THE MATHEMATICIAN'S MIND

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PREFACE TO THE PAPERBACK EDITION

THIS book is a small masterpiece. Its author, Jacques Hadamard (1865–1963), was a distinguished French mathematician who in New York City during World War II turned his attention to the question of how mathematicians invent new ideas.[1] He was inspired by some observations on mathematical creation made by his illustrious predecessor Henri Poincaré in *Science and Méthode.* But Hadamard made his own introspections on the creative process and asked major scientists, mathematicians, and artists for their views as well. He reported some of their insights, including those of the linguist Roman Jakobson, the anthropologist Claude Lévi-Strauss, and the mathematicians George Polya and Norbert Wiener. Perhaps his most famous informant, however, was Albert Einstein, who described his own thinking process. In a letter to Hadamard, Einstein wrote that words seemed to play no role in his mechanism of thought, which instead relied on "certain signs and more or less clear images" (see Appendix II).

The basis for Hadamard's theorizing was the observation that mathematicians, and other creative individuals, often struggle unavailingly for some days on a problem, but subsequently, whilst consciously engaged in another activity, the answer comes to mind in a sudden inspiration. Thus, Poincaré told of a solution that popped into his head from out of nowhere, just as he put his foot on the step of an omnibus from Coutances (see p. 13). Hadamard himself had a similar experience, as did Gauss, Helmholtz, and others.

[1] S. Mandelbrojt wrote: "Few branches of mathematics were uninfluenced by the creative genius of Hadamard." In the *Dictionary of Scientific Biography,* ed. Charles C. Gillespie.

How are we to understand this phenomenon? Hadamard proposed four chronological stages in the process of creation:

(1) Preparation. You work hard on a problem, giving your conscious attention to it.

(2) Incubation. Your conscious preparation sets going an unconscious mechanism that searches for the solution. Poincaré wrote that ideas are like the hooked atoms of Epicurus: preparation sets them in motion and they continue their dance during incubation. The unconscious mechanism evaluates the resulting combinations on aesthetic criteria, but most of them are useless.

(3) Illumination. An idea that satisfies your unconscious criteria suddenly emerges into your consciousness.

(4) Verification. You carry out further conscious work in order to verify your illumination, to formulate it more precisely, and perhaps to follow up on its consequences.

This theory has been enormously influential, and some recent authors take it to be a theory that Hadamard himself formulated.[2] However, as he made clear, the four stages of the creative process were distinguished by Graham Wallace in *The Art of Thought,* which was published in 1926. Wallace, in turn, was anticipated both by Helmholtz and by Poincaré, who suggested that a sudden inspiration was the manifest sign of long, unconscious prior work.

Hadamard had much more to tell us about creation in general and mathematical invention in particular. His book was extraordinarily prescient. In the 1940s, America was in the midst of the dark ages of Behaviorism—the doctrine that psychology should eschew introspection as a method and mental processes as a topic of investigation. Hadamard would have none of this. Moreover, he considered what would now be termed the "modularity" hypothesis: the notion, which he

[2] See, e.g., P. Langley and R. Jones, "A Computational Model of Scientific Insight." In *The Nature of Creativity,* ed. Robert J. Sternberg (New York: Cambridge University Press, 1988), pp. 177–201.

correctly traced back to Gall and the phrenologists, that there are separate mental faculties for each subject—a faculty for mathematics, a faculty for language, and so on. In Hadamard's view, however, modularity did not go far enough. Mental faculties that seemed at first to be simple often turned out to be composite, and here he cited the precursors to modern cognitive neuropsychology, observations of the effects of brain damage on mental competence. There are, he said, distinct components of mathematical ability and distinct styles of mathematicians (see Chapter VII).

Hadamard made a cogent case for the existence of unconscious mental processes. In recognizing a face, he noted, one is sensitive to hundreds of features without being explicitly aware of them. Many parallel unconscious processes must therefore underlie this everyday ability. Hadamard rejected the view that thinking is possible only with the use of language, and he argued that many mathematicians, like Einstein, make use of images and "mental pictures." Levi-Strauss, it seemed, made use of *three-dimensional* mental representations. Hadamard was thus the first to discuss mental imagery during this Behaviorist epoch, and he anticipated its rehabilitation in psychology by some fifteen years.

He had intimations too of many notions that have become standard elements of contemporary cognitive science: the distinction between what the mind does and how it does it; the study of naive physics and of idiot savants; the notion that attention resembles a flashlight with a central focus of full consciousness and a penumbra of elements on the fringe; the hypothesis that creation may depend on lateral thinking, which in his own delightful English Hadamard called "thinking aside"; and the need for a certain degree of disorder (or chaos) in the generation of original ideas—not pure chance on the one hand, and not pure logic on the other.

What has happened in the fifty years of research into mathematical invention since he first published this book? The biggest single change has been made by the computer. Computers have been programmed to assist in the proof of major mathematical theorems, to automate theorem-proving logic, and to model the process of developing interesting mathematical conjectures (with more than a little help from their human friends).[3] If creation is a computational process, then a case can be made that there are only three sorts of creative processes.[4] First, processes that mimic the neo-Darwinian account of the origin of species: a generative stage in which there is a random combination or modification of existing ideas and a critical stage in which knowledge is used to select the more viable possibilities. Evolution depends on repeated iterations of these two stages. Second, neo-Lamarckian processes that use all their knowledge to constrain the generative stage and make a random selection when knowledge fails to select among equally viable alternatives. Such processes seem particularly appropriate for creation in "real time," such as musical improvisation or poetic extemporization. Third, and most plausible for mathematical invention, processes that use knowledge both to generate ideas and to

[3] In 1976, Kenneth Appel and Wolfgang Haken used many hours of computer time to help prove the famous four-color-map theorem; i.e., the different regions of any map on a planar surface can be distinguished from their neighbors using no more than four colors (see "The Solution of the Four-Color-Map Problem," *Scientific American*, September 1977, 108–121). There is a vast literature on fully automated theorem-proving; for a review, see Robert C. Moore's "Automatic Deduction," Overview, Section A of Chapter XII in *The Handbook of Artificial Intelligence. Volume 3,* ed. P. R. Cohen and E. A. Feigenbaum (Reading, Mass.: Addison-Wesley, 1982). Douglas Lenat's program, AM, does not prove theorems, but rather searches for interesting mathematical conjectures. It relies on guidance from its human user (see, e.g., D. B. Lenat and J. S. Brown, "Why AM and Eurisko Appear to Work," *Artificial Intelligence* 23 (1984): 269–294.

[4] This case is made by the present author; see, e.g., the chapter on creation in his book, *The Computer and the Mind* (Cambridge, Mass.: Harvard University Press, 1988).

evaluate viable possibilities. The first stage is presumably unconscious, and the second is conscious.

But is mathematical invention a computable process? Hadamard did not address the issue, but one of his successors, the mathematician and physicist Roger Penrose, gives a negative answer. Penrose argues that consciousness and visual images depend on noncomputational processes.[5] Certain physical processes are not computational—for example, the bleaching of visual pigment in retinal cells when light falls on them—but such processes can at least be modeled in a computer program. According to Penrose, however, the mental processes of mathematical invention cannot even be modeled computationally. This singular state of affairs is possible, but, as yet, there is no decisive evidence either way.

Ironically, the roots of creativity for Hadamard lie not in consciousness, but in the long unconscious work of incubation and in the unconscious aesthetic selection of ideas that thereby pass into consciousness. Latterday cognitive scientists accept the notion of unconscious processes, but Hadamard's particular conception of the unconscious is more problematic.[6] Cognitive scientists argue that conscious performance rests on a raft of unconscious mechanisms that *construct* its contents; that is, ideas do not simply pass like packets from the unconscious system to the conscious system. Your awareness of the meaning of the previous sentence, for example, depends on many unconscious processes that transform sensory information into a conscious construct. In contrast, incubation is supposed to proceed whilst the conscious mind is otherwise engaged on quite different matters, and to deliver to consciousness a fully formed packet of inspiration.

[5] Roger Penrose, *Shadows of the Mind* (New York: Oxford University Press, 1994).

[6] For a thoughtful analysis of creativity and incubation, see David N. Perkins, *The Mind's Best Work* (Cambridge, Mass.: Harvard University Press, 1981).

Can one really ruminate about a profound mathematical problem whilst paying attention to breakfast television? What evidence exists on this matter justifies some skepticism. In one study, chess experts either worked continuously on a chess puzzle or were allowed a two-hour break for incubation to occur. They did not differ in their performance.[7] Hadamard would have surely discounted this study. He would have argued that the duration of the experiment—hours rather than days—and the nature of the task—chess puzzles rather than deep mathematical problems—preclude proper incubation (cf. his remarks in Section III on Catherine Patrick's experiment in which the subjects were required to write poems). The single most important unanswered question about the psychology of creation is accordingly whether incubation, as conceived by Hadamard and his peers, is a genuine phenomenon. The question cannot be answered by introspection, and it has yet to be definitively resolved in the psychological laboratory.

Few psychological works outlast their time; the works that do have several principal holds on our attention. They express themselves vividly, and what they have to say is wise. They convey an insight into psychological phenomena that is not doctrinaire and that has a timeless good sense. And so the reader—even though he or she may know better than the author on some matters—nonetheless comes away from the book with a deeper understanding of mental life. Hadamard's volume is no exception. Since he wrote, the psychological problem of invention in the mathematical field seems to have grown more difficult to solve. Yet the seeds of its solution are more than likely to be found in this book.

P. N. JOHNSON-LAIRD

Princeton, 1995

[7] Robert M. Olton, "Experimental Studies of Incubation: Searching for the Elusive," *Journal of Creative Behavior* 13 (1979): 9–22.

FOREWORD

"Je dirai que j'ai trouvé la démonstration de tel théorème dans telles circonstances; ce théorème aura un nom barbare, que beaucoup d'entre vous ne connaîtront pas; mais cela n'a pas d'importance: ce qui est intéressant pour le psychologue, ce n'est pas le théorème, ce sont les circonstances."

—Henri Poincaré

THIS study, like everything which could be written on mathematical invention, was first inspired by Henri Poincaré's famous lecture before the Société de Psychologie in Paris. I first came back to the subject in a meeting at the Centre de Synthèse in Paris (1937). But a more thorough treatment of it has been given in an extensive course of lectures delivered (1943) at the Ecole Libre des Hautes Etudes, New York City.

I wish to express my gratitude to Princeton University Press, for the interest taken in this work and the careful help brought to its publication.

JACQUES HADAMARD

August 21, 1944
New York, N.Y.

INTRODUCTION

Concerning the title of this study, two remarks are useful. We speak of invention: it would be more correct to speak of discovery. The distinction between these two words is well known: discovery concerns a phenomenon, a law, a being which already existed, but had not been perceived. Columbus discovered America: it existed before him; on the contrary, Franklin invented the lightning rod: before him there had never been any lightning rod.

Such a distinction has proved less evident than appears at first glance. Toricelli has observed that when one inverts a closed tube on the mercury trough, the mercury ascends to a certain determinate height: this is a discovery; but, in doing this, he has invented the barometer; and there are plenty of examples of scientific results which are just as much discoveries as inventions. Franklin's invention of the lightning rod is hardly different from his discovery of the electric nature of thunder. This is a reason why the aforesaid distinction does not truly concern us; and, as a matter of fact, psychological conditions are quite the same for both cases.

On the other hand, our title is "Psychology of Invention in the Mathematical Field," and not "Psychology of Mathematical Invention." It may be useful to keep in mind that mathematical invention is but a case of invention in general, a process which can take place in several domains, whether it be in science, literature, in art or also technology.

Modern philosophers even say more. They have perinescapably imposes on us the path we must follow under penalty of going astray.

This does not preclude many analogies between these two activities, as we shall have occasion to observe. These analogies appeared when, in 1937, at the Centre de Synthèse in Paris, a series of lectures was delivered on invention of various kinds, with the help of the great Genevese psychologist, Claparède. A whole week was devoted to the various kinds of invention, with one session for mathematics. Especially, invention in experimental sciences was treated by Louis de Broglie and Bauer, poetical invention by Paul Valéry. The comparison between the circumstances of invention in these various fields may prove very fruitful.

It is all the more useful, perhaps, to deal with a special case such as the mathematical one, which I shall discuss, since it is the one I know best. Results in one sphere (and we shall see that important achievements have been reached in that field, thanks to a masterly lecture of Henri Poincaré) may always be helpful in order to understand what happens in other ones.

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