The Mathematics *of* Egypt, Mesopotamia, China, India, and Islam

A Sourcebook

The Mathematics *of* Egypt, Mesopotamia, China, India, and Islam

A Sourcebook

Victor Katz, *Editor*

Annette Imhausen
Eleanor Robson
Joseph Dauben
Kim Plofker
J. Lennart Berggren

PRINCETON UNIVERSITY PRESS • PRINCETON AND OXFORD

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540

In the United Kingdom: Princeton University Press, 3 Market Place,
Woodstock, Oxfordshire OX20 1SY

Library of Congress Cataloging-in-Publication Data

The mathematics of Egypt, Mesopotamia, China, India,
and Islam: a sourcebook/Victor Katz, editor.
p. cm.
Includes bibliographical references and index.

ISBN-13: 978-0-691-11485-9 (hardcover: alk. paper)
ISBN-10: 0-691-11485-4 (hardcover: alk. paper)

1. Mathematics, Ancient–Sources. 2. Mathematics–History–Sources.
I. Katz, Victor J.
QA22.M3735 2007
510.9–dc22 2006030851

British Library Cataloging-in-Publication Data is available

This book has been composed in Minion and Univers
Printed on acid-free paper. ∞
press.princeton.edu

Printed in the United States of America

3 5 7 9 10 8 6 4

Contents

Chapter 5
Mathematics in Medieval Islam
J. Lennart Berggren

Preface

Kim Plofker and I conceived of this book at a meeting of the Mathematical Association of America several years ago. We agreed that, since there was now available a fairly large collection of English translations of mathematical texts from Egypt, Mesopotamia, China, India, and Islam, the time was ripe to put together an English sourcebook. This book would have texts gathered together so that they could easily be studied by all those interested in the mathematics of ancient and medieval times. I secured commitments to edit the five sections from outstanding scholars with whom I was familiar, scholars who had already made significant contributions in their fields, and were fluent in both the original languages of the texts and English. There are a growing number of scholars investigating the works of these civilizations, but I believe that this group was the appropriate one to bring this project to fruition.

The editors decided that to the extent possible we would use already existing English translations with the consent of the original publisher, but would, where necessary, produce new ones. In certain cases, we decided that an existing translation from the original language into French or German could be retranslated into English, but the retranslation was always made with reference to the original language. As it turned out, the section editors of both the Egyptian and Mesopotamian sections decided to produce virtually all new translations, because they felt that many previous translations had been somewhat inadequate. The other sections have a mix of original translations and previously translated material.

Each of the five sections of the Sourcebook has a preface, written by the section editor, giving an overview of the sources as well as detailing the historical setting of the mathematics in that civilization. The individual sources themselves also have introductions. In addition, many of the sources contain explanations to help the reader understand the sometimes fairly cryptic texts. In particular, the Chinese and Indian sections have considerably more detailed explanations than the Islamic section, because Islamic mathematicians, being well schooled in Greek mathematics, use "our" techniques of mathematical analysis and proof. Mathematicians from China and India, and from Egypt and Mesopotamia as well, come from traditions far different from ours. So the editors of those sections have spent considerable effort in guiding the reader through this "different" mathematics.

The book is aimed at those having knowledge of mathematics at least equivalent to a U.S. mathematics major. Thus, the intended audience of the book includes students studying mathematics and the history of mathematics, mathematics teachers at all levels, mathematicians, and historians of mathematics.

The editors of this Sourcebook wish to thank our editors at Princeton University Press, David Ireland and Vickie Kearn, who have been extremely supportive in guiding this book from concept to production. We also wish to thank Dale Cotton, our production editor, Dimitri Karetnikov, the illustration specialist, and Alison Anderson, the copy editor, for their friendly and efficient handling of a difficult manuscript. Finally, as always, I want to thank my wife Phyllis for her encouragement and for everything else.

Victor J. Katz
Silver Spring, MD
December, 2005

Permissions

Chapter 1

SECTION IV: The material from the Demotic papyri is reprinted from Richard Parker, *Demotic Mathematical Papyri*, © 1972 by Brown University Press, by permission of University Press of New England, Hanover, NH.

Chapter 3

SECTION IV: The material from the *Zhou bi suan jing* is reprinted from *Astronomy and Mathematics in Ancient China: The Zhou Bi Suan Jing*, by Christopher Cullen, © 1996, with the permission of Cambridge University Press.

SECTION V: The material from the *Nine Chapters on the Mathematical Art* and the *Sea Island Mathematical Manual* is reprinted from *The Nine Chapters on the Mathematical Art: Companion and Commentary*, by Shen Kangshen, John Crossley, and Anthony W.-C. Lun, © 1999, with the permission of Oxford University Press.

SECTION VI.A.: The material from the *Sunzi suan jing* is reprinted from *Fleeting Footsteps: Tracing the Conception of Arithmetic and Algebra in Ancient China*, by Lam Lay Yong and Ang Tian Se, © 1992, with the permission of World Scientific Publishing Co. Pte. Ltd.

SECTION VI.B.: The material from the *Zhang qiujian suan jing* is reprinted from "Zhang Qiujian Suanjing (The Mathematical Classic of Zhang Qiujian): An Overview," by Lam Lay Yong, *Archive for History of Exact Sciences* 50 (1997), 201–240, with the kind permission of Springer Science and Business Media.

SECTION VII.A: The material from chapters 1, 2, 4 of the *Shu shu jiu zhang* is reprinted from *Chinese Mathematics in the Thirteenth Century: The Shu-shu chiu-chang of Ch'in Chiu-shao*, by Ulrich Libbrecht, © 1973, with the permission of MIT Press. The material from chapter 3 of the *Shu shu jiu zhang* is reprinted from *A History of Chinese Mathematics*, by Jean-Claude Martzloff, Stephen Wilson, trans., © 1997, with the kind permission of Springer Science and Business Media.

SECTION VII.B: The material from Li Zhi's *Ce yuan hai jing* is reprinted from "Chinese Polynomial Equations in the Thirteenth Century," in Li Guihao *et al.*, eds., *Explorations in the History of Science and Technology in China*, Chinese Classics Publishing House, Shanghai, © 1982. The material from Li Zhi's *Yigu yanduan* is reprinted from "Li Ye and his Yi Gu Yan Duan (Old Mathematics in Expanded Sections)," by Lam Lay Yong and Ang Tian Se, *Archive for History of Exact Sciences* 29 (1984), 237–265 with the kind permission of Springer Science and Business Media.

SECTION VII.C.: The material from Yang Hui's *Xiangjie Jiuzhang Suanfa* is reprinted from "On The Existing Fragments of Yang Hui's Hsiang Chieh Suan Fa," by Lam Lay Yong, *Archive for History of Exact Sciences* 6 (1969), 82–88 with the kind permission of Springer Science and Business Media. The material from Yang Hui's *Yang Hui Suan Fa* is reprinted from *A Critical Study of the Yang Hui Suan Fa*, by Lam Lay Yong, © 1977, with the permission of Singapore University Press.

SECTION VIII: The material from Matteo Ricci and Xu Guangxi's, "Prefaces" to the First Chinese Edition of Euclid's *Elements* is reprinted from *Euclid in China*, by Peter M. Engelfriet, © 1998 by permission of Brill Academic Publishers.

Chapter 4

SECTION II.B: The material from the *Śulbasūtras* is reprinted from *The Sulbasutras*, by S. N. Sen and A. K. Bag, © 1983 by permission of the Indian National Science Academy.

SECTION II.D.: Figure 2.5 is reprinted from *The Concept of Śūnya*, by A. K. Bag and S. R. Sarma, eds., © 2003 by permission of the Indian National Science Academy.

SECTION III.A.: The material from the *Äryabhaṭīya* is reprinted from *Expounding the Mathematical Seed: a Translation of Bhāskara I on the Mathematical Chapter of the Äryabhaṭīya*, by Agathe Keller, © 2005 by permission of Birkhäuser Verlag AG.

SECTION III.C.: The material from the *Bakhshālī Manuscript* is reprinted from *The Bakhshali Manuscript: An Ancient Indian Mathematical Treatise*, © 1995 by permission of Egbert Forsten Publishing. The material from the *Bījagaṇita* of Bhāskara II with commentary of Sūryadāsa is reprinted from *The Sūryaprakāśa of Sūryadāsa: A Commentary on Bhāskaracarya's Bījagaṇita*, by Pushpa Kumari Jain, © 2001 by permission of the Oriental Institute in Vadodara.

SECTION III.E.: The material from the *Śiṣyadhīvṛddhidatantra* of Lalla is reprinted from *Śiṣyadhivṛddhida Tantra of Lalla*, by Bina Chatterjee, © 1981 by permission of the Indian National Science Academy.

SECTION IV.B.: The material from the *Kriyākramakarī* of Śaṅkara is reprinted from *Studies in Indian Mathematics: Series, Pi and Trigonometry*, by T. Hayashi, T. Kusuba, and M. Yano, Setsuro Ikeyama, trans., by permission of the authors.

SECTION V.A.: The material from the *Gaṇitakaumudī* of Nārāyana Paṇḍita is reprinted from *Combinatorics and Magic Squares in India: A Study of Nārāyana Paṇḍita's "Gaṇitakaumudi", chapters 13-14*, a Brown University dissertation by Takanori Kusuba and is used by permission of the author.

Chapter 5

SECTION II: The material from the Banū Mūsā *On Conics* is reprinted from *Apollonius' Conics Books V-VII: The Arabic Translation of the Lost Greek Original in the Version of the Banu Musa*, by G. J. Toomer, © 1990 with the kind permission of Springer Science and Business Media. The material from ibn al-Haytham's *Completion of the Conics* is reprinted from *Ibn al-Haytham's Completion of the Conics*, by J. P. Hogendijk, © 1985 with the kind permission of Springer Science and Business Media.

SECTION III: The material from al-Khwārizmī's *Treatise on Hindu Calculation* is reprinted from "Thus Spake al-Khwārizmī: A Translation of the Text of Cambridge University Library Ms. Ik.vi.5," by John Crossley and Alan Henry, *Historia Mathematica* 17 (1990), 103–131 with permission from Elsevier Ltd. The material from al-Uqlīdisī's *Chapters on Hindu Arithmetic* is reprinted from *The Arithmetic of al-Uqlidisi*, by A. Saidan, © 1978 by permission of Reidel Publishers. The material from Kūshyār ibn Labān's *Principles of Hindu Reckoning* is reprinted from *Kushyar ibn Laban Principles of Hindu Reckoning*, by M. Levey and M. Petruck, © 1966 by permission of The University of Wisconsin Press.

SECTION IV: The material from al-Samaw'al on al-Karajī and binomial coefficients is reprinted from *The Development of Arabic Mathematics: Between Arithmetic and Algebra*, by Roshdi Rashed, © 1994 with the kind permission of Springer Science and Business Media. The material from the algebra of Omar Khayyam is reprinted from *The Algebra of Omar Khayyam*, by D. Kasir, © 1931 with the permission of Columbia Teachers College Press.

SECTION V: The material from al-Bāghdadī on balanced numbers is reprinted from "Two Problems of Number Theory in Islamic Times," by J. Sesiano, *Archive for History of Exact Sciences* 41, (1991), 235–238 with the kind permission of Springer Science and Business Media.

SECTION VI.A.: The material from al-Kūhī and al-Sābī on centers of gravity is reprinted from "The Correspondence of Abū Sahl al-Kūhī and Abū Isḥāq al-Sābī," by J. Lennart Berggren, *Journal for the History of Arabic Science* 7, (1983), 39–124 with the permission of the *Journal*. The material from al-Kūhī on the astrolabe is reprinted from "Abū Sahl al-Kūhī's Treatise on the Construction of the Astrolabe with Proof: Text, Translation and Commentary," by J. Lennart Berggren, *Physis* 31 (1994), 141–252 with the permission of Leo S. Olschki, publisher. The material from al-Sijzī on the construction of the heptagon is reprinted from "Greek and Arabic Constructions of the Regular Heptagon," by J. P. Hogendijk, *Archive for History of Exact Sciences* 30 (1984), 197–330 with the kind permission of Springer Science and Business Media. The material on the anonymous treatise on the nine-sided figure is reprinted from "An Anonymous Treatise on the Regular Nonagon," by J. L. Berggren, *Journal for the History of Arabic Science* 5 (1981), 37–41 with the permission of the *Journal*. The material from al-Jazarī on constructing geometrical instruments is reprinted from *The Book of Knowledge of Ingenious Mechanical Devices by Ibn al-Razzāz al-Jazarī*, by Donald Hill, © 1974 by permission of Reidel Publishers. The material from al-Tūsī on the theory of parallels is reprinted from *A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space*, by B. A. Rosenfeld, © 1988 with the kind permission of Springer Science and Business Media.

SECTION VI.B.: The material from Abū al-Wafā' on the geometry of artisans is reprinted from "Mathematics and Arts: Connections between Theory and Practice in the Medieval Islamic

World", by Alpay Özdural, *Historia Mathematica* 27, (2000), 171–201 with permission from Elsevier, Ltd. The two patterns for muqarnas are reprinted from *Topkapi Scroll-Geometry and Ornament in Islamic Architecture*, by G. Necipoglu, © 1995 with permission of the Getty Research Institute.

SECTION VII: The material from al-Bīrūnī on geodesy is reprinted from *The Determination of the Coordinates of Cities: Al-Bīrūnī's Taḥdīd al-Amākin*, by Jamil Ali, © 1967 with permission of the American University of Beirut Press, Beirut, Lebanon. The material from al-Kāshī on interpolation using second order differences is reprinted from "A Medieval Interpolation Scheme Using Second Order Differences," by E. S. Kennedy, in *Studies in the Islamic Exact Sciences by E. S. Kennedy, Colleagues, and Former Students*, edited by D. A. King and M. H. Kennedy, © 1983 with permission of the American University of Beirut Press, Beirut, Lebanon.

SECTION VIII: The material from al-Bīrūnī on the number of possibilities for eclipse data is reprinted from *The Determination of the Coordinates of Cities: Al-Bīrūnī's Taḥdīd al-Amākin*, by Jamil Ali, © 1967 with permission of the American University of Beirut Press, Beirut, Lebanon.

SECTION IX: The material from al-Kūhī on the certainty deriving from mathematics is reprinted from "The Correspondence of Abū Sahl al-Kūhī and Abū Isḥāq al-Sābī: A Translation with commentaries," by J. Lennart Berggren, *Journal for the History of Arabic Science* 7, (1983), 39–124 with the permission of the *Journal*. The material from al-Sijzī on heuristics is reprinted from *Al-Sijzī's Treatise on Geometrical Problem Solving*, by J. P. Hogendijk, with the permission of the author.

The Mathematics *of* Egypt, Mesopotamia, China, India, and Islam

A Sourcebook

Introduction

A century ago, mathematics history began with the Greeks, then skipped a thousand years and continued with developments in the European Renaissance. There was sometimes a brief mention that the "Arabs" preserved Greek knowledge during the dark ages so that it was available for translation into Latin beginning in the twelfth century, and perhaps even a note that algebra was initially developed in the lands of Islam before being transmitted to Europe. Indian and Chinese mathematics barely rated a footnote.

Since that time, however, we have learned much. First of all, it turned out that the Greeks had predecessors. There was mathematics both in ancient Egypt and in ancient Mesopotamia. Archaeologists discovered original material from these civilizations and deciphered the ancient texts. In addition, the mathematical ideas stemming from China and India gradually came to the attention of historians. In the nineteenth century, there had been occasional mention of these ideas in fairly obscure sources in the West, and there had even been translations into English or other western languages of certain mathematical texts. But it was only in the late twentieth century that major attempts began to be made to understand the mathematical ideas of these two great civilizations and to try to integrate them into a worldwide history of mathematics. Similarly, the nineteenth century saw numerous translations of Islamic mathematical sources from the Arabic, primarily into French and German. But it was only in the last half of the twentieth century that historians began to put together these mathematical ideas and attempted to develop an accurate history of the mathematics of Islam, a history beyond the long-known preservation of Greek texts and the algebra of al-Khwarizmi. Yet, even as late as 1972, Morris Kline's monumental work *Mathematical Thought from Ancient to Modern Times* contained but 12 pages on Mesopotamia, 9 pages on Egypt, and 17 pages combined on India and the Islamic world (with nothing at all on China) in its total of 1211 pages.

It will be useful, then, to give a brief review of the study of the mathematics of Egypt, Mesopotamia, China, India, and Islam to help put this *Sourcebook* in context.

To begin with, our most important source on Egyptian mathematics, the Rhind Mathematical Papyrus, was discovered, probably in the ruins of a building in Thebes, in the middle of the nineteenth century and bought in Luxor by Alexander Henry Rhind in 1856. Rhind died in 1863 and his executor sold the papyrus, in two pieces, to the British Museum in 1865. Meanwhile, some fragments from the break turned up in New York, having been acquired also in Luxor by the American dealer Edwin Smith in 1862. These are now in the

Brooklyn Museum. The first translation of the Rhind Papyrus was into German in 1877. The first English translation, with commentary, was made in 1923 by Thomas Peet of the University of Liverpool. Similarly, the Moscow Mathematical Papyrus was purchased around 1893 by V. S. Golenishchev and acquired about twenty years later by the Moscow Museum of Fine Arts. The first notice of its contents appeared in a brief discussion by B. A. Turaev, conservator of the Egyptian section of the museum, in 1917. He wrote chiefly about problem 14, the determination of the volume of a frustum of a square pyramid, noting that this showed "the presence in Egyptian mathematics of a problem that is not to be found in Euclid." The first complete edition of the papyrus was published in 1930 in German by W. W. Struve. The first complete English translation was published by Marshall Clagett in 1999.

Thus, by early in the twentieth century, the basic outlines of Egyptian mathematics were well understood—at least the outlines as they could be inferred from these two papyri. And gradually the knowledge of Egyptian mathematics embedded in these papyri and other sources became part of the global story of mathematics, with one of the earliest discussions being in Otto Neugebauer's *Vorlesungen über Geschichte der antiken Mathematischen Wissenschaften* (more usually known as *Vorgriechische Mathematik*) of 1934, and further discussions and analysis by B. L. Van der Waerden in his *Science Awakening* of 1954. A more recent survey is by James Ritter in *Mathematics Across Cultures.*

A similar story can be told about Mesopotamian mathematics. Archaeologists had begun to unearth the clay tablets of Mesopotamia beginning in the middle of the nineteenth century, and it was soon realized that some of the tablets contained mathematical tables or problems. But it was not until 1906 that Hermann Hilprecht, director of the University of Pennsylvania's excavations in what is now Iraq, published a book discussing tablets containing multiplication and reciprocal tables and reviewed the additional sources that had been published earlier, if without much understanding. In 1907, David Eugene Smith brought some of Hilprecht's work to the attention of the mathematical world in an article in the *Bulletin of the American Mathematical Society*, and then incorporated some of these ideas into his 1923 *History of Mathematics.*

Meanwhile, other archaeologists were adding to Hilprecht's work and began publishing some of the Mesopotamian mathematical problems. The study of cuneiform mathematics changed dramatically, however, in the late 1920s, when François Thureau-Dangin and Otto Neugebauer independently began systematic programs of deciphering and publishing these tablets. In particular, Neugebauer published two large collections: *Mathematische Keilschrift-Texte* in 1935–37 and (with Abraham Sachs) *Mathematical Cuneiform Texts* in 1945. He then summarized his work for the more general mathematical public in his 1951 classic, *The Exact Sciences in Antiquity.* Van der Waerden's *Science Awakening* was also influential in publicizing Mesopotamian mathematics. Jens Høyrup's survey of the historiography of Mesopotamian mathematics provides further details.

Virtually the first mention of Chinese mathematics in a European language was in several articles in 1852 by Alexander Wylie entitled "Jottings on the Science of the Chinese: Arithmetic," appearing in the *North China Herald*, a rather obscure Shanghai journal. However, they were translated in part into German by Karl L. Biernatzki and published in *Crelle's Journal* in 1856. Six years later they also appeared in French. It was through these articles that Westerners learned of what is now called the Chinese Remainder problem and how it was initially solved in fourth-century China, as well as about the ten Chinese classics and the Chinese algebra of the thirteenth century. Thus, by the end of the nineteenth century, European historians of mathematics could write about Chinese mathematics, although, since they did not have access to the original material, their works often contained errors.

The first detailed study of Chinese mathematics written in English by a scholar who could read Chinese was *Mathematics in China and Japan*, published in 1913 by the Japanese scholar Yoshio Mikami. Thus David Eugene Smith, who co-authored a work solely on Japanese mathematics with Mikami, could include substantial sections on Chinese mathematics in his own *History* of 1923. Although other historians contributed some material on China during the first half of the twentieth century, it was not until 1959 that a significant new historical study appeared, volume 3 of Joseph Needham's *Science and Civilization in China*, entitled *Mathematics and the Sciences of the Heavens and the Earth.* One of Needham's chief collaborators on this work was Wang Ling, a Chinese researcher who had written a dissertation on the *Nine Chapters* at Cambridge University. Needham's work was followed by the section on China in A. P. Yushkevich's history of medieval mathematics (1961) in Russian, a book that was in turn translated into German in 1964. Since that time, there has been a concerted effort by both Chinese and Western historian of mathematics to make available translations of the major Chinese texts into Western languages.

The knowledge in the West of Indian mathematics occurred much earlier than that of Chinese mathematics, in part because the British ruled much of India from the eighteenth century on. For example, Henry Thomas Colebrooke collected Sanskrit mathematical and astronomical texts in the early nineteenth century and published, in 1817, his *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhascara.* Thus parts of the major texts of two of the most important medieval Indian mathematicians were available in English, along with excerpts from Sanskrit commentaries on these works. Then in 1835, Charles Whish published a paper dealing with the fifteenth-century work in Kerala on infinite series, and Ebenezer Burgess in 1860 published a translation of the *Sūrya-siddhānta*, a major early Indian work on mathematical astronomy. Hendrik Kern in 1874 produced an edition of the *Āryabhaṭīya* of Aryabhata, while George Thibaut wrote a detailed essay on the *Śulbasūtras*, which was published, along with his translation of the *Baudhāyana Śulbasūtra*, in the late 1870s. The research on medieval Indian mathematics by Indian researchers around the same time, including Bāpu Deva Sāstrī, Sudhākara Dvivedī, and S. B. Dikshit, although originally published in Sanskrit or Hindi, paved the way for additional translations into English.

Despite the availability of some Sanskrit mathematical texts in English, it still took many years before Indian contributions to the world of mathematics were recognized in major European historical works. Of course, European scholars knew about the Indian origins of the decimal place-value system. But in part because of a tendency in Europe to attribute Indian mathematical ideas to the Greeks and also because of the sometimes exaggerated claims by Indian historians about Indian accomplishments, a balanced treatment of the history of mathematics in India was difficult to achieve. Probably the best of such works was the *History of Indian Mathematics: A Source Book*, published in two volumes by the Indian mathematicians Bibhutibhusan Datta and Avadhesh Narayan Singh in 1935 and 1938. In recent years, numerous Indian scholars have produced new Sanskrit editions of ancient texts, some of which have never before been published. And new translations, generally into English, are also being produced regularly, both in India and elsewhere.

As to the mathematics of Islam, from the time of the Renaissance Europeans were aware that algebra was not only an Arabic word, but also essentially an Islamic creation. Most early algebra works in Europe in fact recognized that the first algebra works in that continent were translations of the work of al-Khwārizmī and other Islamic authors. There was also some awareness that much of plane and spherical trigonometry could be attributed to Islamic authors. Thus, although the first pure trigonometrical work in Europe, *On Triangles* by Regiomontanus, written around 1463, did not cite Islamic sources, Gerolamo Cardano noted a century later that much of the material there on spherical trigonometry was taken from the twelfth-century work of the Spanish Islamic scholar Jābir ibn Aflaḥ.

By the seventeenth century, European mathematics had in many areas reached, and in some areas surpassed, the level of its Greek and Arabic sources. Nevertheless, given the continuous contact of Europe with Islamic countries, a steady stream of Arabic manuscripts, including mathematical ones, began to arrive in Europe. Leading universities appointed professors of Arabic, and among the sources they read were mathematical works. For example, the work of Ṣadr al-Ṭūsī (the son of Naṣīr al-Dīn al-Ṭūsī) on the parallel postulate, written originally in 1298, was published in Rome in 1594 with a Latin title page. This work was studied by John Wallis in England, who then wrote about its ideas as he developed his own thoughts on the postulate. Still later, Newton's friend, Edmond Halley, translated into Latin Apollonius's *Cutting-off of a Ratio*, a work that had been lost in Greek but had been preserved via an Arabic translation.

Yet in the seventeenth and eighteenth centuries, when Islamic contributions to mathematics may well have helped Europeans develop their own mathematics, most Arabic manuscripts lay unread in libraries around the world. It was not until the mid-nineteenth century that European scholars began an extensive program of translating these mathematical manuscripts. Among those who produced a large number of translations, the names of Heinrich Suter in Switzerland and Franz Woepcke in France stand out. (Their works have recently been collected and republished by the Institut für Geschichte der arabisch-islamischen Wissenschaften.) In the twentieth century, Soviet historians of mathematics began a major program of translations from the Arabic as well. Until the middle of the twentieth century, however, no one in the West had pulled together these translations to try to give a fuller picture of Islamic mathematics. Probably the first serious history of Islamic mathematics was a section of the general history of medieval mathematics written in 1961 by A. P. Yushkevich, already mentioned earlier. This section was translated into French in 1976 and published as a separate work, *Les mathématiques arabes* (*VIII^e–XV^e siècles*). Meanwhile, the translation program continues, and many new works are translated each year from the Arabic, mostly into English or French.

By the end of the twentieth century, all of these scholarly studies and translations of the mathematics of these various civilizations had an impact on the general history of mathematics. Virtually all recent general history textbooks contain significant sections on the mathematics of these five civilizations. As this sourcebook demonstrates, there are many ideas that were developed in these five civilizations that later reappeared elsewhere. The question that then arises is how much effect the mathematics of these civilizations had on what is now world mathematics of the twenty first-century. The answer to this question is very much under debate. We know of many confirmed instances of transmission of mathematical ideas from one of these cultures to Europe or from one of these cultures to another, but there are numerous instances where, although there is circumstantial evidence of transmission, there is no definitive documentary evidence. Whether such will be found as more translations are made and more documents are uncovered in libraries and other institutions around the world is a question for the future to answer.

References

Biernatzki, Karl L. 1856. "Die Arithmetic der Chinesen." *Crelle's Journal* 52: 59–94.

Burgess, Ebenezer. 1860. "The Sūryasiddhānta." *Journal of the American Oriental Society*, 6.

Clagett, Marshall. 1999. *Ancient Egyptian Science: A Source Book.* Vol. 3, *Ancient Egyptian Mathematics.* Philadelphia: American Philosophical Society.

Colebrooke, Henry Thomas. 1817. *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhascara.* London: John Murray.

Datta, B., and A. N. Singh., 1935/38. *History of Hindu Mathematics: A Source Book.* 2 vols. Bombay: Asia Publishing House.

Hilprecht, Hermann V. 1906. *Mathematical, Metrological and Chronological Tablets from the Temple Library of Nippur* Philadelphia: University Museum of Pennsylvania.

Høyrup, Jens. 1996. "Changing Trends in the Historiography of Mesopotamian Mathematics: An Insider's View." *History of Science* 34: 1–32.

Kern, Hendrik. 1874. *The Aryabhatiya, with the Commentary Bhatadipika of Paramadicyara.* Leiden: Brill.

Kline, Morris. 1972. *Mathematical Thought from Ancient to Modern Times.* New York: Oxford University Press.

Mikami, Yoshio. 1913. *The Development of Mathematics in China and Japan.* Leipzig: Teubner.

Needham, Joseph. 1959. *Science and Civilization in China.* Vol. 3, *Mathematics and the Sciences of the Heavens and the Earth.* Cambridge: Cambridge University Press.

Neugebauer, Otto. 1934. *Vorlesungen über Geschichte der antiken Mathematischen Wissenschaften. I. Vorgriechische Mathematik.* Berlin: Springer-Verlag.

———. 1935–37. *Mathematische Keilschrift-Texte.* Berlin: Springer-Verlag.

———. 1945. *Mathematical Cuneiform Texts* (with Abraham Sachs). New Haven: American Oriental Society.

———. 1951. *The Exact Sciences in Antiquity.* Princeton: Princeton University Press.

Peet, Thomas. 1923. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Introduction, Transcription, Translation and Commentary.* London.

Ritter, James. 2000. "Egyptian Mathematics." In Helaine Selin, ed., *Mathematics Across Cultures: The History of Non-Western Mathematics.* Dordrecht: Kluwer.

Smith, David Eugene. 1907. "The Mathematical Tablets of Nippur." *Bulletin of the American Mathematical Society* 13: 392–398.

———. 1923. *History of Mathematics.* 2 vols. Boston: Ginn.

Struve, W. W. 1930. *Mathematische Papyrus des Staatlichen Museums der Schönen Künste in Moskau.* Quellen und Studien zur Geschichte der Mathematik A1. Berlin: Springer-Verlag.

Suter, Heinrich. 1986. *Beiträge zur Geschichte der Mathematik und Astronomie im Islam.* 2 vols. Frankfurt: Institut für Geschichte der arabisch-islamischen Wissenschaften.

Thibaut, George. 1875. "On the Sulba-sutra." *Journal of the Asiatic Society of Bengal.* 44: 227–275.

Thureau-Dangin, François. 1983. *Textes mathématiques babyloniens* (Ex Oriente Lux 1). Leiden: Brill.

Turaev, Boris. A. 1917. "The Volume of the Truncated Pyramid in Egyptian Mathematics." *Ancient Egypt* 3: 100–102.

Van der Waerden, B. L. 1954. *Science Awakening*. New York: Oxford University Press.

Whish, Charles. 1835. "On the Hindu quadrature of a circle." *Transactions of the Royal Asiatic Society of Great Britain and Ireland* 3: 509–23.

Woepcke, Franz. 1986. *Etudes sur les mathématiques arabo-islamiques.* 2 vols. Frankfurt: Institut für Geschichte der arabisch-islamischen Wissenschaften.

Wylie, Alexander. 1852. "Jottings on the Science of the Chinese Arithmetic." *North China Herald*, Aug.–Nov. nos. 108–13, 116–17, 119–21.

Yushkevich, A. P. 1961. *The History of Mathematics in the Middle Ages* (in Russian). Moscow: Fizmatgiz.

———. 1964. *Geschichte der Mathematik in Mittelalter*, Leipzig: Teubner.

———. 1976. *Les mathématiques arabes* (*VIIIe –XVe siècles*). Paris: Vrin.

1 Egyptian Mathematics

Annette Imhausen

Preliminary Remarks

The study of Egyptian mathematics is as fascinating as it can be frustrating. The preserved sources are enough to give us glimpses of a mathematical system that is both similar to some of our school mathematics, and yet in some respects completely different. It is partly this similarity that caused early scholars to interpret Egyptian mathematical texts as a lower level of Western mathematics and, subsequently, to "translate" or rather transform the ancient text into a modern equivalent. This approach has now been widely recognized as unhistorical and mostly an obstacle to deeper insights. Current research attempts to follow a path that is sounder historically and methodologically. Furthermore, writers of new works can rely on progress that has been made in Egyptology (helping us understand the language and context of our texts) as well as in the history of mathematics.

However, learning about Egyptian mathematics will never be an easy task. The main obstacle is the shortage of sources. It has been over 70 years since a substantial new Egyptian mathematical text was discovered. Consequently, we must be extremely careful with our general evaluation of Egyptian mathematics. If we arbitrarily chose six mathematical publications of the past 300 years, what would we be able to say about mathematical achievements between 1700 and 2000 CE? This is exactly our situation for the mathematical texts of the Middle Kingdom (2119–1794/93 BCE). On the positive side, it must be said that the available source material is as yet far from being exhaustively studied, and significant and fascinating new insights are still likely to be gained. Also, the integration of other texts that contain mathematical information helps to fill out the picture. The understanding of Egyptian mathematics depends on our knowledge of the social and cultural context in which it was created, used, and developed. In recent years, the use of other source material, which contains direct or indirect information about Egyptian mathematics, has helped us better understand the extant mathematical texts.

This chapter presents a selection of sources and introduces the characteristic features of Egyptian mathematics. The selection is taken from over 3000 years of history. Consequently, the individual examples have to be taken within their specific context. The introduction following begins with a text *about* mathematics from the New Kingdom (1550–1070/69 BCE) to illustrate the general context of mathematics within Egyptian culture.

To introduce this text, we need to bear in mind that the development and use of mathematical techniques began around 1500 years earlier with the invention of writing and number systems. The available evidence points to administrative needs as the motivation for this development. Quantification and recording of goods also necessitated the development of metrological systems, which can be attested as early as the Old Kingdom and possibly earlier. Metrological systems and mathematical techniques were used and developed by the scribes, that is, the officials working in the administration of Egypt. Scribes were crucial to ensuring the smooth collection and distribution of available goods, thus providing the material basis for a prospering government under the pharaoh. Evidence for mathematical techniques comes from the education and daily work life of these scribes. The most detailed information can be gained from the so-called mathematical texts, papyri that were used in the education of junior scribes. These papyri contain collections of problems and their solutions to prepare the scribes for situations they were likely to face in their later work.

The mathematical texts inform us first of all about different types of mathematical problems. Several groups of problems can be distinguished according to their subject. The majority are concerned with topics from an administrative background. Most scribes were probably occupied with tasks of this kind. This conclusion is supported by illustrations found on the walls of private tombs. Very often, in tombs of high officials, the tomb owner is shown as an inspector in scenes of accounting of cattle or produce, and sometimes several scribes are depicted working together as a group. It is in this practical context that mathematics was developed and practised. Further evidence can be found in three-dimensional models representing scenes of daily life, which regularly include the figure of one or more scribes. Several models depict the filling of granaries, and a scribe is always present to record the respective quantities.

While mathematical papyri are extant from two separate periods only, depictions of scribes as accountants (and therefore using mathematics) are evident from all periods beginning with the Old Kingdom. Additional evidence for the same type of context for mathematics appears during the New Kingdom in the form of literary texts about the scribal profession. These texts include comparisons of a scribe's duties to duties of other professions (soldier, cobbler, farmer, etc.). It is clear that many of the scribes' duties involve mathematical knowledge. The introduction begins with a prominent example from this genre.

Another (and possibly the only other) area in which mathematics played an important role was architecture. Numerous extant remains of buildings demonstrate a level of design and construction that could only have been achieved with the use of mathematics. However, which instruments and techniques were used is not known nor always easy to discern. Past historiography has tended to impose modern concepts on the available material, and it is only recently that a serious reassessment of this subject has been published.[1] Again, detailed accounts of mathematical techniques related to architecture are only extant from the Middle Kingdom on. However, a few sketches from the Old Kingdom have survived as well, which indicate that certain mathematical concepts were present or being developed. These concepts then appeared fully formed in the mathematical texts.

Throughout this chapter Egyptian words appear in what Egyptologists call "transcription." The Egyptian script noted only consonants (although we pronounce some of them as vowels today). For this reason, transcribing hieratic or hieroglyphic texts means to transform the

[1]See [Rossi 2004].

text into letters which are mostly taken from our alphabet and seven additional letters (*ȝ*, *ʿ*, *ḥ*, *ḫ*, *ẖ*, *ṯ*, *ḏ*). In order to be able to read Egyptian, Egyptologists therefore agreed on the convention to insert (in speaking) short "e" sounds where necessary. The pronunciation of the Egyptian transcription alphabet is given below. This is a purely modern convention—how Egyptian was pronounced originally is not known. The Appendix contains a glossary of all Egyptian words in this chapter and their (modern) pronunciation.

Letter	Pronunciation
ȝ	ă
j	i or j
ʿ	ā
w	ŭ, ū, or w
b	b
p	p
f	f
m	m
n	n
r	r
h	h
ḥ	like Arabic ḥ
ḫ	like ch in "loch"
ẖ	like German ch in "ich"
z	voiced s
s	unvoiced s
š	sh
q	emphatic k
k	k
g	g
t	t
ṯ	like ch in "touch"
d	d
ḏ	dg as in "judge"

I. Introduction

The passage below is taken from Papyrus Anastasi I,[2] an Egyptian literary text of the New Kingdom (1550–1070/69 BCE). This composition is a fictional letter, which forms part of a debate between two scribes. The letter begins, as is customary for Egyptian letters, with the writer Hori introducing himself and then addressing the scribe Amenemope (by the shortened form Mapu). After listing the necessary epithets and wishing the addressee well, Hori recounts receiving a letter of Amenemope, which Hori describes as confused and insulting. He then

[2]The hieroglyphic transcription of the various extant sources can be found in [Fischer-Elfert 1983]. An English translation of the complete text is [Gardiner 1911]; this however is based on only ten of the extant 80 sources. The *editio princeps* is [Fischer-Elfert 1986]. The translation given here is my own, which is based on the work by Fischer-Elfert.

proposes a scholarly competition covering various aspects of scribal knowledge. The letter ends with Hori criticizing the letter of his colleague again and suggesting to him that he should sit down and think about the questions of the competition before trying to answer them.

The mathematical section of the letter, translated below, comprises several problems similar to the collections of problems found in mathematical papyri. However, in this letter, the problems are framed by Hori's comments (and sometimes insults), addressed to his colleague Amenemope (Mapu). Hori points out several times the official position which Amenemope claims for himself ("commanding scribe of the soldiers," "royal scribe") and teases him by calling him ironically "vigilant scribe," "scribe keen of wit," "sapient scribe," directly followed by a description of Amenemope's ineptness. In between, Hori describes several situations in which Amenemope is required to use his mathematical knowledge.

Note that while in each case the general problem is easy to grasp, there is not enough information, in fact, for a modern reader to solve these mathematical problems. This is partly due to philological difficulties: even after two editions the text is still far from fully understood. The choice of this extract as the first source text is mainly meant to illustrate the social and cultural context of mathematics in ancient Egypt.

Papyrus Anastasi I, 13,4–18,2

Another topic

Look, you come here and fill me with (the importance of) your office. I will let you know your condition when you say: "I am the commanding scribe of the soldiers." It has been given to you to dig a lake. You come to me to ask about the rations of the soldiers. You say to me: "Calculate it!" I am thrown into your office. Teaching you to do it has fallen upon my shoulders.

I will cause you to be embarrassed, I will explain to you the command of your master—may he live, prosper, and be healthy. Since you are his royal scribe, you are sent under the royal balcony for all kinds of great monuments of Horus, the lord of the two lands. Look, you are the clever scribe who is at the head of the soldiers.

A ramp shall be made of (length) 730 cubits, width 55 cubits, with 120 compartments, filled with reeds and beams. For height: 60 cubits at its top to 30 cubits in its middle, and an inclination (*sqd*) of 15 cubits, its base 5 cubits. Its amount of bricks needed shall be asked from the overseer of the troops. All the assembled scribes lack someone (i.e., a scribe) who knows them (i.e., the number of bricks). They trust in you, saying: "You are a clever scribe my friend. Decide for us quickly. Look, your name has come forward. One shall find someone in this place to magnify the other thirty. Let it not be said of you: there is something that you don't know. Answer for us the number (lit. its need) of bricks." Look, its measurements are before you. Each one of its compartments is of 30 cubits (in length) and a width of 7 cubits.

Hey Mapu, vigilant scribe, who is at the head of the soldiers, distinguished when you stand at the great gates, bowing beautifully under the balcony. A dispatch has come from the crown prince to the area of Ka to please the heart of the Horus of Gold, to calm the raging lion. An obelisk has been newly made, graven with the name of his majesty—may he live, prosper, and be healthy—of 110 cubits in the length of its shaft, its pedestal of 10 cubits, the circumference of its base makes 7 cubits on all its sides, its narrowing towards the summit 1 cubit, its pyramidion 1 cubit in height, its point 2 digits.

Add them up in order to make it from parts. You shall give every man to its transport, those who shall be sent to the Red Mountain. Look, they are waiting for them. Prepare the way for the crown prince Mes-Iten. Approach; decide for us the amount of men who will be in front of it. Do not let them repeat writing while the monument is in the quarry. Answer quickly, do not hesitate! Look, it is you who is looking for them for yourself. Get going! Look, if you hurry, I will cause your heart to rejoice.

I used to [...] under the top like you. Let us fight together. My heart is apt, my fingers listen. They are clever, when you go astray. Go, don't cry, your helper is behind you. I let you say: "There is a royal scribe with Horus, the mighty bull." And you shall order men to make chests into which to put letters that I will have written you secretly. Look, it is you who shall take them for yourself. You have caused my hands and fingers to be trained like a bull at a feast until every feast in eternity.

You are told: "Empty the magazine that has been loaded with sand under the monument for your lord—may he live, prosper, and be healthy—which has been brought from the Red Mountain. It makes 30 cubits stretched upon the ground with a width of 20 cubits, passing chambers filled with sand from the riverbank. The walls of its chambers have a breadth of 4 to 4 to 4 cubits. It has a height of 50 cubits in total. [...] You are commanded to find out what is before it. How many men will it take to remove it in 6 hours if their minds are apt? Their desire to remove it will be small if (a break at) noon does not come. You shall give the troops a break to receive their cakes, in order to establish the monument in its place. One wishes to see it beautiful.

O scribe, keen of wit, to whom nothing whatsoever is unknown. Flame in the darkness before the soldiers, you are the light for them. You are sent on an expedition to Phoenicia at the head of the victorious army to smite those rebels called Nearin. The bow-troops who are before you amount to 1900, Sherden 520, Kehek 1600, Meshwesh <100>, Tehesi 880, sum 5000 in all, not counting their officers. A complimentary gift has been brought to you and placed before you: bread, cattle, and wine. The number of men is too great for you: the provision is too small for them. Sweet Kemeh bread: 300, cakes: 1800, goats of various sorts: 120, wine: 30. The troops are too numerous; the provisions are underrated like this what you take from them (i.e., the inhabitants). You receive (it); it is placed in the camp. The soldiers are prepared and ready. Register it quickly, the share of every man to his hand. The Bedouins look on in secret. O learned scribe, midday has come, the camp is hot. They say: 'It is time to start'. Do not make the commander angry! Long is the march before us. What is it, that there is no bread at all? Our night quarters are far off. What is it, Mapu, this beating we are receiving (lit. of us)? Nay, but you are a clever scribe. You cease to give (us) food when only one hour of the day has passed? The scribe of the ruler—may he live, prosper, and be healthy—is lacking. Were you brought to punish us? This is not good. If Pa-Mose hears of it, he will write to degrade you."

The extract above shows that mathematics constituted an important part in a scribe's education and daily life. Furthermore, it illustrates the kind of mathematics that was practiced in Egypt. The passages cited refer to mathematical knowledge that a scribe should have in order to handle his daily work: accounting of grain, land, and labor in pharaonic Egypt. There have been several attempts to reconstruct actual mathematical exercises from the examples referred to in this source. All of them have met difficulties, which are caused not only by the numerous philological problems but also by the fact that the problems are deliberately "underdetermined." These examples were not intended to be actual mathematical problems that the Egyptian reader (i.e., scribe) should solve, but they were meant to remind him of types of mathematical problems he encountered in his own education.

TIMELINE [a]	EXTANT MATHEMATICAL TEXTS [b]	SCRIPT
Archaic Period Dyn. 1–2 (c. 300–2686 BCE)		
Old Kingdom Dyn. 3–8 (2686–2160 BCE)		HIERATIC
First Intermediate Period Dyn. 9–10 (2160–2025 BCE)		
Middle Kingdom Dyn. 11–12 (2025–1773 BCE)	***pMoscow (E4676)*** ***Math. Leather Roll (BM10250)*** ***Lahun Fragments pBerlin 6619*** ***Cairo Wooden Boards pRhind (BM10057–8)***	
Second Intermediate Period Dyn. 13–17 (1773–550 BCE		
New Kingdom Dyn. 18–20 (1550–1069 BCE)	***Ostracon Senmut 153*** ***Ostracon Turin 57170***	
Third Intermediate Period Dyn. 21-25 (1069–656 BCE)		
Late Period Dyn. 26–31 (664–332 BCE)		
Greek/Roman Period (332 BCE–395 CE)	*pCairo JE 89127–30* *PCairo JE 89137–43* *pBM 10399* *pBM10520* *pBM10794* *pCarlsberg 30*	DEMOTIC

[a]Dates according to [Shaw 2000].
[b]In this column Hieratic texts are listed in bold and italic, while Demotic texts are listed in italic.

Educational texts are the main source of our knowledge today about Egyptian mathematics. As already mentioned, there are very few sources available. These are listed in the table above. (Note that only mathematical texts, i.e., texts which teach mathematics, are included here, and therefore *pAnastasi I* is not listed.) Egyptian mathematical texts belong to two distinct groups: table texts and problem texts. Examples of both groups will be presented in this chapter. These are complemented by administrative texts that show mathematical practices in daily life.

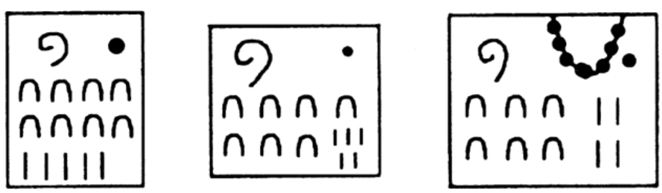
The following paragraphs present the Egyptian number system, arithmetical techniques, Egyptian fraction reckoning, and metrology, in order to make the sources more easily accessible.

I.a. Invention of writing and number systems

The earliest evidence of written texts in Egypt at the end of the fourth millennium BCE consists of records of names (persons and places) as well as commodities and their quantities. They show the same number system as is used in later times in Egypt, a decimal system without positional notation, i.e. with a new sign for every power of 10:

1 10 100 1000 10,000 100,000 1,000,000

Naqada tablets CG 14101, 14102, 14103



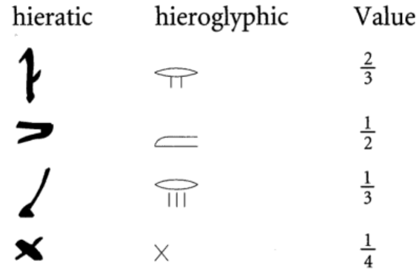
14101 14102 14103

These predynastic tablets were probably attached to some commodity (there is a hole in each of the tablets), and represented a numeric quantity related to this commodity. The number written on the first tablet is 185; the sign for 100 is written once, followed by the sign for 10 eight times, and the sign for 1 five times. The second tablet shows the number 175, and the third tablet 164. In addition, a necklace is drawn on the third tablet. This is interpreted as a tablet attached to a necklace of 164 pearls.

Parallel with the hieroglyphic script, which throughout Egyptian history was mainly used on stone monuments, a second, simplified script evolved, written with ink and a reed pen on papyrus, ostraca, leather, or wood. This cursive form of writing is known as "hieratic script." The individual signs often resemble their hieroglyphic counterparts. Over time the hieratic script became more and more cursive, and groups of signs were combined into so-called ligatures.

Hieroglyphic script could be written in any direction suitable to the purpose of the inscription, although the normal direction of writing is from right to left. Thus, the orientation of the individual symbols, such as the glyph for 100, varies. Compare the glyph for 100 in the table above and in the illustrated tablets. Hieratic, however, is always written from right to left. While hieroglyphic script is highly standardized, hieratic varies widely depending on the handwriting of the individual scribe. Therefore, it is customary in Egyptology to provide a hieroglyphic transcription of the hieratic source text.

Notation for fractions: Egyptian mathematics used unit fractions (i.e., $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc.) almost exclusively; the single exception is $\frac{2}{3}$. In hieratic, the number that is the denominator is written with a dot above it to mark it as a fraction. In hieroglyphic writing the dot is replaced by the hieroglyph ⊂⊃ ("part"). The most commonly used fractions $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ were written by special signs:



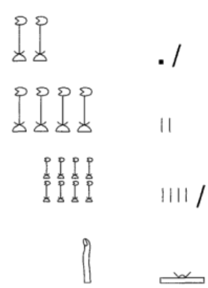
More difficult fractions like $\frac{3}{4}$ or $\frac{5}{6}$ were represented by sums of unit fractions written in direct juxtaposition, e.g., $\frac{3}{4} = \frac{1}{2}\,\frac{1}{4}$ (hieroglyphic ⊂ ×); $\frac{5}{6} = \frac{2}{3}\,\frac{1}{6}$ (hieroglyphic ⊂⊃ ⊂⊃III). In transcription, fractions are rendered by the denominator with an overbar, e.g., $\frac{1}{2}$ is written as $\bar{2}$. The fraction $\frac{2}{3}$ is written as $\bar{\bar{3}}$.

I.b. Arithmetic

Calculation with integers: the mathematical texts contain terms for addition, subtraction, multiplication, division, halving, squaring, and the extraction of a square root. Only multiplication and division were performed as written calculations. Both of these were carried out using a variety of techniques the choice of which depended on the numerical values involved. The following example of the multiplication of 2000 and 5 is taken from a problem of the *Rhind Mathematical Papyrus* (remember that the hieratic original, and therefore this hieroglyphic transcription, are read from right to left):

Rhind Mathematical Papyrus, problem 52

\ .	2000
2	4000
\ 4	8000
Total	10,000



The text is written in two columns. It starts with a dot in the first column and the number that shall be multiplied in the second column. The first line is doubled in the second line.

Therefore we see "2" in the first column and "4000" in the second column, the third line is twice the second ("4" in the first column, "8000" in the second column).

The first column is then searched for numbers that add up to the multiplicative factor 5 (the dot in the first line counts as "1"). This can be achieved in this example by adding the first and third lines. These lines are marked with a checkmark (\). The result of the multiplication is obtained by adding the marked lines of the second column. If the multiplicative factor exceeds 10, the procedure is slightly modified, as can be followed in the example below from problem 69 of the *Rhind Mathematical Papyrus*. The multiplication of 80 and 14 is performed as follows:

Rhind Mathematical Papyrus, problem 69

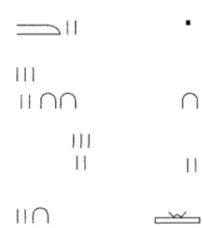
.	80
\ 10	800
2	160
\ 4	320
Total	1120

After the initial line, we move directly to 10; then the remaining lines are carried out in the usual way, starting with double of the first line.

Divisions are performed in exactly the same way, with the roles of first and second column switched. The following example is taken from problem 76 of the *Rhind Mathematical Papyrus*. The division that is performed is $30 \div 2\frac{1}{2}$.

Rhind Mathematical Papyrus, problem 76

.	$2\bar{2}$
\ 10	25
\ 2	5
Total	12



Again we find two columns. This time the divisor is subsequently either doubled or multiplied by 10. Then the second column is searched for numbers that add up to the dividend 30. The respective lines are marked. The addition of the first column of these lines leads to the result of the division.

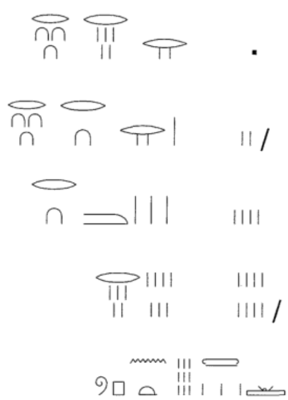
Calculation with fractions: the last example of the division included a fraction ($2\bar{2}$); however, in this example it had little effect on the performance of the operation. From previous examples of multiplication (and division) it is obvious that doubling is an operation which has to be performed frequently. If fractions are involved, the fraction has to be doubled. If the fraction is a single unit fraction with an even denominator, halving the denominator easily does this, e.g., double of $\overline{64}$ is $\overline{32}$ ($\frac{2}{64} = \frac{1}{32}$). If, however, the denominator is odd (or a series of unit fractions is to be doubled), the result is not as easily found. For this reason the so-called $2 \div N$ table was created. This table lists the doubles of odd unit fractions. Examples of this table can be found in the section of table texts following this introduction. Obviously, the level of difficulty in carrying out these operations usually rises considerably as soon as fractions are involved.

The layout of a multiplication with fractions is the same as the layout of the multiplication of integers. The following example—the result of which is unfortunately partly destroyed—is taken from problem 6 of the *Rhind Mathematical Papyrus*, the multiplication of $\bar{\bar{3}}\ \bar{5}\ \overline{30}$ with 10.

Rhind Mathematical Papyrus, problem 6

·	$\bar{\bar{3}}\ \bar{5}\ \overline{30}$
\2	$1\bar{\bar{3}}\ \overline{10}\ \overline{30}$
4	$3\bar{2}\ \overline{10}$
\8	$7\bar{5}$

Total 9 loaves of bread. This is it.

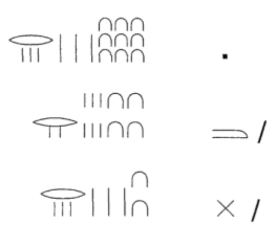


Note that the multiplication with ten in this case is not performed directly, but explicitly carried out through doubling and addition.

Divisions with a divisor greater than the dividend use a series of halvings starting either with $\frac{2}{3}$ ($\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \ldots$) or with $\frac{1}{2}$ ($\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$). For instance the division $70 \div 93\frac{1}{3}$ in problem 58 of the *Rhind Mathematical Papyrus* is performed as follows:

Rhind Mathematical Papyrus, problem 58

·	$93\ \bar{3}$
\ $\bar{2}$	$46\ \bar{\bar{3}}$
\ $\bar{4}$	$23\ \bar{3}$
[Total	$\bar{2}\ \bar{4}$]



In more difficult numerical cases the division is first carried out as a division with remainder. The remainder is then handled separately.

I.c. Metrology

Note that the following overview is by no means a complete survey of Egyptian metrology, but includes only those units which are used in the sources of this chapter.

The approximate values given here are derived from the approximation that 1 cubit ≈ 52.5 cm, which is used in standard textbooks. It must be noted however, that this was determined as an average of cubit rods of so-called votive cubits, that is, cubits that have been placed in a tomb or temple as ritual objects. (Some other cubit rods have been unearthed that bear signs of actually having been used by architects and workers.) A valid "standard cubit" throughout Egypt did not exist. Naturally, the same holds for area and volume measures.

Length measures

1 *ḫt*	= 100 cubits	≈ 52.5 m
1 cubit	= 7 palms	≈ 52.5 cm
1 palm	= 4 digits	≈ 7.5 cm
1 digit		≈ 19 mm

Area measures

1 *ḫꜣ-tꜣ*	= 10 *sṯꜣ.t*	≈ 27562.5 m^2
1 *sṯꜣ.t*	= (1 *ḫt*)^2	≈ 2756.25 m^2
1 area-cubit	= 1 cubit × 100 cubit	≈ 27.56 m^2

Volume measures

1 *ẖꜣr*	= 16 *ḥqꜣ.t*	≈ 76.8 l[3]
1 *ẖꜣr*	= 20 *ḥqꜣ.t* = 2/3 cubic-cubit	≈ 96.5 l[4]
1 *ḥqꜣ.t*	= 10 *hnw*	≈ 4.8 l
1 *hnw*	= 32 *rꜣ*	≈ 0.48 l
1 *rꜣ*		≈ 15 ml

II. Hieratic Mathematical Texts

Egyptian mathematical texts can be assigned to two groups: table texts and problem texts. Table texts include tables for fraction reckoning (e.g., the 2 ÷ *N* table, which will be the first source text below, and the table found on the *Mathematical Leather Roll*) as well as tables for the conversion of measures (e.g., *Rhind Mathematical Papyrus*, Nos. 47, 80, and 81). Problem texts state a mathematical problem and then indicate its solution by means of step-by-step instructions. For this reason, they are also called procedure texts.

---

[3]This is the New Kingdom value, from Dynasty 20 onward. In the Old Kingdom and Middle Kingdom, the value was 1 *ẖꜣr* = 10 *ḥqꜣ.t*.

[4]This is the value found in the *Rhind Mathematical Papyrus*.

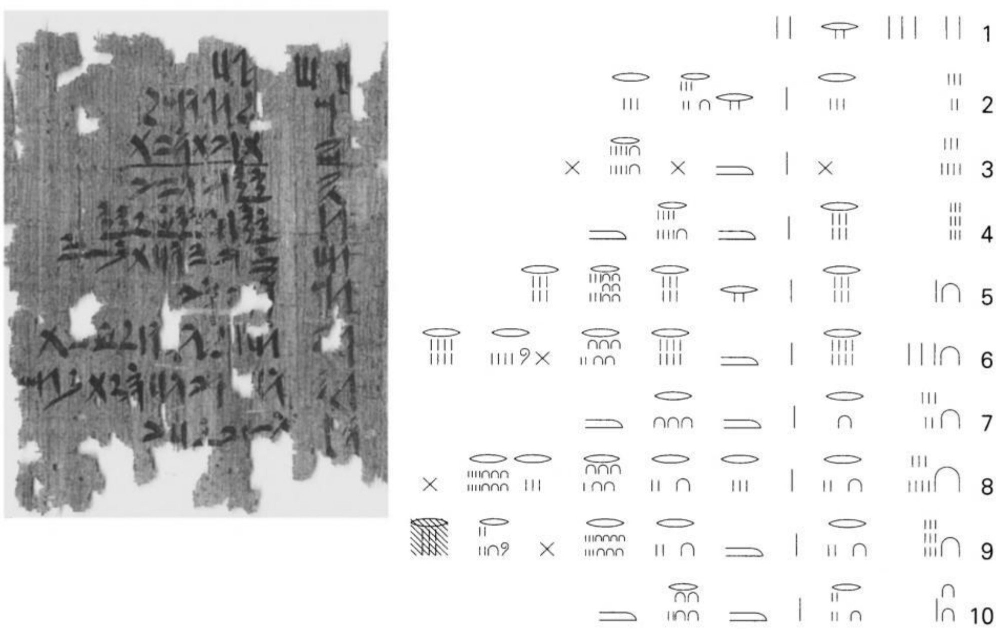
The extant hieratic source texts (in order of their publication) are

- *Rhind Mathematical Papyrus* (BM 10057–10058)
- *Lahun Mathematical Fragments* (7 fragments: UC32114, UC32118B, UC32134, UC32159–32162)
- *Papyrus Berlin* 6619 (2 fragments)
- *Cairo Wooden Boards* (CG 25367 and 25368)
- *Mathematical Leather Roll* (BM 10250)
- *Moscow Mathematical Papyrus* (E4674)
- *Ostracon Senmut* 153
- *Ostracon Turin* 57170

Most of these texts were bought on the antiquities market, and therefore we do not know their exact provenance. An exception is the group of mathematical fragments from Lahun, which were discovered by William Matthew Flinders Petrie when he excavated the Middle Kingdom pyramid town of Lahun.

II.a. Table texts

UC 32159



Reprinted by permission of Petrie Museum of Eqyptian Archaeology, University College, London.

The photograph shows a part of the so-called 2 ÷ *N* table from one of the Lahun fragments. The hieroglyphic transcription of the fragment on the photo is given next to it. This table was used to aid fraction reckoning. Remember that Egyptian fraction reckoning used only unit fractions and the fraction $\frac{2}{3}$. As multiplication consisted of repeated doubling, multiplication of fractions often involved the doubling of fractions. This can easily be done if the denominator is even. To double a unit fraction with an even denominator, its denominator has to be halved, e.g., $2 \times \frac{1}{8} = \frac{1}{4}$.

However the doubling of a fraction with an odd denominator always consists of a series of two or more unit fractions, which are not self-evident. Furthermore, there are often several possible representations; however, Egyptian mathematical texts consistently used only one, which can be found in the $2 \div N$ table. Below is the transcription of our example into numbers:

	Column I	Column II					
1	2 3	$\overline{\overline{3}}$	2				
2	5	$\overline{3}$	$1\overline{\overline{3}}$	$\overline{15}$	$\overline{3}$		
3	7	$\overline{4}$	$1\overline{2}\,\overline{4}$	$\overline{28}$	$\overline{4}$		
4	9	$\overline{6}$	$1\overline{2}$	$\overline{18}$	$\overline{2}$		
5	11	$\overline{6}$	$1\overline{\overline{3}}\,\overline{6}$	$\overline{66}$	$\overline{6}$		
6	13	$\overline{8}$	$1\overline{2}\,\overline{8}$	$\overline{52}$	$\overline{4}$	$\overline{104}$	$\overline{8}$
7	15	$\overline{10}$	$1\overline{2}$	$\overline{30}$	$\overline{2}$		
8	17	$\overline{12}$	$1\overline{3}\,\overline{12}$	$\overline{51}$	$\overline{3}$	$\overline{68}$	$\overline{4}$
9	19	$\overline{12}$	$1\overline{2}\,\overline{12}$	$\overline{76}$	$\overline{4}$	$\overline{114}$	[$\overline{6}$]
10	21	$\overline{14}$	$1\overline{2}$	$\overline{42}$	$\overline{2}$		

The numbers are grouped in two columns. The first column contains the divisor N (in the first line only it shows both dividend 2 and divisor 3). The second column shows alternatingly fractions of the divisor and their value (as a series of unit fractions). For example, the second line starts with the divisor 5 in the first column; therefore it is $2 \div 5$ that is expressed as a series of unit fractions. It is followed in the second column by $\overline{3}$, $1\overline{\overline{3}}$, $\overline{15}$, and $\overline{3}$. This has to be read as $\overline{3}$ of 5 is $1\overline{\overline{3}}$ and $\overline{15}$ of 5 is $\overline{3}$. Since $1\overline{\overline{3}}$ plus $\overline{3}$ equals 2, the series of unit fractions to represent $2 \div 5$ is $\overline{3}\;\overline{15}$.

The Recto of the *Rhind Mathematical Papyrus* contains the $2 \div N$ table for $N = 3$ to $N = 101$. Here, the solutions are marked in red ink, rendered as **bold** in the transcription below. There have been several attempts to explain the choices of representations in the $2 \div N$ table. These attempts were mostly based on modern mathematical formulas, and none of them gives a convincing explication of the values we find in the table. It is probable that the table was constructed based on experiences in handling fractions. Several "guidelines" for the selection of suitable fractions can be discerned. The author tried to keep the number of fractions to represent $2 \div N$ small; we generally find representations composed of two or three fractions only. Another guiding rule seems to be the choice of fractions with a small denominator over a bigger denominator, and the choice of denominators that can be decomposed into several components.

Rhind Mathematical Papyrus, 2 ÷ N Table

N	*2 ÷ N*	*N*	*2 ÷ N*
3	$\overline{\overline{\mathbf{3}}}$ 2	53	$\overline{\mathbf{30}}$ 1 $\overline{\overline{3}}$ $\overline{10}$ $\overline{\mathbf{318}}$ $\overline{6}$ $\overline{\mathbf{795}}$ $\overline{15}$
5	$\overline{\mathbf{3}}$ 1 $\overline{\overline{3}}$ $\overline{\mathbf{15}}$ $\overline{3}$	55	$\overline{\mathbf{30}}$ 1 $\overline{\overline{3}}$ $\overline{6}$ $\overline{\mathbf{330}}$ $\overline{6}$
7	$\overline{\mathbf{4}}$ 1 $\overline{2}$ $\overline{4}$ $\overline{\mathbf{28}}$ $\overline{4}$	57	$\overline{\mathbf{38}}$ 1 $\overline{2}$ $\overline{\mathbf{114}}$ $\overline{2}$
9	$\overline{\mathbf{6}}$ 1 $\overline{2}$ $\overline{\mathbf{18}}$ $\overline{2}$	59	$\overline{\mathbf{36}}$ 1 $\overline{2}$ $\overline{12}$ $\overline{18}$ $\overline{\mathbf{236}}$ $\overline{4}$ $\overline{\mathbf{531}}$ $\overline{9}$
11	$\overline{\mathbf{6}}$ 1 $\overline{\overline{3}}$ $\overline{6}$ $\overline{\mathbf{66}}$ $\overline{6}$	61	$\overline{\mathbf{40}}$ 1 $\overline{2}$ $\overline{40}$ $\overline{\mathbf{244}}$ $\overline{4}$ $\overline{\mathbf{488}}$ $\overline{8}$ $\overline{\mathbf{610}}$ $\overline{10}$
13	$\overline{\mathbf{8}}$ 1 $\overline{2}$ $\overline{8}$ $\overline{\mathbf{52}}$ $\overline{4}$ $\overline{\mathbf{104}}$ $\overline{8}$	63	$\overline{\mathbf{42}}$ 1 $\overline{2}$ $\overline{\mathbf{126}}$ $\overline{2}$
15	$\overline{\mathbf{10}}$ 1 $\overline{2}$ $\overline{\mathbf{30}}$ $\overline{2}$	65	$\overline{\mathbf{39}}$ 1 $\overline{\overline{3}}$ $\overline{\mathbf{195}}$ $\overline{3}$
17	$\overline{\mathbf{12}}$ 1 $\overline{3}$ $\overline{12}$ $\overline{\mathbf{51}}$ $\overline{3}$ $\overline{\mathbf{68}}$ $\overline{4}$	67	$\overline{\mathbf{40}}$ 1 $\overline{2}$ $\overline{8}$ $\overline{20}$ $\overline{\mathbf{335}}$ $\overline{5}$ $\overline{\mathbf{536}}$ $\overline{8}$
19	$\overline{\mathbf{12}}$ 1 $\overline{2}$ $\overline{12}$ $\overline{\mathbf{76}}$ $\overline{4}$ $\overline{\mathbf{114}}$ $\overline{6}$	69	$\overline{\mathbf{46}}$ 1 $\overline{2}$ $\overline{\mathbf{138}}$ $\overline{2}$
21	$\overline{\mathbf{14}}$ 1 $\overline{2}$ $\overline{\mathbf{42}}$ $\overline{2}$	71	$\overline{\mathbf{40}}$ 1 $\overline{2}$ $\overline{4}$ $\overline{40}$ $\overline{\mathbf{568}}$ $\overline{8}$ $\overline{\mathbf{710}}$ $\overline{10}$
23	$\overline{\mathbf{12}}$ 1 $\overline{\overline{3}}$ $\overline{4}$ $\overline{\mathbf{276}}$ $\overline{12}$	73	$\overline{\mathbf{60}}$ 1 $\overline{6}$ $\overline{20}$ $\overline{\mathbf{219}}$ $\overline{3}$ $\overline{\mathbf{292}}$ $\overline{4}$ $\overline{\mathbf{365}}$ $\overline{5}$
25	$\overline{\mathbf{15}}$ 1 $\overline{\overline{3}}$ $\overline{\mathbf{75}}$ $\overline{3}$	75	$\overline{\mathbf{50}}$ 1 $\overline{2}$ $\overline{\mathbf{150}}$ $\overline{2}$
27	$\overline{\mathbf{18}}$ 1 $\overline{2}$ $\overline{\mathbf{54}}$ $\overline{2}$	77	$\overline{\mathbf{44}}$ 1 $\overline{2}$ $\overline{4}$ $\overline{\mathbf{308}}$ $\overline{4}$
29	$\overline{\mathbf{24}}$ 1 $\overline{6}$ $\overline{24}$ $\overline{\mathbf{58}}$ $\overline{2}$ $\overline{\mathbf{174}}$ $\overline{6}$ $\overline{\mathbf{232}}$ $\overline{8}$	79	$\overline{\mathbf{60}}$ 1 $\overline{4}$ $\overline{15}$ $\overline{\mathbf{237}}$ $\overline{3}$ $\overline{\mathbf{316}}$ $\overline{4}$ $\overline{\mathbf{790}}$ $\overline{10}$
31	$\overline{\mathbf{20}}$ 1 $\overline{2}$ $\overline{20}$ $\overline{\mathbf{124}}$ $\overline{4}$ $\overline{155}$ $\overline{5}$	81	$\overline{\mathbf{54}}$ 1 $\overline{2}$ $\overline{\mathbf{162}}$ $\overline{2}$
33	$\overline{\mathbf{22}}$ 1 $\overline{2}$ $\overline{\mathbf{66}}$ $\overline{2}$	83	$\overline{\mathbf{60}}$ 1 $\overline{3}$ $\overline{20}$ $\overline{\mathbf{332}}$ $\overline{4}$ $\overline{\mathbf{415}}$ $\overline{5}$ $\overline{\mathbf{498}}$ $\overline{6}$
35	$\overline{\mathbf{30}}$ 1 $\overline{6}$ $\overline{\mathbf{42}}$ $\overline{\overline{3}}$ $\overline{6}$	85	$\overline{\mathbf{51}}$ 1 $\overline{\overline{3}}$ $\overline{\mathbf{255}}$ $\overline{3}$
37	$\overline{\mathbf{24}}$ 1 $\overline{2}$ $\overline{24}$ $\overline{\mathbf{111}}$ $\overline{3}$ $\overline{\mathbf{296}}$ $\overline{8}$	87	$\overline{\mathbf{58}}$ 1 $\overline{2}$ $\overline{\mathbf{174}}$ $\overline{2}$
39	$\overline{\mathbf{26}}$ 1 $\overline{2}$ $\overline{\mathbf{78}}$ $\overline{2}$	89	$\overline{\mathbf{60}}$ 1 $\overline{3}$ $\overline{10}$ $\overline{20}$ $\overline{\mathbf{356}}$ $\overline{4}$ $\overline{\mathbf{534}}$ $\overline{6}$ $\overline{\mathbf{890}}$ $\overline{10}$
41	$\overline{\mathbf{24}}$ 1 $\overline{\overline{3}}$ $\overline{24}$ $\overline{\mathbf{246}}$ $\overline{6}$ $\overline{\mathbf{328}}$ $\overline{8}$	91	$\overline{\mathbf{70}}$ 1 $\overline{5}$ $\overline{10}$ $\overline{\mathbf{130}}$ $\overline{\overline{3}}$ $\overline{30}$
43	$\overline{\mathbf{42}}$ 1 $\overline{42}$ $\overline{\mathbf{86}}$ $\overline{2}$ $\overline{\mathbf{129}}$ $\overline{3}$ $\overline{\mathbf{301}}$ $\overline{7}$	93	$\overline{\mathbf{62}}$ 1 $\overline{2}$ $\overline{\mathbf{186}}$ $\overline{2}$
45	$\overline{\mathbf{30}}$ 1 $\overline{2}$ $\overline{\mathbf{90}}$ $\overline{2}$	95	$\overline{\mathbf{60}}$ 1 $\overline{2}$ $\overline{12}$ $\overline{\mathbf{380}}$ $\overline{4}$ $\overline{\mathbf{570}}$ $\overline{6}$
47	$\overline{\mathbf{30}}$ 1 $\overline{2}$ $\overline{15}$ $\overline{\mathbf{141}}$ $\overline{3}$ $\overline{\mathbf{470}}$ $\overline{10}$	97	$\overline{\mathbf{56}}$ 1 $\overline{2}$ $\overline{8}$ $\overline{14}$ $\overline{28}$ $\overline{\mathbf{679}}$ $\overline{7}$ $\overline{\mathbf{776}}$ $\overline{8}$
49	$\overline{\mathbf{28}}$ 1 $\overline{2}$ $\overline{4}$ $\overline{\mathbf{196}}$ $\overline{4}$	99	$\overline{\mathbf{66}}$ 1 $\overline{2}$ $\overline{\mathbf{198}}$ $\overline{2}$
51	$\overline{\mathbf{34}}$ 1 $\overline{2}$ $\overline{\mathbf{102}}$ $\overline{2}$	101	$\overline{\mathbf{101}}$ 1 $\overline{\mathbf{202}}$ $\overline{2}$ $\overline{\mathbf{303}}$ $\overline{3}$ $\overline{\mathbf{606}}$ $\overline{6}$

Mathematical Leather Roll

Column 1		Column 2		Column 3		Column 4	
$\overline{10}\ \overline{40}$	it is $\overline{8}$	$\overline{30}\ \overline{45}\ \overline{90}$	it is $\overline{15}$	$\overline{10}\ \overline{40}$	it is $\overline{8}$	$\overline{18}\ \overline{36}$	it is $\overline{12}$
$\overline{5}\ \overline{20}$	it is $\overline{4}$	$\overline{24}\ \overline{48}$	it is $\overline{16}$	$\overline{5}\ \overline{20}$	it is $\overline{4}$	$\overline{21}\ \overline{42}$	it is $\overline{14}$
$\overline{4}\ \overline{12}$	it is $\overline{3}$	$\overline{18}\ \overline{36}$	it is $\overline{12}$	$\overline{4}\ \overline{12}$	it is $\overline{3}$	$\overline{45}\ \overline{90}$	it is $\overline{30}$
$[\overline{10}]\ \overline{10}$	it is $\overline{5}$	$\overline{21}\ \overline{42}$	it is $\overline{14}$	$\overline{10}\ \overline{10}$	it is $\overline{5}$	$\overline{30}\ \overline{60}$	it is $\overline{20}$
$[\overline{6}\ \overline{6}]$	it is $\overline{3}$	$\overline{45}\ \overline{90}$	it is $\overline{30}$	$\overline{6}\ \overline{6}$	it is $\overline{3}$	$\overline{15}\ \overline{30}$	it is $\overline{10}$
$[\overline{6}\ \overline{6}\ \overline{6}]$	it is $\overline{2}$	$\overline{30}\ \overline{60}$	it is $\overline{20}$	$\overline{6}\ \overline{6}\ \overline{6}$	it is $\overline{2}$	$\overline{48}\ \overline{96}$	it is $\overline{32}$
$[\overline{3}\ \overline{3}]$	it is $\overline{\overline{3}}$	$\overline{15}\ \overline{30}$	it is $\overline{10}$	$\overline{3}\ \overline{3}$	it is $\overline{\overline{3}}$	$\overline{96}\ \overline{192}$	$\overline{64}$
$[\overline{25}]\ \overline{15}\ \overline{75}\ \overline{200}$	it is $\overline{8}$	$\overline{48}\ \overline{96}$	it is $\overline{32}$	$\overline{25}\ \overline{15}\ \overline{75}\ \overline{200}$	it is $\overline{8}$		
$[\overline{50}]\ \overline{30}\ \overline{150}\ \overline{400}$	it is $\overline{16}$	$\overline{96}\ \overline{192}$	$\overline{64}$	$\overline{50}\ \overline{30}\ \overline{150}\ \overline{400}$	it is $\overline{16}$		
$\overline{25}\ \overline{50}\ \overline{150}$	it is $\overline{6}$			$\overline{25}\ \overline{50}\ \overline{150}$	it is $\overline{6}$		
$[\overline{9}\ \overline{18}]$	it is $\overline{6}$			$\overline{9}\ \overline{18}$	it is $\overline{6}$		
$[\overline{7}\ \overline{14}]\ \overline{28}$	it is $\overline{4}$			$\overline{7}\ \overline{14}\ \overline{28}$	it is $\overline{4}$		
$[\overline{12}\ \overline{24}]$	it is $[\overline{8}]$			$\overline{12}\ \overline{24}$	it is $\overline{8}$		
$\overline{14}\ \overline{21}\ \overline{42}$	it is $[\overline{7}]$			$\overline{14}\ \overline{21}\ \overline{42}$	it is $\overline{7}$		
$[\overline{18}\ \overline{27}]\ \overline{54}$	it is $[\overline{9}]$			$\overline{18}\ \overline{27}\ \overline{54}$	it is $\overline{9}$		
$[\overline{12}\ \overline{33}]\ \overline{66}$	it is $[\overline{11}]$			$\overline{12}\ \overline{33}\ \overline{66}$	it is $\overline{11}$		
$[\overline{28}\ \overline{49}]\ \overline{196}$	it is $[\overline{13}]$			$\overline{28}\ \overline{49}\ \overline{196}$	it is $\overline{13}$		
				$\overline{30}\ \overline{45}\ \overline{90}$	it is $\overline{15}$		
				$\overline{2}[\overline{4}]\overline{4}[\overline{8}]$	it is $[\overline{1}]\overline{6}$		

The *Mathematical Leather Roll* is another aid for fraction reckoning. It contains 26 sums of unit fractions which equal a single unit fraction. The 26 sums have been noted in two columns, followed by another two columns with the same 26 sums. The numeric transcription given above shows the arrangement of the sums of the source.

Apart from fraction reckoning, tables were also needed for the conversion of different measuring units. An example of these tables can be found in the *Rhind Mathematical Papyrus*, No. 81. Here, two systems of volume measures, *ḥqꜣ.t* and *hnw*, are compared. *ḥqꜣ.t* is the basic measuring unit for grain, with 1 *ḥqꜣ.t* equaling 10 *hnw*. The *ḥqꜣ.t* was used with a system of submultiples, which were written by distinctive signs:

	$\frac{1}{2}$ *ḥqꜣ.t*
	$\frac{1}{4}$ *ḥqꜣ.t*
	$\frac{1}{8}$ *ḥqꜣ.t*
	$\frac{1}{16}$ *ḥqꜣ.t*
	$\frac{1}{32}$ *ḥqꜣ.t*
	$\frac{1}{64}$ *ḥqꜣ.t*

In older literature about Egyptian mathematics these signs are often interpreted as hieratic versions of the hieroglyphic parts of the eye of the Egyptian god Horus. However, texts from the early third millennium as well as depictions in tombs of the Old Kingdom, which show the same signs prove that the eye of Horus was not connected to the origins of the hieratic signs.[5] 1 *ḥqꜣ.t* also equals 32 *rꜣ*, the smallest unit for measuring volumes.

The table found in No. 81 of the *Rhind Mathematical Papyrus* is divided into three parts. Each part is introduced by an Egyptian particle (in the translation rendered as "now"). The first section of the table, arranged in two columns, lists the submultiples of the *ḥqꜣ.t* as *hnw*. Due to the values of the submultiples, each line is half of its predecessor. The following two sections are both laid out in three columns. The first column gives combinations of the submultiples of the *ḥqꜣ.t* and *rꜣ.w*. The second column lists the respective volume in *hnw*. The last column contains the volumes as fractions of the *ḥqꜣ.t*, this time not written in the style of submultiples but as a pure numeric fraction of the unit *ḥqꜣ.t*.

The source text of these last two sections shows a rather large number of errors. Out of 82 entries 11 are wrong. Some of these errors seem to be simple writing errors; some follow from using a faulty entry in a previous line or column to calculate the new entry. The table is given here with all the original (sometimes wrong) values followed by footnotes that give the correct value and—if possible—an explanation for the error. It is difficult to account for the large number of mistakes in this table. The *Rhind Mathematical Papyrus* (of which this table is a part) is a collection of tables and problems, mostly organized in a carefully thought out sequence. It was presumably the manual of a teacher.

[5]See [Ritter 2002] for a detailed discussion.

Rhind Mathematical Papyrus, No. 81

Another reckoning of the hnw

Now	$\overline{2}$	*ḥqꜣ.t*	5
	$\overline{4}$	*ḥqꜣ.t*	2 $\overline{2}$
	$\overline{8}$	*ḥqꜣ.t*	1 $\overline{4}$
	$\overline{16}$	*ḥqꜣ.t*	$\overline{2}\ \overline{8}$
	$\overline{32}$	*ḥqꜣ.t*	$\overline{4}\ \overline{16}$
	$\overline{64}$	*ḥqꜣ.t*	$\overline{8}\ \overline{32}$

Now	$\overline{2}\ \overline{4}\ \overline{8}$	*ḥqꜣ.t*		as *hnw*, it is 8 $\overline{2}\ \overline{4}$	
	$\overline{2}\ \overline{4}$	*ḥqꜣ.t*		· it is 7 $\overline{2}$	
	$\overline{2}\ \overline{8}\ \overline{32}$	*ḥqꜣ.t*	3 $\overline{3}$ *rꜣ.w*	· 6 $\overline{2}\ \overline{16}$[6]	it is $\overline{\overline{3}}$ of a *ḥqꜣ.t*
	$\overline{2}\ \overline{8}$	*ḥqꜣ.t*		· 6 $\overline{4}$	it is $\overline{5}$ of a *ḥqꜣ.t*[7]
	$\overline{4}\ \overline{8}$	*ḥqꜣ.t*		· 3 $\overline{2}\ \overline{4}$	it is 3 of a *ḥqꜣ.t*[8]
	$\overline{4}\ \overline{32}\ \overline{64}$	*ḥqꜣ.t*	1 $\overline{\overline{3}}$ *rꜣ.w*	· 3 $\overline{4}\ \overline{8}$[9]	it is $\overline{7}$ of a *ḥqꜣ.t*[10]
	$\overline{4}$	*ḥqꜣ.t*		· 2 $\overline{2}$	it is $\overline{4}$ of a *ḥqꜣ.t*
	$\overline{8}\ \overline{16}$	*ḥqꜣ.t*	4 *rꜣ.w*	· 2	it is $\overline{5}$ of a *ḥqꜣ.t*
	[$\overline{8}\ \overline{32}$]	*ḥqꜣ.t*	3 $\overline{3}$ *rꜣ.w*	· [1 $\overline{\overline{3}}$]	it is [$\overline{6}$ of] a *ḥqꜣ.t*

Now	$\overline{8}\ \overline{16}$	*ḥqꜣ.t*	4 *rꜣ.w*	it is 2 *hnw*	**it is $\overline{5}$ of a *ḥqꜣ.t***
	$\overline{16}\ \overline{32}$	*ḥqꜣ.t*	2 *rꜣ.w*	it is 1 *hnw*	**it is $\overline{10}$ of a *ḥqꜣ.t***
	$\overline{32}\ \overline{64}$	*ḥqꜣ.t*	1 *rꜣ.w*	it is $\overline{2}$ *hnw*	**it is $\overline{20}$ of a *ḥqꜣ.t***
	$\overline{64}$	*ḥqꜣ.t*	3 *rꜣ.w*	it is $\overline{4}$ *hnw*	**it is $\overline{40}$ of a *ḥqꜣ.t***
	$\overline{16}$	*ḥqꜣ.t*	1 $\overline{3}$ *rꜣ.w*	it is $\overline{\overline{3}}$ *hnw*	**it is $\overline{30}$ of a *ḥqꜣ.t***[11]
	$\overline{32}$	*ḥqꜣ.t*	$\overline{\overline{3}}$ *rꜣ.w*	it is $\overline{3}$ *hnw*	**it is $\overline{60}$ of a *ḥqꜣ.t***[12]

---

[6]Correct value: 6$\overline{\overline{3}}$ *hnw*. Possible explanation for the mistake: $\overline{2}\ \overline{8}\ \overline{32}$ *ḥqꜣ.t* = 6 $\overline{2}\ \overline{16}$ *hnw*, therefore it is likely that 3$\overline{3}$ *rꜣ.w* of the first column were forgotten when the second column was determined.

[7]Correct value: $\overline{2}\ \overline{8}$ of a *ḥqꜣ.t*.

[8]Correct value: $\overline{4}\ \overline{8}$ of a *ḥqꜣ.t*.

[9]Correct value: 3 $\overline{48}$ *hnw*. Possible explanation for the mistake: What I read as $\overline{4}$ may be a very badly written $\overline{40}$, but it seems more probable to read $\overline{4}$.

[10]Correct value: $\overline{4}\ \overline{32}\ \overline{48}$ of a *ḥqꜣ.t*.

[11]Correct value: $\overline{15}$ of a *ḥqꜣ.t*.

[12]Correct value: $\overline{30}$ of a *ḥqꜣ.t*. Possible explanation for the mistake: Calculation based on wrong entry in previous line of this column.

64	*ḥqꜣ.t*	3 *rꜣ.w*	it is 5 *hnw*[13]	**it is 50 of a *ḥqꜣ.t*[14]**
$\overline{2}$	*ḥqꜣ.t*		it is 5 *hnw*	**it is $\overline{2}$ of a *ḥqꜣ.t***
$\overline{4}$	*ḥqꜣ.t*		it is 2 $\overline{2}$ *hnw*	**it is $\overline{4}$ of a *ḥqꜣ.t***
$\overline{2}\ \overline{4}$	*ḥqꜣ.t*		it is 7 $\overline{2}$ *hnw*	**it is $\overline{2}\ \overline{4}$ of a *ḥqꜣ.t***
$\overline{2}\ \overline{4}\ \overline{8}$	*ḥqꜣ.t*		it is 8 $\overline{2}$ *hnw*[15]	**it is $\overline{2}\ \overline{4}\ \overline{8}$ of a *ḥqꜣ.t***
$\overline{2}\ \overline{8}$	*ḥqꜣ.t*		it is 6 $\overline{4}$ *hnw*	**it is $\overline{2}\ \overline{8}$ of a *ḥqꜣ.t***
$\overline{4}\ \overline{8}$	*ḥqꜣ.t*		it is $\overline{2}\ \overline{4}$ *hnw*[16]	**it is $\overline{4}\ \overline{8}$ of a *ḥqꜣ.t***
$\overline{2}\ \overline{8}\ \overline{32}$	*ḥqꜣ.t*	3 $\overline{3}$ *rꜣ.w*	it is 6 $\overline{\overline{3}}$ *hnw*	**it is $\overline{\overline{3}}$ of a *ḥqꜣ.t***
$\overline{4}\ \overline{16}\ \overline{64}$	*ḥqꜣ.t*	1 $\overline{\overline{3}}$ *rꜣ.w*	it is 3 $\overline{3}$ *hnw*	**it is $\overline{3}$ of a *ḥqꜣ.t***
$\overline{8}$	*ḥqꜣ.t*		it is 1 $\overline{4}$ *hnw*	**it is $\overline{8}$ of a *ḥqꜣ.t***

II.b. Problem texts

The extant hieratic mathematical texts contain approximately 100 problems, most of which come from the *Rhind* and *Moscow Mathematical Papyri*. The problems can generally be assigned to three groups:

- pure mathematical problems teaching basic techniques
- practical problems, which contain an additional layer of knowledge from their respective practical setting
- non-utilitarian problems, which are phrased with a pseudo-daily life setting without having a practical application (only very few examples extant)

The following sections present selected problems of all three groups. Because problems are often phrased elliptically, occasionally other examples from the same problem type must be read in order to understand the problem. This will be seen from the first two examples (*Rhind Mathematical Papyrus*, problems 26 and 27). Unfortunately, due to the scarcity of source material, many problem types exist only in a few examples or even only in one.

The individual sources share a number of common features. They can generally be described as rhetorical, numeric, and algorithmic. "Rhetoric" refers to the texts being written without the use of any symbolism (like +, –, √). The complete procedure is written as a prose text, in which all mathematical operations are expressed verbally. "Numeric" describes the absence of variables (like *x* and *y*). The individual problems always use concrete numbers. Nevertheless, it is quite obvious that general procedures were taught through these concrete examples without being limited to specific numeric values. "Algorithmic" refers to the way mathematical knowledge was taught in Egypt—by means of procedures. The solutions to the problems are given as step by step instructions which lead to the numeric result of the given problem.

---

[13]Correct value: $\overline{6}$ *hnw*.

[14]Correct value: $\overline{60}$ of a *ḥqꜣ.t*.

[15]Correct value: 8 $\overline{2}\ \overline{4}$ *hnw*. Possible explanation for the mistake: the author forgot to write $\overline{4}$.

[16]Correct value 3 $\overline{2}\ \overline{4}$ *hnw*. Possible explanation for the mistake: the author forgot to write 3.

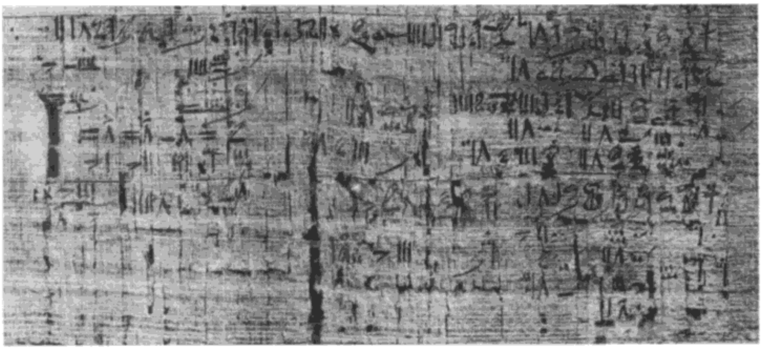
While the problem texts show these similarities, each source also shows some characteristics which make it distinct from the others. For instance, the examples from the *Rhind Mathematical Papyrus* usually include problems, the instructions for their solution, verification of results, and calculations related to the instructions or calculations as part of the verification. The examples from the *Moscow Mathematical Papyrus* only note the problem and the instructions for its solution. Furthermore, the two texts show slightly different ways of expressing these instructions. Since it is by no means self-evident to a modern reader how to read (and understand) these texts, the first example will be discussed in full detail. For the examples of practical problems, a basic knowledge of their respective backgrounds is often essential to understand the mathematical procedure. Therefore the commentary to those problems may contain an overview of their setting.

A note on language and translations

The problem texts show a high level of uniformity in grammar and wording. The individual parts of a problem, that is, title, announcement of its data, instructions for its solution, announcement and verification of the result are clearly marked through different formalisms. This will be mirrored in the translations given in this chapter. The individual termini for mathematical objects and operations were developed from daily life language. Thus the Egyptian *w3ḥ* ("to put down") became the terminus for "to add." In my translations I have used modern mathematical expressions wherever it is clear that the same concept is expressed. This can be assumed for all of the basic arithmetic operations. However, scholars have not yet determined if there are, as in the Mesopotamian case, subtle differences between apparent synonyms. The use of different grammatical structures to distinguish individual parts of a problem text can be summarized as shown in the following table.

Section of the problem text	*Grammatical markers*
Title	infinitive construction
Announcement of given data	2nd person construction, directly addressing the pupil
Instructions for solution	2nd person, imperative or *sḏm.ḫr.f* (see below)
Announcement of intermediate results	*sḏm.ḫr.f* (3rd person)
Announcement of final result	nominal constructions
Working	purely numerical

The instructions use a special verb form called the *sḏm.ḫr.f*, which indicates a necessary consequence from a previously stated condition. It is found not only in mathematical texts but also in medical texts. In mathematical texts it is used in the instructions as well as in announcing intermediate results. In translations this was traditionally rendered by "you are to..." in instructions and by the present tense "it becomes" in the announcement of intermediate results. This practice ignores the fact that the verb form used in both cases is the same, and should, consequently, be translated as such. In my translations I have used "shall" to express *sḏm.ḫr.f*.



Rhind Mathematical Papyrus Problems 26 and 27.
Reprinted by permission of The British Museum.

Rhind Mathematical Papyrus, Problem 26

A quantity, its $\overline{4}$ (is added) to it *so that 15 results*

Calculate with 4.
You shall calculate its $\overline{4}$ as 1. Total 5.
Divide 15 by 5.
\ .   5
\ 2   10
3 shall result.
Multiply 3 times 4.
.   3
2   6
\ 4   12
12 shall result.
.   12
$\overline{4}$ 3 Total 15.

The quantity 12
its $\overline{4}$ 3, **total 15.**

This problem belongs to the group of ꜥḥꜥ-problems, named after the characteristic term used in the title of each of these problems. ꜥḥꜥ is the Egyptian word for "quantity" or "number." The ꜥḥꜥ-problems, as can be seen from the example above, teach the procedure for determining an unknown quantity (ꜥḥꜥ) from a given relation with a known result. This example presents a quantity to be determined, which becomes 15 if its fourth is added to it. The text of the problem can be divided into three sections:

- title and given data
- procedure to solve the problem
- verification

The beginning of the problem is marked by the use of red ink (rendered as bold print in the transliteration). The procedure is then given as a sequence of instructions, sometimes followed by their respective calculations. For example, after the instruction "divide 15 by 5" we see the actual operation carried out. Once the result is obtained a verification is executed, first in the form of a calculation and then indicated by the use of red ink, as a complete statement.

In order to achieve a close reading of the source text, the individual steps of the solution have to be followed as such. We can make this procedure clearer if we rewrite the given instructions using our basic mathematical symbolism (+, −, ×, ÷). The procedure stated in the problem looks as follows after this rewriting ([ ] indicate ellipses in the text):

data		$\overline{4}$
		15
sequence of instructions	**1**	$[1 \div \overline{4}] = 4$
	2	$4 \times \overline{4} = 1$
	3	$4 + 1 = 5$
	4	$15 \div 5 = 3$
	5	$3 \times 4 = 12$
verification	$\mathbf{v_1}$	$12 \times \overline{4} = 3$
	$\mathbf{v_2}$	$12 + 3 = 15$

The text starts by announcing the given data of the problem: $\overline{4}$ and 15. In the rewritten form they are noted above the sequence of instructions. The instructions begin with "Calculate with 4." Since 4 is the inverse of the first datum ($\overline{4}$), there must have been one step in the calculation that has not been noted in the source text, namely the calculation of the inverse of $\overline{4}$. In the rewritten procedure above, we include this as step **1**. To indicate that it was not noted in the source text, we use square brackets ($[1 \div \overline{4}]$). Step **2** is the multiplication of the result of step **1** with the first datum ($4 \times \overline{4}$). Step **3** adds the result of steps **1** and **2**: $4 + 1$. Step **4** uses the second datum (15) and the result of step **3**: $15 \div 5$. Step **5** finally is the multiplication of the results of steps **1** and **4**: 3×4.

By following the procedure in this rewritten form several observations can be made. The basic structure of the text is sequential; results obtained in one step may be used in later step(s). Thus the result of **1** is used in **2**, **3**, and **5**; the result of **2** is used in **3**, the result of **3** is used in **4**, and the result of **4** is used in **5**. Data can be used at any time in the procedure. In this example the first datum ($\overline{4}$) appears in steps **1** and **2**; the second datum (15) in step **4**. Other numbers appearing in the instructions are either inherent to the specific mathematical operation carried out (e.g., the number 1 in the calculation of the inverse), or to the procedure itself (we will see an example of this later). The scribe must have known these numbers; they were learned with the sequence of operations of the procedure.

The different categories of "numbers" can be made even more obvious by rewriting the procedure again, this time indicating the data as D_1 ($=\overline{4}$) and D_2 (=15), and the result of step number n by $\boldsymbol{n}$, and the constants as before by their numerical value:

	D_1
	D_2
1	$[1 \div D_1]$
2	$\mathbf{1} \times D_1$
3	$\mathbf{1} + \mathbf{2}$
4	$D_2 \div \mathbf{3}$
5	$\mathbf{4} \times \mathbf{1}$
$\mathbf{v_1}$	$\mathbf{5} \times D_1$
$\mathbf{v_2}$	$\mathbf{5} + \mathbf{v_1} = D_2$

Again the sequential character is obvious. Rewriting procedure texts in this way enables a modern reader to compare the procedure of different problems more easily, as well as to see similarities between individual examples.

The solution of this example uses the so-called method of false position. A wrong solution (= 4) is assumed. In order to make this wrong solution suitable for the following calculations, it is determined here as the inverse of the first datum. The unknown (false solution) and its fractional part are then added (= 5). This is compared to the given (correct) result (= 15). Since the result obtained with the assumed number is three times smaller than the given result, the assumed number has to be multiplied by 3 to obtain the correct solution.

Rhind Mathematical Papyrus, Problem 27

A quantity, its $\overline{5}$ (is added) to it *so that 21 results*

.	5	
$\overline{5}$	1	Total 6.
\\ .	6	
\\ 2	12	
\\ $\overline{2}$	3	Total 21.
\\ .	3$\overline{2}$	
2	7	
\\ 4	14 (sic! source text 15)	

The quantity 17 $\overline{2}$,

its $\overline{5}$ .3$\overline{2}$ **Total 21.**

Problem 27 also belongs to the group of *ꜥḥꜥ*-problems. Indeed, it is very similar to its predecessor, problem 26. However, after the title, which again includes the given data, only three calculations are noted, and not a single instruction. A comparison with the calculations of problem 26 reveals that the procedure of solving this problem is identical. This can best be seen if we rewrite the procedure in the same way as we have done in problem 26. The rewritten procedure shows the similarity (operations are reconstructed based on the calculations):

	No. 27				*No. 26*
	$\overline{5}$		D_1		$\overline{4}$
	21		D_2		15
1	$[1 \div \overline{5}] = 5$	**1**	$[1 \div D_1]$	**1**	$[1 \div \overline{4}] = 4$
2	$5 \times \overline{5} = 1$	**2**	$\mathbf{1} \times D_1$	**2**	$4 \times \overline{4} = 1$
3	$5 + 1 = 6$	**3**	$\mathbf{1} + \mathbf{2}$	**3**	$4 + 1 = 5$
4	$21 \div 6 = 3\overline{2}$	**4**	$D_2 \div \mathbf{3}$	**4**	$15 \div 5 = 3$
5	$3\overline{2} \times 5 = 17\overline{2}$	**5**	$\mathbf{4} \times \mathbf{1}$	**5**	$3 \times 4 = 12$
$\mathbf{v_1}$	$17\overline{2} \times \overline{5} = 3\overline{2}$	$\mathbf{v_1}$	$\mathbf{5} \times D_1$	$\mathbf{v_1}$	$12 \times \overline{4} = 3$
$\mathbf{v_2}$	$17\overline{2} + 3\overline{2} = 21$	$\mathbf{v_2}$	$\mathbf{5} + \mathbf{v_1} = D_2$	$\mathbf{v_2}$	$12 + 3 = 15$

Moscow Mathematical Papyrus, Problem 25

Method of calculating a quantity calculated times 2
together with (it, i.e., the quantity), it has come to 9.
Which is the quantity that was asked for?
You shall calculate the sum of this quantity and this 2.
3 shall result.
You shall divide 9 by this 3.
3 times shall result.
Look, 3 is that which was asked for.
What has been found by you is correct.

This example from the *Moscow Mathematical Papyrus* shows several differences to the style of the *Rhind Mathematical Papyrus.* Only the instructions were noted, no calculation was written down. Also, after the statement of the solution, no verification is carried out; instead we find a note stating that the solution is correct.

The title indicates that it is another example of an *ʿḥʿ*-problem. However, in this example, instead of adding a fractional part of the unknown quantity to itself, a multiple of it must be added. Consequently, the procedure to solve this problem differs from the two previous examples.

	2		D_1
	[1]		D_2
	9		D_3
1	$1 + 2 = 3$	**1**	$D_1 + D_2$
2	$9 \div 3 = 3$	**2**	$D_3 \div \mathbf{1}$

Rhind Mathematical Papyrus, Problem 50

***Method of calculating** a circular **area** of 9 ḫt*
What is its amount as area?
You shall subtract its (i.e., the diameter's) $\overline{9}$ as 1,
while the remainder is 8.
You shall multiply 8 times 8.
It shall result as 64.
It is its amount as area: 64 *sṯꜣ.t*.

Calculation how it results: (9 *ḫt*)

·	9
its $\overline{9}$	1

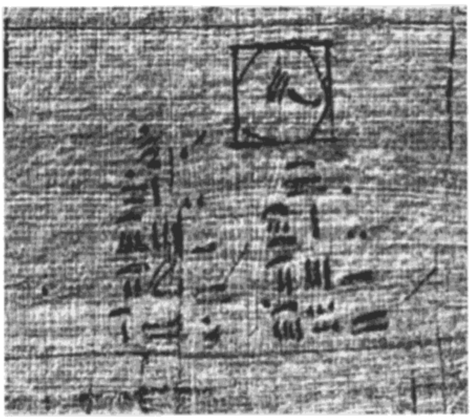
subtraction from it, remainder: 8

·	8
2	16
4	32
\8	64

Its amount as area: 64 *sṯꜣ.t*.

This problem teaches the Egyptian algorithm to calculate the area of the circle of diameter 9 *ḫt*: one ninth of the diameter is subtracted from it, and the remainder is squared. The procedure uses the diameter (given in this example as 9 *ḫt*) and the constant $\bar{9}$. The source text of problem 50 shows another feature found in some of the Egyptian problem texts. There is a drawing of the calculated object, a little bigger than the column breadth in which it is written. The drawing of the circle has its characteristic dimension, its diameter, written inside it. This type of drawing has been named an *in-line-drawing* by Jim Ritter. As in other drawings of Egyptian mathematical texts, they are sufficiently accurate to show the "idea" of the represented object. However, they are not technical drawings and the information we can gain from them is limited.

Rhind Mathematical Papyrus, Problem 48



.	8 *sṯꜣ.t*	\ .	9 *sṯꜣ.t*
2	16 *sṯꜣ.t*	2	18 *sṯꜣ.t*
4	32 *sṯꜣ.t*	4	36 *sṯꜣ.t*
\8	64 *sṯꜣ.t*	\8	72 *sṯꜣ.t*
		Total:	81 *sṯꜣ.t.*

Rhind Mathematical Papyrus, Problem 48. Reprinted by permission of The British Museum.

The text of this problem comprises a drawing (into which the number 9 is inscribed) and two calculations. The calculations can easily be identified as two multiplications, namely 8 times 8 *sṯꜣ.t* and 9 times 9 *sṯꜣ.t*. The drawing shows a square of base 9 (9 is the number written inside it) with another geometric figure inscribed into it. The second calculation (9 times 9 *sṯꜣ.t*) determines the area of the square. The first calculation can be interpreted as the calculation of the area of a circle of diameter 9, as in the previous example of problem 50. Only the last of the three steps of the algorithm was written down in the form of its working. This suits the drawing which we can identify as a circle inscribed into a square. Again, as in the case of the *in-line-drawing* of the previous problem, the sketch is sufficiently accurate to give an idea of the objects; however, it is far from being a technical drawing.[17]

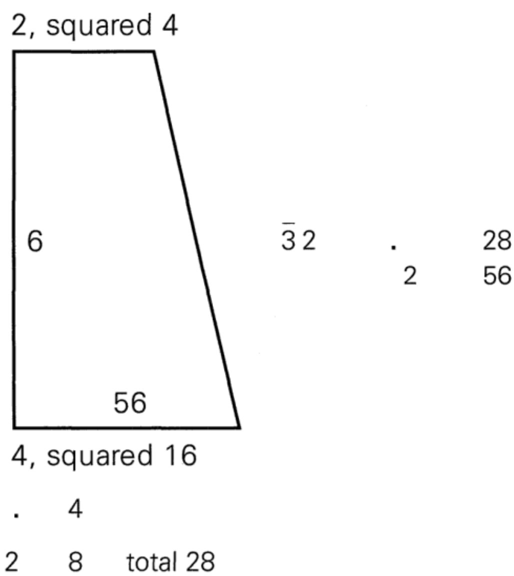
[17]Previous interpretations of this problem, which tried to use this drawing alone to establish how the Egyptian method to calculate the area of a circle was developed, ignored this (as well as the two calculations referring to the drawing). It is not possible from the extant sources to follow the development of mathematical techniques. What we see are techniques presented in a form suitable for teaching junior scribes, and not the research notes of advanced scribes.

The resemblance to the procedure of calculating the area of a circle has already been mentioned. Note, however, that if D_1 is the diameter of the object (which it must be according to the result of this problem), then the first step is the calculation of double the diameter. If we accept the second datum ʿḏ as being $4\bar{2}$, we can interpret the first five steps of the algorithm as the calculation of half the circumference of a circle of diameter (*tp-rꜣ*) $4\bar{2}$. The last step is then the multiplication of base and height to obtain the area.

Moscow Mathematical Papyrus, Problem 14

Method of calculating a ⏢.

If you are told ⏢ of 6 as height, of 4 as lower side, and of 2 as upper side.
You shall square these 4. 16 shall result.
You shall double 4. 8 shall result.
You shall square these 2. 4 shall result.
You shall add the 16 and the 8 and the 4. 28 shall result.
You shall calculate $\bar{3}$ of 6. 2 shall result.
You shall calculate 28 times 2. 56 shall result.
Look, belonging to it is 56.
What has been found by you is correct.



This problem, again from the *Moscow Mathematical Papyrus*, teaches the method for calculating the volume of a truncated pyramid. The truncated pyramid is not designated by an Egyptian term, but rather by its *in-line-drawing* ⏢. After the instructions, a sketch drawing is made and, exceptional for the *Moscow Papyrus*, calculations are noted. The sketch includes the data of the object, and the results of operations performed with these data. Thus, the lower side is indicated as 4, followed by its square 16 (used in the calculation). The same is done for the upper side (2, and its square 4) and the height (6, and its third 2). The multiplication 2×4 is indicated below the drawing, followed by the total of 16, 4, and 8 (28). Next to the result of the calculation of a third of the height (2), the final multiplication carried out in this problem (2×28) is noted. The result of the problem, the volume of the truncated pyramid (56), is indicated inside the drawing.

> If one transforms this procedure into a modern formula, the result is
>
> $$V = \tfrac{1}{3}(a^2 + 2ab + b^2),$$
>
> which is the correct formula for the calculation of the volume of a truncated square pyramid of upper base *a*, lower base *b*, and height *h* (not an approximation).

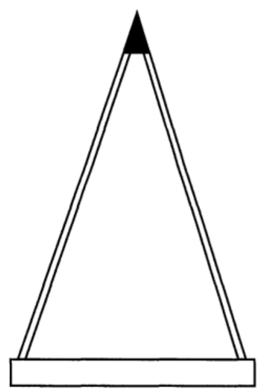
There have been several attempts to determine how this procedure was discovered by the Egyptians. However, these are all only more or less likely speculations.[19] The mathematical texts themselves give no indications how the procedures taught in them were found, nor do the administrative texts.

Rhind Mathematical Papyrus, Problem 56

***Method of reckoning** a pyramid: 360 as base, 250 as height of it.*
Let me know its *sqd*.
You shall calculate the half of 360. It shall result as 180.
You shall divide 180 by 250. $\bar{2}\,\bar{5}\,\overline{50}$ of one cubit shall result.
One cubit is 7 palms. You shall multiply by 7.

.	7
$\bar{2}$	$3\,\bar{2}$
$\bar{5}$	$1\,\bar{3}\,\overline{15}$
$\overline{50}$	$\overline{10}\,\overline{25}$

Its *sqd* is $5\,\overline{25}$ palms.



Problem 56 is one example of the six problems from the *Rhind Papyrus* relating to pyramids. All six of the problems teach the relation between base, height, and the slope of the sides. The Egyptians used the term *sqd* to describe the slope of the walls. The *sqd* measures how many palms an inclined plane retreats on a vertical height of one cubit. It is always measured in palms or palms and digits. Consequently, the *sqd* of a pyramid can be calculated as

$$sqd \text{ [palms]} = 7 \text{ [palms]} \cdot \frac{\frac{1}{2} base \text{ [cubits]}}{height \text{ [cubits]}}$$

[19]See, for example [Gillings 1964], [Gunn/Peet 1929, 178–185], [Thomas 1931], and [Vetter 1933].

This problem first gives the base and height of a pyramid. Its *sqd* is to be determined. The text of the problem is accompanied by a sketch of a pyramid. The numerical values of base and height are written next to this drawing.

Rhind Mathematical Papyrus, Problem 41

Method of calculating a circular granary of 9, *10.*
You shall subtract $\overline{9}$ of 9 as 1, remainder 8.
Multiply 8 times 8. 64 shall result.
You shall multiply 64 times 10. It shall result as 640.
Add its half to it. It shall result as 960.

It is its amount in ḫꜣr.
You shall calculate $\overline{20}$ of 960 as 48.
This is its content in quadruple *ḥqꜣ.t*: grain 48 *ḥqꜣ.t*
Method of its procedure.

.	8	.	64
2	16	\10	640
4	32	\$\overline{2}$	320
\8	64	total	960
		$\overline{10}$	96
		$\overline{20}$	48

Rhind Papyrus problem 41 and the following example (problem 42) teach the calculation of the volume of a granary with a circular base. Egyptian tomb decorations as well as archaeological finds describe two types of granaries, those with a circular base that look like cones, and those with a rectangular base. The conic granaries are treated as cylinders in the examples of the mathematical problems. Consequently, the calculation of their volume consists of the Egyptian procedure for determining the area of the circle and the multiplication of this area by the given height. However, this is not the end of the procedure of this problem, as rewriting the algorithm shows:

	9
	10
1	$\overline{9} \times 9 = 1$
2	$9 - 1 = 8$
3	$8 \times 8 = 64$
4	$64 \times 10 = 640$
5	$\overline{2} \times 640 = 320$
6	$640 + 320 = 960$
7	$\overline{20} \times 960 = 48$

The dimensions of the granary are given in cubits (not explicitly stated in this problem). Therefore the resulting volume in **4** is obtained in cubic cubits. This needs to be transferred into the volume units usually used with large amounts of grain, *ḫꜣr* (obtained in step **6** as 960), and hundreds of *ḥqꜣ.t* (obtained in step **7** as 48). As in previous examples of the *Rhind Papyrus*, we find the actual calculations carried out at the end of the problem.

Lahun Fragment UC 32160 (Griffith, Petrie Papyri IV.3), column 1–2

12

(1365 $\overline{3}$) 8

$\overline{\overline{3}}$	8	\\.	256
$\overline{3}$	4	2	512
total	16	\\4	1024
\\.	16	[\\]$\overline{3}$	85 $\overline{3}$
\\10	160		total 1365 $\overline{3}$
\\ 5	80		
total	256		

The text of this fragment does not have a problem and instructions for its solution. Instead, we find a drawing and several calculations. They belonged to a problem and its procedure—which were written either on a separate papyrus or on a now lost part of this papyrus. Three calculations are associated with the drawing. They are written in two columns under and next to it. The first calculation (ll. 1–3) is the multiplication $1\overline{3} \times 12 = 16$; the second calculation (ll. 4–7) the multiplication $16 \times 16 = 256$; and the second column holds the calculation $5\overline{3} \times 256 = 1365\overline{3}$.

From these calculations and the numerical values given in the drawing (without knowing anything else about the problem) we can reconstruct the following procedure (steps are indicated as ***n*′**, since there may be steps before the ones reconstructed here):

	12		D_1
	8		D_2
1′	$1\overline{3} \times 12 = 16$	**1′**	$1\overline{3} \times D_1$
2′	$16 \times 16 = 256$	**2′**	**1′** × **1′**
3′	$256 \times 5\overline{3} = 1365\overline{3}$	**3′**	**2′** × $5\overline{3}$

At this point it is unclear if $1\overline{3}$ and $5\overline{3}$ are further data, derived from the given data, or constants inherent to the problem. Also, the second datum (8), which is known from the *in-line-drawing*, does not appear in this procedure.

A comparison with the problems of the *Rhind Mathematical Papyrus* brings us to problem 43 where similar multiplications are carried out: $1\overline{3} \times 8 = 10\overline{\overline{3}}$ and subsequently $10\overline{\overline{3}} \times 10\overline{\overline{3}} = 113\overline{\overline{3}}\,\overline{9}$. Following this is the calculation of $113\overline{\overline{3}}\,\overline{9} \times 4$, 4 being $\overline{\overline{3}}$ of another datum of problem 43. This also fits our procedure, for the $5\overline{3}$—which we meet in the last step of our procedure—is indeed $\overline{\overline{3}}$ of 8. Therefore we can reconstruct the following procedure:

	12		D_1
	8		D_2
1	$1\overline{3} \times 12 = 16$	**1**	$1\overline{3} \times D_1$
2	$16 \times 16 = 256$	**2**	**1** × **1**
3	$\overline{\overline{3}} \times 8 = 5\overline{3}$	**3**	$\overline{\overline{3}} \times D_2$
4	$256 \times 5\overline{3} = 1365\overline{3}$	**4**	**2** × **3**

Problem 43 of the *Rhind Mathematical Papyrus* is the calculation of the volume of a granary with a circular base from its given diameter and height. Unfortunately the text of this problem is corrupt; therefore it was not chosen as an example in this selection. It teaches an alternative method of determining the volume of a circular based granary, the result of which is expressed directly as an amount in *ẖꜣr*.

The drawing found at the beginning of the calculations of the Lahun fragment suits this interpretation: indicated are the circular shape of the base, the diameter (12), which is written above it, the height (8), which is written on its side, and its content in *ẖꜣr* ($1365\bar{3}$), which is written inside it. Thus the author appears to be calculating the volume in *ẖꜣr* of a cylinder of diameter d and height h in the form $\left(\frac{4}{3}d\right)^2\left(\frac{2}{3}h\right)=\frac{32}{27}d^2h$. This is evidently equivalent to using the algorithm of problems 48 and 50 of the *Rhind Mathematical Papyrus* for calculating the area of a circle of diameter d, which we have expressed in the modern formula $A=\frac{1}{4}\left(\frac{256}{81}\right)d^2$.

Rhind Mathematical Papyrus, Problem 65

Method of calculating 100 loaves of bread for 10 men

a sailor, a commander, and a watchman as doubles.
Its procedure:
You shall add these beneficiaries (ration receivers). 13 shall result.
Divide the 100 loaves by 13. $7\,\bar{\bar{3}}\,\overline{39}$ shall result.
You shall say:
This is the ration for these 7 men, and the sailor, commander, and watchman as doubles.

. $7\,\bar{\bar{3}}\,\overline{39}$	the sailor	$15\,\bar{3}\,\overline{26}\,\overline{78}$
. $7\,\bar{\bar{3}}\,\overline{39}$	the commander	$15\,\bar{3}\,\overline{26}\,\overline{78}$
. $7\,\bar{\bar{3}}\,\overline{39}$	the watchman	$15\,\bar{3}\,\overline{26}\,\overline{78}$
. $7\,\bar{\bar{3}}\,\overline{39}$		
. $7\,\bar{\bar{3}}\,\overline{39}$	total: 100.	
. $7\,\bar{\bar{3}}\,\overline{39}$		
. $7\,\bar{\bar{3}}\,\overline{39}$		

The Egyptian ration system was based on the distribution of quantities of grain, and of bread and beer. It constituted the core of Egyptian administration, and it must have been a frequent task for a scribe to calculate amounts of food for various beneficiaries (see also the respective section of the *Anastasi I Papyrus* in the introduction of this chapter).

This problem has 100 loaves of bread to be distributed among 10 men. Three of them shall receive double the amount the others do. The solution determines a "corresponding" number of recipients who would all get the same ration. That is, the three persons receiving the double amount are counted twice. In the Egyptian text, this procedure is called "adding the beneficiaries." The basic ration is then calculated by the division of the given loaves by the number of "recipients." For those who get the double share, the basic ration must be doubled. The verification at the end adds up the individual rations.

sandals consists of the preparation of the leather, its cutting, and finally the finishing process of the shoe, which incorporates putting together the leather pieces and, eventually, applying some kind of decoration to the leather.

The title of the problem provides the information that the work-rate of a cobbler must be calculated. The following lines state that the cobbler has to cut leather for ten pairs of sandals as his work-rate per day if he only cuts the leather; but if the leather has been already cut, he has to finish five pairs per day. What is looked for is the amount of his contribution, given the condition that the cobbler does both parts of the sandal production, cutting and finishing the shoe. The following instructions given to solve the problem use the fact that the contribution of the cobbler, if he only cuts leather, is twice the contribution if he finishes the sandals. If he has to cut ten pairs or finish five as his daily work-rate, it will take three days work to amount to ten pairs of sandals. What was wanted was the number of sandals per day if he does both. This is obtained by dividing 10 by 3, which results in $3\frac{1}{3}$ pairs of sandals.

III. Mathematics in Administrative Texts

The following section introduces several examples of administrative documents, another genre of texts that incorporate mathematical knowledge. Contrary to the mathematical problem texts, in this type of text only data and results are noted. The mathematical operations that were executed to obtain the results from the data are "missing." Consequently, it is not always possible to reconstruct the way the results were obtained. Furthermore, many of these texts are badly preserved, so that some parts of data or results are no longer extant. Finally, not every administrative text with numerical information is mathematically interesting; many of them are simple lists of data.

III.a. Middle Kingdom texts: The Reisner papyri

The *Reisner Papyri* provide us with the exceptional case of an example of a reasonably well-preserved text including some mathematically interesting passages, of which two will be discussed here. These papyri were found during excavations at Naga ed-Deir conducted by the Egyptologist George A. Reisner in 1901–1904: "Among the later tombs on the slope below the cliff, one contained four rolls of hieratic papyri, badly worm eaten; and another yielded a set of poisoned arrows."[22]

The *Reisner I Papyrus*, from which the following two examples are taken, measures approximately 3.50 meters, and it has a height of 31.6 cm. As in other accounting papyri of the Old and Middle Kingdoms the papyrus is divided by several horizontal lines which were meant to help the scribe in aligning his entries. Based on the palaeography, the regnal years found in the text (without the name of the ruler), and personal names, the text has been assigned to the reign of Senusret I (1956–1911 BCE).

The documents are accounts of building construction and carpentry workshops, including lists of workmen arranged in groups, as well as calculations related to construction projects and the necessary workers to perform them. Several accounts refer to a dockyard workshop,

[22]See [Reisner 1904, p. 108]—A photo of the papyri in situ can be found in [Simpson 1963, frontispiece].

including lists of copper tools (axes, adzes, saws, etc.) by units of copperweight, presumably for recasting. For the purpose of editing and reference, the editor W. K. Simpson divided the document into several sections designated by letters of the alphabet. The chosen division reflects the scribes' own divisions to a major extent. A number of these sections belong together due to their subject. Five sections of the *Reisner I Papyrus* constitute records of the construction of a building, presumably a temple (sections G, H, I, J, and K, of which section I is the first of our examples here).[23] They contain four accounts followed by a summary. The layout of these accounts is quite clear. On the left side of the text (= the right side of the translation) we find six columns that list lengths, widths, depths, units, the product of the four previous columns, and the respective number of workers (which is left blank in Section I). The right side of the text (= the left side of the translation) indicates dates and circumstances of the work recorded otherwise by numeric entries alone. In section I this contains seven dates, as well as various details about the place of work and the material involved. The dates are given in the usual way as *x*. (month) of a season, *y* day. The Egyptian calendar distinguished three seasons *3ḥ.t* (inundation), *prt* (emergence of the crops), and *šmw* (harvest). Each season had four months of 30 days. Furthermore, there were five additional days (epagomenals) at the end of the year.

Unfortunately, some of the terms for materials are not known from any other source, and the occurrences in this text are not sufficient to establish their meaning. Therefore, the translation given here is only a provisional one.[24] It is interesting to see that all of the activities are characterized by lengths, widths, depths, and a subsequent multiplier (units), because these structure the necessary data to calculate the total volume. The individual actions described seem to indicate that this section of the text is concerned with the production of bricks.

The account lists three dimensions, a number of units, and the product of these. The last column (enlistees) is left blank. Almost all measurements are given as cubits, except for three, explicitly marked as palms. In three instances the products have been miscalculated (marked in my translation by sic! with the correct value given in brackets). In addition there are three numbers in red (indicated in bold in the translation) written in between the columns. Their meaning and relation to the other entries in this section are not clear.

The second example, *Papyrus Reisner I*, section O, records workers' compensation over a period of 72 days. The heading of the first column gives a date, presumably the beginning of the period covered in the table, as well as the information where the work took place. The first column consists of a list of 20 names. Following this are six columns, each of which includes a 12-day period. The entries in these columns are divided into the record of days the individual worker worked in the respective 12 days and the amount of *trsst*-bread[25] he received as compensation for this work. Within the entries of the record of days worked, black numbers indicate actual days worked, red numbers indicate absence. In the first column we find 12, 12, and 2 in red, in the second column 3, 2, and 3, and in the third column 8, 7, 11, and 11. These days of absence are added, and the totals are noted in black at the bottom of the text: 26 for the first column, 8 for the second column and 47 (miscalculated for 37) for the third column.

[23] A useful introduction into the subject is [Arnold 1991].

[24] For further information see the extensive lexicographical commentary in [Simpson 1963].

[25] The Egyptians had a variety of bread and cake types. What kind of bread/cake the individual names designate is not know. The bread used in this text is called *trsst* in Egyptian. It is attested in only one other source, which also deals with rations.

A Construction Account: Papyrus Reisner I, Section I

	4. *prt*, day 15	lengths	widths	depths	<units>	product	equaling enlistees
	given [to him] in molding the ground: the great chamber	12	5	$\overline{2}$	1	30	
	[given to him in] the august [chamber]	[1]5	5	$\overline{2}$	1	37$\overline{2}$	
ḫ3[26]	[..........] given to him in the eastern chapel of the glorious chambers	8	5	[$\overline{2}$]	1	20	
	[..........given] to him in ... for ...	[1]8	1[1]	[$\overline{3}$]	1	132	
	the western *mḫ3w*[27]	32	4	[$\overline{4}$]	1	32	
ḫ3	the eastern *mḫ3w*	52	3 **298[..]**	[$\overline{4}$]	1	39	
	[1. *šmw*], day 28, given to him as fill: the great chamber	24	5 palms	$\overline{2}$	1	8; 4 palms	
	[given to] him carrying *srft*[28]	26	6	5 palms	1	111; 3 palms	
	[..]	20	5	5 palms	1	71; 3 palms	
	given to him in loosening brick clay	27	7	2	1	378	
	2. *šmw*, day 1, given to him in removing water from the field	8	7	2	1	112	
	2. *šmw*, day 2, given to him as builders in the tower	1$\overline{2}$	1$\overline{2}$	2	2	9	
		2$\overline{2}$	1$\overline{2}$	1$\overline{2}$	2	11$\overline{4}$	
		3$\overline{2}$	2$\overline{2}$	1$\overline{2}$	2	25$\overline{4}$ (sic! 26$\overline{4}$)	
		4	2$\overline{2}$ **81$\overline{2}$**	1$\overline{2}$	2	36 (sic! 30)	
	2. *šmw*, day [..], completed for him in brick-clay of the fields	10	5$\overline{2}$	<..>$\overline{4}$	1	55 (sic! 13$\overline{2}$$\overline{4}$)	
		16	5$\overline{2}$	[...]	1	75$\overline{4}$	
		8	6 **556$\overline{4}$**	**[1]**	1	48	
	completed for him in large size brick						

[26]*ḫ3*: mark used to call attention to related items.
[27]*mḫ3w*: not known from any other text.
[28]*srft*: material used in construction; lighter in weight than stone or sand. There is a detailed discussion in [Simpson, 1963, pp. 74–75].

Recompensation of Workers: Papyrus Reisner I, Section O

	12 days	*trsst*	12 days	*trsst*	12 days	*trsst*	12 days	*trsst*	12 days	*trsst*	12 days	*trsst*
*Year 24, **2nd month of šmw, day 21***												
List of enlistees who are in This:												
Nakhti's son Se-ankhi-nedjes	12	94	12	80	11	80	12	88	10	80	15	120
Senmutef's son Zi-n-Wosret **Redi-wi-Sobek**	10	**7** 56	12	96	11	244			10	80	15	
Senbebu's son Zi-n-Min	**12**	\	12	88	11	80	12	94	10	80	15	120
Se-ankhi's son Anhur-nakhte	12	**8** 64	12	88	11	80	12	62	10	80	15	80
Wosre's son Ikeki	12	**2** 80	9 **3**	72	11	88	12	48	10	80	15	120
Senet's son Kemni	12	94	12	96	11	88	12	80	10	80	15	120
Iri's son Mentu-hotep	12	90	12	88	11	88	12	88	10	80	15	120
Renef-ankhu's son Si-Anhur[29]	**12**	\	12	88	11	56	12	88	10	80	15	120
Hedjenenu's son Gem-mutef	12	94	10 **2**	80	11	88	12	88	10	80	5	40
Ankhu's son Anhur-nakhte	12	94	12	84	11	88	12	92	10	80	15	120
Inyotef's son Sefkhy	12	94	12	96	11	88	12	80	10	80	15	120
Sobek-wosre's son Sobek-nakhte	12	94	12	96 60	11	88	12	88	10	80	15	130
Sobeknakhte's son Sobek-nofre	12	94	12	96	11	88	12	92	10	80	15	120
Nakht-aa's son Nakht-tjen	12	94	12	96	11	88	12	92	10	80	15	120
Sobek-nakhte's son Shemai	12	94	9 **3**	72	11	88	12	88	10	80	15	120
Seankhi's son Sobek-hotep	10 **2**	80	12	96	3 **8**	24	\	\				
Irne's son Yu	12	94	12	96	11	88	12	84	10	80	15	120
Zi-n-Wosret's son Sehetep-ib	12	94	12	96	4 **7**	32	\	\				
Ameny's son Neferkhau	12	64	5	40	\ **11**		\	\				
his brother Sefkhy	12	94	12	96	\ **11**		\	\				
total	26	**1610**	8	**1546**	47	**1564**	**1530**	**100**		***1280***		***1710***

[29]Note that the original source text is "messier" than this translation: In this line, as well as in the next, and in the third line from the bottom further names (Nakhti in this line, Wehemy and Si-nefer in the next, and Merernedjes further below) have been inserted in between the columns that list the days worked and respective *trsst* loaves.

Since the number of days worked and the compensation in bread are both noted, the number of *trsst* loaves divided by the number of days worked results in the number of *trsst* loaves given to a worker per day. We would expect this to be constant; however, the number of *trsst* loaves per day for the individual persons varies. Sometimes it even varies for one individual on different days. It is hard to say what might have caused this variation, since the only information we get here is the actual numbers of *trsst* loaves given to the workers. It is noteworthy, however, that the most frequent ratio is that of 8 *trsst* loaves per day.

The totals given in red at the bottom of the sheet (1610, 1546, 1564, 1530/100, 1280, and 1710) equal only in one instance (1280) the sum of values given in the respective columns above. Therefore—if we don't assume that the scribe simply didn't know how to add—it seems that values from another source were also used in establishing the totals.

III.b. New Kingdom texts: Ostraca from Deir el Medina[30]

No mathematical texts from the New Kingdom are extant. Instead, there is a significant number of administrative and other texts which show the application of mathematical knowledge. The majority of these documents come from the area of Thebes.

The tombs of the pharaohs of the New Kingdom are located in a desert valley, on the western bank of the Nile River opposite Luxor. This valley is today known as the Valley of the Kings. The workers (quarrymen, plasterers, draftsmen, and painters) who constructed and decorated these tombs lived in a village located in another nearby desert valley. This site is today known as Deir el-Medina. The village was founded in the eighteenth dynasty and was inhabited until the end of the twentieth dynasty. Apart from the remains of this settlement, tens of thousands of documents have been discovered there, including (among others) letters, administrative and legal documents, magic and religious texts, as well as texts and sketches relating to the construction of tombs. These sources give us some insight into daily life at a workers' settlement during the New Kingdom. The government provided the workers at Deir el-Medina with all necessary commodities for their life as well as equipment for their work. Wages consisted mostly of grain. They were higher than what could have been consumed, and thus supposedly included a surplus used for exchange.

The Egyptian week had ten days, the last two of which were free. In addition there seem to have been long weekends and afternoons that were free as well. The working day lasted for approximately four hours in the morning and four hours in the afternoon with a break around lunch. The crews were divided into two groups, one for the right side of the tomb and one for the left side. Each group had its own foreman and assistant. The work on the tomb began with the selection of a site in the valley, probably under the eyes of a royal commission. Then a plan of the tomb was drawn, presumably on papyrus. The typical tomb consisted of several descending corridors and a number of rooms, the last of which usually contained the sarcophagus. The construction of the tomb usually began with quarrying the corridors and rooms. The plasterers worked behind the quarrymen and covered the uneven walls with a layer of gypsum and whitewash to enable their further decoration. Then the proposed texts and designs would be drafted and checked by a master. Finally, these texts were painted or

[30] A more detailed overview of daily life in Deir el Medina and the construction of tombs in the New Kingdom can be found in [McDowell 1999] and [Bierbrier 1982].

texts (easily) into their hieroglyphic counterparts. The grammar is the intermediate stage between Late Egyptian and Coptic.

Compared to earlier mathematical texts, the demotic sources show some features which are a continuation of Egyptian tradition, but also several changes, some of which may be due to a Mesopotamian influence. As in the earlier hieratic corpus, we can distinguish between table texts and problem texts. The problem texts can still be characterized formally as numeric, rhetorical, and algorithmic. However, the verb-form *sḏm.ḫr.f*, characteristic of the hieratic mathematical texts, is no longer used (in fact, it is no longer existent in demotic at all). Furthermore the terminology for individual mathematical operations has sometimes changed. And, as will become obvious from the following examples, the problem types as well as the algorithms for their solution have been modified.

The extant demotic mathematical papyri comprise eight sources.[37] Five of them, which contain 72 short problems and tables, were published in [Parker 1972]. Two more were published by the same author in two articles, and another, unidentified example was edited by Eugene Revillout. All of these publications are more than 30 years old and, due to development in the study of demotic, are in need of reworking. However, for the time being, Parker's translations, based on the knowledge of the entire corpus, are the best available, and so are used in the following few examples. The problem numbers given are those of [Parker 1972].

IV.b. Table texts

BM 10520 (No. 54)

64
128
192
256
320
384
448
512
576
640
704
768
832
896
960
1024

Written as one column of numbers only, this "table" comprises the multiples of 64 from 64 (= 1 × 64) up to 1024 (= 16 × 64).

[37]Detailed references for the individual papyri and their publications can be found in [Fowler 1999, 258 and note 80].

BM 10794 (No. 67)

The method of taking [$\overline{150}$ to 10]

1 to $\overline{150}$
2 to $\overline{90}$ $\overline{450}$
3 to $\overline{60}$ $\overline{300}$
4 to $\overline{45}$ $\overline{225}$
5 to $\overline{30}$
6 to $\overline{30}$ [$\overline{150}$]
7 to $\overline{30}$ $\overline{90}$ [$\overline{450}$]
8 to $\overline{20}$ $\overline{300}$
9 to $\overline{30}$ [$\overline{45}$ $\overline{225}$]
10 to $\overline{15}$

One can follow the building of this table according to certain groups; for example, from the calculation of $2 \div 150 = \overline{90}\ \overline{450}$ it is easy to get to $4 \div 150$ by simply halving the respective denominators $\overline{90}$ and $\overline{450}$. This is also possible for the pairs $3 \div 150$ and $6 \div 150$ as well as $5 \div 150$ and $10 \div 150$. A second technique, which can be observed from $6 \div 90$ onward, uses the addition of previous solutions, $7 \div 150 = 5 \div 150 + 2 \div 150 = \overline{30} + \overline{90}\ \overline{450} = \overline{30}\,\overline{90}\ \overline{450}$. This technique may have been used for $6 \div 150$, $7 \div 150$, $8 \div 150$, $9 \div 150$, and $10 \div 150$ (in the cases of $8 \div 150$ and $10 \div 150$ with a further step, combining two of the unit fractions). Note that $6 \div 150$ and $10 \div 150$ could have been found by both techniques alike.

IV.c. Problem texts

pCairo JE 89127–30, 89137–43 (No. 7)[38]

The things you (should) know about the articles of cloth. Viz.
If it is said to you: "Have sailcloth made for the ships,"
and it is said to you: "Give 1000 cloth-cubits to one sail;
have the height of the sail be (in the ratio) 1 to 1 $\overline{2}$ the width,"
(here is) the way of doing it. Viz.
Find its half, when it happens that the ratio is 1 to 1 $\overline{2}$: result 1500.
Cause that it reduce to its square root: result 38 $\overline{\overline{3}}$ $\overline{20}$.
You shall say: "The height of the sail is 3[8] $\overline{\overline{3}}$ $\overline{20}$ cubits."
You shall take to it $\overline{\overline{3}}$—since it happens that it is [to 1 $\overline{2}$] that 1 makes [a ratio]:
result 25 $\overline{\overline{3}}$ $\overline{10}$ $\overline{90}$.
It is the width.

The problem calculates length and width of a rectangular sail from its given area and ratio of length and width. As we have done in the earlier examples, it is possible here, too, to rewrite the algorithm in a more symbolic form.

[38]Translation of [Parker 1972, 19].

	1000		D_1
	$1\bar{2}$		D_2
1	$1\bar{2} \times 1000 = 1500$	**1**	$D_2 \times D_1$
2	$\sqrt{1500} = 38\ \bar{\bar{3}}\ \overline{20}$	**2**	$\sqrt{\mathbf{1}}$
3	$\bar{\bar{3}} \times 38\ \bar{\bar{3}}\ \overline{20} = 25\ \bar{\bar{3}}\ \overline{10}\ \overline{90}$	**3**	$\overline{D_2} \times \mathbf{2}$

The solution determines the height from the given area (D_1) and ratio (D_2) in steps (1) and (2); then the width is calculated using the height and the given ratio (3).

To understand these instructions, a look at a modern solution may help. It must be stressed, however, that this *is not a translation of the source into modern terminology*, nor is it an explanation how the Egyptian mathematician arrived at this particular algorithm. It is merely meant as a help for the modern reader to understand what is going on in the algorithm.

> MODERN SOLUTION If we designate the height as x and the width as y, the information given in the data can be expressed as follows:
>
> (I) $x \times y = 1000$
>
> (II) $x \div y = \frac{3}{2}$ or (II*) $y = x \times \frac{2}{3}$
>
> (II*) in (I) : $x^2 \times \frac{2}{3} = 1000$ and thus $x^2 = \frac{3}{2} \times 1000$
>
> (remember that $\frac{3}{2} \times 1000$ is what is calculated as the first step of the algorithm above). From this we obtain $x = \sqrt{\frac{3}{2} \times 1000}$ (which is calculated in the second step of the algorithm); and finally $y = \frac{2}{3} \times x$ (the last step of the algorithm).

The following problem is well known from Mesopotamian sources. We are given the length of an erect pole (leaning against a wall). The pole's foot is then moved outward a given distance, and it has to be determined how far the top of the pole has been lowered. It is generally assumed that a transmission of mathematical and astronomical knowledge from Mesopotamia to Egypt occurred, possibly in the times of Persian rule of Egypt.[39] However, a detailed study of this assumed transmission has yet to be done.

pCairo JE 89127–30, 89137–43 (No. 26)[40]

A pole which is 10 cubits [when erect to (the) top].
[If the number] of its foot (moved) outward is 8 cubits,
wh[at is the lowering of its top from it?]
You shall reckon 10, 10 times: result 100.
You shall reckon 8, 8 times: result 6[4].
Take it [from 1]00: [result] 36.
Cause that it reduce to its square root: result 6.
Ta[ke it from 10: remainder 4].
You shall say:
"Four cubits is [the] number [of the lowering of its top from it]"

---

[39]See [Høyrup 2002, 405–6] and [Parker 1972, 5–6].
[40]Translation of [Parker 1972, 35–37].

As in the previous problem we can easily follow the algorithm, which we can also compare to the modern solution:

	10		D_1
	8		D_2
1	$10 \times 10 = 100$	**1**	$D_1 \times D_1$
2	$8 \times 8 = 64$	**2**	$D_2 \times D_2$
3	$100 - 64 = 36$	**3**	**1 – 2**
4	$\sqrt{36} = 6$	**4**	$\sqrt{\mathbf{3}}$
5	$10 - 6 = 4$	**5**	$D_1 -$ **4**

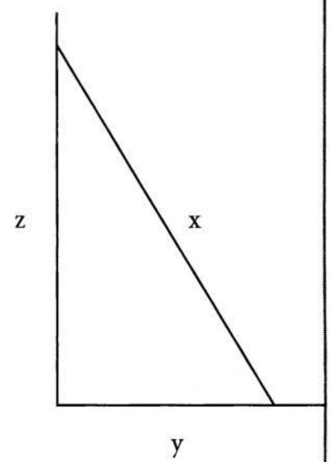
Modern solution The pole (of length x), the height which it measures against the wall once the foot is moved out (z), and the distance which the foot was moved out (y) constitute a right triangle.

Therefore

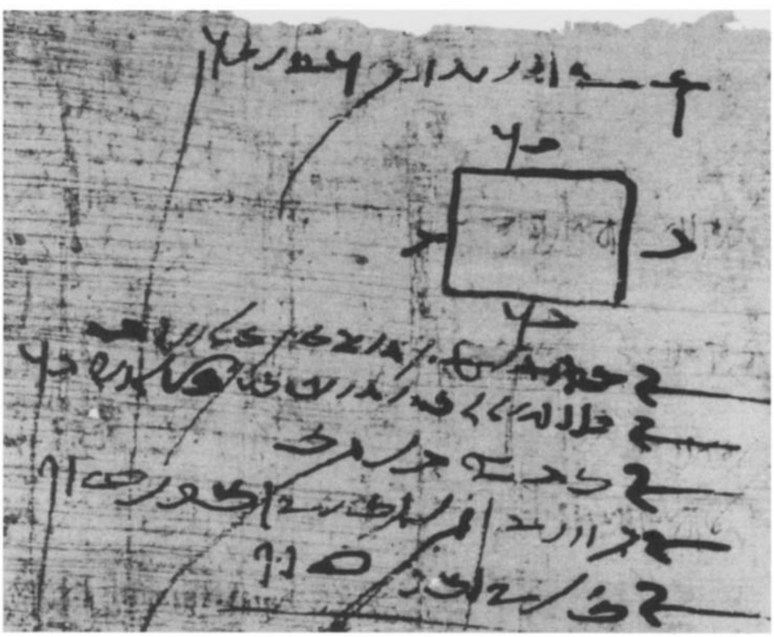
$$z^2 = x^2 - y^2.$$

(The algorithm calculates x^2 and y^2 in the first two steps, and then the difference (z^2) in the third step.)

We are looking for "the lowering of its top from it," which is in our modern terminology $x - z$. (The algorithm moves on to extract the square root ($= z$), and finally calculates the difference $x - z$.)

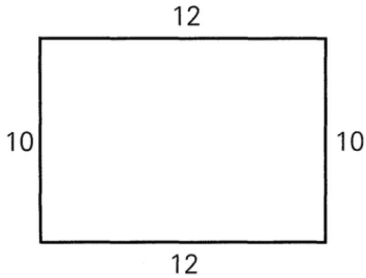


pBM 10520 (No. 64)[41]



[41]Translation of [Parker 1972, 71].

A piece (of land).
Its plan.



You shall add the south and the north: (result) 20. Viz.
Its half, 10.
You shall add the east and the west: (result) 24. Viz.
Its half, 12.
You shall reckon 10 <12> times: result: 120.
You shall carry 100 into 120 in order to bring another (formulation):
result $1\overline{5}$ (land) cubits.
You shall say:
"In order to bring another (formulation), $1\overline{5}$ (land) cubits."

This problem calculates the area of a rectangular field of sides 10 and 12. We already know an earlier Egyptian calculation of the area of a rectangle from the *Rhind Mathematical Papyrus* (problem 49). The algorithm of that problem multiplies the two sides (as we would expect) to obtain the area. The algorithm of our problem, however, begins with calculating the sum of the opposing sides, which is then halved. The results of this, since the geometrical shape is a rectangle, is of course again the (given) length of the sides. These are then multiplied and finally the result is calculated in a typical unit for measuring fields, the (land)-cubit, a strip of 1 cubit × 100 cubits. An advantage of this method is its applicability to approximate areas of all types of quadrilaterals, especially irregular ones, for which it is generally used.

Hieratic Text: Robins/Shute, 1987, pl. 19–20
MMP, No. 15
Hieratic Text: Struve, 1930, Col. XXX
MMP, No. 23
Hieratic Text: Struve, 1930, Col. XLII

Mathematics in administrative texts

Middle Kingdom texts: the Reisner papyri

Reisner I, Section I
Hieratic Text: Simpson, 1963, pl. 15
Reisner I, Section O
Hieratic Text: Simpson, 1963, pl. 21

New Kingdom texts: Ostraca from Deir el Medina

Ostracon IFAO 1206
Hieratic Text: Wimmer, 2000, pl. XLVI and XLVIII

Mathematics in the late period

Table texts

BM 10520, No. 54
Demotic Text: Parker, 1972, pl. 19
BM 10794, No. 67
Demotic Text: Parker, 1972, pl. 24

Problem texts

Cairo JE 89127–30, 89137–43, No. 7
Demotic Text: Parker, 1972, pl. 2
Cairo JE 89127–30, 89137–43, No. 26
Demotic Text: Parker, 1972, pl. 9

V.c. References

Arnold, Dieter. 1991. *Building in Egypt.* New York, Oxford: Oxford University Press.

Bierbrier, Morris L. 1982. *The Tomb-builders of the Pharaohs.* London: British Museum Publications.

British Museum. 1842. *Select papyri in the hieratic character from the collections of the British Museum.* London: British Museum.

Collier, Mark, and Stephen Quirke. 2002. *The UCL Lahun papyri: letters.* Oxford: Archaeopress.

Faltings, Dina. 1998. *Die Keramik der Lebensmittelproduktion im Alten Reich: Ikonographie und Archäologie eines Gebrauchsartikels.* Studien zur Archäologie und Geschichte Altägyptens 14. Heidelberg: Heidelberger Orientverlag.

Fischer-Elfert, Hans Werner. 1983. *Die satirische Streitschrift des Papyrus Anastasi I.* 2nd ed. Kleine Ägyptische Texte. Wiesbaden: Otto Harrassowitz.

Fischer-Elfert, Hans Werner. 1986. *Die satirische Streitschrift des Papyrus Anastasi: Übersetzung und Kommentar.* Ägyptologische Abhandlungen 44. Wiesbaden: Otto Harrassowitz.

Fowler, David H. 1999. *The mathematics of Plato's Academy: a new reconstruction.* Oxford: Oxford University Press.

Gardiner, Alan H. 1911. *Egyptian hieratic texts. series I: literary texts of the New Kingdom.* Leipzig: J.C. Hinrichs.

Gillings, Richard J. 1964. "The volume of a truncated pyramid in Ancient Egypt." *Mathematics Teacher* 57: 552–55.

Glanville, Stephen R. K. 1927. "The mathematical leather roll in the British Museum." *Journal of Egyptian Archaeology* 13: 232–239.

Griffith, Francis L. 1898. *The Petrie papyri: Hieratic papyri from Kahun and Gurob.* London: Quaritch.

Gunn, Battiscomb, and Thomas E. Peet. 1929. "Four geometrical problems from the Moscow Mathematical Papyrus." *Journal of Egyptian Archaeology* 15: 167–85.

Hoffmann, Friedhelm. 1996. "Die Aufgabe 10 des Moskauer mathematischen Papyrus." *Zeitschrift für Ägyptische Sprache und Altertumskunde* 123: 19–26.

Høyrup, Jens. 2002. *Lengths, widths, surfaces: a portrait of Old Babylonian algebra and its kin.* Berlin: Springer.

McDowell, Andrea G. (1999). *Village life in ancient Egypt: Laundry lists and love songs.* Oxford: Oxford University Press.

Quibell, James E. 1904. *Archaic objects.* Catalogue général des antiquités égyptiennes du Musée du Caire. nos. 11001–12000 et 14001–14754. Cairo: Institut Français d'Archéologie Orientale.

Parker, Richard A. 1972. *Demotic mathematical papyri.* Providence: Brown University Press.

Peet, Thomas E. 1923. *The Rhind Mathematical Papyrus, British Museum 10057 and 10058.* London: Hodder & Stoughton.

Reisner, George A. 1904. "Work of the expedition of the University of California at Naga-Ed-Der." *Annales du Service des Antiquités de l'Égypte* 5(1904).

Ritter, James. 2002. "Closing the Eye of Horus." In John M. Steele and Annette Imhausen (eds.), *Under One Sky: Astronomy and Mathematics in the Ancient Near East.* Alter Orient und Altes Testament 297. Münster: Ugarit Verlag.

Robins, Gay, and Charles Shute. 1987. *The Rhind Mathematical Papyrus: An Ancient Egyptian Text.* London: British Museum Publications.

Rossi, Corinna. 2004. *Architecture and Mathematics in Ancient Egypt.* Cambridge: Cambridge University Press.

Simpson, William Kelly. 1963. *Papyrus Reisner I: The Records of a Building Project in the Reign of Sesostris I.* Boston: Museum of Fine Arts.

Shaw, Ian (ed.) 2000. *The Oxford History of Ancient Egypt.* Oxford: Oxford University Press.

Struve, Wasili W. 1930. *Mathematischer Papyrus des staatlichen Museums der schönen Künste in Moskau.* Otto Neugebauer, Julius Stenzel, Otto Toeplitz (eds.), Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik, Abteilung A: Quellen 1. Berlin: Julius Springer.

Thomas, W. R. 1931. "Moscow Mathematical Papyrus, No. 14." *Journal of Egyptian Archaeology* 17: 50–52.

Vetter, Quido. 1933. "Problem 14 of the Moscow Mathematical Papyrus." *Journal of Egyptian Archaeology* 19: 16–18.

Wimmer, Stefan. 2000. "Welches Jahr 29?". In R. J. Demarée and A. Egberts (eds.), *Deir el-Medina in the Third Millennium AD* (Egyptologische Uitgaven 14). Leiden: Nederlands Instituut Voor Het Nabije Oosten.

2 Mesopotamian Mathematics

Eleanor Robson

I. Introduction

It is tempting to think that, because it all happened such a long time ago, there is little new to say about mathematical developments in ancient Mesopotamia (southern Iraq and neighbouring areas). The standard histories of mathematics all tell much the same story: of the sexagesimal place value system, approximations to $\sqrt{2}$, Pythagorean triples, and a little algebra. Those achievements, impressive as they are, are but a tiny fraction of what could and should be told. Many thousands of Mesopotamian mathematical documents survive, written on clay tablets in the cuneiform script; hundreds have been edited and published since the beginning of the twentieth century. New translations of over sixty of them are presented here, spanning some three thousand years and thousands of square kilometers. Some are just a few lines of calculation or a roughly sketched diagram; others are long compilations of mathematical problems or highly sophisticated arithmetical tables.

In fact, as we shall see in this introduction, modern scholarly understanding of Mesopotamian mathematics is changing and improving at a rate unparalleled since the 1930s. Not only are dozens of new sources published every year, causing constant reevaluation of the historical corpus, but recent developments in the scripts, languages, history and archaeology of the region have stimulated exciting new lines of research that contextualise ancient mathematical practice in a way that was unimaginable even ten years ago. This introduction briefly sketches answers to three questions, showing why and how this section of the Sourcebook has been compiled:

- How have Western views of Mesopotamian mathematics changed over the last two millennia?
- Who wrote mathematics in ancient Mesopotamia, why, and how?
- How have the translations presented here been chosen and produced?

At the end of the introduction there are also practical notes on sources, metrology, and bibliography.

I.a. Mesopotamian mathematics through Western eyes

Myths and rumors

The mathematical achievements and interests of ancient Iraq have undergone periodic reassessments over the last two thousand years, ever since writers in the Greek tradition began to make comments and observations. But the most thorough overhauls have come since the rediscovery of cuneiform culture in the second half of the nineteenth century and again in the last decades of the twentieth.

For the most part Greek writers took Egypt to be the birthplace of mathematics, but they credited the Babylonian, or Chaldaean, priests of southern Iraq with astrology and the ability to make predictions from the stars. Strabo, for instance, names three Babylonian astronomers, two of whom have since been identified in cuneiform sources too. The transmission to Classical science of Babylonian observational data, values of periodicities, and even the sexagesimal place value system for fractional values in astronomical calculations, all confirm that there were direct contacts during the Persian and Hellenistic periods (c. 550–150 BCE). There are no early Greek traditions about Babylonian mathematics, however. It was only around 300 CE that Iamblichus claimed that Pythagoras had spent time in Babylon—nearly a millennium earlier, in the sixth century BCE. Without corroboration from earlier or more

TABLE 2.1: The Rediscovery of Mesopotamian Mathematics since 1800

Date	*Scholarly developments*[7]	*Political events*[8]
pre-1800	Traditional Biblical narratives, Greek stories, and travellers' tales of ancient Mesopotamia	1638–1918: Iraq under Ottoman rule
1820	1819: Tiny display of undeciphered cuneiform tablets at the British Museum	
1840	1842: Anglo-French archaeological rediscovery of ancient Assyria (northern Iraq)	
1860	1857: Akkadian cuneiform officially deciphered at the Royal Asiatic Society, London	
1880	1880s–: Mass recovery of cuneiform tablets in Babylonia (southern Iraq)	1881: Museum of the Ancient Orient founded in Istanbul, the Ottoman capital
1900	1900: First drawings of Old Babylonian mathematical tablets published 1903–: Progress in understanding sexagesimal numeration and tables	
1920	1910s–: Widespread adoption of scientific excavation methods 1916: First decipherment of an Old Babylonian mathematical problem 1922: First publication of Late Babylonian mathematics	1914–18: First World War stops excavation 1920: Formation of modern Iraqi state 1932: Iraqi independence from British Mandate
1940	1930–45: "Golden age" of translation; major editions by Neugebauer, Sachs, and Thureau-Dangin	1940s: Iraqi excavations begin
1960	1956: First volume of the *Chicago Assyrian Dictionary* published 1961–: Publication of mathematics from Susa (southwest Iran) and western peripheries	1968: Ba'athist coup in Iraq
1980	1976–: Increased study of third-millennium mathematics in Sumerian 1984: First volume of the *Pennsylvania Sumerian Dictionary* published	1970s–80s: Large-scale damming projects on the Euphrates, Tigris and Diyala rivers drive rescue excavations in archaeologically unexplored areas of Iraq
2000	1990–: Høyrup develops discourse analysis of Old Babylonian mathematics 1997–: Study of archaeological and social context of Mesopotamian mathematics	1990–: Gulf War ends excavations; UN sanctions result in widespread archaeological looting 2003: Iraq War results in end of Ba'athist regime; pillage of museums and archaeological sites

[7]For more details of the mathematical historiography see [Høyrup 1996]; for archaeology the best account is still [Lloyd 1980].

[8]For the political background see [Tripp 2002].

picture is much more complicated, inviting us to study it on its own terms, not simply as a precursor to something else. Farther afield, there has been widespread critical examination of academia's imperialist legacy. Influenced by Edward Said's seminal *Orientalism*, post-colonialist historians have highlighted and challenged European and American scholarship's appropriation of the Middle East's past for the West. Similarly, an influential anthropological critique of the categories "us/them" and "sameness/otherness" has drawn attention to the familiarizing strategies used by nineteenth- and twentieth-century scholars to domesticate mainstream historical interpretations of ancient Mesopotamian culture toward the West and away from the Middle East.[9]

It is hoped that the wide range of mathematical sources translated in this chapter reflects these changing attitudes: they have been chosen not as extraordinary or surprising examples of modernity with which you are invited to identify. Rather they represent the typical products of scribal culture, reflecting a wide variety of textual genres from the simplistic to the sophisticated, from rote-learned tables and rough calculations to carefully constructed word problems. But inevitably, as research methodologies develop and multiply, and ever more sources are discovered and deciphered, the selections and interpretations given here will come to seem as dated as those of a hundred years ago. Scholarly fashions come and go, but the tablets themselves endure.

I.b. Mathematics and scribal culture in ancient Iraq

Mathematics is not created out of nothing—it is written by individuals operating within the social and intellectual norms and conventions of the societies in which they dwell. Thus coming to grips with another culture's mathematics is not simply a matter of translating one notation into another. Instead we need to explore the personal, intellectual, and social circumstances under which it was written. Paradoxically, of all ancient and "other" mathematics, Mesopotamia provides us with the most potential for contextualising interpretation. First, relatively imperishable clay tablets leave us with a written legacy of primary sources many times greater than other ancient societies such as Egypt (or compare the Greek tradition, where original documents are almost nonexistent). Second, Iraq has some incredibly well preserved and carefully recorded urban archaeology, including many sites where mathematical tablets have been found. To study its mathematics, then, only as mathematics and not as the product of a person's body, brain, and culture, we willfully ignore a historical source of unparalleled richness that has the potential to help us understand the interconnections between mathematics and other aspects of culture and society that no other ancient civilization can match.

Rather than outlining here such a history of Mesopotamian mathematics (Table 2), we shall look instead at three different archaeological contexts in which mathematics has been found. As we shall see, we have to abandon modern notions of universal literacy and numeracy; in this world reading, writing, and calculating were almost exclusively professional activities to which even the wealthy and powerful did not necessarily have access. Writing was used only by temples, palaces, and affluent families, primarily to record property ownership and the rights to income. It will thus be important for us to distinguish between numeracy, which uses mathematics for the accounting and administration of assets, and mathematics, as an intellectual activity for teaching, learning, and creating new mathematical skills and ideas.[10]

---

[9]See, e.g., [Fowler 1999; Said 1978; Bahrani 1998].

[10]See [Robson 2001a] for a discussion of contexualization; and [Robson forthcoming] for a social history of mathematics in ancient Iraq.

TABLE 2.2: Overview of Mathematical Developments in, Ancient Mesopotamia

Date	*Mathematical developments*	*Socio-political background*[11]
4000 BCE	Pre-3200: Preliterate token-based accounting	Increasing urbanization in southern Iraq
3500 BCE	3200: Literate numeracy; the first school mathematics	Uruk period/Early Bronze Age Sumerian language
3000 BCE	Sophisticated accounting and quantitative planning	Early Dynastic period: city states
2500 BCE	School mathematics; c. 2050: first attestation of the mature sexagismal place value system	Akkadian language Territorial empires of Akkad and Ur
2000 BCE	c. 1850–1650: widespread evidence of "pure" mathematics in scribal training: line geometry, concrete algebra, quantity surveying	City states; empire of Babylon: Middle Bronze Age, or Old Babylonian period
1500 BCE	Cuneiform culture and sexagesimal numeracy spread from southern Iraq	"Amarna age" of international diplomatic contact across the Middle East; Late Bronze Age
1000 BCE	800 BCE–: quantitive methods in Assyrian scholarship	Assyrian empire; Aramaic language and the alphabet; Iron Age
500 BCE	400 BCE–: mathematics in the temples of Uruk and Babylon	Persian and Seleucid empires: Late Babylonian period
0 BCE/CE	75 CE: the last known datable cuneiform tablet; transmission of mathematical knowledge and practice to other languages	Parthian empire

Inana's temple in Uruk, 3200 BCE

By the late fourth millennium, the flat marshlands of the south of modern-day Iraq were teeming not only with wildlife but also with people. The region's population centers, which were the largest the world had yet known, were sustained through a sophisticated network of socioeconomic interactions. In earlier societies most fit and healthy individuals had been economically active providers and producers as well as consumers. Now new social relations of unprecedented complexity relied also on managers, administrators, and organizers, who earned their living and prestige not through production but through the oversight and control of the community as a whole. This new social class dispensed justice, managed communal building and agricultural projects, and took the lead in religious life, and it did so through the institution of the temple.

The temple was at the literal and metaphorical heart of every large Sumerian settlement. Made of mud brick and whitewashed in brilliant limestone plaster, it dominated the flat marshland landscape as a conspicuous emblem of the city's wealth, prestige, and functionality. At one level the home of the city's patron god or goddess, it was also the economic powerhouse of the city and its hinterland. Through offerings and large-scale use of labor, whether forced or voluntary, temples came to own vast tracts of arable land, where barley was grown and huge herds of sheep and goats were tended. All of these assets had to be managed in order to provide for the god and his or her followers; this is the first known context in which

[11]For a general overview of Mesopotamian history, see [Kuhrt 1995; Roaf 1990; or Van De Mieroop 2004].

writing was used. Managers at the goddess Inana's temple E-ana in Uruk, some time in the thirty-third century BCE, adapted a long-used system of accounting with clay tokens in order to record, manage, and predict the wealth of their employer. Onto flattened clay surfaces they impressed stylized outlines of accounting tokens to represent numbers, and scratched pictograms and other symbols to stand for the objects they were counting and accounting for. The accountants used a dozen or so different numeration and metrological systems, depending on the type of commodity they were dealing with. As trainees they thus had to learn how to deal with various number bases and conversions between different measurement systems as well as how to write some 1200 symbols representing all the different categories and subtypes of objects, people, animals, land, and other assets they managed. The world's earliest known piece of school mathematics is an exercise in calculating the areas of two fields, yielding conspicuously round answers (see pp. 73–74). We do not know exactly who wrote it, or exactly when, but it was found along with about 5,000 other tablets, mostly temple accounts but including several hundred other school exercises, mostly vocabulary lists. The tablets had been thrown away when they were no longer needed, along with other rubbish discarded by the administrators, and then reused as building rubble when Inana's temple was rebuilt and refurbished some time before 3000 BCE.[12]

A scribal school in Nippur, 1740 BCE

A few hundred miles north of Uruk, and about a millennium and a half later, the city of Nippur was an important religious center of the kingdom of Babylonia, which covered most of south and central Iraq. Home of the great god Enlil, it was where kings traditionally came to receive Enlil's blessings and permission to rule. Some 100 meters south of his great temple complex E-kur was an unassuming small house in a block of other small houses, which had been built out of mud brick in the late nineteenth century BCE. Now some eighty years later it was occupied by a priest who ran a small school in his tiny front courtyard, where he had built a bench and a bitumen-lined bin in which to soak and recycle the tablets that his students wrote.

Here he taught one or two students, perhaps his own sons. For their elementary education he followed a system used by other teachers in the city: first the basics of impressing wedge-shaped marks in the clay with a reed stylus, then the careful copying, repetition, and memorization of a standard sequence of visually simple cuneiform signs. Next his young charges learned how to write the Sumerian words for various objects, grouped according to the materials they were made of, just as their long-ago predecessors had in Uruk. Then came a series of more abstract exercises in understanding the complexities of Sumerian and the cuneiform script, including a set series of multiplication tables and metrological lists (cf. pp. 82–90). Only then were the students ready to write whole sentences of Sumerian, and to take their first steps in Sumerian literature, a subject in which this particular teacher had a special interest. The literary compositions were not simply stories; they educated the young men in the myths and belief system of the temple, and inculcated in them a strong self-identity as professionally literate and numerate scribes. (Translations of some of these literary works are scattered throughout this chapter.) The seemingly endless rote-learning was interrupted occasionally by the opportunity to practice mathematical calculations—but the students' results were not always correct.

---

[12]See [Nissen et al. 1993, 1–46] for more detail on the Uruk accounting system.

Sometime after 1739 BCE, when the house needed to be repaired, nearly 1500 school tablets were used as bricks and building material, becoming embedded in the fabric of the house for the rest of its life.[13]

A scholarly household in Uruk, 420 BCE

By the late fifth century BCE Babylonia was a province of the enormous Persian Empire, though so far the end of native rule after millennia had made little impact on the day-to-day lives of the inhabitants. On the eastern outskirts of Uruk, inside the city wall, was a large mud-brick house arranged around a central courtyard. It was occupied by a family of scholars with close associations to the sky god Anu's temple Resh in the city center, who traced their descent from an ancestor called Shangû-Ninurta, "Chief administrator of the god Ninurta." The men of the family made their living as incantation-priests and healers, using a mixture of herbal remedies, incantations, and supernatural diagnostic and prognostic techniques to care for their patients.

The family owned a library of nearly two hundred scholarly and professional works on clay tablets and waxed wooden writing boards, stored in large terracotta jars in a special room off the courtyard. The older men taught the younger male members of the family to write Sumerian and Akkadian and to calculate. The boys' early efforts were recycled in a bituminized area of the courtyard near the tablet room. By their late teens they were ready to copy the sophisticated reference materials of their trade, signing and dating their tablets and dedicating them to an older family member. Part of their profession involved a deep understanding of the complex ominous calendar of auspicious, inauspicious, and evil days so that their medication and ritual could be as effective as possible. This led to an interest in astronomy and mathematics, which involved not only copying arithmetical tables and collections of word problems from earlier originals but also checking that the tables were indeed correct (see pp. 167–170).[14] When the family vacated the house, for reasons unknown, they left the library *in situ*. The abandoned house eventually collapsed or was demolished, crushing the storage jars and their contents, and a new house built over the ruins of the old.

These three examples show, then, that mathematics was not a leisure pursuit in ancient Iraq; nor was it an exclusive professional activity supported by institutional patronage. Rather, it was a fundamental part of the process of becoming professionally literate and numerate, whether as accountant, priest, or scholar. All the mathematics presented here should be understood in such a pedagogical, and usually domestic, context.

I.c. From tablet to translation

Abandonment and discovery

After their useful life was over, most clay tablets were dunked into water and recycled without ever being baked for posterity. This is true not only of trainee scribes' school exercises, but also of archival documents belonging to temples, palaces, and families. With the development of libraries in the mid-second millennium, reference copies began to be kept of important scholarly works, both in repositories attached to temples or palaces, and in smaller family

[13]See [Robson 2001, 2002] for more on the school known as House F in Nippur.

[14]See [Robson forthcoming, chap. 8] for more on the Shangû-Ninurta family, and other mathematically inclined occupants of their house.

Thus I have distinguished consistently between, for instance, symmetrical and asymmetrical addition: X *u* Y *kamārum* "to sum X and Y" versus X *ana* Y *wasābum* "to add X to Y." The latter is a very physical operation; its subtractive counterpart is X *ina* Y *nasāhum* "to take away X from Y." Similarly several sorts of multiplication are distinguished within the Old Babylonian mathematical corpus. Simple numerical multiplication of the sort found in tables is represented by the logogram A.RÁ "times." Geometrical multiplication, X *u* Y *šutakūlum*, in which lines are multiplied to form areas, literally "to make X and Y hold each other," I have translated as "to combine X and Y." Repetition of the same entity N times, X *ana* N *esēpum*, has become "to copy X N times." Finally, there is a more general verb, X *ana* Y *našûm*, literally "to raise X to Y," for which I have retained the phrase "to multiply X by Y."

I have also chosen to translate the names of some geometrical figures more literally, as their semantic range is not identical to ours. Thus *mithartum*, usually translated "square," may mean the area or side of a square, or even square root. I have thus chosen the term "square-side." Similarly *kippatum*, "circle," may stand for the area of a circle or its circumference; I have not distinguished them in the translations. On the other hand I have not (yet) been able to find a satisfactory single translation for *siliptum*, both "rectangle" and "diagonal," so I have translated it with one or the other term as appropriate. Then there is a range of geometrical figures which have no direct counterpart in modern mathematics: *santakkum* "wedge" is a three-sided figure, not strictly a triangle as one or more sides may be curved. A "crescent moon," *uskarum*, is a circle cut by a chord; it usually means a semicircle but may be a larger or smaller segment than that. Irregular quadrilaterals that may have one or more curved sides are called *pūt alpim*, "ox's brow," while the *apsamikkum*, a concave square figure composed of four inverted quarter-segments of the circumference of a circle, I have more loosely rendered as "cow's nose." The shape delimited by two one-third segments of a circle is an "ox's eye," *īn alpim*, while that composed of two one-quarter segments is a "barge," *makurrum*. A length running across the interior of a geometrical figure is called *tallum*; most often it means a diameter, but it can have other meanings too, so I translate it here as "dividing line." Finally, in geometrical algebra a line may be given an extra dimension, always of length 1, in order to convert it to an area: this is called a *wasītum*, or "projection."

In any event, it is important to remember that, however faithfully I have attempted to render the original sources into English, what you will be reading are *my translations*, and thus interpretations which are open to doubt and challenge. They are not the primary sources themselves, but an early twenty-first-century representation of them which is necessarily far removed from the originals. You will be reading alphabetic texts from a printed book in a familiar language, perhaps seated at a desk in a library or an office, a physical experience far removed from squatting on the ground in bright sunlight to pore over a clay tablet held in your hand. It will never be possible to fully comprehend this mathematics as it was meant to be read, for we cannot entirely escape our own twenty-first-century lives and brains and training, however hard we try (nor, perhaps, should we want to). But even if the enterprise is ultimately doomed to failure, that does not mean it is not rewarding and satisfying to try.

I.d. Explananda

Sources

The tablets translated here come from the following collections. For the most part I have based my translations on personal inspection of the tablets themselves, or of good photographs. This was not possible, for obvious reasons, in the case of the tablets from the Iraq Museum (IM), Baghdad, where published scale drawings (hand copies) have had to suffice.

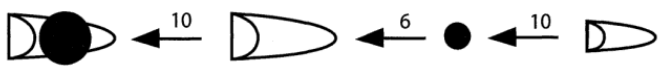
A	The Oriental Institute of the University of Chicago, Chicago.
AO, AOT	The Louvre, Paris; including excavations at Telloh (AOT). Tablets studied courtesy of Béatrice André-Salvini of the Département des Antiquités Orientales.
Ash	The Ashmolean Museum, Oxford. Tablets studied courtesy of the late Roger Moorey, former Keeper of Antiquities.
BM, *UET*	The British Museum, London; including the excavations at Ur (*UET*). Tablets studied courtesy of Christopher Walker, formerly Senior Assistant Keeper of the Ancient Near East, and the Trustees of the British Museum.
CBS, UM	Collection of the Babylonian Section, University of Pennsylvania Museum of Archaeology and Anthropology, Philadelphia. Tablets studied courtesy of Steve Tinney, Curator of the Babylonian Section.
HS	The Hilprecht Collection, University of Jena.
IM, Db, Haddad, W	The Iraq Museum, Baghdad; including excavations at Tell Dhiba'i (Db); Tell Haddad, ancient Me-Turan (Haddad); and Uruk (W).
Plimpton	The George A Plimpton Collection, Columbia University. Tablets studied courtesy of Jane Siegel of the Rare Book and Manuscript Library.
Strasbourg	Cuneiform collection of the Bibliothèque Nationale et Universitaire, Strasbourg.
TSS	Istanbul Arkeoloji Müzelerinde, Istanbul; excavations at Shuruppag.
VAT	Vorderasiatisches Museum, Berlin. Tablets studied by Jeremy Black courtesy of Joachim Marzahn, curator.
YBC	Yale Babylonian Collection, New Haven. Tablets studied courtesy of Ulla Jeyes, curator; and with the generous assistance of Paul-Alain Beaulieu.

Many cuneiform tablets are also in the hands of private collectors. However, as the export of Iraqi antiquities has been banned for many decades, and the market in them illegal in all member states of the United Nations since May 2003, the study or purchase of privately owned material is tantamount to handling stolen goods, except in the extremely rare cases where a long pedigree of ownership can be proved. Furthermore, the illicit market in cuneiform tablets encourages the continued plundering of badly protected, fragile archaeological sites and the consequent destruction of vital historical data for future generations. There is expected to be an upsurge in the underground international trade in cuneiform tablets as a result of the devastating archaeological looting in the aftermath of the 2003 Iraq War. Tablets for sale in shops, at auction, or on the web should be reported to the relevant national art theft police authority or to Interpol so that they may be confiscated and repatriated.

Metrology[19]

Early metrology The accountants of late fourth-millennium Uruk used at least twelve different metrological systems, depending on what they were measuring or counting. For instance, when counting discrete objects, their notation distinguished between the living and the dead, and between fish and cheese. However, identical symbols were used in different systems with different meanings. Although these systems were reformed and simplified over the coming centuries, some of the notational ambiguity remained in Early Dynastic metrology, as did the bundling of number and unit into a single sign.

Area units: ≈ 3.9 km^2 ≈ 64.8 ha ≈ 6.48 ha ≈ 2.16 ha ≈ 3600 m^2



Length units: ≈ 3.6 km ≈ 360 m ≈ 60 m ≈ 6 m

Length measures

1 rod	= 2 reeds	≈ 6 meters
1 reed	= 3 seed-cubits	≈ 3 meters
1 seed-cubit	= 2 cubits	≈ 1 meter
1 cubit	= 2 half-cubits	≈ 50 cm
1 cubit	= 3 double-hands	
1 double-hand	= 10 fingers	≈ 17 cm
1 finger		≈ 17 mm

Area measures

1 rod × 1 rod	= 1 sar	≈ 36 m^2
100 sar	= 1 iku	≈ 3600 m^2
1 eshe	= 6 iku	≈ 2.16 ha
1 bur	= 3 eshe = 18 iku	≈ 6.5 ha
1 shekel	= 1/60 sar	≈ 0.6 m^2

Capacity measures

1 **sila**	≈ 1 liter
1 ban	= 10 sila (10 liters)
1 bariga	= 6 ban (60 liters)
1 lidga	= 4 bariga (240 liters)
1 (great) gur	= 2 lidga (480 liters)
1 granary	= 2400 (great) gur (1,152,000 liters)

Metrology from the twenty-first to the sixteenth centuries Standard units of calculation (which are often implicit within Old Babylonian problems) are shown in **bold**.

[19]Following [Nissen et al. 1993, 30–31; Powell 1990; George 1993: 119].

Length measure

1 finger	≈ 17 mm
1 cubit	= 30 fingers (0.5 m)
1 **rod**	= 12 cubits (6 m)
1 chain	= 1 00 cubits or 5 rods (30 m)
1 cable	= 1 00 rods (360 m)
1 league	= 30 00 rods or 30 cables (10.8 km)

(1 **cubit** is the standard unit of height.)

Area and volume measure

1 area **sar**	= 1 rod square (36 m^2)
1 volume **sar**	= 1 area sar × 1 cubit (18 m^3)
1 ubu	= 50 sar (1800 m^2 or 900 m^3)
1 iku	= 2 ubu = 1 40 sar (3600 m^2 or 1800 m^3)
1 eshe	= 6 iku (2.16 ha or 108,000 m^3)
1 bur	= 3 eshe (6.48 ha or 324,000 m^3)

Units from the iku upward were not written explicitly, but used special unit-specific notation. These writings are indicated in the translations by putting the units in brackets, thus: "2 (bur) 1 (eshe) 3 (iku) area." Special, absolute value signs were used to write multiples of the bur: 10, 60, 600, 3600, and occasionally the "big 3600," or 3600 × 3600.

Capacity

1 **sila**	≈ 1 liter. The sila may be divided into 60 shekels.
1 ban	= 10 sila (10 liters)
1 bariga	= 6 ban (60 liters)
1 gur	= 5 bariga (300 liters)

Ban and usually bariga units were not written explicitly, but used special unit-specific notation. These writings are indicated in the translations by putting the units in brackets, thus: "1 gur 2 (bariga) 3 (ban) 4 sila." Multiples of the gur were written with the sexagesimal place value system.

Weight

1 grain	≈ 0.05 g
1 shekel	= 3 00 grains (8.3 g)
1 **mina**	= 1 00 shekels (0.5 kg)
1 talent	= 1 00 minas (30 kg)

Multiples of the talent were written with the sexagesimal place value system.

Brick measure

1 brick **sar**	= 720 bricks
Size of a small, unbaked brick: 15 × 10 × 5 fingers	
Number of small bricks in 1 volume sar: 1 26 24 bricks	= 7;12 brick sar
Size of a square, baked brick: 20 × 20 × 5 fingers	
Number of square bricks in 1 volume sar: 32 24 bricks	= 2;42 brick sar

First-millennium metrologies

Arû measure for lengths and areas

1 (big) finger	≈ 3 cm		
1 *arû* cubit	= 24 (big) fingers (0.75 m)		
1 rod	= 12 cubits (9 m)	1 rod × 1 rod	= 1 sar (≈ 81 m^2)
		1 ubu	= 50 sar (0.4 ha)
		1 iku	= 2 ubu (0.81 ha)
		1 eshe	= 6 iku
		1 bur	= 3 eshe = 18 iku
		1 shar	= 60 bur

Arû "seed measure" for areas

1 sila	≈ 270 m^2	
1 ban	= 10 sila (0.27 ha)	
1 bariga	= 6 ban (1.62 ha)	= 2 iku
1 gur	= 5 bariga (8.1 ha)	= 10 iku

Cable "reed measure" for lengths and areas

1 finger	≈ 2 cm	
1 cable-cubit	= 24 fingers (0.5 m)	
1 rod	= 12 cubits (6 m)	1 rod × 1 rod = 1 sar (36 m^2)
1 chain	= 5 rods or 50 cubits (30 m)	
1 cable	= 2 chains or 10 rods (60 m)	1 cable × 1 cable = 25 sar (900 m^2)
		1 iku = 1 40 sar (0.36 ha)

Cable "seed measure" for area

1 grain	= 70 cm^2
1 nindan	≈ 1800 grams (7.5 m^2)
1 sila	= 10 nindan (75 m^2)
1 ban	= 6 sila (450 m^2)
1 bariga	= 6 ban (0.27 ha)
1 gur	= 5 bariga (1.35 ha)

"Reed measure" for lengths and areas

1 finger	≈ 2 cm	1 finger × 1 finger	= 1 small finger (4 cm^2)
		1 cubit × 1 finger	= 1 grain (100 cm^2)
		1 reed × 1 finger	= 1 (area) finger (730 cm^2)
1 cubit	= 24 fingers (0.5 m)	1 cubit × 1 cubit	= 1 small cubit (0.25 m^2)
		1 reed × 1 cubit	= 1 (area) cubit (1.75 m^2)
1 reed	= 7 cubits (3.5 m)	1 reed × 1 reed	= 1 (area) reed (12.25 m^2)
1 rod	= 2 reeds or 14 cubits (7 m)		