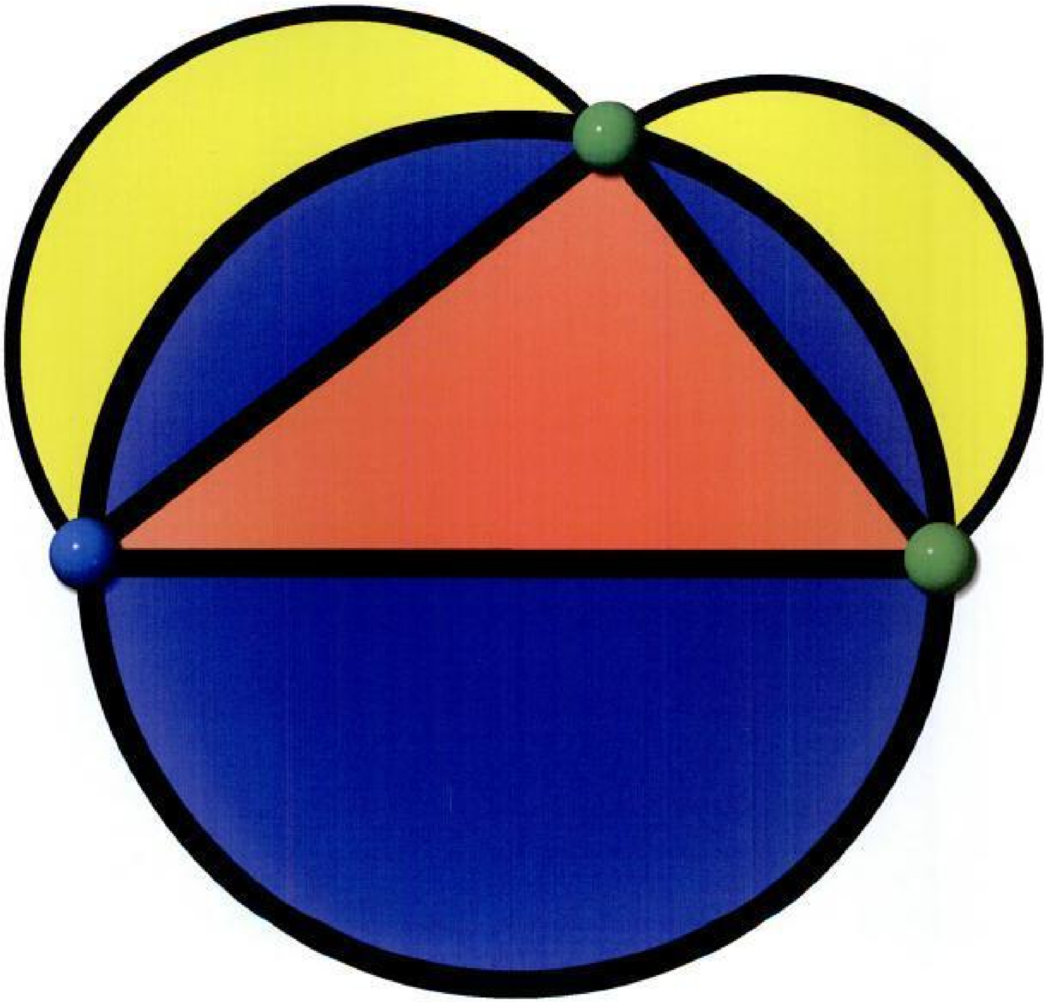
For the ancient Greeks, finding the quadrature meant using a straightedge and compass to construct a square equal in area to a given shape. If such a construction is possible, the shape is said to be "quadrable" (or "squareable"). The Greeks had accomplished the quadrature of polygons, but curved forms were more difficult. In fact, it must have seemed unlikely, at first, that curved objects could be quadrable at all.

Hippocrates is also famous for compiling the first-known organized work on geometry, nearly a century before Euclid. Euclid may have used some of Hippocrates' ideas in his own work, *Elements*. Hippocrates' writings were significant because they provided a common framework upon which other mathematicians could build.

Hippocrates' lune quest was actually part of a research effort to achieve the "quadrature of the circle"—that is, to construct a square with the same area as a circle. Mathematicians had tried to solve the problem of "squaring the circle" for more than 2,000 years, until Ferdinand von Lindemann in 1882 proved that it is impossible. Today, we know that only five types of lune exist that are quadrable. Three of these were discovered by Hippocrates, and two more kinds were found in the mid-1770s.

SEE ALSO Pythagorean Theorem and Triangles (c. 600 B.C.), Euclid's *Elements* (300 B.C.), Descartes' *La Géométrie* (1637), and Transcendental Numbers (1844).

The two lunes (the yellow crescent-shaped areas) associated with the sides of a right triangle have a combined area equal to that of the triangle. Ancient Greek mathematicians were enchanted by the elegance of these kinds of geometrical findings.



c. 350 B.C.

Platonic Solids

Plato (c. 428 B.C.–c. 348 B.C.)

A *Platonic solid* is a convex multifaceted 3-D object whose faces are all identical polygons, with sides of equal length and angles of equal degrees. A Platonic solid also has the same number of faces meeting at every vertex. The best-known example of a Platonic solid is the cube, whose faces are six identical squares.

The ancient Greeks recognized and proved that only five Platonic solids can be constructed: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. For example, the icosahedron has 20 faces, all in the shape of equilateral triangles.

Plato described the five Platonic solids in *Timaeus* in around 350 B.C. He was not only awestruck by their beauty and symmetry, but he also believed that the shapes described the structures of the four basic elements thought to compose the cosmos. In particular, the tetrahedron was the shape that represented fire, perhaps because of the polyhedron's sharp edges. The octahedron was air. Water was made up of icosahedra, which are smoother than the other Platonic solids. Earth consisted of cubes, which look sturdy and solid. Plato decided that God used the dodecahedron for arranging the constellations in the heavens.

Pythagoras of Samos—the famous mathematician and mystic who lived in the time of Buddha and Confucius, around 550 B.C.—probably knew of three of the five Platonic solids (the cube, tetrahedron, and dodecahedron). Slightly rounded versions of the Platonic solids made of stone have been discovered in areas inhabited by the late Neolithic people of Scotland at least 1,000 years before Plato. The German astronomer Johannes Kepler (1571–1630) constructed models of Platonic solids nested within one another in an attempt to describe the orbits of the planets about the sun. Although Kepler's theories were wrong, he was one of the first scientists to insist on a geometrical explanation for celestial phenomena.

SEE ALSO Pythagoras Founds Mathematical Brotherhood (c. 530 B.C.), Archimedean Semi-Regular Polyhedra (c. 240 B.C.), Euler's Formula for Polyhedra (1751), Icosian Game (1857), Pick's Theorem (1899), Geodesic Dome (1922), Császár Polyhedron (1949), Szilassi Polyhedron (1977), Spidrons (1979), and Solving of the Holyhedron (1999).

A traditional dodecahedron is a polyhedron with 12 pentagonal faces. Shown here is Paul Nylander's graphical approximation of a hyperbolic dodecahedron, which uses a portion of a sphere for each face.

Aristotle's *Organon*

Aristotle (384 B.C.–322 B.C.)

Aristotle was a Greek philosopher and scientist, a pupil of Plato, and a teacher of Alexander the Great. The *Organon* (*Instrument*) refers to the collection of six of Aristotle's works on logic: *Categories*, *Prior Analytics*, *De Interpretatione*, *Posterior Analytics*, *Sophistical Refutations*, and *Topics*. Andronicus of Rhodes determined the ordering of the six works around 40 B.C. Although Plato (c. 428–348 B.C.) and Socrates (c. 470–399 B.C.) delved into logical themes, Aristotle actually systematized the study of logic, which dominated scientific reasoning in the Western world for 2,000 years.

The goal of the *Organon* is not to tell readers what is true, but rather to give approaches for how to investigate truth and how to make sense of the world. The primary tool in Aristotle's tool kit is the syllogism, a three-step argument, such as "All women are mortal; Cleopatra is a woman; therefore, Cleopatra is mortal." If the two premises are true, we know that the conclusion must be true. Aristotle also made a distinction between particulars and universals (general categories). *Cleopatra* is a particular term. *Woman* and *mortal* are universal terms. When universals are used, they are preceded by "all," "some," or "no." Aristotle analyzed many possible kinds of syllogisms and showed which of them are valid.

Aristotle also extended his analysis to syllogisms that involved modal logic—that is, statements containing the words "possibly" or "necessarily." Modern mathematical logic can depart from Aristotle's methodologies or extend his work into other kinds of sentence structures, including ones that express more complex relationships or ones that involve more than one quantifier, such as "No women like all women who dislike some women." Nevertheless, Aristotle's systematic attempt at developing logic is considered to be one of humankind's greatest achievements, providing an early impetus for fields of mathematics that are in close partnership with logic and even influencing theologians in their quest to understand reality.

SEE ALSO Euclid's *Elements* (300 B.C.), Boolean Algebra (1854), Venn Diagrams (1880), *Principia Mathematica* (1910–1913), Gödel's Theorem (1931), and Fuzzy Logic (1965).

Italian Renaissance artist Raphael depicts Aristotle (right), holding his Ethics, *next to Plato. This Vatican fresco,* The School of Athens, *was painted between 1510 and 1511.*


ETI

c. 320 B.C.

Aristotle's Wheel Paradox

Aristotle (384 B.C.–322 B.C.)

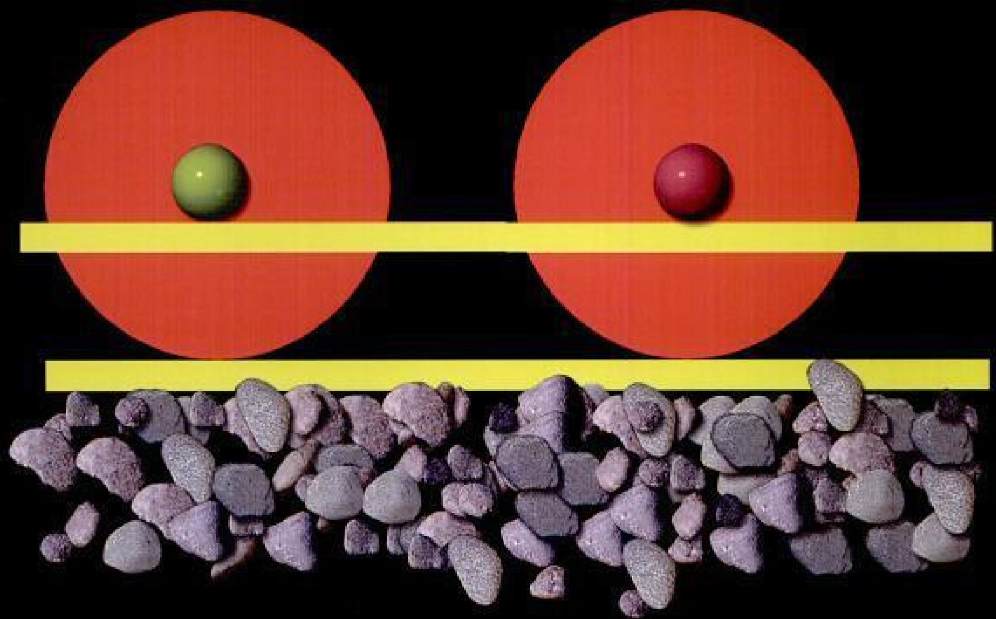
The paradox of Aristotle's wheel is mentioned in the ancient Greek text *Mechanica*. The problem has haunted some of the greatest mathematicians for centuries. Consider a small wheel mounted on a large wheel, diagrammed as two concentric circles. A one-to-one correspondence exists between points on the larger circle and those on the smaller circle; that is, for each point in the large circle, there is exactly one point on the small circle, and vice versa. Thus, the wheel assembly might be expected to travel the same horizontal distance regardless of whether it is rolled on a rod that touches the smaller wheel or rolled along the bottom wheel that touches the road. But how can this be? After all, we know that the two circumferences of the circles are different.

Today, mathematicians know that a one-to-one correspondence of points doesn't mean that two curves must have the same length. Georg Cantor (1845–1918) showed that the number, or cardinality, of points on line segments of any length is the same. He called this **Transfinite Number** of points the "continuum." For example, all the points in a segment from zero to one can even be put in one-to-one correspondence with all points of an infinite line. Of course, before the work of Cantor, mathematicians had quite a difficult time with this problem. Also note that, from a physical standpoint, if the large wheel did roll along the road, the smaller wheel would skip and be dragged along the line that touches its surface.

The precise date and authorship of *Mechanica* may forever be shrouded in mystery. Although often attributed as the work of Aristotle, many scholars doubt that *Mechanica*, the oldest-known textbook on engineering, was actually written by Aristotle. Another possible candidate for authorship is Aristotle's student Straton of Lampsacus (also known as Strato Physicus), who died around 270 B.C.

SEE ALSO Zeno's Paradoxes (c. 445 B.C.), St. Petersburg Paradox (1738), Cantor's Transfinite Numbers (1874), Barber Paradox (1901), Banach-Tarski Paradox (1924), Hilbert's Grand Hotel (1925), Birthday Paradox (1939), Coastline Paradox (c. 1950), Newcomb's Paradox (1960), Continuum Hypothesis Undecidability (1963), and Parrondo's Paradox (1999).

Consider a small wheel glued to a large wheel. Describe the motion of the wheel assembly as it moves from right to left along a rod that touches the smaller wheel and a road that touches the bottom wheel.



300 B.C.

Euclid's *Elements*

Euclid of Alexandria (c. 325 B.C.–c. 270 B.C.)

The geometer Euclid of Alexandria lived in Hellenistic Egypt, and his book *Elements* is one of the most successful textbooks in the history of mathematics. His presentation of plane geometry is based on theorems that can all be derived from just five simple axioms, or postulates, one of which is that only one straight line can be drawn between any two points. Given a point and a line, another famous postulate suggests that only one line through the point is parallel to the first line. In the 1800s, mathematicians finally explored **Non-Euclidean Geometries**, in which the parallel postulate was no longer always required. Euclid's methodical approach of proving mathematical theorems by logical reasoning not only laid the foundations of geometry but also shaped countless other areas concerning logic and mathematical proofs.

Elements consists of 13 books that cover two- and three-dimensional geometries, proportions, and the theory of numbers. *Elements* was one of the first books to be printed after the invention of the printing press and was used for centuries as part of university curricula. More than 1,000 editions of *Elements* have been published since its original printing in 1482. Although Euclid was probably not the first to prove the various results in *Elements*, his clear organization and style made the work of lasting significance. Mathematical historian Thomas Heath called *Elements* "the greatest mathematical textbook of all time." Scientists like Galileo Galilei and Isaac Newton were strongly influenced by *Elements*. Philosopher and logician Bertrand Russell wrote, "At the age of eleven, I began Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world." The poet Edna St. Vincent Millay wrote, "Euclid alone has looked on Beauty bare."

SEE ALSO Pythagorean Theorem and Triangles (c. 600 B.C.), Quadrature of the Lune (c. 440 B.C.), Aristotle's *Organon* (c. 350 B.C.), Descartes' *La Géométrie* (1637), Non-Euclidean Geometry (1829), and Weeks Manifold (1985).

This is the frontispiece of Adelard of Bath's translation of Euclid's Elements, *c. 1310. This translation from Arabic to Latin is the oldest surviving Latin translation of* Elements.

Archimedes: Sand, Cattle & Stomachion

Archimedes of Syracuse (c. 287 B.C.–c. 212 B.C.)

In 1941, mathematician G. H. Hardy wrote, "Archimedes will be remembered when [playwright] Aeschylus is forgotten, because languages die and mathematical ideas do not. 'Immortality' may be a silly word, but probably a mathematician has the best chance of whatever it may mean." Indeed, Archimedes, the ancient Greek geometer, is often regarded as the greatest mathematician and scientist of antiquity and one of the four greatest mathematicians to have walked the Earth—together with Isaac Newton, Carl Friedrich Gauss, and Leonhard Euler. Interestingly, Archimedes sometimes sent his colleagues false theorems in order to trap them when they stole his ideas.

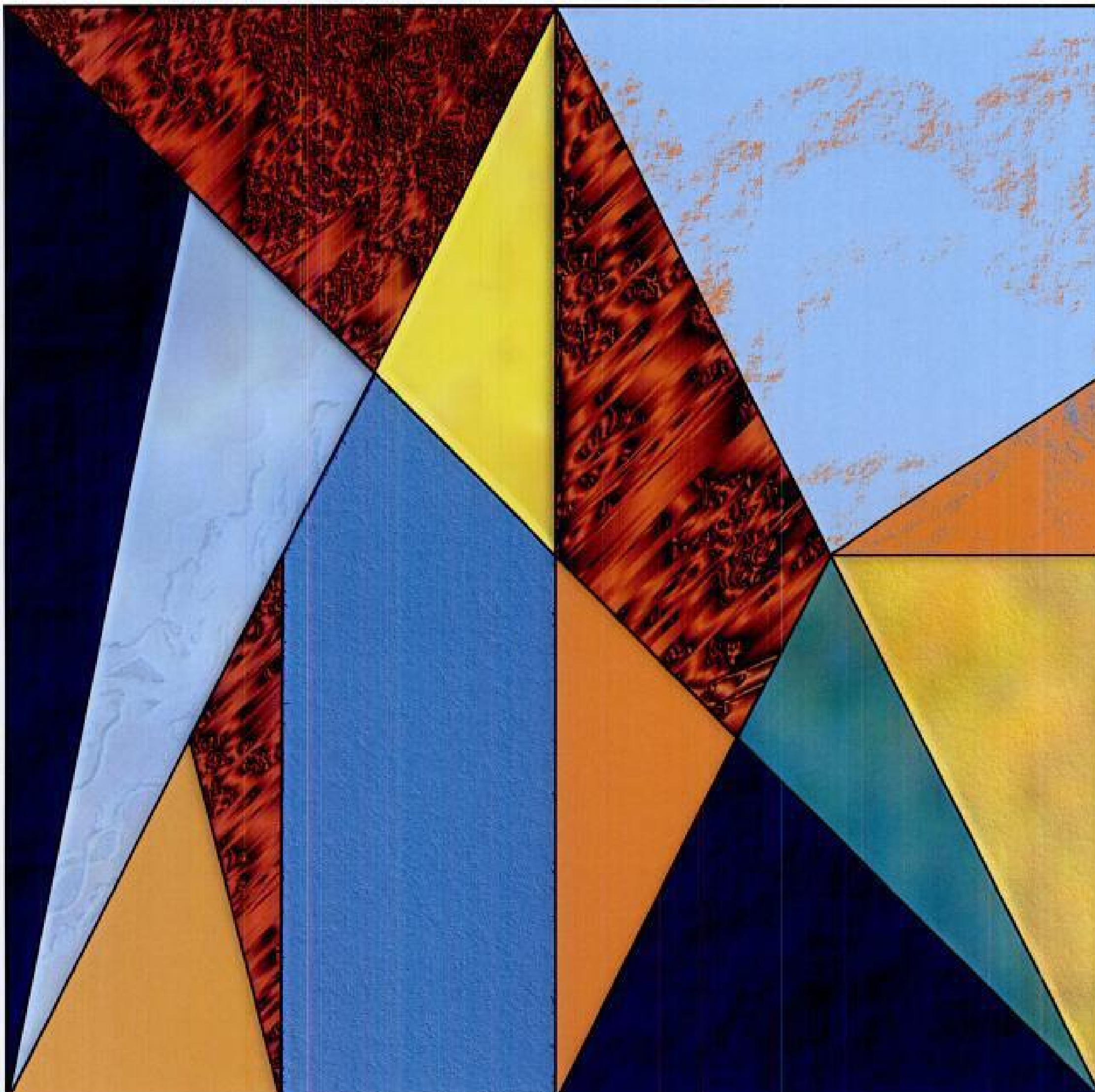
In addition to many other mathematical ideas, he is famous for his contemplation of tremendously large numbers. In his book *The Sand Reckoner*, Archimedes estimated that 8×10^{63} grains of sand would fill the universe.

More amazingly, the number $7.760271406486818269530232833213\ldots \times 10^{202544}$ is the solution to one version of Archimedes' famous "cattle problem," which involves computing the total number of cattle in a puzzle concerning four hypothetical herds of different colors. Archimedes wrote that anyone who could solve the problem would be "crowned with glory" and would be "adjudged perfect in this species of wisdom." Not until 1880 did mathematicians have an approximate answer. A more precise number was first calculated in 1965 by Canadian mathematicians Hugh C. Williams, R. A. German, and C. Robert Zarnke using an IBM 7040 computer.

In 2003, math historians discovered long lost information on the *Stomachion of Archimedes*. In particular, an ancient parchment, overwritten by monks nearly a thousand years ago, describes Archimedes' Stomachion, a puzzle involving combinatorics. *Combinatorics* is a field of math dealing with the number of ways a given problem can be solved. The goal of the Stomachion is to determine in how many ways the 14 pieces shown here can be put together to make a square. In 2003, four mathematicians determined that the number is 17,152.

SEE ALSO π (c. 250 B.C.), Euler's Polygon Division Problem (1751), Googol (c. 1920), and Ramsey Theory (1928).

For Archimedes' Stomachion puzzle, one goal is to determine in how many ways the 14 pieces shown here can be put together to make a square. In 2003, four mathematicians determined that the number is 17,152. (Rendering by Teja Krašek.)



MATHEMATICS

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