THE MATHS BOOK

DK

CONTENTS

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FOREWORD

Summarizing all of mathematics in one book is a daunting and indeed impossible task. Humankind has been exploring and discovering mathematics for millennia. Practically, we have relied on maths to advance our species, with early arithmetic and geometry providing the foundations for the first cities and civilizations. And philosophically, we have used mathematics as an exercise in pure thought to explore patterns and logic.

As a subject, mathematics is surprisingly hard to pin down with one catch-all definition. “Mathematics” is not simply, as many people think, “stuff to do with numbers”. That would exclude a huge range of mathematical topics, including much of the geometry and topology covered in this book. Of course numbers are still very useful tools to understand even the most esoteric areas of mathematics, but the point is that they are not the most interesting aspect of it. Focusing just on numbers misses the wood for the threes.

For the record, my own definition of maths as “the sort of things that mathematicians enjoy doing”, while delightfully circular, is largely unhelpful. *Big Ideas: Simply Explained* is actually not a bad definition. Mathematics could be seen as the attempt to find the simplest explanations for the biggest ideas. It is the endeavour of finding and summarizing patterns. Some of those patterns involve the practical triangles required to build pyramids and divide land; other patterns attempt to classify all of the 26 sporadic groups of abstract algebra. These are very different problems in terms of both usefulness and complexity, but both types of pattern have become the obsession of mathematicians throughout the ages.

There is no definitive way to organize all of mathematics, but looking at it chronologically is not a bad way to go. This book uses the historical journey of humans discovering maths as a way to classify it and wrangle it into a linear progression. Which is a valiant but difficult effort. Our current mathematical body of knowledge has been built up by a haphazard and diverse range of people across time and cultures.

So something like the short section on magic squares covers thousands of years and the span of the globe. Magic squares – arrangements of numbers where the sum in each row, column, and diagonal is always the same – are one of the oldest areas of recreational mathematics. Starting in the 9th century BCE in China, the story then bounces around via Indian texts from 100 CE, Arab scholars in the Middle Ages, Europe during the Renaissance, and finally, modern Sudoku-style puzzles. Across a mere two pages this book has to cover 3,000 years of history ending with geomagic squares in 2001. And even in this small niche of mathematics, there will be many magic square developments that there was simply not enough room to include. The whole book should be viewed as a curated tour of mathematical highlights.

Studying even just a sample of mathematics is a great reminder of how much humans have achieved. But it also highlights where mathematics could do better: things like the glaring omission of women from the history of mathematics cannot be ignored. A lot of talent has been squandered over the centuries, and a lot of credit has not been appropriately given. But I hope that we are now improving the diversity of mathematicians, and encouraging all humans to discover and learn about mathematics.

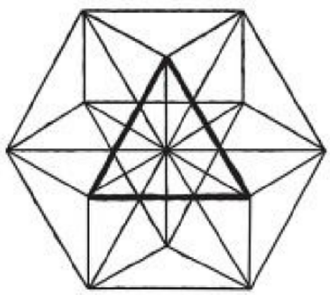
Because going forward, the body of mathematics will continue to grow. Had this book been written a century earlier it would have been much the same up until about page 280. And then it would have ended. No ring theory from Emmy Noether, no computing from Alan Turing, and no six degrees of separation from Kevin Bacon. And no doubt that will be true again 100 years from now. The edition printed a century from now will carry on past page 325: covering patterns totally alien to us. And because anyone can do maths, there is no telling who will discover this new maths, and where or when. To make the biggest advancement in mathematics during the 21st century, we need to include all people. I hope this book helps inspire everyone to get involved.

MATT PARKER

Matt Parker

INTRODUCTION

The history of mathematics reaches back to prehistory, when early humans found ways to count and quantify things. In doing so, they began to identify certain patterns and rules in the concepts of numbers, sizes, and shapes. They discovered the basic principles of addition and subtraction – for example, that two things (whether pebbles, berries, or mammoths) when added to another two invariably resulted in four things. While such ideas may seem obvious to us today, they were profound insights for their time. They also demonstrate that the history of mathematics is above all a story of discovery rather than invention. Although it was human curiosity and intuition that recognized the underlying principles of mathematics, and human ingenuity that later provided various means of recording and notating them, those principles themselves are not a human invention. The fact that 2 + 2 = 4 is true, independent of human existence; the rules of mathematics, like the laws of physics, are universal, eternal, and unchanging. When mathematicians first showed that the angles of any triangle in a flat plane when added together come to 180°, a straight line, this was not their invention: they had simply discovered a fact that had always been (and will always be) true.



Early applications

The process of mathematical discovery began in prehistoric times, with the development of ways of counting things people needed to quantify. At its simplest, this was done by cutting tally marks in a bone or stick, a rudimentary but reliable means of recording numbers of things. In time, words and symbols were assigned to the numbers and the first systems of numerals began to evolve, a means of expressing operations such as acquisition of additional items, or depletion of a stock, the basic operations of arithmetic.

As hunter-gatherers turned to trade and farming, and societies became more sophisticated, arithmetical operations and a numeral system became essential tools in all kinds of transactions. To enable trade, stock-taking, and taxes in uncountable goods such as oil, flour, or plots of land, systems of measurement were developed, putting a numerical value on dimensions such as weight and length. Calculations also became more complex, developing the concepts of multiplication and division from addition and subtraction – allowing the area of land to be calculated, for example.

In the early civilizations, these new discoveries in mathematics, and specifically the measurement of objects in space, became the foundation of the field of geometry, knowledge that could be used in building and tool-making. In using these measurements for practical purposes, people found that certain patterns were emerging, which could in turn prove useful. A simple but accurate builder's square can be made from a triangle with sides of three, four, and five units. Without that accurate tool and knowledge, the roads, canals, ziggurats, and pyramids of ancient Mesopotamia and Egypt could not have been built.

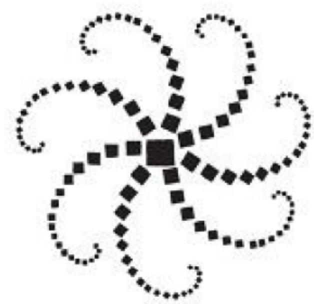
As new applications for these mathematical discoveries were found – in astronomy, navigation, engineering, book-keeping, taxation, and so on – further patterns and ideas emerged. The ancient civilizations each established the foundations of mathematics through this interdependent process of application and discovery, but also developed a fascination with mathematics for its own sake, so-called pure mathematics. From the middle of the first millennium BCE, the first pure mathematicians began to appear in Greece, and slightly later in India and China, building on the legacy of the practical pioneers of the subject – the engineers, astronomers, and explorers of earlier civilizations.

Although these early mathematicians were not so concerned with the practical applications of their discoveries, they did not restrict their studies to mathematics alone. In their exploration of the properties of numbers, shapes, and processes, they discovered universal rules and patterns that raised metaphysical questions about the nature of the cosmos, and even suggested that these patterns had mystical properties. Often mathematics was therefore seen as a complementary discipline to philosophy – many of the greatest mathematicians through the ages have also been philosophers, and vice versa – and the links between the two subjects have persisted to the present day.

It is impossible to be a mathematician without being a poet of the soul.

Sofya Kovalevskaya

Russian mathematician



Arithmetic and algebra

So began the history of mathematics as we understand it today – the discoveries, conjectures, and insights of mathematicians that form the bulk of this book. As well as the individual thinkers and their ideas, it is a story of societies and cultures, a continuously developing thread of thought from the ancient civilizations of Mesopotamia and Egypt, through Greece, China, India, and the Islamic empire to

Renaissance Europe and into the modern world. As it evolved, mathematics was also seen to comprise several distinct but interconnected fields of study.

The first field to emerge, and in many ways the most fundamental, is the study of numbers and quantities, which we now call arithmetic, from the Greek word *arithmos* ("number"). At its most basic, it is concerned with counting and assigning numerical values to things, but also the operations, such as addition, subtraction, multiplication, and division, that can be applied to numbers. From the simple concept of a system of numbers comes the study of the properties of numbers, and even the study of the very concept itself. Certain numbers – such as the constants π, e, or the prime and irrational numbers – hold a special fascination and have become the subject of considerable study.

Another major field in mathematics is algebra, which is the study of structure, the way that mathematics is organized, and therefore has some relevance in every other field. What marks algebra from arithmetic is the use of symbols, such as letters, to represent variables (unknown numbers). In its basic form, algebra is the study of the underlying rules of how those symbols are used in mathematics – in equations, for example. Methods of solving equations, even quite complex quadratic equations, had been discovered as early as the ancient Babylonians, but it was medieval mathematicians of the Islamic Golden Age who pioneered the use of symbols to simplify the process, giving us the word "algebra", which is derived from the Arabic *al-jabr*. More recent developments in algebra have extended the idea of abstraction into the study of algebraic structure, known as abstract algebra.

Geometry is knowledge of the eternally existent.

Pythagoras

Ancient Greek mathematician

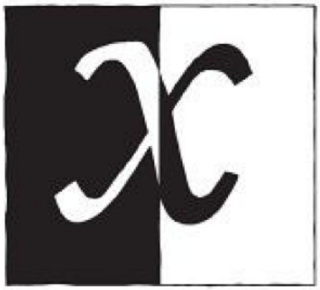
Geometry and calculus

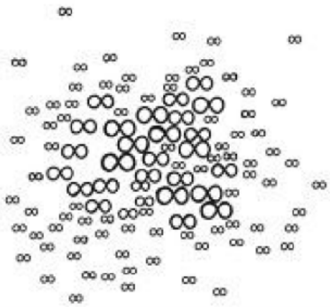
A third major field of mathematics, geometry, is concerned with the concept of space, and the relationships of objects in space: the study of the shape, size and position of figures. It evolved from the very practical business of describing the physical dimensions of things, in engineering and construction projects, measuring and apportioning plots of land, and astronomical observations for navigation and compiling calendars. A particular branch of geometry, trigonometry (the study of the properties of triangles), proved to be especially useful in these pursuits. Perhaps because of its very concrete nature, for many ancient civilizations, geometry was the cornerstone of mathematics, and provided a means of problem-solving and proof in other fields.

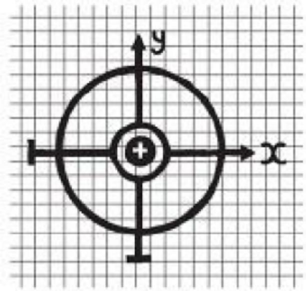
This was particularly true of ancient Greece, where geometry and mathematics were virtually synonymous. The legacy of great mathematical philosophers such as Pythagoras, Plato, and Aristotle was consolidated by Euclid, whose principles of mathematics based on a combination of geometry and logic were accepted as the subject's foundation for some 2,000 years. In the 19th century, however, alternatives to classical Euclidean geometry were proposed, opening up new areas of study, including topology, which examines the nature and properties not only of objects in space, but of space itself.

Since the Classical period, mathematics had been concerned with static situations, or how things are at any given moment. It failed to offer a means of measuring or calculating continuous change. Calculus, developed independently by Gottfried Leibniz and Isaac Newton in the 17th century, provided an answer to this problem. The two branches of calculus, integral and differential, offered a method of analysing such things as the slope of curves on a graph and the area beneath them as a way of describing and calculating change.

The discovery of calculus opened up a field of analysis that later became particularly relevant to, for example, the theories of quantum mechanics and chaos theory in the 20th century.







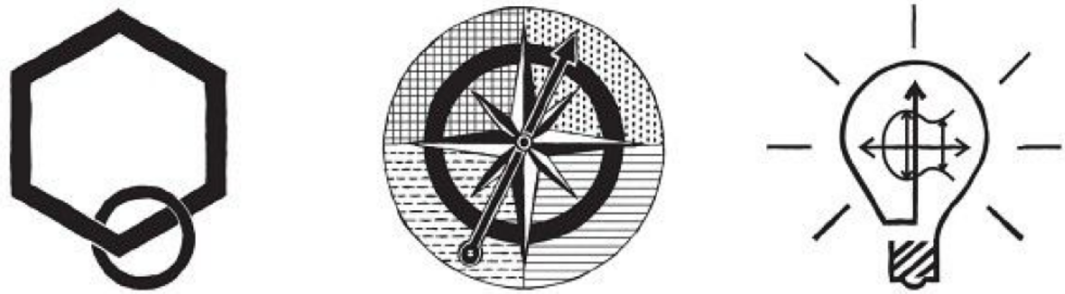
Revisiting logic

The late 19th and early 20th centuries saw the emergence of another field of mathematics – the foundations of mathematics. This revived the link between philosophy and mathematics. Just as Euclid had done in the 3rd century BCE, scholars including Gottlob Frege and Bertrand Russell sought to discover the logical foundations on which mathematical principles are based. Their work inspired a re-examination of the nature of mathematics itself, how it works, and what its limits are. This study of basic mathematical concepts is perhaps the most abstract field, a sort of meta-mathematics, yet an essential adjunct to every other field of modern mathematics.

In mathematics, the art of asking questions is more valuable than solving problems.

Georg Cantor

German mathematician



New technology, new ideas

The various fields of mathematics – arithmetic, algebra, geometry, calculus, and foundations – are worthy of study for their own sake, and the popular image of academic mathematics is that of an almost incomprehensible abstraction. But applications for mathematical discoveries have usually been found, and advances in science and technology have driven innovations in mathematical thinking.

A prime example is the symbiotic relationship between mathematics and computers. Originally developed as a mechanical means of doing the "donkey work" of calculation to provide tables for mathematicians, astronomers and so on, the actual construction of computers required new mathematical thinking. It was mathematicians, as much as engineers, who provided the means of building mechanical, and then electronic computing devices, which in turn could be used as tools in the discovery of new mathematical ideas. No doubt, new applications for mathematical theorems will be found in the future too – and with numerous problems still unsolved, it seems that there is no end to the mathematical discoveries to be made.

The story of mathematics is one of exploration of these different fields, and the discovery of new ones. But it is also the story of the explorers, the mathematicians who set out with a definite aim in mind, to find answers to unsolved problems, or to travel into unknown territory in search of new ideas – and those who simply stumbled upon an idea in the course of their mathematical journey, and were inspired to see where it would lead. Sometimes the discovery would come as a game-changing revelation, providing a way into unexplored fields; at other times it was a case of “standing on the shoulders of giants”, developing the ideas of previous thinkers, or finding practical applications for them.

This book presents many of the “big ideas” in mathematics, from the earliest discoveries to the present day, explaining them in layperson’s language, where they came from, who discovered them, and what makes them significant. Some may be familiar, others less so. With an understanding of these ideas, and an insight into the people and societies in which they were discovered, we can gain an appreciation of not only the ubiquity and usefulness of mathematics, but also the elegance and beauty that mathematicians find in the subject.

Mathematics, rightly viewed, possesses not only truth, but supreme beauty.

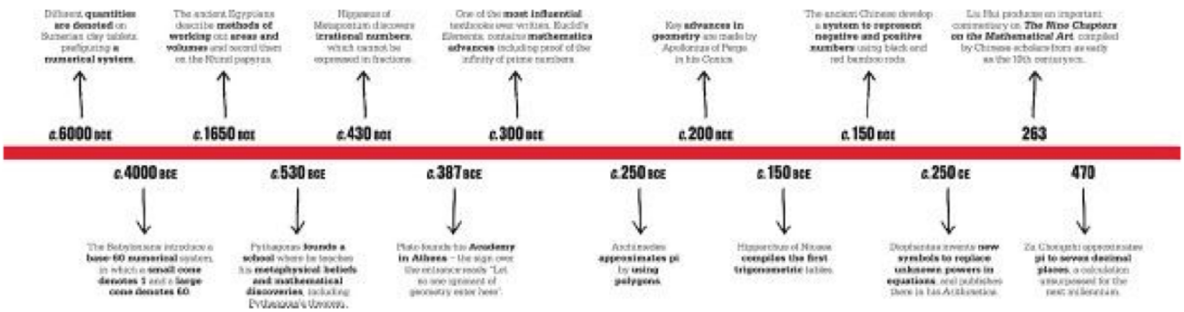
Bertrand Russell

British philosopher and mathematician

INTRODUCTION

As early as 40,000 years ago, humans were making tally marks on wood and bone as a means of counting. They undoubtedly had a rudimentary sense of number and arithmetic, but the history of mathematics only properly began with the development of numerical systems in early civilizations. The first of these emerged in the sixth millennium BCE, in Mesopotamia, western Asia, home to the world's earliest agriculture and cities. Here, the Sumerians elaborated on the concept of tally marks, using different symbols to denote different quantities, which the Babylonians then developed into a sophisticated numerical system of cuneiform (wedge-shaped) characters. From about 4000 BCE, the Babylonians used elementary geometry and algebra to solve practical problems – such as building, engineering, and calculating land divisions – alongside the arithmetical skills they used to conduct commerce and levy taxes.

A similar story emerges in the slightly later civilization of the ancient Egyptians. Their trade and taxation required a sophisticated numerical system, and their building and engineering works relied on both a means of measurement and some knowledge of geometry and algebra. The Egyptians were also able to use their mathematical skills in conjunction with observations of the heavens to calculate and predict astronomical and seasonal cycles and construct calendars for the religious and agricultural year. They established the study of the principles of arithmetic and geometry as early as 2000 BCE.



Greek rigour

The 6th century BCE onwards saw a rapid rise in the influence of ancient Greece across the eastern Mediterranean. Greek scholars quickly assimilated the mathematical ideas of the Babylonians and Egyptians. The Greeks used a numerical system of base-10 (with ten symbols) derived from the Egyptians. Geometry in particular chimed with Greek culture, which idolized beauty of form and symmetry. Mathematics became a cornerstone of Classical Greek thinking, reflected in its art, architecture, and even philosophy. The almost mystical qualities of geometry and numbers inspired Pythagoras and his followers to establish a cult-like community, dedicated to studying the mathematical principles they believed were the foundations of the Universe and everything in it.

Centuries before Pythagoras, the Egyptians had used a triangle with sides of 3, 4, and 5 units as a building tool to ensure corners were square. They had come across this idea by observation, and then applied it as a rule of thumb, whereas the Pythagoreans set about rigorously showing the principle, offering a proof that it is true for all right-angled triangles. It is this notion of proof and rigour that is the Greeks' greatest contribution to mathematics.

Plato's Academy in Athens was dedicated to the study of philosophy and mathematics, and Plato himself described the five Platonic solids (the tetrahedron, cube, octahedron, dodecahedron, and icosahedron). Other philosophers, notably Zeno of Elea, applied logic to the foundations of mathematics, exposing the problems of infinity and change. They even explored the strange phenomenon of irrational numbers. Plato's pupil Aristotle, with his methodical analysis of logical forms, identified the difference between inductive reasoning (such as inferring a rule of thumb from observations) and deductive reasoning (using logical steps to reach a certain conclusion from established premises, or axioms).

From this basis, Euclid laid out the principles of mathematical proof from axiomatic truths in his *Elements*, a treatise that was the foundation of mathematics for the next two millennia. With similar rigour, Diophantus pioneered the use of symbols to represent unknown numbers in his equations; this was the first step towards the symbolic notation of algebra.

A new dawn in the East

Greek dominance was eventually eclipsed by the rise of the Roman Empire. The Romans regarded mathematics as a practical tool rather than worthy of study. At the same time, the ancient civilizations of India and China independently developed their own numerical systems. Chinese mathematics in particular flourished between the 2nd and 5th centuries CE, thanks largely to the work of Liu Hui in revising and expanding the classic texts of Chinese mathematics.

NUMERALS TAKE THEIR PLACES

POSITIONAL NUMBERS

IN CONTEXT

KEY CIVILIZATION

Babylonians

FIELD

Arithmetic

BEFORE

40,000 years ago Stone Age people in Europe and Africa count using tally marks on wood or bone.

6000–5000 BCE Sumerians develop early calculation systems to measure land and to study the night sky.

4000–3000 BCE Babylonians use a small clay cone for 1 and a large cone for 60, together with a clay ball for 10, as their base-60 system evolves.

AFTER

2nd century CE The Chinese use an abacus in their base-10 positional number system.

7th century In India, Brahmagupta establishes zero as a number in its own right and not just a placeholder.

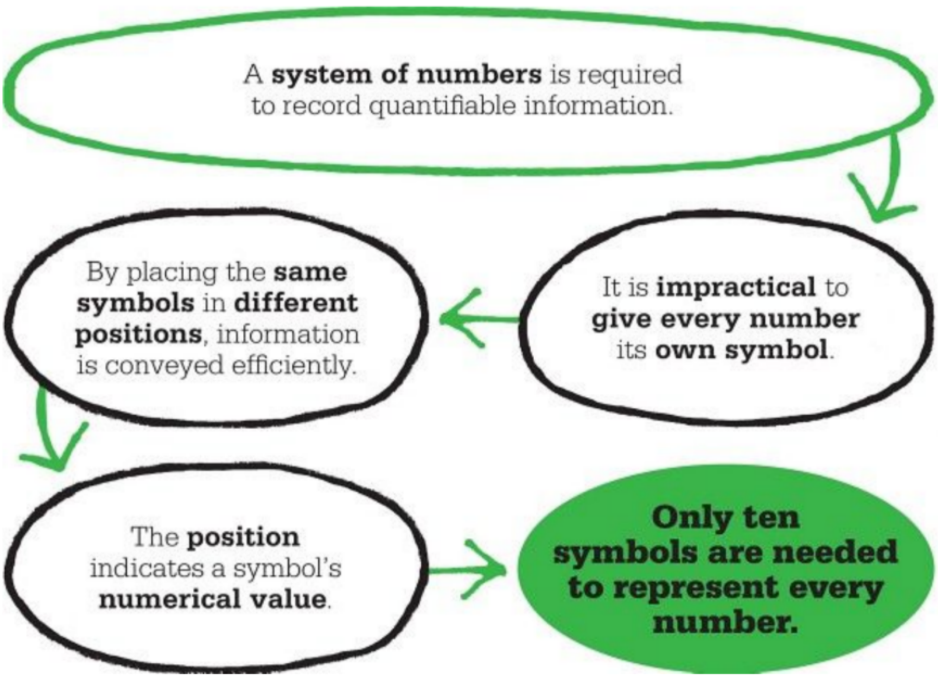
It is given to us to calculate, to weigh, to measure, to observe; this is natural philosophy.

Voltaire

French philosopher

The first people known to have used an advanced numeration system were the Sumerians of Mesopotamia, an ancient civilization living between the Tigris and Euphrates rivers in what is present-day Iraq. Sumerian clay tablets from as early as the 6th millennium BCE include symbols denoting different quantities. The Sumerians, followed by the Babylonians, needed efficient mathematical tools in order to administer their empires.

What distinguished the Babylonians from neighbours such as Egypt was their use of a positional (place value) number system. In such systems, the value of a number is indicated both by its symbol and its position. Today, for instance, in the decimal system, the position of a digit in a number indicates whether its value is in ones (less than 10), tens, hundreds, or more. Such systems make calculation more efficient because a small set of symbols can represent a huge range of values. By contrast, the ancient Egyptians used separate symbols for ones, tens, hundreds, thousands, and above, and had no place value system. Representing larger numbers could require 50 or more hieroglyphs.

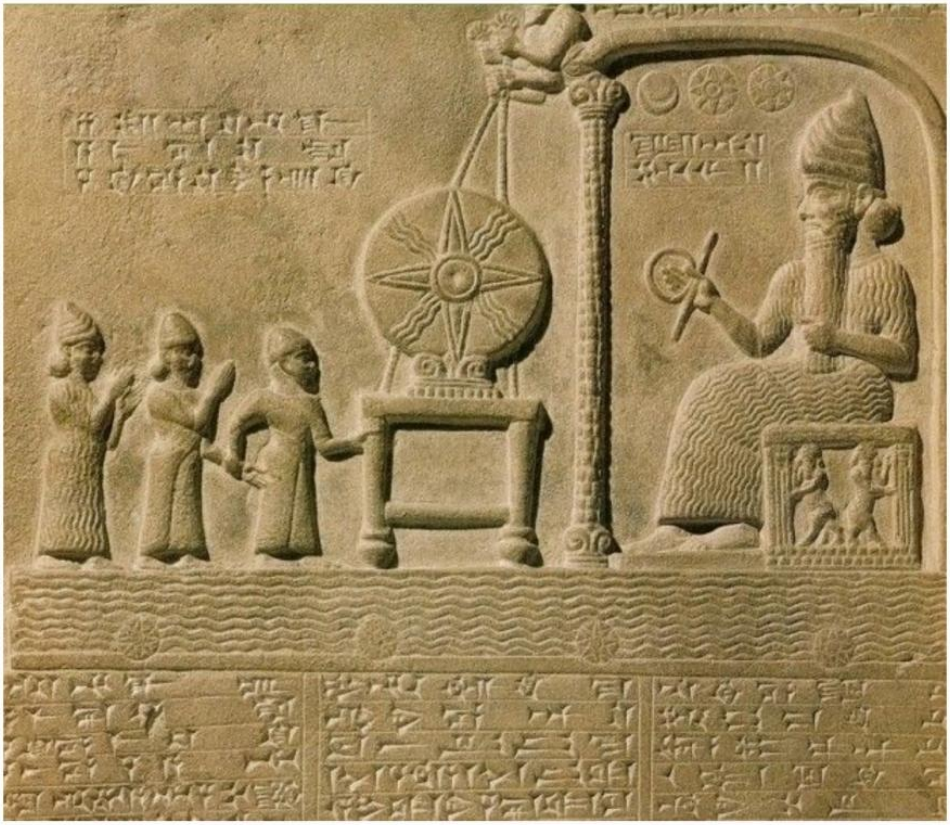


Using different bases

The Hindu–Arabic numeration that is employed today is a base-10 (decimal) system. It requires only 10 symbols – nine digits (1, 2, 3, 4, 5, 6, 7, 8, 9) and a zero as a placeholder. As in the Babylonian system, the position of a digit indicates its value, and the smallest value digit is always to the right. In a base-10 system, a two-digit number, such as 22, indicates $(2 \times 10^1) + 2$; the value of the 2 on the left is ten times that of the 2 on the right. Placing digits after the number 22 will create hundreds, thousands, and larger powers of 10. A symbol after a whole number (the standard notation now is a decimal point) can also separate it from its fractional parts, each representing a tenth of the place value of the preceding figure. The Babylonians worked with a more complex sexagesimal (base-60) number system that was probably inherited from the earlier Sumerians and is still used across the world today for measuring time, degrees in a circle (360° = 6 × 60), and geographic coordinates. Why they used 60 as a number base is still not known for sure. It may have been chosen because it can be divided by many other numbers – 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30. The Babylonians also based their calendar year on the solar year (365.24 days); the number of days in a year was 360 (6 × 60) with additional days for festivals.

In the Babylonian sexagesimal system, a single symbol was used alone and repeated up to nine times to represent symbols for 1 to 9. For 10, a different symbol was used, placed to the left of the one symbol, and repeated two to five times in numbers up to 59. At 60 (60 × 1), the original symbol for one was reused but placed further to the left than the symbol for 1. Because it was a base-60 system, two such symbols signified 61, while three such symbols indicated 3,661, that is, $60 \times 60\ (60^2) + 60 + 1$.

The base-60 system had obvious drawbacks. It necessarily requires many more symbols than a base-10 system. For centuries, the sexagesimal system also had no place value holders, and nothing to separate whole numbers from fractional parts. By around 300 BCE, however, the Babylonians used two wedges to indicate no value, much as we use a placeholder zero today; this was possibly the earliest use of zero.



The Babylonian sun-god Shamash awards a rod and a coiled rope, ancient measuring devices, to newly trained surveyors, on a clay tablet dating from around 1000 BCE.

Other counting systems

In Mesoamerica, on the other side of the world, the Mayan civilization developed its own advanced numeration system in the 1st millennium BCE – apparently in complete isolation. Theirs was a base-20 (vigesimal) number system, which probably evolved from a simple counting method using fingers and toes. In fact, base-20 number systems were used across the world, in Europe, Africa, and Asia. Language often contains remnants of this system. For example, in French, 80 is expressed as *quatre-vingt* (4 × 20); Welsh and Irish also express some numbers as multiples of 20, while in English a score is 20. In the Bible, for instance, Psalm 90 talks of a human lifespan being “threescore years and ten” or as great as “fourscore years”.

From around 500 BCE until the 16th century when Hindu–Arabic numbers were officially adopted in China, the Chinese used rod numerals to represent numbers. This was the first decimal place value system. By alternating quantities of vertical rods with horizontal rods, this system could indicate ones, tens, hundreds, thousands, and more powers of 10, much as the decimal system does today. For example, 45 was written with four horizontal bars representing 4×10^1 (40) and five vertical bars for 5×1 (5). However, four vertical rods followed by five vertical rods indicated 405 (4×100, or 10^2) $+ 5 \times 1$ – the absence of horizontal rods meant there were no tens in the number. Calculations were carried out by manipulating the rods on a counting board. Positive and negative numbers were represented by red and black rods respectively or different cross-sections (triangular and rectangular). Rod numerals are still used occasionally in China, just as Roman numerals are sometimes used in Western society.

The Chinese place value system is reflected in the Chinese abacus (suanpan). Dating back to at least 200 BCE, it is one of the oldest bead-counting devices, although the Romans used something similar. The Chinese version, which is still used today, has a central bar and a varying number of vertical wires to separate ones from tens, hundreds, or more. In each column, there are two beads above the bar worth five each and five beads below the bar worth one each.

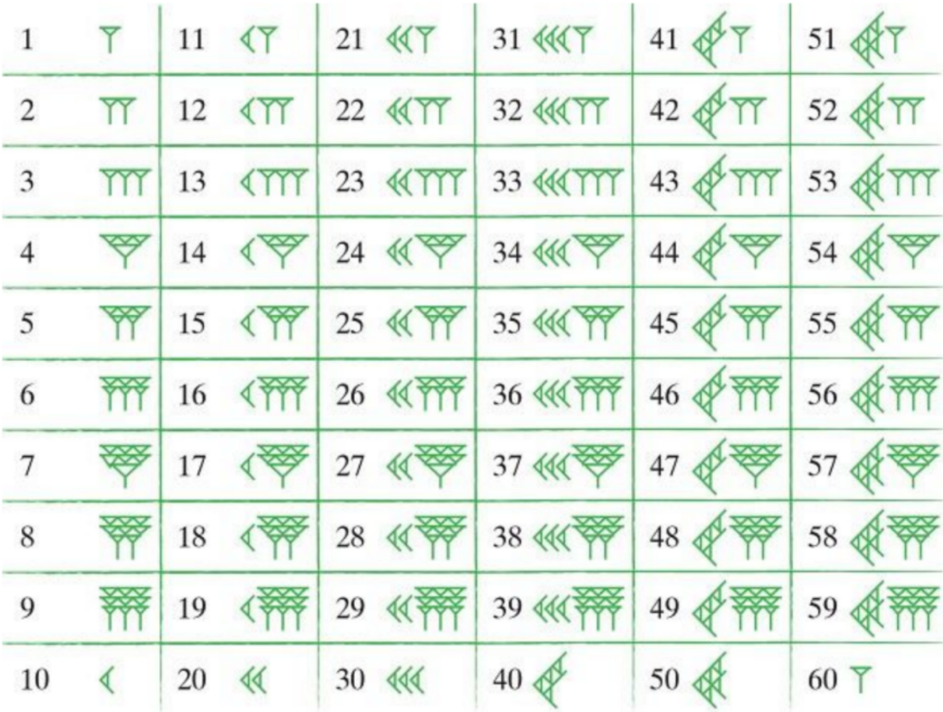
The Japanese adopted the Chinese abacus in the 14th century and developed their own abacus, the soroban, which has one bead worth five above the central bar and four beads each worth one below the bar in each column. Japan still uses the soroban today: there are even contests in which young people demonstrate their ability to perform soroban calculations mentally, a skill known as *anzan*.

Cuneiform

Cuneiform, a word derived from the Latin *cuneus* ("wedge") to describe the shape of the symbols, was inscribed into wet clay, stone, or metal.

In the late 19th century, academics deciphered the "cuneiform" (wedge-shaped) markings on clay tablets recovered from Babylonian sites in and around Iraq. Such marks, denoting letters and words as well as an advanced number system, were etched in wet clay with either end of a stylus. Like the Egyptians, the Babylonians needed scribes to administer their complex society, and many of the tablets bearing mathematical records are thought to be from training schools for scribes.

A great deal has now been discovered about Babylonian mathematics, which extended to multiplication, division, geometry, fractions, square roots, cube roots, equations, and other forms, because – unlike Egyptian papyrus scrolls – the clay tablets have survived well. Several thousand, mostly dating from between 1800 and 1600 BCE, are housed in museums around the world.



The Babylonian base-60 number system was built from two symbols – the single unit symbol, used alone and combined for numbers 1 to 9, and the 10 symbol, repeated for 20, 30, 40, and 50.

The Babylonian and Assyrian civilizations have perished... yet Babylonian mathematics is still interesting, and the Babylonian scale of 60 is still used in astronomy.

G. H. Hardy

British mathematician

Modern numeration

The Hindu–Arabic decimal system used throughout the world today has its origins in India. In the 1st to 4th centuries CE, the use of nine symbols along with zero was developed to allow any number to be written efficiently, through the use of place value. The system was adopted and refined by Arab mathematicians in the 9th century.

They introduced the decimal point, so that the system could also express fractions of whole numbers.

Three centuries later, Leonardo of Pisa (Fibonacci) popularized the use of Hindu–Arabic numerals in Europe through his book *Liber Abaci* (1202). Yet the debate about whether to use the new system rather than Roman numerals and traditional counting methods lasted for several hundred years, before its adoption paved the way for modern mathematical advances.

With the advent of electronic computers, other number bases became important – particularly binary, a number system with base 2. Unlike the base-10 system with its 10 symbols, binary has just two: 1 and 0. It is a positional system but instead of multiplying by 10, each column is multiplied by 2, also expressed as 2^1, 2^2, 2^3 and upwards. In binary, the number 111 means $1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$, that is $4 + 2 + 1$, or 7 in our decimal number system.

In binary, as in all modern number systems whatever their base, the principles of place value are always the same. Place value – the Babylonian legacy – remains a powerful, easily understood, and efficient way to represent large numbers.

The fact that we work in 10s as opposed to any other number is purely a consequence of our anatomy. We use our ten fingers to count.

Marcus du Sautoy
British mathematician

Ebisu, the Japanese god of fishermen and one of the seven gods of fortune, uses a soroban to calculate his profits in *The Red Snapper's Dream* by Utagawa Toyohiro.

Mayan numeral system

The Dresden Codex, the oldest surviving Mayan book, dating from the 13th or 14th century, illustrates Mayan number symbols and glyphs.

The Mayans, who lived in Central America from around 2000 BCE, used a base-20 (vigesimal) number system from around 1000 BCE to perform astronomical and calendar calculations. Like the Babylonians, they used a calendar of 360 days plus festivals, to make 365.24 days based on the solar year; their calendars helped them work out the growing cycles of crops.

The Mayan system employed symbols: a dot representing one and a bar representing five. By using combinations of dots over bars they could generate numerals up to 19. Numbers larger than 19 were written vertically, with the lowest numbers at the bottom, and there is evidence of Mayan calculations up to hundreds of millions. An inscription from 36 BCE shows that they used a shell-shaped symbol to denote zero, which was widely used by the 4th century.

The Mayans' number system was in use in Central America until the Spanish conquests in the 16th century. Its influence, however, never spread further.

See also: The Rhind papyrus • The abacus • Negative numbers • Zero • The Fibonacci sequence • Decimals

THE SQUARE AS THE HIGHEST POWER

QUADRATIC EQUATIONS

IN CONTEXT

KEY CIVILIZATIONS

Egyptians (*c.* 2000 BCE), **Babylonians** (*c.* 1600 BCE)

FIELD

Algebra

BEFORE

***c.* 2000 BCE** The Berlin papyrus records a quadratic equation solved in ancient Egypt.

AFTER

7th century CE The Indian mathematician Brahmagupta solves quadratic equations using only positive integers.

10th century CE Egyptian scholar Abu Kamil Shuja ibn Aslam uses negative and irrational numbers to solve quadratic equations.

1545 Italian mathematician Gerolamo Cardano publishes his *Ars Magna*, setting out the rules of algebra.

Quadratic equations are those involving unknown numbers to the power of 2 but not to a higher power; they contain x^2 but not x^3, x^4, and so on. One of the main rudiments of mathematics is the ability to use equations to work out solutions to real-world problems. Where those problems involve areas or paths of curves such as parabolas, quadratic equations become very useful, and describe physical phenomena, such as the flight of a ball or a rocket.

Quadratic **equations** contain the **power of 2**, so are used when calculating with **two dimensions**.

The **number of dimensions** is equal to the **maximum number of real solutions** an equation has.

There is a maximum of **two real solutions** for a **quadratic equation**, **three** for a **cubic equation**, and so on.

If a quadratic equation, or any equation, is **set equal to zero** (e.g. $x^2 + 3x + 2 = 0$), the **solutions are called roots**.

In a quadratic equation, these **two roots** are the points where a quadratic curve **crosses** the ***x*-axis** on a graph.

Ancient roots

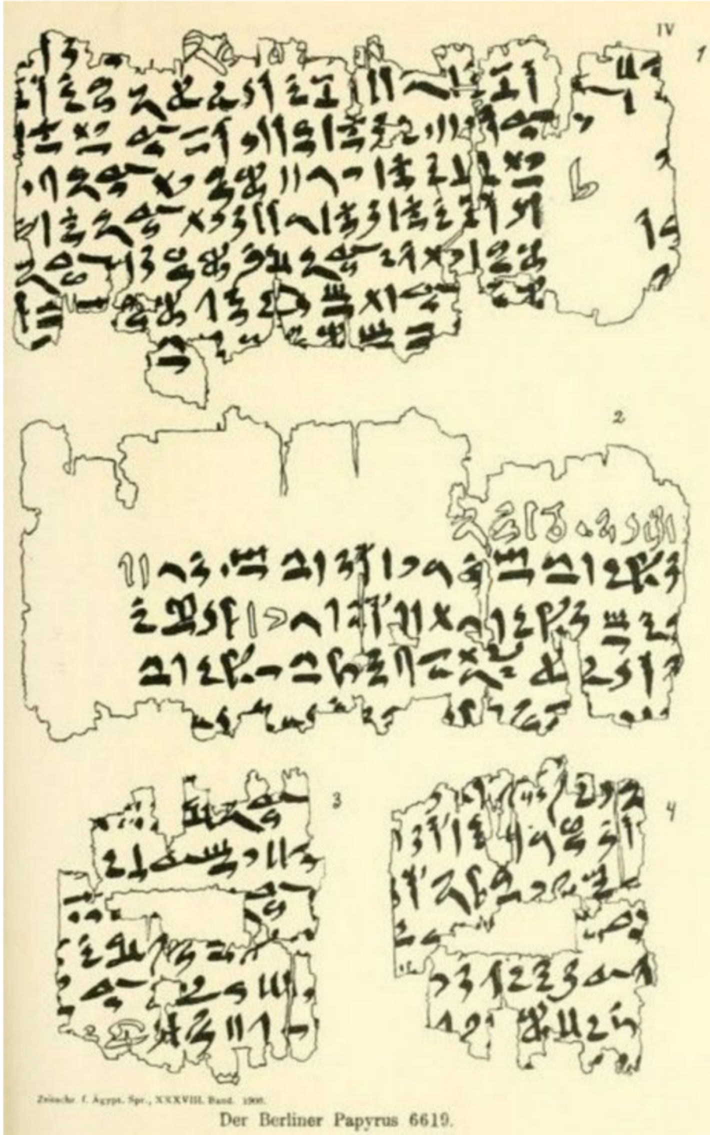
The history of quadratic equations extends across the world. It is likely that these equations first arose from the need to subdivide land for inheritance purposes, or to solve problems involving addition and multiplication.

One of the oldest surviving examples of a quadratic equation comes from the ancient Egyptian text known as the Berlin papyrus (*c.* 2000 BCE). The problem contains the following information: the area of a square of 100 cubits is equal to that of two smaller squares. The side of one of the smaller squares is equal to one half plus a quarter of the side of the other. In modern notation, this translates into two simultaneous equations: $x^2 + y^2 = 100$ and $x = (1/2 + 1/4)y = 3/4\, y$. These can be simplified to the quadratic equation $(3/4\, y)^2 + y^2 = 100$ to find the length of a side on each square.

The Egyptians used a method called "false position" to determine the solution. In this method, the mathematician selects a convenient number that is usually easy to calculate, then works out what the solution to the equation would be using that number. The result shows how to adjust the number to give the correct solution the equation. For example, in the Berlin papyrus problem, the simplest length to use for the larger of the two small squares is 4, because the problem deals with quarters. For the side of the smallest square, 3 is used because this length is $3/4$ of the side of the other small square. Two squares created using these false position numbers would have areas of 16 and 9 respectively, which when added together give a total area of 25. This is only $1/4$ of 100, so the areas must be quadrupled to match the Berlin papyrus equation. The lengths therefore must be doubled from the false positions of 4 and 3 to reach the solutions: 8 and 6.

Other early records of quadratic equations are found in Babylonian clay tablets, where the diagonal of a square is given to five decimal places. The Babylonian tablet YBC 7289 (*c.* 1800–1600 BCE) shows a method of working out the quadratic equation $x^2 = 2$ by drawing rectangles and trimming them down into squares. In the 7th century CE, Indian mathematician Brahmagupta wrote a formula for solving quadratic equations that could be applied to equations in the form $ax^2 + bx = c$. Mathematicians at the time did not use letters or symbols, so he wrote his solution in words, but it was similar to the modern formula shown above.

In the 8th century, Persian mathematician al-Khwarizmi employed a geometric solution for quadratic equations known as completing the square. Until the 10th century, geometric methods were were often used, as quadratic equations were used to solve real-world problems involving land rather than abstract algebraic challenges.



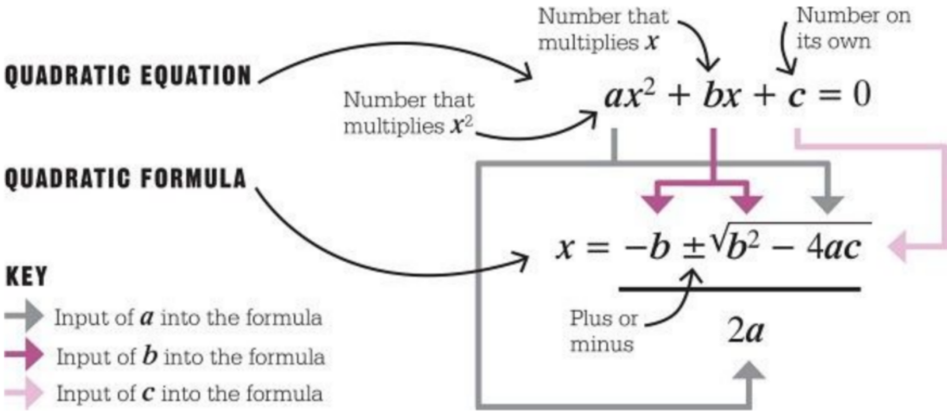
The Berlin papyrus was copied and published by German Egyptologist Hans Schack-Schackenburg in 1900. It contains two mathematical problems, one of which is a quadratic equation.

Negative solutions

Indian, Persian, and Arab scholars thus far had used only positive numbers. When solving the equation $x^2 + 10x = 39$, they gave the solution as 3. However, this is one of two correct solutions to the problem; -13 is the other. If x is -13, $x^2 = 169$ and $10x = -130$. Adding a negative number gives the same result as subtracting its equivalent positive number, so $169 + -130 = 169 - 130 = 39$.

In the 10th century, Egyptian scholar Abu Kamil Shuja ibn Aslam made use of negative numbers and algebraic irrational numbers (such as the square root of 2) as both solutions and coefficients (numbers multiplying an unknown quantity). By the 16th century, most mathematicians accepted negative solutions and were comfortable with surds (irrational roots – those that cannot be expressed exactly as a decimal). They had also started using numbers and symbols, rather than writing equations in words. Mathematicians now utilized the plus or minus symbol, ±, in solving quadratic equations. With the equation $x^2 = 2$, the solution is not just $x = \sqrt{2}$ but $x = \pm\sqrt{2}$. The plus or minus symbol is included because two negative numbers multiplied together make a positive number. While $\sqrt{2} \times \sqrt{2} = 2$, it is also true that $-\sqrt{2} \times -\sqrt{2} = 2$.

In 1545, Italian scholar Gerolamo Cardano published his *Ars Magna* (*The Great Art, or the Rules of Algebra*) in which he explored the problem: "What pair of numbers have a sum of ten and product of 40?" He found that the problem led to a quadratic equation which, when he completed the square, gave $\sqrt{(-15)}$. No numbers available to mathematicians at the time gave a negative number when multiplied by themselves, but Cardano suggested suspending belief and working with the square root of negative 15 to find the equation's two solutions. Numbers such as $\sqrt{(-15)}$ would later be known as "imaginary" numbers.



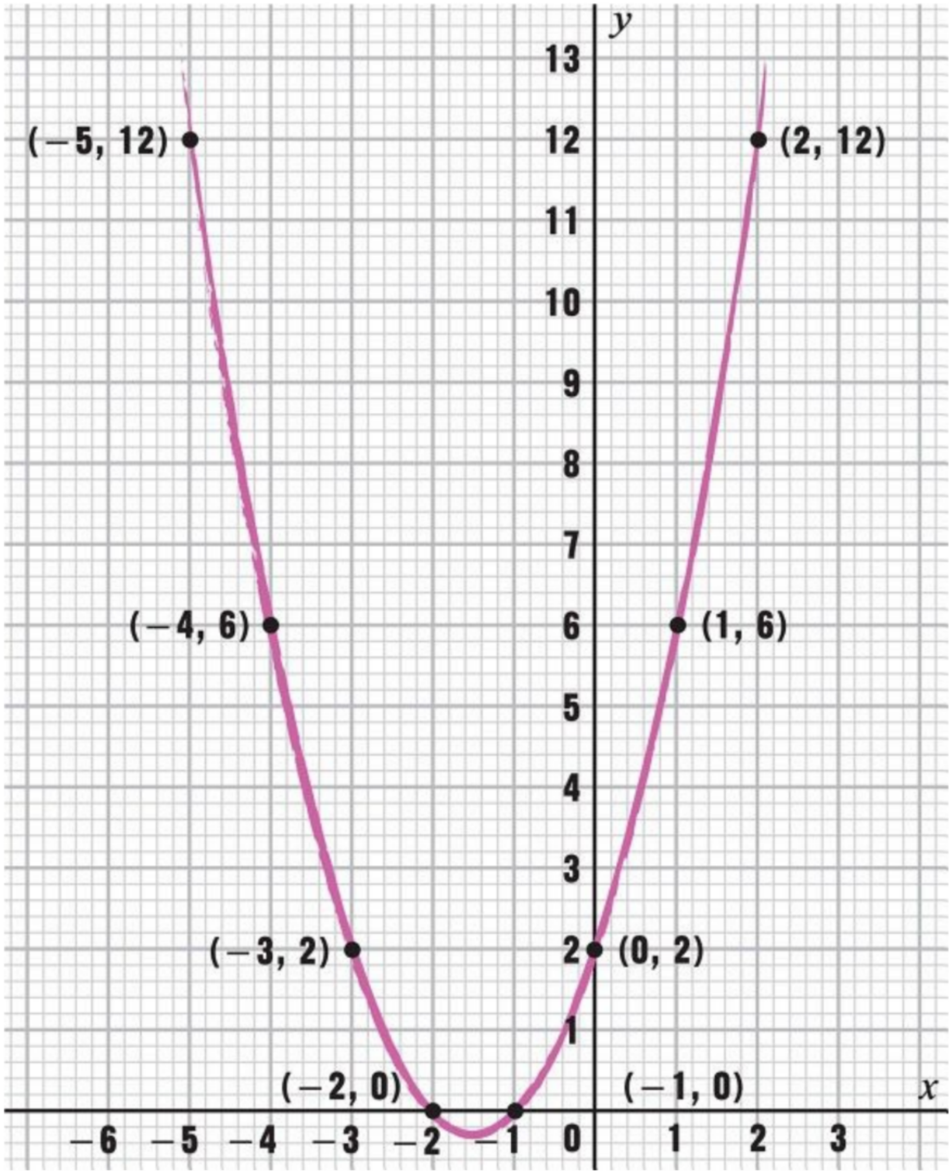
The quadratic formula is a way to solve quadratic equations. By modern convention, quadratic equations include a number, ***a***, multiplied by x^2; a number, ***b***, multiplied by ***x***; and a number, ***c***, on its own. The illustration below shows how the formula uses ***a***, ***b***, and ***c*** to find the value of ***x***. Quadratic equations often equal 0, because this makes them easy to work out on a graph; the ***x*** solutions are the points where the curve crosses the ***x*** axis.

Politics is for the present, but an equation is for eternity.

Albert Einstein

Structure of equations

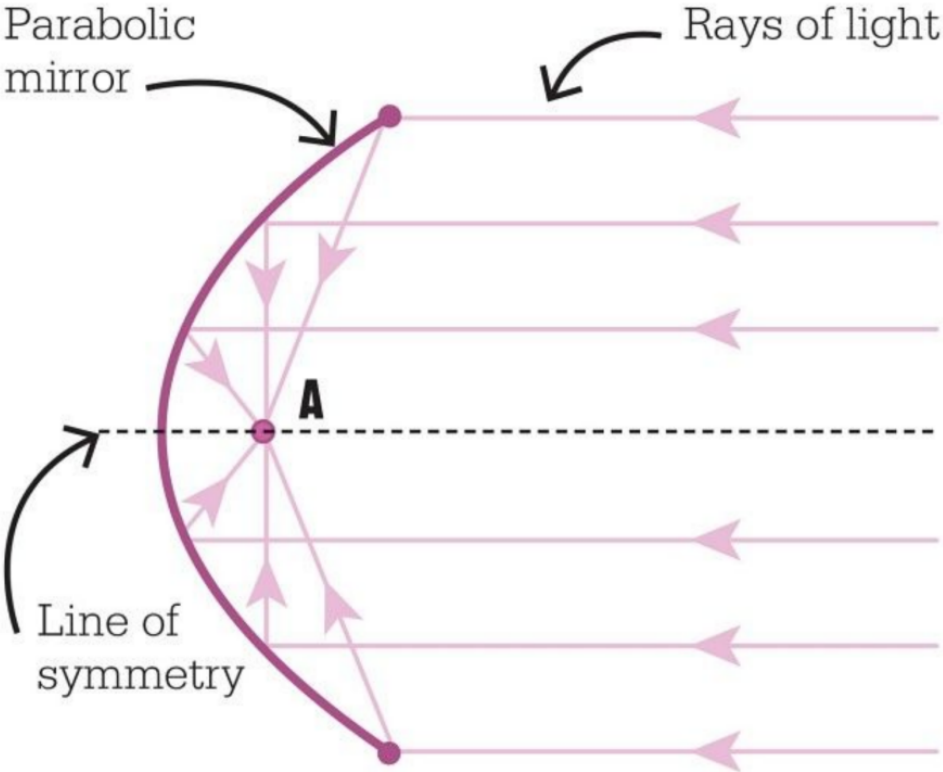
Modern quadratic equations usually look like $ax^2 + bx + c = 0$. The letters ***a***, ***b***, and ***c*** represent known numbers, while ***x*** represents the unknown number. Equations contain variables (symbols for numbers that are unknown), coefficients, constants (those that do not multiply variables), and operators (symbols such as the plus and equals sign). Terms are the parts separated by operators; they can be a number or variable, or a combination of both. The modern quadratic equation has four terms: ax^2, ***bx***, ***c***, and 0.



A graph of the quadratic function $\boldsymbol{y} = \boldsymbol{ax}^2 + \boldsymbol{bx} + \boldsymbol{c}$ creates a U-shaped curve called a parabola. This graph plots the points (in black) of the quadratic function where $\boldsymbol{a} = 1$, $\boldsymbol{b} = 3$, and $\boldsymbol{c} = 2$. This expresses the quadratic equation $\boldsymbol{x}^2 + 3\boldsymbol{x} + 2 = 0$. The solutions for $\boldsymbol{x}$ are where $\boldsymbol{y} = 0$ and the curve crosses the $\boldsymbol{x}$ axis. These are -2 and -1.

Parabolas

A function is a group of terms that defines a relationship between variables (often x and y). The quadratic function is generally written as $y = ax^2 + bx + c$, which, on a graph, produces a curve called a parabola. When real (not imaginary) solutions to $ax^2 + bx + c = 0$ exist, they will be the roots – the points where the parabola crosses the x axis. Not all parabolas cut the x axis in two places. If the parabola touches the x axis only once, this means that there are coincident roots (the solutions are equal to each other). The simplest equation of this form is $y = x^2$. If the parabola does not touch or cross the x axis, there are no real roots. Parabolas prove useful in the real world because of their reflective. properties. Satellite dishes are parabolic for this reason. Signals received by the dish will reflect off the parabola and be directed to one single point – the receiver.



Parabolic objects have special reflective properties. With a parabolic mirror, any ray of light parallel to its line of symmetry will reflect off the surface to the same fixed point (A).

The Rhind papyrus in the British Museum in London provides an intriguing account of mathematics in ancient Egypt. Named after Scottish antiquarian Alexander Henry Rhind, who purchased the papyrus in Egypt in 1858, it was copied from earlier documents by a scribe, Ahmose, more than 3,500 years ago. It measures 32 cm ($12^1/_2$ in) by 200 cm ($78^1/_2$ in) and includes 84 problems concerned with arithmetic, algebra, geometry, and measurement. The problems, recorded in this and other ancient Egyptian artefacts such as the earlier Moscow papyrus, illustrated techniques for working out areas, proportions, and volumes.

The Eye of Horus, an Egyptian god, was a symbol of power and protection. Parts of it were also used to denote fractions whose denominators were powers of 2. The eyeball, for example, represents $^1/_4$, while the eyebrow is $^1/_8$.

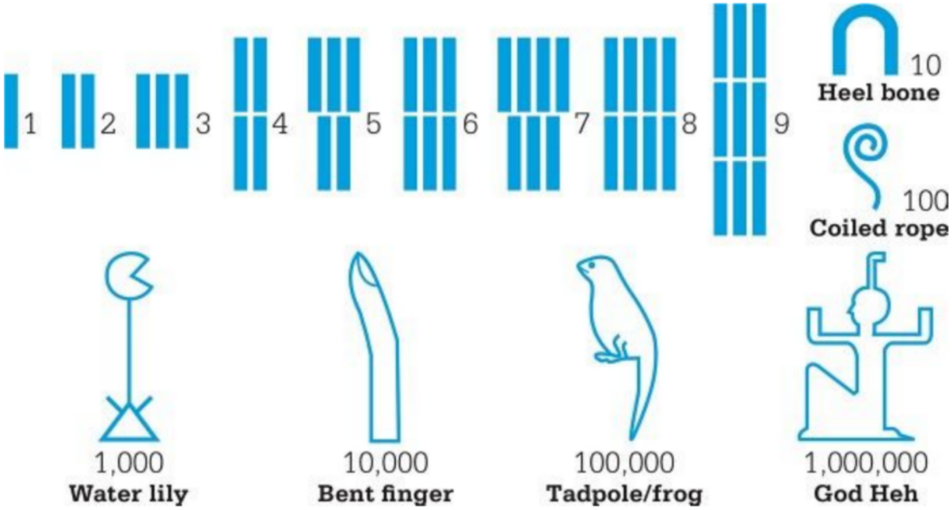
Representing concepts

The Egyptian number system was the first decimal system. It used strokes for single digits and a different symbol for each power of 10. The symbols were then repeated to create other numbers. A fraction was shown as a number with a dot above it. The Egyptian concept of a fraction was closest to a unit fraction – that is, $^1/_n$, where n is a whole number. When a fraction was doubled, it had to be rewritten as one unit fraction added to another unit fraction; for example, $^2/_3$ in modern notation would be $^1/_2 + ^1/_6$ in Egyptian notation (not $^1/_3 + ^1/_3$ because the Egyptians did not allow repeats of the same fraction).

The 84 problems in the Rhind papyrus illustrate the mathematical methods in common use in ancient Egypt. Problem 24, for example, asks what quantity, if added to its seventh part, becomes 19. This translates as $\boldsymbol{x} + {}^x/_7 = 19$. The approach applied to problem 24 is known as "false position". This technique – used well into the Middle Ages – is based on trial and improvement, choosing the simplest, or "false", value for a variable and adjusting the value using a scaling factor (the required quantity divided by the result).

In the workings for problem 24, one-seventh is easiest to find for the number 7, so 7 is used first as a "false" value for the variable. The result of the calculation – 7 plus $^7/_7$ (or 1) – is 8, not 19, so a scaling factor is needed. To find how far the guess of 7 is from the required quantity, 19 is divided by 8 (the "false" answer). This produces a result of $2 + ^1/_4 + ^1/_8$ (not $2^3/_8$, as Egyptian multiplication was based on doubling and halving fractions), which is the scaling factor that should be applied. So, 7 (the original "false" value) is multiplied by $2 + ^1/_4 + ^1/_8$ (the scaling factor) to give the quantity $16 + ^1/_2 + ^1/_8$ (or $16^5/_8$).

Many problems in the papyrus deal with working out shares of commodities or land. Problem 41 asks for the volume of a cylindrical store with a diameter of 9 cubits and a height of 10 cubits. The method finds the area of a square whose side length is $^8/_9$ of the diameter, and then multiplies this by the height. The figure of $^8/_9$ is used as an approximation for the proportion of the area of a square that would be taken up by a circle if it were drawn within the square. This method is used in problem 50 to find the area of a circle: subtract $^1/_9$ from the diameter of the circle, and find the area of the square with the resulting side length.



Ancient Egyptians used vertical lines to denote the numbers 1 to 9. Powers of ten, particularly those inscribed on stone, were depicted as hieroglyphs – picture symbols.

Level of accuracy

Since the Ancient Greeks, the area of a circle has been found by multiplying the square of its radius ($\boldsymbol{r}^2$) with the number pi (π), written as $\pi \boldsymbol{r}^2$. The ancient Egyptians had no concept of pi, but the calculations in the Rhind papyrus show that they were close to its value. Their circle area calculation – with the circle diameter as twice the radius ($2\boldsymbol{r}$) – can be expressed as $(^8/_9 \times 2\boldsymbol{r})^2$, which, simplified, is $^{256}/_{81}\,\boldsymbol{r}^2$, giving an equivalent for pi of $^{256}/_{81}$. As a decimal, this is about 0.6 per cent greater than the true value of pi.

Instruction books

The Rhind papyrus scribe used the hieratic system of writing numerals. This cursive style was more compact and practical than drawing complex hieroglyphs.

The Rhind and Moscow papyri are the most complete mathematical documents to survive from the height of the ancient Egyptian civilization. They were painstakingly copied by scribes well versed in arithmetic, geometry, and mensuration (the study of measurements), and are likely to have been used for training of other scribes. Although they captured probably the most advanced mathematical knowledge of the time, they were not seen as works of scholarship. Instead, they were instruction manuals for use in trade, accounting, construction, and other activities that involved measurement and calculation.

Egyptian engineers, for example, used mathematics in the building of pyramids. The Rhind papyrus includes a calculation for the slope of a pyramid, using the *seked* – a measure for the horizontal distance travelled by a slope for each drop of 1 cubit. The steeper the side of a pyramid, the fewer the *sekeds*.

See also: Positional numbers • Pythagoras • Calculating pi • Algebra • Decimals

THE SUM IS THE SAME IN EVERY DIRECTION

MAGIC SQUARES

IN CONTEXT

KEY CIVILIZATION

Ancient Chinese

FIELD

Number theory

BEFORE

9th century BCE The Chinese *I Ching* (*Book of Changes*) lays out trigrams and hexagrams of numbers for use in divination.

AFTER

1782 Leonhard Euler writes about Latin squares in his *Recherches sur une nouvelle espèce de carrés magiques* (*Investigations on a new type of magic square*).

1979 The first Sudoku-style puzzle is published by Dell Magazines in New York.

2001 British electronics engineer Lee Sallows invents magic squares called “geomagic squares”, which contain geometric shapes rather than numbers.

There are thousands of ways in which to arrange the numbers 1 to 9 in a three-by-three grid. Only eight of these produce a magic square, where the sum of the numbers square does not exist because it would only work if all the numbers were identical. As the orders increase, so do the quantities of magic squares. Order four produces 880 magic squares – with a magic total of 34. There are hundreds of millions of order five magic squares, while the quantity of order six magic squares has not yet been calculated.

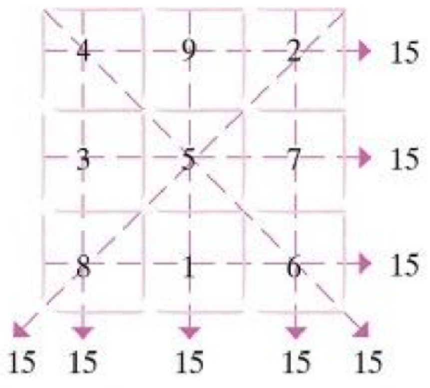
Magic squares have been an enduring source of fascination for mathematicians. The 15th-century Italian mathematician Luca Pacioli, author of *De viribus quantitatis* (*On the Power of Numbers*), collected magic squares. In 18th-century Switzerland, Leonhard Euler also became interested in them, and devised a form that he named Latin squares. The rows and columns in a Latin square contain figures or symbols that appear only once in each row and column.

One derivation of the Latin square – Sudoku – has become a popular puzzle. Devised in the US in the 1970s (where it was called Number Place), Sudoku took off in Japan in the 1980s, acquiring its now-familiar name, which means "single digits". A Sudoku puzzle is a nine-by-nine Latin square with the added restriction that subdivisions of the square must also contain all nine numbers.

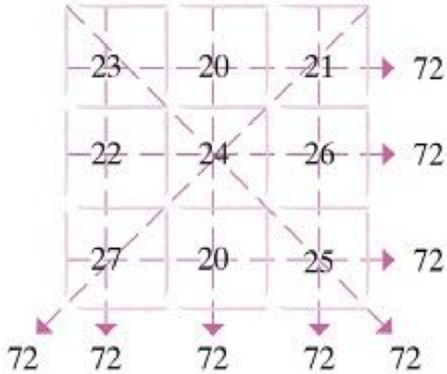
The most magically magical of any magic square ever made by a magician.

Benjamin Franklin

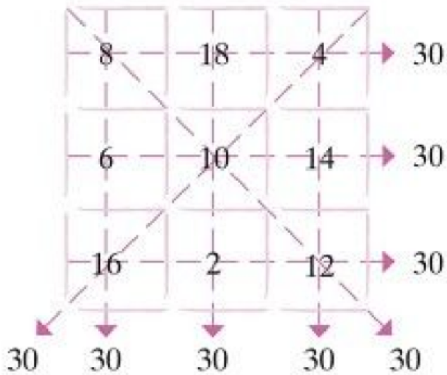
Talking about a magic square that he discovered



The Lo Shu magic square has a magic total of 15.



Here, 19 is added to each of the numbers in the Lo Shu square; the magic total is 72.



Here, all the numbers in the Lo Shu square have been doubled; the magic total is 30.

Once you have one magic square you can add the same quantity to every number in the square and still end up with a magic square. Similarly, if you multiply all the numbers by the same quantity you still have a magic square.

See also: Irrational numbers • Eratosthenes' sieve • Negative numbers • The Fibonacci sequence • The golden ratio • Mersenne primes • Pascal's triangle

NUMBER IS THE CAUSE OF GODS AND DAEMONS

PYTHAGORAS

IN CONTEXT

KEY FIGURE

Pythagoras (*c.* 570 BCE–495 BCE)

FIELD

Applied geometry

BEFORE

***c.* 1800** BCE The columns of cuneiform numbers on the Plimpton 322 clay tablet from Babylon include some numbers related to Pythagorean triples.

6th century BCE Greek philosopher Thales of Miletus proposes a non-mythological explanation of the Universe – pioneering the idea that nature can be interpreted by reason.

AFTER

***c.* 380** BCE In the tenth book of his *Republic*, Plato espouses Pythagoras's theory of the transmigration of souls.

***c.* 300** BCE Euclid produces a formula to find sets of primitive Pythagorean triples.

The 6th-century BCE Greek philosopher Pythagoras is also antiquity's most famous mathematician. Whether or not he was responsible for all the many achievements attributed to him in maths, science, astronomy, music, and medicine, there is no doubt that he founded an exclusive community that lived for the pursuit of mathematics and philosophy, and regarded numbers as the sacred building blocks of the Universe.

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First published in Great Britain in 2019
by Dorling Kindersley Limited,
80 Strand, London, WC2R 0RL

A CIP catalogue record for this book is
available from the British Library.

ISBN: 9780241350362

This digital edition published in 2019

eISBN: 9780241425893

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