THE MATHEMATICS OF *Everyday Life*

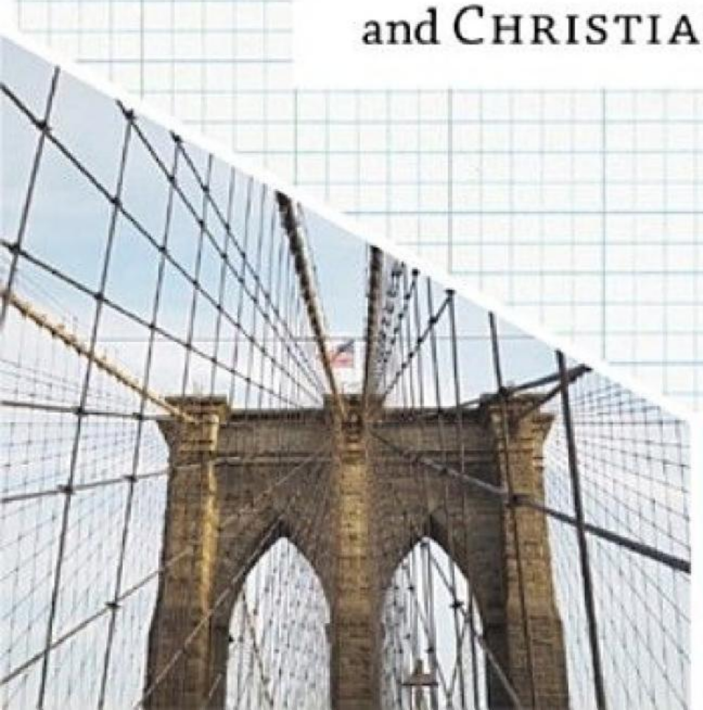

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Figure 1.9.

As previously mentioned, many mathematicians spent a time analyzing the problem of a perpetual calendar and devoted considerable attention to calculating the date of Easter Sunday. All church holidays fall on a specific date. The ecclesiastical rule regarding Easter is, however, rather complicated. Easter must fall on the Sunday after the first full moon that occurs after the vernal equinox. Easter Sunday, therefore, is a movable feast that may fall as early as March 22 or as late as April 25. The following procedure to find Easter Sunday in any year from 1900 to 1999 is based on a method developed by the famous German mathematician Carl Friedrich Gauss (1777–1855).

1. Find the remainder when the year is divided by 4. Call this remainder *a*.
2. Find the remainder when the year is divided by 7. Call this remainder *b*.
3. Find the remainder when the year is divided by 19. Multiply this remainder by 19, add 24, and again find the remainder when the total is divided by 30. Call this remainder *c*.
4. Then add $2a + 4b + 6c + 3$. Divide this total by 7 and call the remainder *d*.

The sum of *c* and *d* will give the number of days after March 22 on which Easter Sunday will fall. As an example, let's try to determine what the date of Easter Sunday in 1921 was.

1. $\frac{21}{4} = 5 +$ remainder of 1
2. $\frac{21}{7} = 3 +$ remainder of 0
3. $\frac{21}{19} = 1 +$ remainder of 2; $\frac{2 \cdot 19 + 24}{30} = 2 +$ remainder of 2
4. $\frac{2 + 0 + 12 + 3}{7} = 2 +$ remainder of 3
5. 2 + 3 = 5 days after March 22 is March 27.

(Note: The method above gives the date accurately except for the years 1954 and 1981. These years it gives a date exactly 1 week late, the correct Easters being April 18 and 19, respectively.)

We hope that this discussion gives you some insight into the complications of our calendar. For instance, we now understand why George Washington was born February 11, 1732, but we celebrate it on February 22. These curiosities await all who have a better understanding of the mathematical nature of our calendar, and now you perhaps won't take the calendar for granted!

Now with an insight into how numbers were developed over the past centuries and how we use them today, we are ready to embark on a journey of exploring the use of mathematics in our everyday life, where numbers still play an essential role.

CHAPTER 2

MATHEMATICS IN OUR EVERYDAY LIVES—ARITHMETIC SHORTCUTS AND THINKING MATHEMATICALLY

Almost every day we find ourselves in situations in which we apply, consciously or unconsciously, mathematical knowledge. Basic abilities in arithmetic are among those mathematical competencies we use most frequently. We may estimate the time we need to get from one place to another, combining different means of transportation, or calculating the total cost of the products in our shopping cart before we go to the register. Of course, one could argue that mastering arithmetic is nowadays superfluous and old-fashioned, since the cellphone practically everybody carries around is a smartphone, which can be used as a calculator. However, simple calculations can often be done much faster without the assistance of a calculator. It also keeps the brain fit, and using our own built-in "computer" is definitely more enjoyable than commanding an electronic device. But, most important, arithmetic offers an accessible playground for everyone to learn and develop mathematical thinking and problem-solving strategies, which can, in fact, be very helpful for making decisions in our everyday lives. We shall begin with some arithmetic shortcuts that can allow you to look at numbers and make some mental calculations—even faster than the calculator can.

ARITHMETIC WITH THE NUMBERS 9 AND 11

You may have wondered why certain numbers carry some special properties that allow a simplification of arithmetic processes, or, in other words, some arithmetic tricks. Let's consider some of these here. Both the number 11 and the number 9 are situated on either side of the number 10—which is the base of our number system—and therefore have very interesting properties. These properties give these numbers some unusual benefits in calculation. Let us begin by examining how we can determine divisibility by the numbers 9 and 11.

There are times in everyday-life situations when it is useful to know if a given number is divisible by 9 or by 3—especially if it can be done mentally! For example, suppose a restaurant bill of $71.22 needs to be split into three equal parts. Before actually doing the division, the thought about whether or not it is possible to split the bill equally into three parts may come into question. Wouldn't it be nice if there were some mental arithmetic shortcut for determining this? Well, here comes mathematics to the rescue. We are going to provide you with a rule to determine if a number is divisible by 3 or divisible by 9. The rule, simply stated, is:

If the sum of the digits of a number is divisible by 3 (or 9), then the original number is divisible by 3 (or 9).

Perhaps an example would be best to introduce this technique. Consider the number 276,357. Let's test it for divisibility by 3 (or 9). The sum of the digits is 2 + 7 + 6 + 3 + 5 + 7 = 30, which is divisible by 3, but not by 9. Therefore, the original number (276,357) is divisible by 3, but not by 9. However, the number 14,688 is divisible by both 3 and 9, since the sum of its digits is 1 + 4 + 6 + 8 + 8 = 27, which is divisible by both 3 and 9.

Just to make sure you are comfortable with this procedure, we will consider another example. Is the number 457,875 divisible by 3 or 9? The sum of the digits is 4 + 5 + 7 + 8 + 7 + 5 = 36, which is divisible by 9 (and then, of course, by 3 as well), so the number 457,875 is divisible by 3 and by 9. If by some remote chance it is not immediately clear to you whether the sum of the digits is divisible by 3 or 9 (perhaps because it might still be too large a sum), then take the sum of the digits of this just-found sum and continue the process until you can visually make a determination of divisibility by 3 or 9.

Now that you are an expert at determining whether a number is divisible by 3 or 9, we can go back to the original question about the divisibility of the restaurant bill of $71.22. Can it be divided into three equal parts? Because 7 + 1 + 2 + 2 = 12, and 12 is divisible by 3, then we can conclude that $71.22 is divisible by 3. (Notice that we need not be concerned with the decimal, since it is the number comprised of the digits with which we are concerned.)

In case you are interested as to why this rule actually works, here is a brief explanation using very simple algebra. Consider the base-10 number $N = ab{,}cde$, where the letters a, b, c, d, and e represent the digits and, therefore, the value of the number can be expressed as follows:

$N = 10^4a + 10^3b + 10^2c + 10d + e = (9 + 1)^4a + (9 + 1)^3b + (9 + 1)^2c + (9 + 1)d + e.$
Gathering those multiples of 9, we get

$N = [9M + (1)^4]a + [9M + (1)^3]b + [9M + (1)^2]c + [9 + (1)]d + e$ (where $9M$ indicates a different multiple of 9 each time).

Factoring these multiples of 9, we get $N = 9M[a + b + c + d] + a + b + c + d + e$, which implies that the entire expression will be divisible by 9 when the sum of the digits $a + b + c + d + e$ is divisible by 9.

Let us now consider if there is an analogous special property for divisibility by 11. If you have a calculator at hand, the problem is easily solved. But that is not always the case. Besides, there is such a clever “rule” for testing for divisibility by 11—one that it is worth knowing just for its charm.

The rule is quite simple:

If the difference of the sums of the alternate digits is divisible by 11, then the original number is also divisible by 11.

This sounds a bit complicated, but it really isn't. Let us take this rule a piece at a time. “The sums of the alternate digits” means you begin at one end of the number, taking the first, third, and fifth, digit (and so on), and add them. Then you add the remaining (even-placed) digits. Subtract the two sums, and then inspect this resulting difference for divisibility by 11.

Perhaps it might be best to demonstrate this through an example. Suppose we test 768,614 for divisibility by 11. The sums of the alternate digits are: $7 + 8 + 1 = 16$ and $6 + 6 + 4 = 16$.

The difference of these two sums is $16 - 16 = 0$, which is divisible by 11. (Remember, $\frac{0}{11} = 0$.) Therefore, we can conclude that 768,614 is divisible by 11.

Another example might be helpful to firm up an understanding of this procedure. To determine whether 918,082 is divisible by 11, we once again find the sums of the alternate digits: $9 + 8 + 8 = 25$ and $1 + 0 + 2 = 3$. The difference of the two sums is $25 - 3 = 22$, which is divisible by 11, and so the number 918,082 is divisible by 11. Here we have an example of a technique that not only can be helpful but also demonstrates the power and consistency of mathematics.

We would be remiss if we did not provide a justification for this ratherunexpected technique for determining whether a number is divisible by 11. Here is a brief discussion about why this rule works as it does. Consider the base-10 number $N = ab{,}cde$, where the letters a, b, c, d, and e represent the digits and, therefore, the value of the number can be expressed as

$$N = 10^4a + 10^3b + 10^2c + 10d + e = (11 - 1)^4a + (11 - 1)^3b + (11 - 1)^2c + (11 - 1)d + e.$$

If we let $11M$ represent a number which is a multiple of 11 (and it can be a different number each time, but still a multiple of 11), we can express the above equation as: $N = [11M + (-1)^4]a + [11M + (-1)^3]b + [11M + (-1)^2]c + [11 + (-1)]d + e$ or $N = 11M[a + b + c + d] + a - b + c - d + e$, which implies that divisibility by 11 of the number N is dependent on the divisibility of $a - b + c - d + e$, which written another way, is $(a + c + e) - (b + d)$, which is actually the difference of the sums of the alternate digits.

Having now considered rules for divisibility by these two special numbers, 9 and 11, let's consider other properties that these numbers have, to simplify our arithmetic processes. Perhaps one of the simplest mathematical tricks is to multiply by 11, mentally! This trick often gets a rise out of the unsuspecting mathematics-phobic person, because it is so simple that it is even easier than doing it on a calculator.

The rule is very simple:

To multiply a two-digit number by 11, just add the two digits and place this sum between the two digits.

Let's try using this technique. Suppose you need to multiply 45 by 11. According to the rule, add 4 and 5 and place this sum between the 4 and 5 to get 495.

This can become a bit more difficult when the sum of the two digits you are adding results in a two-digit number. We no longer have a single digit to place between the two original digits. So, if the sum of the two digits is greater than 9, then we place the units digit between the two digits of the number being multiplied by 11 and "carry" the tens digit to be added to the hundreds digit of the multiplicand. (Recall: The multiplicand is the number that is multiplied by another number, the multiplier—in this case, 11.) Let's try this procedure by finding the product of $78 \cdot 11$. We first get the sum of the digits: $7 + 8 = 15$. We place the 5 between the 7 and 8, and then we add the 1 to the 7, to get [7 + 1][5][8], or 858.

It is fair to ask whether this technique also holds true when a number consisting of more than two digits is multiplied by 11. Let's go straight for a larger number such as 12,345 and multiply it by 11. Here we retain the first and last digit, then we begin at the right-side digit and add every pair of digits, going to the left: 1[1 + 2][2 + 3][3 + 4][4 + 5]5 = 135,795.

As was the case earlier, if the sum of two digits is greater than 9, then we place the units digit appropriately and carry the tens digit. To better understand how this is done, consider the following multiplication, 456,789 · 11:

Follow along as we carry the process step-by-step:

4[4 + 5][5 + 6][6 + 7][7 + 8][8 + 9]9
4[4 + 5][5 + 6][6 + 7][7 + 8][17]9
4[4 + 5][5 + 6][6 + 7][7 + 8 + *1*][7]9
4[4 + 5][5 + 6][6 + 7][16][7]9
4[4 + 5][5 + 6][6 + 7 + *1*][6][7]9
4[4 + 5][5 + 6][14][6][7]9
4[4 + 5][5 + 6 + *1*][4][6][7]9
4[4 + 5][12][4][6][7]9
4[4 + 5 + *1*][2][4][6][7]9
4[10][2][4][6][7]9
[4 + *1*][0][2][4][6][7]9
[5][0][2][4][6][7]9
5,024,679

This technique for multiplying by 11 might well be shared with your friends. Not only will they be impressed with your cleverness, they may also appreciate knowing this shortcut.

We now revert back to the number 9, as we search for a technique for multiplying any number by 9. Although this technique may be a bit cumbersome, especially when compared to using a calculator, this algorithm provides some insights into number theory, which is the basis for our understanding arithmetic processes. The number 9 has another unusual feature that enables us to use a surprising multiplication algorithm. Don't be distracted by the rather-complicated appearance. Just know that we present it here to indicate a multiplication property provided by the number 9. This procedure is intended for multiplying numbers of two digits or more by 9. It is best to discuss the procedure as we apply it to the multiplication 76,354 · 9.

Step 1	Subtract the units digit of the	10 – 4 = **6**

	multiplicand from 10.	
Step 2	Subtract each of the remaining digits (beginning with the tens digit) from 9, and then add this result to the previous digit in the multiplicand, read it from right to left. (For any two-digit sums, carry the tens digit to the next sum.)	9 − 5 = 4, 4 + 4 = **8** 9 − 3 = 6, 6 + 5 = 11, **1** 9 − 6 = 3, 3 + 3 = 6, 6 + 1 = **7** 9 − 7 = 2, 2 + 6 = **8**
Step 3	Subtract 1 from the left-most digit of the multiplicand.	7 − 1 = **6**
Step 4	List the results in reverse order to get the desired product.	**687,186**

Table 2.1.

Although the technique here is not one that would be used to do the multiplication, we merely offer it for your amusement and so that the number 9 does not feel neglected in the multiplication process as compared to the number 11.

To compensate for the inability of the 9 to compete with the 11 in the process of multiplication, we shall present a procedure that the 9 provides beyond that of the 11.

HOW 9s CAN CHECK YOUR ARITHMETIC

As we mentioned earlier, the first occurrence in Western Europe of the Hindu-Arabic numerals we use today was in the book *Liber Abaci*, which was written in 1202 by Leonardo of Pisa (otherwise known as Fibonacci). As a young boy, Fibonacci traveled with his father, who directed a trading post in Bugia (in modern-day Algeria). Fibonacci traveled extensively along the Mediterranean coast, where he met merchants and became fascinated with the number system they used to do arithmetic. Recall that the publication of *Liber Abaci* provided the first use of these numerals in Europe. Before that, roman numerals were used extensively. They were, clearly, much more cumbersome to use for calculation than these ten numerals he had experienced in the Arabic world and which had originated in India.

Fascinated as Fibonacci was by the arithmetic calculations used in the Islamic world, in his book he introduced the system of "casting out nines"—which refers to a calculation check to determine if your result is possibly correct. The process requires subtracting a specific number of groups of 9 from the sum of the digits of the result (or, in other words, taking bundles of 9 away from the sum). Although this technique might come in handy, the nice thing about it is that it again demonstrates a hidden magic in ordinary arithmetic.

Before we discuss this arithmetic-checking procedure, we will consider how the remainder of a division by 9 compares to removing groups of 9 from the digit sum of the number. Let us find the remainder, when, say, 8,768 is divided by 9. Using a calculator, we find that the quotient is 974 with a remainder of 2.

This remainder can also be obtained by "casting out nines" from the digit sum of the number 8,768: This means that we will find the sum of the digits; and if the sum is more than a single digit, we shall repeat the procedure with the obtained sum. In the case of our given number, 8,768, the digit sum is 29 (8 + 7 + 6 + 8 = 29). Since this result is not a single-digit number, we will repeat the process with the number 29. Again, the casting-out-nines procedure is used to obtain 2 + 9 = 11; and, again, repeating this procedure for 11, we get 1 + 1 = 2, which is the same remainder as when we earlier divided 8,768 by 9.

We can now take this process of casting out nines to another application, that of checking multiplication. Perhaps it is best to see it applied. We would like to see if the following multiplication is correct: 734 · 879 = 645,186. We can check this by division, but that would be somewhat lengthy. We can also see if this product could be correct by casting out nines. To do that, we will take each of the factors and the product and then add the digits of each number—continuing this process as before until a single digit results:

For 734: 7 + 3 + 4 = 14; then 1 + 4 = 5.
For 879: 8 + 7 + 9 = 24; then 2 + 4 = 6.
For 645,186: 6 + 4 + 5 + 1 + 8 + 6 = 30; then 3 + 0 = 3.

The product of the first two factors is 5 · 6 = 30, which yields 3 by casting out nines (3 + 0 = 3); because this is the same as what we obtained when we added the digits of the product 645,186 (30), the answer could be correct.

For practice, we will do another casting-out-nines "check" for the following multiplication:

56,589 · 983,678 = 55,665,354,342

For 56,589:	5 + 6 + 5 + 8 + 9 = 33;	3 + 3 = 6
For 983,678:	9 + 8 + 3 + 6 + 7 + 8 = 41;	4 + 1 = 5

For 55,665,354,342: $5 + 5 + 6 + 6 + 5 + 3 + 5 + 4 + 3 + 4 + 2 = 48$;

$4 + 8 = 12$; $1 + 2 = 3$

To check for possibly having the correct product, we multiply the results from our first two factors: $6 \cdot 5 = 30$, or $3 + 0 = 3$, which matches the 3 resulting from the product digit.

A similar procedure can be used to check for the likelihood of a correct sum or quotient, simply by taking the sum (or quotient) and casting out nines, then taking the sum (or quotient) of these "remainders" and comparing it with the remainder of the sum (or quotient). They should be equal if the answer is correct.

As we deal with the base-10 throughout our lives, it is good to see how the numbers on either side of the 10 have special properties because of their position relative to 10. Once again, because our culture has selected 10 as the base for our number system, we have had an opportunity to explore the peculiarities that evolve from this arrangement.

RULES FOR DIVISIBILITY

Having discussed some of the peculiarities of the numbers 9 and 11 (which included rules for divisibility), it is appropriate for us now to consider rules for divisibility by other numbers. We can easily determine when a number is divisible by 2 or by 5, simply by looking at the last digit (i.e., the units digit) of the number. That is, if the last digit is an even number (such as 2, 4, 6, 8, 0, and so on), then the number will be divisible by 2. A fun fact that is not as well known is that if the number formed by the last two digits of a given number is divisible by 4, then the number itself is divisible by 4. Also, if the number formed by the last three digits is divisible by 8, then the number itself is divisible by 8. You ought to be able to extend this rule to divisibility by higher powers of 2 as well.

Similarly, for 5, if the last digit of the number being inspected for divisibility is either a 0 or 5, then the number itself will be divisible by 5. If the number formed by the last two digits is divisible by 25, then the number itself is divisible by 25. The similarity of this rule to the previous one results from the fact that 2 and 5 are the prime factors of 10, which is the base of our decimal number system.

With the proliferation of the calculator, there is no longer a crying need to be able to detect by which numbers a given number is divisible. You can simply do the division on a calculator. Yet, for a better appreciation of mathematics, divisibility rules provide an interesting window into the nature of numbers and their properties. For this reason (among others), the topic of divisibility still finds a place on the mathematics-learning spectrum.

Most perplexing has always been to establish rules for divisibility by prime numbers (which are numbers whose only factors are 1 and the number itself). This is especially true of the rule for divisibility by 7, which follows a series of very nifty divisibility rules for the numbers 2 through 6. As you will soon see, the techniques for some of the divisibility rules for prime numbers are almost as cumbersome as an actual division algorithm, yet they are fun, and, believe it or not, they can come in handy. Let us consider the rule for divisibility by 7 and then, as we inspect it, let's see how this can be generalized for other prime numbers.

> *The rule for divisibility by 7:* **Delete the last digit from the given number, and then subtract twice this deleted digit from the remaining number. If the result is divisible by 7, the original number is divisible by 7. This process may be repeated if the result is too large for simple inspection of divisibility of 7.**

Let's try an example to see how this rule works. Suppose we want to test the number 876,547 for divisibility by 7. Begin with 876,547 and delete its units digit, 7, and subtract its double, 14, from the remaining number: 87,654 – 14 = 87,640. Since we cannot yet visually inspect the resulting number for divisibility by 7 we continue the process.

We take the resulting number 87,640 and delete its units digit, 0, and subtract its double, still 0, from the remaining number; we get 8,764 – 0 = 8,764. This did not change the resulting number, 8,764, as we seek to check for divisibility by 7, so we continue the process.

Again, we take the resulting number 8,764 and delete its units digit, 4, and subtract its double, 8, from the remaining number; we get 876 – 8 = 868. Since we still cannot visually inspect the resulting number for divisibility by 7, we continue the process.

Continue with the resulting number 868 and delete its units digit, 8, and subtract its double, 16, from the remaining number. Doing this, we get 86 – 16 = 70, which is clearly divisible by 7. Therefore, the number 876,547 is divisible by 7.

Before we continue with our discussion of divisibility of prime numbers, you ought to practice this rule with a few randomly selected numbers and then check your results with a calculator.

Why does this rather-strange procedure actually work? The beauty of mathematics is that it clearly explains why some amazing procedures actually work. This will all make sense to you after you see what is happening with this procedure.

To justify the technique of determining divisibility by 7, consider the various possible terminal digits (that you are "dropping") and the corresponding subtraction that is actually being done by dropping the last digit. In the chart below you will see how dropping the terminal digit and doubling it to get the units digit of the number being subtracted gives us in each case a multiple of 7. That is, you have taken "bundles of 7" away from the original number. Therefore, if the remaining number is divisible by 7, then so is the original number, because you have separated the original number into two parts, each of which is divisible by 7, and, therefore, the entire number must be divisible by 7.

Terminal Digit	**Number Subtracted from Original**
1	$20 + 1 = 21 = 3 \cdot 7$
2	$40 + 2 = 42 = 6 \cdot 7$
3	$60 + 3 = 63 = 9 \cdot 7$
4	$80 + 4 = 84 = 12 \cdot 7$
5	$100 + 5 = 105 = 15 \cdot 7$
6	$120 + 6 = 126 = 18 \cdot 7$
7	$140 + 7 = 147 = 21 \cdot 7$
8	$160 + 8 = 168 = 24 \cdot 7$
9	$180 + 9 = 189 = 27 \cdot 7$

Table 2.2.

Now that we have a better understanding of why this works for divisibility by 7, let's examine the "trick" for divisibility by 13.

The rule for divisibility by 13: **This is similar to the rule for testing divisibility by 7, except that the 7 is replaced by 13 and, instead of subtracting twice the deleted digit, we subtract nine times the deleted digit each time.**

Let's check for divisibility by 13 for the number 5,616. Begin with 5,616 and delete its units digit, 6, then multiply it by 9 to get 54, which is then subtracted from the remaining number: 561 – 54 = 507. Since we still cannot visually inspect the resulting number for divisibility by 13, we continue the process. Take the resulting number, 507, and delete its units digit and subtract nine times this digit from the remaining number: 50 – 63 = –13. We see that –13 is divisible by 13, and, therefore, the original number is divisible by 13.

You might be wondering why we take the unit digit and multiply it by 9. To determine the "multiplier," 9 in this case, we sought the smallest multiple of 13 that ends in a 1. That was 91, where the tens digit is 9 times the units digit. Once again, consider the various possible terminal digits and the corresponding subtractions in the following table.

Terminal Digit	**Number Subtracted from Original**
1	$90 + 1 = 91 = 7 \cdot 13$
2	$180 + 2 = 182 = 14 \cdot 13$
3	$270 + 3 = 273 = 21 \cdot 13$
4	$360 + 4 = 364 = 28 \cdot 13$
5	$450 + 5 = 455 = 35 \cdot 13$
6	$540 + 6 = 546 = 42 \cdot 13$
7	$630 + 7 = 637 = 49 \cdot 13$
8	$720 + 8 = 728 = 56 \cdot 13$
9	$810 + 9 = 819 = 63 \cdot 13$

Table 2.3.

In each case, a multiple of 13 is being subtracted one or more times from the original number. Hence, if the remaining number is divisible by 13, then the original number is divisible by 13. Let's move on to another prime number.

Divisibility by 17: **Delete the units digit and subtract five times the deleted digit each time from the remaining number until you reach a number small enough to determine its divisibility by 17.**

We justify the rule for divisibility by 17 as we did the rules for divisibility by 7 and 13. Each step of the procedure subtracts a "bundle of 17s" from the original number until we reduce the number to a manageable size and can make a visual inspection to determine divisibility by 17.

The patterns developed in the preceding three divisibility rules (for 7, 13, and 17) should lead you to develop similar rules for testing divisibility by larger primes. The following chart presents the "multipliers" of the deleted digits for various primes.

To Test Divisibility By . . .	7	11	13	17	19	23	29	31	37	41	43	47
Multiplier	2	1	9	5	17	16	26	3	11	4	30	14

Table 2.4.

You may want to extend this chart. It's fun, and it will increase your perception of mathematics. You may also want to extend your knowledge of divisibility rules to include composite (i.e., non-prime) numbers. Why the following rule refers to relatively prime factors[1] and not just any factors is something that will sharpen your understanding of number properties. Perhaps the easiest response to this question is that relatively prime factors have independent divisibility rules, whereas other factors may not.

> *Divisibility by composite numbers:* **A given number is divisible by a composite number if it is divisible by each of its relatively prime factors.** The chart below offers illustrations of this rule. You should complete the chart to 48.

To Be Divisible By . . .	6	10	12	15	18	21	24	26	28
The Number Must Be Divisible By	2,3	2,5	3,4	3,5	2,9	3,7	3,8	2,13	4,7

Table 2.5.

At this juncture, you have not only a rather-comprehensive list of rules for testing divisibility but also an interesting insight into elementary number theory. Practice using these rules (to instill greater familiarity) and try to develop rules to test divisibility by other numbers in base-10 and to generalize these rules to other bases. Unfortunately, a lack of space prevents a more detailed development here. Yet we hope that these above examples have whet your appetite.

A QUICK METHOD TO MULTIPLY BY FACTORS OF POWERS OF 10

We all know that multiplying by powers of 10 is relatively easy. You need only place the appropriate number of zeros onto the number being multiplied by the power of ten. That is, 685 times 1,000 is 685,000. However, multiplying by factors of powers of 10 is just a bit more involved, but, in many cases, it can also be done mentally. Let's consider multiplying 16 by 25 (which is a factor of 100).

Since $25 = \frac{100}{4}$, $16 \cdot 25 = 16 \cdot \frac{100}{4} = \frac{16}{4} \cdot 100 = 4 \cdot 100 = 400$.

Nothing like a little practice to solidify a new algorithm. Following are a few more such examples:

$38 \cdot 25 = 38 \cdot \frac{100}{4} = \frac{38}{4} \cdot 100 = \frac{19}{2} \cdot 100 = 9.5 \cdot 100 = 950$

$1.7 \cdot 25 = \frac{17}{10} \cdot \frac{100}{4} = \frac{17}{4} \cdot \frac{100}{10} = 4.25 \cdot 10 = 42.5$.

In the previous line, in our last step we were able to break up the fraction $\frac{17}{4}$ as follows:

$\frac{17}{4} = \frac{16}{4} + \frac{1}{4} = 4 + 0.25 = 4.25$.

In an analogous fashion, we can multiply numbers by 125, since $125 = \frac{1000}{8}$.

Here are some examples of multiplication by 125 done mentally!

$32 \cdot 125 = 32 \cdot \frac{1000}{8} = \frac{32}{8} \cdot 1000 = 4 \cdot 1000 = 4000$

$78 \cdot 125 = 78 \cdot \frac{1000}{8} = \frac{78}{8} \cdot 1000 = \frac{39}{4} \cdot 1000 = 9.75 \cdot 1000 = 9750$

When multiplying by 50, you can use $50 = \frac{100}{2}$; when multiplying by 20, you can use $20 = \frac{100}{5}$.

Practice with these special numbers will clearly be helpful to you, since with some practice you will be able to do many calculations faster than the time it takes to find and then turn on your calculator!

ARITHMETIC WITH NUMBERS OF TERMINAL DIGIT 5

We offer now some other possibilities for mental calculation. Not that we want to detract you from using a calculator, but we wish merely to give you a sense of understanding and appreciation of number relationships that are part of our everyday lives.

Suppose we want to square 45. That is, $45^2 = 45 \cdot 45 = 2025$.

The process requires three steps:

Step 1	Multiply the multiples of 10 that are one higher and one lower than the number to be squared.	$40 \cdot 50 = 2000$
Step 2	Square the units digit 5.	$5 \cdot 5 = 25$
Step 3	Add the two results from steps 1 and 2.	$2000 + 25 = 2025$

Table 2.6.

In case you are curious why this works, we offer a short proof of this by using elementary algebra. Recall that the square of a binomial $(u + v)^2 = u^2 + 2uv + v^2$. We let $(10a + 5)$ be the multiple of 5 to be squared.

Then we have $(10a + 5)^2 = 100a^2 + 100a + 25 = 100a(a + 1) + 25 = a(a + 1) \cdot 100 + 25$, which can be rewritten as $10a \cdot 10(a + 1) + 25$, which shows algebraically just what we did in the three steps above.

Here is an example of how to interpret this for an actual multiplication, say, $175 \cdot 175$:

Where $a = 17$, we get:

$x^2 = 10a \cdot 10\,(a + 1) + 25 = 10 \cdot 17 \cdot 10(17 + 1) + 25 = \mathbf{170 \cdot 180 + 5 \cdot 5 =} 30600 + 25 = 30{,}625$.

Another way of looking at this technique is by considering the pattern that evolves below. Look at this list and see if you can identify the pattern of the two-digit numbers squared.

$05^2 = \mathbf{25}$

$15^2 = 2\mathbf{25}$

$25^2 = 6\mathbf{25}$

$35^2 = 12\mathbf{25}$

$45^2 = 20\mathbf{25}$

$55^2 = 30\mathbf{25}$

$65^2 = 42\mathbf{25}$

$75^2 = 56\mathbf{25}$

$85^2 = 72\mathbf{25}$

$95^2 = 90\mathbf{25}$

You will notice that in each case the square ends with 25 and the preceding digits are determined as follows:

$05^2 = \mathbf{00}25,\ 0 = 0 \cdot 1$

$15^2 = \mathbf{02}25,\ 2 = 1 \cdot 2$

$25^2 = \mathbf{06}25,\ 6 = 2 \cdot 3$

$35^2 = \mathbf{12}25,\ 12 = 3 \cdot 4$

$45^2 = \mathbf{20}25,\ 20 = 4 \cdot 5$

$55^2 = \mathbf{30}25,\ 30 = 5 \cdot 6$

$65^2 = \mathbf{42}25,\ 42 = 6 \cdot 7$

$75^2 = \mathbf{56}25,\ 56 = 7 \cdot 8$

$85^2 = \mathbf{72}25,\ 72 = 8 \cdot 9$

$95^2 = \mathbf{90}25,\ 90 = 9 \cdot 10$

The same rule can be extended to three-digit numbers and beyond. Take, for example, $235^2 = 552\mathbf{25}$; the three digits preceding the last two digits (25), which are $552 = 23 \cdot 24$. The advantage of mental arithmetic tends to lose its attractiveness when we exceed two-digit numbers, since we must multiply two two-digit numbers—something usually not easily done mentally.

MULTIPLYING TWO-DIGIT NUMBERS LESS THAN 20

Aside from the electronic calculator, there are many methods for multiplying two-digit numbers up to 20. Here are some of these methods, which might also provide some insight to other methods.

Multiply 18 · 17 mentally. Here we seek to "extract" a multiple of 10 first—this time for the number $17 = 10 + 7$.

$$18 \cdot 17 = 18 \cdot 10 + 18 \cdot 7 = 18 \cdot 10 + 10 \cdot 7 + 8 \cdot 7 = 180 + 70 + 56 = 306$$

Or we can break up the number $18 = 10 + 8$, and work the method as follows:

$$18 \cdot 17 = 10 \cdot 17 + 8 \cdot 17 = 10 \cdot 17 + 10 \cdot 8 + 8 \cdot 7 = 170 + 80 + 56 = 306$$

Another method of multiplication would seek to obtain simpler factors—first for $18 = 20 - 2$:

$$18 \cdot 17 = (20 - 2) \cdot 17 = 20 \cdot 17 - 2 \cdot 17 = 340 - 34 = 306$$

Or, if we choose to use 17, $17 = 20 - 3$:

$18 \cdot 17 = 18 \cdot (20 - 3) = 18 \cdot 20 - 18 \cdot 3 = 360 - 54 = 306$

We can also use an entirely different method, albeit more of a novelty than a real help for quick multiplication, to multiply two numbers less than 20, such as $18 \cdot 17$, as follows:

Step 1	Select one of the two numbers you are multiplying (say, 18) and to it add the units digit of the other number (17).	$18 + 7 = 25$
Step 2	Place a zero at the end of this number.	250
Step 3	Multiply the two units digits of the two original numbers.	$8 \cdot 7 = 56$
Step 4	Add the results of steps 2 and 3.	$250 + 56 = 306$

Table 2.7.

You might wish to try this technique with other two-digit numbers up to 20, probably more for entertainment than for practical use.

In the previous cases, we used the properties of binomial multiplication. Here we will use this in a more general way. Perhaps you can recall the following binomial multiplication: $(u + v) \cdot (u - v) = u^2 - uv + uv - v^2 = u^2 - v^2$, where u and v can take on any values that would be convenient to us. When we can apply this to the multiplication of $93 \cdot 87$, we notice that the two numbers are symmetrically distanced from 90. This allows us to do the following:

$93 \cdot 87 = (90 + 3) \cdot (90 - 3) = 90^2 - 3^2 = 8100 - 9 = 8091$.

Here are a few further examples to help see the procedure in action.

$42 \cdot 38 = (40 + 2) \cdot (40 - 2) = \mathbf{40}^2 - \mathbf{2}^2 = 1600 - 4 = 1596$

$64 \cdot 56 = (60 + 4) \cdot (60 - 4) = \mathbf{60}^2 - \mathbf{4}^2 = 3600 - 16 = 3584$

As strange as this procedure may seem to be, with a little practice, it could come in quite handy when faced with this sort of multiplication problem.

Here are a few more examples to guide you along:

$67 \cdot 63 = (65 + 2) \cdot (65 - 2) = \mathbf{65^2} - \mathbf{2^2} = [60 \cdot 70 + 5 \cdot 5] - 4 = 4225 - 4 = 4221$

$26 \cdot 24 = (25 + 1) \cdot (25 - 1) = \mathbf{25^2} - \mathbf{1^2} = [20 \cdot 30 + 5 \cdot 5] - 1 = 625 - 1 = 624$

MENTAL ARITHMETIC CAN BE MORE CHALLENGING—BUT USEFUL!

Here is another possible calculating shortcut that can be done mentally, of course, with some practice. Take, for example, the multiplication: $95 \cdot 97$.

Step 1	Add the numbers (95 + 97).	= 192
Step 2	Delete the hundreds digit.	= 92
Step 3	Add two zeros onto the number	= 9200
Step 4	$(100 - 95) \cdot (100 - 97) = 5 \cdot 3$	= 15
Step 5	Add the last two numbers	= 9215

Table 2.8.

This technique also works when seeking the product of two numbers that are farther apart. Let's examine $89 \cdot 73$:

$89 + 73 = 162$

~~1~~62 (Delete the hundreds digit.)

Add on two zeros = 6200

Then add $(100 - 89) \cdot (100 - 73) = 11 \cdot 27 = 297$ to get 6497.

For those who might be curious why this technique works, we can show you the simple algebraic justification:

We begin with the two-digit numbers represented by $100 - a$ and $100 - b$, where a and b are less than 100.

Step 1: $(100 - a) + (100 - b) = 200 - a - b$

Step 2: Delete the hundreds digit—which means subtracting 100 from the number:

$(200 - a - b) - 100 = 100 - a - b$

Step 3: Add on two zeros, which means multiplying by 100:

$(100 - a - b) \cdot 100 = 10{,}000 - 100a - 100b$

Step 4: $a \cdot b$

Step 5: Add the last two results:

$10{,}000 - 100a - 100b + a \cdot b$

$= 100 \cdot (100 - a) - (100b - ab)$

$= 100 \cdot (100 - a) - b \cdot (100 - a) = (100 - a) \cdot (100 - b)$, which is what we set out to show. Now you just need to practice this method to master it!

On paper you might think that some of the methods we are presenting as mental arithmetic are more complicated than doing the work with the traditional algorithms. Yet, with practice, some of these methods for two-digit numbers will become simpler to do mentally than writing them on paper and following the conventional steps.

ARITHMETIC WITH LOGICAL THINKING

In today's world, most people seem to neglect arithmetic shortcuts. With the ever-present electronic calculator, many people stop thinking arithmetically, and quantitatively. Yet there are times when we truly have to think, and neither the calculator nor any arithmetic shortcuts will help us find the required answer. We will begin with a little bit of logic that shows us the kind of thinking that we need to do in today's world.

There are times when simple arithmetic is not enough to answer a question. These are times when logical reasoning must be used to buttress the arithmetic. Let's consider one such example.

A woman passes her neighbor's house with her three sons, and her neighbor asks her how old her three sons are. She responds that, coincidentally, the product of their ages is 36, and the sum of their ages is the same number as his address. He looks puzzled as he stares at the house number, which is 13, and then he becomes even more puzzled when the woman tells him that she almost forgot one essential piece of information: her oldest son's name is Max. This really baffles him. How can this man determine the ages of the woman's sons? (We are only dealing with integer ages.)

To determine the ages of the three sons, the man considers which three numbers have a product of 36 and a sum of 13. They are (1, 6, 6) and (2, 2, 9).

So, when the woman tells the man that she left out an essential piece of information, it must have been to differentiate between the two sums of 13. When she mentioned that her oldest son is named Max, she was indicating that there is only *one* older son, thus eliminating the possibility of twins (or perhaps children of the same age having been born in the same year) of age 6. Hence, the ages of the three sons are 2, 2, and 9. Here arithmetic calculation alone did not help us answer the question; we needed to use some logical thinking. Let us now consider some clever methods for mental calculation.

USING THE FIBONACCI NUMBERS TO CONVERT KILOMETERS TO AND FROM MILES

When we say that mathematics is always available to help in your everyday life, we can demonstrate it very nicely by showing you a clever way to convert miles to and from kilometers. Most of the world uses kilometers to measure distance, while the United States still holds on to the mile to measure distance. This requires a conversion of units when one travels in a country in which the measure of distance is not the one to which we are accustomed. Such conversions can be done with specially designed calculators or by some "trick" method. That is where the famous Fibonacci numbers come in. Just to refresh your memory, the first few Fibonacci numbers are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,...

Before we discuss the conversion process between these two units of measure, let's look at their origin. The mile derives its name from the Latin word for 1,000, *mille*, as it represented the distance that a Roman legion could march in 1,000 paces (which is 2,000 steps). One of these paces was about 5 feet, so the Roman mile was about 5,000 feet. The Romans marked off these miles with stones along the many roads they built in Europe—hence the name, "milestones"! The name *statute mile* (our usual measure of distance in the United States today) goes back to Queen Elizabeth I of England (1533–1603) who redefined the mile from 5,000 feet to 8 furlongs (5,280 feet) by statute in 1593. A furlong is a measure of distance within Imperial units and US customary units. Although its definition has varied historically, in modern terms it equals 660 feet, and is therefore equal to 201.168 meters. There are eight furlongs in a mile. The name "furlong" derives from the Old English words *furh* ("furrow") and *lang* ("long"). It originally referred to the length of the furrow in one acre of a ploughed open field (a medieval communal field that was divided into strips). The term is used today for distances horses run at a race track.

The metric system dates back to 1790, when the French National Assembly (during the French Revolution) requested that the French Academy of Sciences establish a standard of measure based on the decimal system, which they did. The unit of length they called a "meter" is derived from the Greek word *metron*, which means "measure." Its length was determined to be one ten-millionth ($1 \cdot 10^{-7}$) of the distance from the North Pole to the equator along a meridian going near Dunkirk, France, and Barcelona, Spain. Clearly, the metric system is better suited for scientific use than is the American system of measure. By an Act of Congress in 1866, it became "lawful throughout the United States of America to employ the weights and measures of the metric system in all contracts, dealings, or court proceedings."[2] Although it has not been used very often, there is curiously no such law establishing the use of our mile system.

To convert miles to and from kilometers, we need to see how one mile relates to the kilometer. The statute mile is exactly 1609.344 meters long. Translated into kilometers, this is 1.609344 kilometers. One the other hand, one kilometer is 0.621371192 miles long. The nature of these two numbers (reciprocals that differ by almost 1) might remind us of the golden ratio, which is approximately 1.618, and its reciprocal, which is approximately 0.618. Remember, it is the only number whose reciprocal differs from it by exactly 1. This would tell us that the Fibonacci numbers, the ratio of whose consecutive members approaches the golden ratio, might come into play here.

Let's see what length 5 miles would be in kilometers: 5 times 1.609344 = 8.04672 ≈ 8.

We could also check to see what the equivalent of 8 kilometers would be in miles: 8 times 0.621371192 = 4.970969536 ≈ 5. This allows us to conclude that approximately 5 miles is equal to 8 kilometers. Here we have two of our Fibonacci numbers.

As mentioned above, the ratio of a Fibonacci number to the one before it is approximately , which is the symbol used to note the golden ratio. Therefore, since the relationship between miles and kilometers is very close to the golden ratio, they appear to be almost in the relationship of consecutive Fibonacci numbers. Using this relationship, we would be able to approximately convert 13 kilometers to miles by replacing 13 with the previous Fibonacci number, 8. This would reveal to us that 13 kilometers is equivalent to about 8 miles. Similarly, 5 kilometers is about 3 miles, and 2 kilometers is roughly 1 mile. The higher Fibonacci numbers will give us a more accurate estimate, since the ratio of these larger consecutive Fibonacci numbers gets closer to .

Now suppose you want to convert 20 kilometers to miles. We have selected 20 because it is *not* a Fibonacci number. We can express 20 as a sum of Fibonacci numbers and convert each number separately and then add them. Thus, 20 kilometers = 13 kilometers + 5 kilometers + 2 kilometers. By replacing each of these Fibonacci numbers with the one lower, we have 13 replaced by 8, 5 replaced by 3, and 2 replaced by 1. This, therefore, reveals that 20 kilometers is approximately equal to 8 + 3 + 1 = 12 miles. (Of course, if we would like to have a faster and perhaps less accurate estimate, we notice that 20 is close to the Fibonacci number 21. Using that number gives us 13 miles, a reasonable estimate done more quickly.)

Representing integers as sums of Fibonacci numbers is not a trivial matter. We can see that every natural number can be expressed as the sum of other Fibonacci numbers without repeating any one of them in the sum. Let's take the first few Fibonacci numbers to demonstrate this property as shown in table 2.9:

n	**The Sum of Fibonacci Numbers Equal to *n***
1	1
2	2
3	3

4	1 + 3
5	5
6	1 + 5
7	2 + 5
8	8
9	1 + 8
10	2 + 8
11	3 + 8
12	1 + 3 + 8
13	13
14	1 + 13
15	2 + 13
16	3 + 13

Table 2.9.

You should begin to see patterns and also note that we used the fewest number of Fibonacci numbers in each sum in the table above. For example, we could also have represented 13 as the sum of 2 + 3 + 8, or as 5 + 8. Try to express larger natural numbers as the sum of Fibonacci numbers. Each time, ask yourself if you have used the fewest numbers in your sum.

To use this process to achieve the reverse, that is, to convert miles to kilometers, we write the number of miles as a sum of Fibonacci numbers and then replace each by the next *larger* Fibonacci number. Converting 20 miles to kilometers, therefore, gives us a sum as 20 miles = 13 miles + 5 miles + 2 miles. Now, replacing each of the Fibonacci numbers with their next larger in the sequence, we arrive at 20 miles = 21 kilometers + 8 kilometers + 3 kilometers = 32 kilometers.

To use this procedure, we are not restricted to use the Fibonacci representation of a number that uses the fewest Fibonacci numbers. You can use any combination of Fibonacci numbers whose sum is the number you are converting. For instance, 40 kilometers is $2 \cdot 20$, and we have just seen that 20 kilometers is 12 miles. Therefore, 40 kilometers is $2 \cdot 12 = 24$ miles (approximately). It should be noted that the larger the Fibonacci numbers being used, the more accurate the estimated conversion will be.

Consequently, we have another example of how some more sophisticated mathematics can be helpful in resolving a common, everyday problem.

THINKING "OUTSIDE THE BOX"

While grappling with a problem, very often we are asked by friends and colleagues to "think outside the box." Essentially, what is being suggested is to avoid trying to solve a problem in the traditional and expected fashion, and instead to look at the problem from a different point of view. Practically by definition, this could be considered a counterintuitive way of thinking. This can even be true when a rather-simple problem is posed and the straightforward solution becomes a bit complicated. You could say that many people look at a problem in a psychologically traditional manner: the way it is presented and played out. To illustrate this point, we offer a problem here to convince you of an alternate method of thinking. Try the problem yourself (don't look below at the solution), and see whether you fall into the "majority-solvers" group. The solution offered later will probably enchant you, as well as provide future guidance to you.

> *The problem:* **A single-elimination (one loss and the team is eliminated) basketball tournament has 25 teams competing. How many games must be played until there is a single tournament champion?**

Typically, the common way to approach this problem is to simulate the tournament, by selecting 24 teams to play in the first round (with one team drawing a bye). This will eliminate 12 teams (12 games have now been played). Similarly, of the remaining 13 teams, 6 play against another 6, leaving 7 teams in the tournament (18 games have been played now). In the next round, of the 7 remaining teams, 3 can be eliminated (21 games have so far been played). The four remaining teams play and eliminate 2 teams, leaving 2 teams for the championship game (23 games have now been played). This championship game is the 24th game.

A much simpler way to solve this problem, one that most people do not naturally come up with as a first attempt, is to focus only on the losers and not on the winners (as we have done above). We then ask the key question: "How many losers must there be in the tournament with 25 teams in order for there to be one winner?" The answer is simple: 24 losers. How many games must be played to get 24 losers? Naturally, 24. So there you have the answer, very simply done. Now most people will ask themselves, "Why didn't I think of that?" The answer is it was contrary to the type of training and experience we have had. Becoming aware of the strategy of looking at the problem from a different point of view may sometimes reap nice benefits, as was the case here.

Another way of looking at a solution to this problem—albeit quite similar to the previous solution—is to create an artificial situation where of the 25 teams, we will make 24 of these teams high-school-level players, and the 25th team a professional basketball team, such as the New York Knicks, which we will assume is superior to all of the other teams and will easily defeat each one. In this artificially construed situation, we would have each of the 24 high-school teams playing the Knicks, and, as expected, they would lose the game. Hence, after 24 games, a champion (in this case, the New York Knicks) is achieved.

In everyday life, we are often faced with simple issues that can be addressed by considering taking a look at the situation from an alternative point of view, just as we did in the above discussions. Suppose, for instance, you are asked to determine the number of people in attendance at a meeting of an association. Counting the members present would be unwieldy, since there may be many empty seats spread throughout the auditorium. Absentees all called in prior to the meeting to be excused. Therefore, you can solve the problem of determining the number present by subtracting the number of absentees from the total membership of the association. This exemplifies approaching the problem from a point of view different from simply counting or systematically "estimating" the attendance. If there are a small number of empty seats, counting the empty seats would be another way of determining the attendance (assuming you know what the total number of seats in the room is).

In any competitive sports event, the immediate tendency is to plan to use your strengths or strategies directly. An alternative point of view would be to assess and evaluate your competitor's strengths and weaknesses, and then generate your strategy from that assessment. Rather than viewing the impending contest and developing your game plan from your own vantage point, you could just as easily adopt a different point of view and assess it by a consideration of the competition.

Of course, this strategy has uses beyond the sports arena. It is interesting to note that in any form of negotiations, rather than only considering your own point of view, it is important to anticipate what position your "opponent" will take. Looking at the situation from this other point of view might help you find an appropriate direction for your own stance at these negotiations.

Another way of looking at this problem-solving strategy is to consider a detective investigating a case. She can sometimes select the guilty party from among several suspects not necessarily by proving that this one person committed the crime, but rather by adopting a different point of view and establishing that all of the other suspects had valid alibis, for instance. It is a process of elimination. Naturally, more substantial arguments would be necessary for a conviction, but at least this process would establish a direction for the investigation.

It is hoped that this logical demonstration will motivate you to consider alternative methods of solution even when faced with what appears to be a rather simple problem, and thereby establish a much more elegant method of solution.

SOLVING PROBLEMS BY CONSIDERING EXTREMES

Sometimes a clever solution to a simple mathematics problem reinforces a clever thinking procedure. One such strategy is sometimes used subconsciously by many people. We are referring to making a decision based on the process of using extremes. We often use extremes camouflaged in the phrase "the worst-case scenario." This approach might be considered as we decide which way to pursue an issue. Such a use of an extreme generally brings us to a good decision.

For instance, one can observe that the windshield of a car appears to get wetter the faster a car is moving in a rainstorm. This could lead some people to conclude that the car would not get as wet if it were to move slower. This leads to the natural next question that can be asked, namely, is it better to walk slowly or to run in a rainstorm so that you can minimize how much rain soaks your clothing? Setting aside the amount of wetness that the front of your body might get from the storm, let us consider two extreme cases for the top of the head: first, going infinitely fast, and, second, going so slowly as to practically be stationary. In the first case, there will be a certain amount of wetness on the top of the head. But, if we proceed at a speed of practically 0 mph, we would get drenched! Therefore, we conclude that the faster you move, the dryer you stay.

The previous illustration of this rather-useful problem-solving technique, *using extreme cases*, demonstrates how we use this strategy to clearly sort out an otherwise-cumbersome problem situation. We also use this same strategy of considering extremes in more everyday situations. A person who plans to buy an item where bargaining plays a part, such as in the process of buying a house or purchasing an item at a garage sale, must determine a strategy to make the seller an offer. He must decide what the lowest (extreme) price ought to be and what the highest (extreme) price might be, and then orient himself from there. In general, we often consider the extreme values of anything we plan to purchase and then make our decision about which price to settle on based on the extreme situations.

This is also relevant when it comes to scheduling. When you are budgeting time, you must consider extreme cases to be sure that time allocations are adequate. For example, allowing the maximum amount of time for each of a series of tasks would then enable an assurance of when the series of tasks would certainly be completed.

Extreme cases are also utilized when we seek to test a product, say, stereo speakers. We would want to test them at an extremely low volume and at an extremely high volume. We would then take for granted (with a modicum of justification), that speakers that pass the extreme-conditions test would also function properly between these extreme situations.

In mathematics, using extremes can be particularly useful. We often hear a problem that dramatically makes that point. A word about the problem before we present it. This is a problem that is very easy to understand. However, the beauty of the problem is the elegant solution that involves the use of considering an extreme. After you read the problem and consider a method of solution, allow yourself to perhaps struggle a bit before giving up (if you need to), and then consider the elegant solution provided here that is based on using an extreme situation. Here is the problem:

> *The problem:* **We have two one-gallon bottles. One bottle contains a quart of red wine and the other bottle contains a quart of white wine. We take a tablespoonful of red wine and pour it into the white wine. Then we take a tablespoon of this new mixture (white wine and red wine) and pour it into the bottle of red wine. Is there more red wine in the white-wine bottle, or more white wine in the red-wine bottle?**

To solve the problem, we can figure this out in any of the usual ways—often referred to in the high-school context as "mixture problems"—or we can use some clever logical reasoning and look at the problem's solution as follows: With the first "transport" of wine, there is only red wine on the tablespoon. On the second "transport" of wine, there is as much white wine on the spoon as there remains red wine in the "white-wine bottle." This may require some abstract thinking, but you should "get it" soon.

The simplest solution to understand, and the one that demonstrates a very powerful strategy, is that of *using extremes*. Let us now employ this strategy for the above problem. To do this, we will consider the tablespoonful quantity to be a bit larger. Clearly the outcome of this problem is independent of the quantity transported. Therefore, let us use an *extremely* large quantity. We will let this quantity actually be the *entire* one quart—the extreme amount. Following the instructions given in the problem statement, we will take this entire amount (one quart of red wine), and pour it into the white-wine bottle. This mixture is now 50 percent white wine and 50 percent red wine. We then pour one quart of this mixture back into the red-wine bottle. The mixture is now the same in both bottles. Therefore, we can conclude that there is as much white wine in the red-wine bottle as there is red wine in the white-wine bottle, and the problem is solved!

We can consider another form of an extreme case, where the spoon doing the wine transporting has a zero quantity. In this case, the conclusion follows immediately: There is as much red wine in the white-wine bottle (none) as there is white wine in the red-wine bottle (none). Once again, by using extremes we very easily solved the problem in a rather-elegant fashion.

Another problem that can be rather easily solved by using an extreme condition is the following:

> *The problem:* **A car is driving along a highway at a constant speed of 55 miles per hour. The driver notices a second car, exactly ½ mile behind him. The second car passes the first, exactly 1 minute later. How fast was the second car traveling, assuming its speed is constant?**

Although this problem could be easily solved by using the traditional procedures taught in elementary algebra classes, it can be much more easily disposed of by considering an extreme situation. Assume that the first car is going *extremely* slowly, that is, at 0 mph. Under these conditions, the second car travels ½ mile in one minute to catch the first car, which is to say that the second car would have to be traveling at a speed of 30 mph. Therefore, when the first car is moving at 0 mph, we find that the second car is traveling 30 mph faster than the first car. If, on the other hand, the first car is traveling at 55 mph (as was stated in the original problem), then the second car must be traveling at 30 + 55 = 85 mph (within the legal limit, we hope!).

We offer these problems merely as a demonstration of the power of thinking from the point of view of extremes, something that is used frequently in solving mathematical problems, as well as in the making of proper decisions in our everyday life.

THE WORKING-BACKWARD STRATEGY IN PROBLEM SOLVING

Often, without being directly aware that we are using a working-backward strategy, we find it to be a rather-useful approach. For example, the best approach to determine the most efficient route from one city to another depends on whether the starting point or the destination (endpoint) has numerous access roads. When there are fewer roads leading from the starting point, the forward method is usually superior. However, when there are many roads leading from the starting point and only one or two from the destination, an efficient way to plan the trip is to locate this final destination on a map, determine which roads lead most directly back toward the starting position, and then determine to which larger road that "last" access road leads. Progressively, by continuing in this way (i.e., *by working backward*), you land on a familiar road that is easily reachable from the starting point. At this step, you will have mapped out the trip in a very systematic way.

Let's consider a different example: A person who has an appointment in a distant city must determine the flight she will take to arrive comfortably on time for her meeting, yet not too far in advance. She begins by examining the airline schedule, starting with the arrival time closest to her appointment. Will she arrive in time? Is she "cutting it too close"? If so, she examines the next-earlier arrival time. Is this time all right? What if there is a weather delay? When is the next-earlier flight? Thus, by working backward, the traveler can decide the most appropriate flight to take to get to her appointment on time.

Let's examine another case of working backward. When a high-school freshman announces to a guidance counselor his desire to be admitted to a major Ivy League school and wants to know what courses to take, the counselor will usually look at what the potential college requires. At that point, the counselor begins to build the student's program for the next four years by working backward from the Advanced Placement courses that are usually taken in the senior year. However, to be prepared to take these courses, the student must first take some of the more-basic courses as a freshman, sophomore, and junior. For example, to take the Advanced Placement examination in calculus during senior year, the student must take the appropriate prerequisites. As we have stated previously, when there is a single final goal and we are interested in discovering the path to that endpoint from a starting point, we have a good opportunity to use the working-backward strategy.

The strategy game of Nim is another excellent example of when it is appropriate to use the working-backward strategy. In one version of the game, two players are faced with 32 toothpicks placed in a pile between them. Each player in turn takes 1, 2, or 3 toothpicks from the pile. The player who takes the final toothpick is the winner. Players develop a winning strategy by working backward from 32 (i.e., to win, the player must pick up the 28th toothpick, the 24th toothpick, etc.). Proceeding in this manner from the final goal of 32, we find that the player who wins picks the 28th, 24th, 20th, 16th, 12th, 8th, and 4th toothpicks. Thus, a winning strategy is to permit the opponent to go first and proceed as we have shown.

Although many problems may require some reverse reasoning (even if only to a minor extent), there are some problems whose solutions are dramatically facilitated by working backward. Consider the following problem:

> *The problem:* **The sum of two numbers is 12, and the product of the same two numbers is 4. Find the sum of the reciprocals of the two numbers.**

The common approach is to immediately generate two equations $x + y = 12$ and $xy = 4$, where x and y represent the two numbers, and then to solve this pair of equations simultaneously by substitution. Assuming that the problem solver is aware of the quadratic formula (which is a staple topic in the high-school curriculum), the correct result will be a pair of rather unpleasant-looking values for x and y, namely, $x = 6 + 4\sqrt{2}$ and $y = 6 - 4\sqrt{2}$ Then we must find the reciprocals of these numbers and, finally, their sum. Can this problem be solved in this manner? Yes, of course! However, this rather-complicated solution process can be made much simpler by starting from the end of the problem, namely, with what we wish to find, $\frac{1}{x}+\frac{1}{y}$. One might logically ask, "What do we usually do when we see two fractions that are to be added?" If we compute the sum in the usual way, we obtain $\frac{x+y}{xy}$. However, since we were told at the outset that $x + y = 12$ and $xy = 4$, this fraction is a value $\frac{12}{4} = 3$, and the problem is solved! This is a dramatic example of how working backward trivializes a mathematical problem that done conventionally would be significantly more difficult.

Another everyday problem that requires working backward could be the following:

> *The problem:* **Charles has an 11-liter can and a 5-liter can. How can he measure out exactly 7 liters of water?**

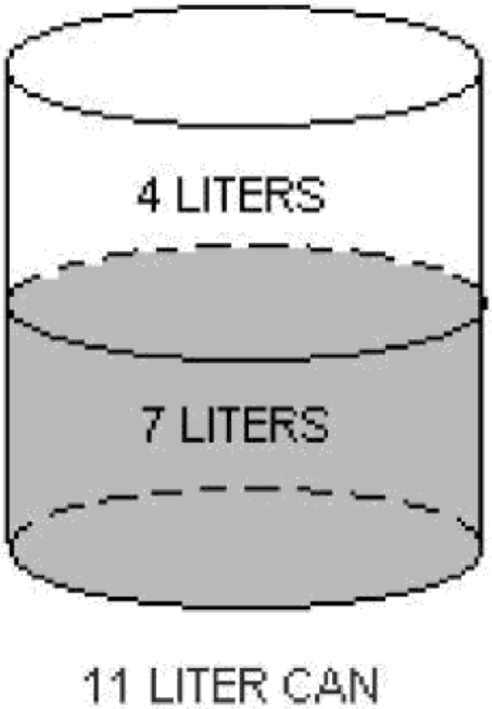

Figure 2.1.

Typically, a common procedure to resolve this problem is to keep "pouring" back and forth in an attempt to arrive at the correct answer, a sort of "unintelligent" guessing and testing. However, the problem can be solved in a more organized manner by using the working-backward strategy. To employ the strategy, we realize that we need to end up with 7 liters in the 11-liter can, leaving a total of 4 empty liters in the can. (See fig. 2.1) But where do 4 empty liters come from?

To obtain 4 liters, we must leave 1 liter in the 5-liter can. Now, how can we obtain 1 liter in the 5-liter can? Fill the 11-liter can and pour from it twice into the 5-liter can (fig. 2.2), discarding the water. This leaves 1 liter in the 11-liter can. Pour the 1 liter into the 5-liter can.

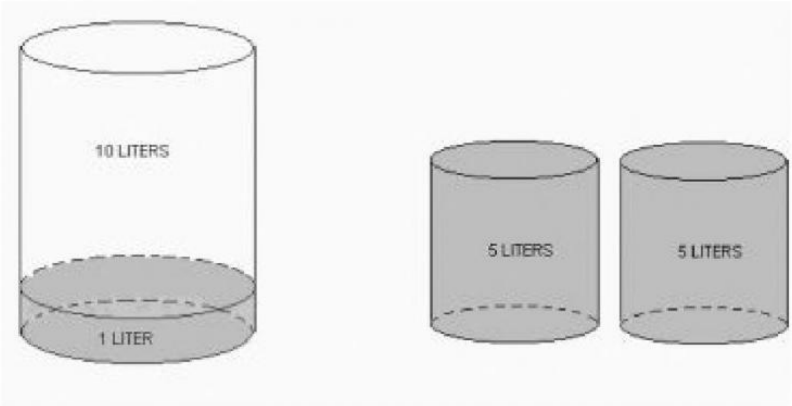

Figure 2.2.

Then fill the 11-liter can and pour off the 4 liters needed to fill the 5-liter can. (See fig. 2.3.) This leaves the required 7 liters in the 11-liter can. And the problem is solved!

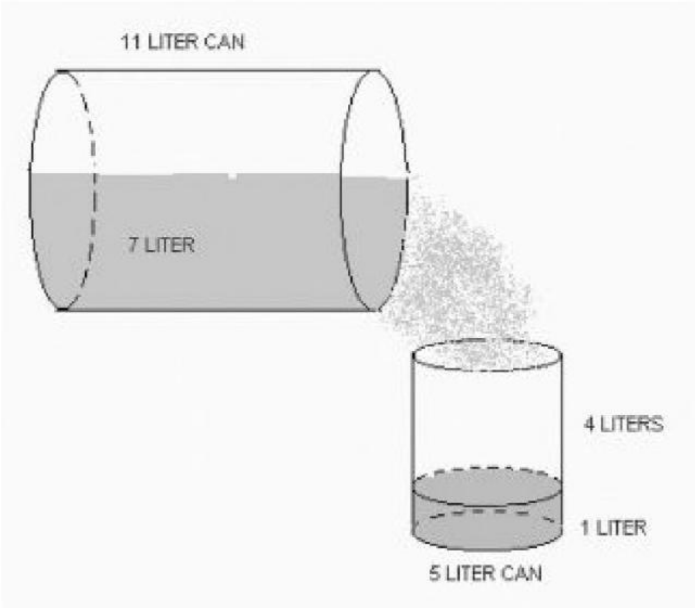

Figure 2.3.

Note that problems of this sort do not always have a solution. That is, if you wish to construct additional problems of this type, you must bear in mind that a solution exists only when the difference of multiples of the capacities of the two given cans can be made equal to the desired quantity. In this problem, $2 \cdot 11 - 3 \cdot 5 = 7$.

This concept can lead to a discussion of parity. We know that the sum of two like parities will always be even (i.e., even + even = even, and odd + odd = even), whereas the sum of two unlike parities will always be odd, odd + even = odd. Thus, if two even quantities are given, they can never yield an odd quantity. For example, given a 10-liter can and a 2-liter can, it is not possible to measure out an odd quantity of liters, since $n \cdot 10 - m \cdot 2 = (5n - m) \cdot 2$, which is even for all pairs of integers n and m. On the other hand, if at least one of the two cans holds an odd quantity of liters, we can measure out both odd and even quantities of liters (but not arbitrary ones).

Now that we have explored arithmetic shortcuts and clever methods of thinking logically, we are ready to apply our skills to everyday-life encounters.

CHAPTER 3

MATHEMATICAL APPEARANCES AND APPLICATIONS IN EVERYDAY-LIFE PROBLEMS

There is hardly a moment in our daily travels through our regular experiences when, in some form or another, mathematics does not present itself. In this chapter, we will expose some of these experiences as well as make you aware of how knowledge of some simple mathematics can facilitate your daily experiences and help you solve everyday life problems.

SHOPPING WITH MATHEMATICAL SUPPORT

Most supermarkets today provide the unit cost of an item. This is very helpful in that it allows the consumer to decide whether it makes sense to buy two 12 oz. jars of mayonnaise costing $1.35 per jar, or one 30 oz. jar of the same brand of mayonnaise costing $3.49. We have been trained to think that the larger quantity is generally the better price value. However, there is a neat little trick to determining which option offers the better price per ounce.

First we need to establish the price per ounce for each of the two different-sized jars:

For the 12-ounce jars, the price per ounce is: $\frac{\$1.35}{12}$.

For the 30-ounce jar, the price per ounce is: $\frac{\$3.49}{30}$.

To compare the two fractions, $\frac{1.35}{12}$? $\frac{3.49}{30}$, in order to see which is larger, we can implement a neat little algorithm to accomplish this task. We will cross multiply, writing the products under the fraction whose numerator was used. (See table 3.1)

Fraction to Determine Price Per Ounce	$\frac{1.35}{12}$	$\frac{3.49}{30}$