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Pure mathematics consists entirely of such asseverations as that, if such and such a proposition is true of anything, *then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true. . . . If our hypothesis is about* anything *and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.*

—BERTRAND RUSSELL

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By PHILIP E. B. JOURDAIN

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INTRODUCTION

AN eminent mathematician once remarked that he was never satisfied with his knowledge of a mathematical theory until he could explain it to the next man he met in the street. That is hardly exaggerated; however, we must remember that a satisfactory explanation entails duties on both sides. Any one of us has the right to ask of a mathematician, "What is the use of mathematics?" Any one may, I think and will try to show, rightly suppose that a satisfactory answer, if such an answer is anyhow possible, can be given in quite simple terms. Even men of a most abstract science, such as mathematics or philosophy, are chiefly adapted for the ends of ordinary life; when they think, they think, at the bottom, like other men. They are often more highly trained, and have a technical facility for thinking that comes partly from practice and partly from the use of the contrivances for correct and rapid thought given by the signs and rules for dealing with them that mathematics and modern logic provide. But there is no real reason why, with patience, an ordinary person should not understand, speaking broadly, what mathematicians do, why they do it, and what, so far as we know at present, mathematics is.

Patience, then, is what may rightly be demanded of the inquirer. And this really implies that the question is not merely a rhetorical one—an expression of irritation or scepticism put in the form of a question for the sake of some fancied effect. If Mr. A. dislikes the higher mathematics because he rightly perceives that they will not help him in the grocery business, he asks disgustedly, "What's the use of mathematics?" and does not wait for an answer, but turns his attention to grumbling at the lateness of his dinner. Now, we will admit at once that higher mathematics is of no more use in the grocery trade than the grocery trade is in the navigation of a ship; but that is no reason why we should condemn mathematics as entirely useless. I remember reading a speech made by an eminent surgeon, who wished, laudably enough, to spread the cause of elementary surgical instruction. "The higher mathematics," said he with great satisfaction to himself, "do not help you to bind up a broken leg!" Obviously they do not; but it is equally obvious that surgery does not help us to add up accounts; . . . or even to think logically, or to accomplish the closely allied feat of seeing a joke.

To the question about the use of mathematics we may reply by pointing out two obvious consequences of one of the applications of mathematics: mathematics prevents much loss of life at sea, and increases the commercial prosperity of nations. Only a few men—a few intelligent philosophers and more amateur philosophers who are not highly intelligent—would doubt if these two things were indeed benefits. Still, probably, all of us act as if we thought that they were. Now, I do not mean that mathematicians go about with life-belts or serve behind counters; they do not usually do so. What I mean I will now try to explain.

Natural science is occupied very largely with the prevention of waste of the labour of thought and muscle when we want to call up, for some purpose or other, certain facts of experience. Facts are sometimes quite useful. For instance, it is useful for a sailor to know the positions of the stars and sun on the nights and days when he is out of sight of land. Otherwise, he cannot find his whereabouts. Now, some people connected with a national institution publish periodically a *Nautical Almanac* which contains the positions of stars and other celestial things you see through telescopes, for every day and night years and years ahead. This *Almanac*, then, obviously increases the possibilities of trade beyond coasting-trade, and makes travel by ship, when land cannot be sighted, much safer; and there would be no *Nautical Almanac* if it were not for the science of astronomy; and there would be no practicable science of astronomy if we could not organise the observations we make of sun and moon and stars, and put hundreds of observations in a convenient form and in a little space—in short, if we could not economise our mental or bodily activity by remembering or carrying about two or three little formulæ instead of fat books full of details; and, lastly, we could not economise this activity if it were not for mathematics.

Just as it is with astronomy, so it is with all other sciences—both those of Nature and mathematical science: the very essence of them is the prevention of waste of the energies of muscle and memory. There are plenty of things in the unknown parts of science to work our brains at, and we can only do so efficiently if we organise our thinking properly, and consequently do not waste our energies.

The purpose of this little volume is not to give—like a text-book—a collection of mathematical methods and examples, but to do, firstly, what text-books do not do: to show how and why these methods grew up. All these methods are simply means, contrived with the conscious or unconscious end of economy of thought-labour, for the convenient handling of long and complicated chains of reasoning. This reasoning, when applied to foretell natural events, on the basis of the applications of mathematics, as sketched in the fourth chapter, often gives striking results. But the methods of mathematics, though often suggested by natural events, are purely logical. Here the word "logical" means something more than the traditional doctrine consisting of a series of extracts from the science of reasoning, made by the genius of Aristotle and frozen into a hard body of doctrine by the lack of genius of his school. Modern logic is a science which has grown up with mathematics, and, after a period in which it moulded itself on the model of mathematics, has shown that not only the reasonings but also conceptions of mathematics are logical in their nature.

In this book I shall not pay very much attention to the details of the elementary arithmetic, geometry, and algebra of the many text-books, but shall be concerned with the discussion of those conceptions—such as that of negative number—which are used and not sufficiently discussed in these books. Then, too, I shall give a somewhat full account of the development of analytical methods and certain examinations of principles.

I hope that I shall succeed in showing that the process of mathematical discovery is a living and a growing thing. Some mathematicians have lived long lives full of calm and unwavering faith—for faith in mathematics, as

I will show, has always been needed—some have lived short lives full of burning zeal, and so on; and in the faith of mathematicians there has been much error.

Now we come to the second object of this book. In the historical part we shall see that the actual reasonings made by mathematicians in building up their methods have often not been in accordance with logical rules. How, then, can we say that the reasonings of mathematics are logical in their nature? The answer is that the one word "mathematics" is habitually used in two senses, and so, as explained in the last chapter, I have distinguished between "mathematics," the methods used to discover certain truths, and "Mathematics," the truths discovered. When we have passed through the stage of finding out, by external evidence or conjecture, how mathematics grew up with problems suggested by natural events, like the falling of a stone, and then how something very abstract and intangible but very real separated out of these problems, we can turn our attention to the problem of the nature of Mathematics without troubling ourselves any more as to how, historically, it gradually appeared to us quite clearly that there is such a thing at all as Mathematics—something which exists apart from its application to natural science. History has an immense value in being suggestive to the investigator, but it is, logically speaking, irrelevant. Suppose that you are a mathematician; what you eat will have an important influence on your discoveries, but you would at once see how absurd it would be to make, say, the momentous discovery that 2 added to 3 makes 5 depend on an orgy of mutton cutlets or bread and jam. The methods of work and daily life of mathematicians, the connecting threads of suggestion that run through their work, and the influence on their work of the allied work of others, all interest the investigator because these things give him examples of research and suggest new ideas to him; but these reasons are psychological and not logical.

But it is as true as it is natural that we should find that the best way to become acquainted with new ideas is to study the way in which knowledge about them grew up. This, then, is what we will do in the first place, and it is here that I must bring my own views forward. Briefly stated, they are these. Every great advance in mathematics with which we shall be concerned here has arisen out of the needs shown in natural science or out of the need felt to connect together, in one methodically arranged whole, analogous mathematical processes used to describe different natural phenomena. The application of logic to our system of descriptions, which we may make either from the motive of satisfying an intellectual need (often as strong, in its way, as hunger) or with the practical end in view of satisfying ourselves that there are no hidden sources of error that may ultimately lead us astray in calculating future or past natural events, leads at once to those modern refinements of method that are regarded with disfavour by the old-fashioned mathematicians.

In modern times appeared clearly—what had only been vaguely suspected before—the true nature of Mathematics. Of this I will try to give some account, and show that, since mathematics is logical and not psychological in its nature, all those petty questions—sometimes amusing and often tedious—of history, persons, and nations are irrelevant to Mathematics in itself. Mathematics has required centuries of excavation, and the process of excavation is not, of course, and never will be, complete. But we see enough now of what has been excavated clearly to distinguish between it and the tools which have been or are used for excavation. This confusion, it should be noticed, was never made by the excavators themselves, but only by some of the philosophical onlookers who reflected on what was being done. I hope and expect that our reflections will not lead to this confusion.

CHAPTER I

THE GROWTH OF MATHEMATICAL SCIENCE IN ANCIENT TIMES

IN the history of the human race, inventions like those of the wheel, the lever, and the wedge were made very early—judging from the pictures on ancient Egyptian and Assyrian monuments. These inventions were made on the basis of an instinctive and unreflecting knowledge of the processes of nature, and with the sole end of satisfaction of bodily needs. Primitive men had to build huts in order to protect themselves against the weather, and, for this purpose, had to lift and transport heavy weights, and so on. Later, by reflection on such inventions themselves, possibly for the purposes of instruction of the younger members of a tribe or the newly-joined members of a guild, these isolated inventions were classified according to some analogy. Thus we see the same elements occurring in the relation of a wheel to its axle and the relation of the arm of a lever to its fulcrum—the same weights at the same distance from the axle or fulcrum, as the case may be, exert the same power, and we can thus class both instruments together in virtue of an analogy. Here what we call "scientific" classification begins. We can well imagine that this pursuit of science is attractive in itself; besides helping us to communicate facts in a comprehensive, compact, and reasonably connected way, it arouses a purely intellectual interest. It would be foolish to deny the obvious importance to us of our bodily needs; but we must clearly realise two things:—(1) The intellectual need is very strong, and is as much a fact as hunger or thirst; sometimes it is even stronger than bodily needs—Newton, for instance, often forgot to take food when he was engaged with his discoveries. (2) Practical results of value often follow from the satisfaction of intellectual needs. It was the satisfaction of certain intellectual needs in the cases of Maxwell and Hertz that ultimately led to wireless telegraphy; it was the satisfaction of some of Faraday's intellectual needs that made the dynamo and the electric telegraph possible. But many of the results of strivings after intellectual satisfaction have as yet no obvious bearing on the satisfaction of our bodily needs. However, it is impossible to tell whether or no they will always be barren in this way. This gives us a new point of view from which to consider the question, "What is the use of mathematics?" To condemn branches of mathematics because their results cannot obviously be applied to some practical purpose is short-sighted.

The formation of science is peculiar to human beings among animals. The lower animals sometimes, but rarely, make isolated discoveries, but never seem to reflect on these inventions in themselves with a view to rational classification in the interest either of the intellect or of the indirect furtherance of practical ends. Perhaps the greatest difference between man and the lower animals is that men are capable of taking circuitous paths for the attainment of their ends, while the lower animals have their minds so filled up with their needs that they try to seize the object they want, or remove that which annoys them, in a direct way. Thus, monkeys often vainly snatch at things they want, while even savage men use catapults or snares or the consciously observed properties of flung stones.

The communication of knowledge is the first occasion that compels distinct reflection, as everybody can still observe in himself. Further, that which the old members of a guild mechanically pursue strikes a new member as strange, and thus an impulse is given to fresh reflection and investigation.

When we wish to bring to the knowledge of a person any phenomena or processes of nature, we have the choice of two methods: we may allow the person to observe matters for himself, when instruction comes to an end; or, we may describe to him the phenomena in some way, so as to save him the trouble of personally making anew each experiment. To describe an event—like the falling of a stone to the earth—in the most comprehensive and compact manner requires that we should discover what is constant and what is variable in the processes of nature; that we should discover the same law in the moulding of a tear and in the motions of the planets. This is the very essence of nearly all science, and we will return to this point later on.

We have thus some idea of what is known as "the economical function of science." This sounds as if science were governed by the same laws as the management of a business; and so, in a way, it is. But whereas the aims of a business are not, at least directly, concerned with the satisfaction of intellectual needs, science—including natural science, logic, and mathematics—uses business methods consciously for such ends. The methods are far wider in range, more reasonably thought out, and more intelligently applied than ordinary business methods, but the principle is the same. And this may strike some people as strange, but it is nevertheless true: there appears more and more as time goes on a great and compelling beauty in these business methods of science.

The economical function appears most plainly in very ancient and modern science. In the beginning, all economy had in immediate view the satisfaction simply of bodily wants. With the artisan, and still more so with the investigator, the most concise and simplest possible knowledge of a given province of natural phenomena—a knowledge that is attained with the least intellectual expenditure—naturally becomes in itself an aim; but though knowledge was at first a means to an end, yet, when the mental motives connected therewith are once developed and demand their satisfaction, all thought of its original purpose disappears. It is one great object of science to replace, or save the trouble of making, experiments, by the reproduction and anticipation of facts in thought. Memory is handier than experience, and often answers the same purpose. Science is communicated by instruction, in order that one man may profit by the experience of another and be spared the trouble of accumulating it for himself; and thus, to spare the efforts of posterity, the experiences of whole generations are stored up in libraries. And further, yet another function of this economy is the preparation for fresh investigation.[1]

The economical character of ancient Greek geometry is not so apparent as that of the modern algebraical sciences. We shall be able to appreciate this fact when we have gained some ideas on the historical development of ancient and modern mathematical studies.

The generally accepted account of the origin and early development of geometry is that the ancient Egyptians were obliged to invent it in order to restore the landmarks which had been destroyed by the periodical inundations of the Nile. These inundations swept away the landmarks in the valley of the river, and, by altering the course of the river, increased or decreased the taxable value of the adjoining lands, rendered a tolerably accurate system of surveying indispensable, and thus led to a systematic study of the subject by the priests. Proclus (412–485 A.D.), who wrote a summary of the early history of geometry, tells this story, which is also told by Herodotus, and observes that it is by no means strange that the invention of the sciences should have originated in practical needs, and that, further, the transition from perception with the senses to reflection,

[1] *Cf.* pp. 5, 13, 15, 16, 42, 71.

and from reflection to knowledge, is to be expected. Indeed, the very name "geometry"—which is derived from two Greek words meaning *measurement of the earth*—seems to indicate that geometry was not indigenous to Greece, and that it arose from the necessity of surveying. For the Greek geometricians, as we shall see, seem always to have dealt with geometry as an abstract science—to have considered *lines* and *circles* and *spheres* and so on, and not the rough pictures of these abstract ideas that we see in the world around us—and to have sought for propositions which should be absolutely true, and not mere approximations. The name does not therefore refer to this practice.

However, the history of mathematics cannot with certainty be traced back to any school or period before that of the Ionian Greeks. It seems that the Egyptians' geometrical knowledge was of a wholly practical nature. For example, the Egyptians were very particular about the exact orientation of their temples; and they had therefore to obtain with accuracy a north and south line, as also an east and west line. By observing the points on the horizon where a star rose and set, and taking a plane midway between them, they could obtain a north and south line. To get an east and west line, which had to be drawn at right angles to this, certain people were employed who used a rope ABCD, divided by knots or marks at B and C, so that the lengths AB, BC, CD were in the proportion 3:4:5. The length BC was placed along the north and south line, and pegs P and Q inserted at the knots B and C. The piece BA (keeping it stretched all the time) was then rotated round the peg P, and similarly the piece CD was rotated round the peg Q, until the ends A and D coincided; the point thus indicated was marked by a peg R. The result was to form a triangle PQR whose angle at P was a right angle, and the line PR would give an east and west line. A similar method is constantly used at the present time by practical engineers, and by gardeners in marking tennis courts, for measuring a right angle. This method seems also to have been known to the Chinese nearly three thousand years ago, but the Chinese made no serious attempt to classify or extend the few rules of arithmetic or geometry with which they were acquainted, or to explain the causes of the phenomena which they observed.

The geometrical theorem of which a particular case is involved in the method just described is well known to readers of the first book of Euclid's *Elements*. The Egyptians must probably have known that this theorem is true for a right-angled triangle when the sides containing the right angle are equal, for this is obvious if a floor be paved with tiles of that shape. But these facts cannot be said to show that geometry was then studied as a science. Our real knowledge of the nature of Egyptian geometry depends mainly on the Rhind papyrus.

The ancient Egyptian papyrus from the collection of Rhind, which was written by an Egyptian priest named Ahmes considerably more than a thousand years before Christ, and which is now in the British Museum, contains a fairly complete applied mathematics, in which the measurement of figures and solids plays the principal part; there are no theorems properly so called; everything is stated in the form of problems, not in general terms, but in distinct numbers. For example: to measure a rectangle the sides of which contain two and ten units of length; to find the surface of a circular area whose diameter is six units. We find also in it indications for the measurement of solids, particularly of pyramids, whole and truncated. The arithmetical problems dealt with in this papyrus—which, by the way, is headed "Directions for knowing all dark things"—contain some very interesting things. In modern language, we should say that the first part deals with the reduction of fractions whose numerators are 2 to a sum of fractions each of whose numerators is 1. Thus $2/29$ is stated to be the sum of $1/24$, $1/58$, $1/174$, and $1/232$. Probably Ahmes had no rule for forming the component fractions, and the answers given represent the accumulated experiences of previous writers. In one solitary case, however, he has indicated his method, for, after having asserted that $2/3$ is the sum of $1/2$ and $1/6$, he added that therefore two-thirds of one-fifth is equal to the sum of a half of a fifth and a sixth of a fifth, that is, to $1/10 + 1/30$.

That so much attention should have been paid to fractions may be explained by the fact that in early times their treatment presented considerable difficulty. The Egyptians and Greeks simplified the problem by reducing a fraction to the sum of several fractions, in each of which the numerator was unity, so that they had to consider only the various denominators: the sole exception to this rule being the fraction $2/3$. This remained the Greek practice until the sixth century of our era. The Romans, on the other hand, generally kept the denominator equal to twelve, expressing the fraction (approximately) as so many twelfths.

In Ahmes' treatment of multiplication, he seems to have relied on repeated additions. Thus, to multiply a certain number, which we will denote by the letter "a," by 13, he first multiplied by 2 and got $2a$, then he doubled the results and got $4a$, then he again doubled the result and got $8a$, and lastly he added together a, $4a$, and $8a$.

Now, we have used the sign "a" to stand for any number: not a particular number like 3, but *any* one. This is what Ahmes did, and what we learn to do in what we call "algebra." When Ahmes wished to find a number such that it, added to its seventh, makes 19, he symbolised the number by the sign we translate "heap." He had also signs for our "+," "−," and "=".[2] Nowadays we can write Ahmes' problem as: Find the number x

[2] In this book I shall take great care in distinguishing signs from what they signify. Thus 2 is to be distinguished from "2": by "2" I mean the *sign*, and the sign written without inverted commas indicates the thing signified. There has been, and is, much confusion, not only with beginners but with eminent mathematicians between a sign

such that $x + \frac{x}{7} = 19$. Ahmes gave the answer in the form $16 + \frac{1}{2} + \frac{1}{8}$.

We shall find that algebra was hardly touched by those Greeks who made of geometry such an important science, partly, perhaps, because the almost universal use of the abacus [3] rendered it easy for them to add and subtract without any knowledge of theoretical arithmetic. And here we must remember that the principal reason why Ahmes' arithmetical problems seem so easy to us is because of our use from childhood of the system of notation introduced into Europe by the Arabs, who originally obtained it from either the Greeks or the Hindoos. In this system an integral number is denoted by a succession of digits, each digit representing the product of that digit and a power of ten, and the number being equal to the sum of these products. Thus, by means of the local value attached to nine symbols and a symbol for zero, any number in the decimal scale of notation can be expressed. It is important to realise that the long and strenuous work of the most gifted minds was necessary to provide us with simple and expressive notation which, in nearly all parts of mathematics, enables even the less gifted of us to reproduce theorems which needed the greatest genius to discover. Each improvement in notation seems, to the uninitiated, but a small thing: and yet, in a calculation, the pen sometimes seems to be more intelligent than the user. Our notation is an instance of that great spirit of economy which spares waste of labour on what is already systematised, so that all our strength can be concentrated either upon what is known but unsystematised, or upon what is unknown.

Let us now consider the transformation of Egyptian geometry in Greek hands. Thales of Miletus (about 640–546 B.C.), who, during the early part of his life, was engaged partly in commerce and partly in public affairs, visited Egypt and first brought this knowledge into Greece. He discovered many things himself, and communicated the beginnings of many to his successors. We cannot form any exact idea as to how Thales presented his geometrical teaching. We infer, however, from Proclus that it consisted of a number of isolated propositions which were not arranged in a logical sequence, but that the proofs were deductive, so that the theorems were not a mere statement of an induction from a large number of special instances, as probably was the case with the Egyptian geometri-

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and what is signified by it. Many have even maintained that *numbers* are the *signs* used to represent them. Often, for the sake of brevity, I shall use the word in inverted commas (say "*a*") as short for "what we call '*a*,'" but the context will make plain what is meant.

[3] The principle of the abacus is that a number is represented by counters in a series of grooves, or beads strung on parallel wires; as many counters being put on the first groove as there are units, as many on the second as there are tens, and so on. The rules to be followed in addition, subtraction, multiplication, and division are given in various old works on arithmetic.

cians. The deductive character which he thus gave to the science is his chief claim to distinction. Pythagoras (born about 580 B.C.) changed geometry into the form of an abstract science, regarding its principles in a purely abstract manner, and investigated its theorems from the immaterial and intellectual point of view. Among the successors of these men, the best known are Archytas of Tarentum (428–347 B.C.), Plato (429–348 B.C.), Hippocrates of Chios (born about 470 B.C.), Menaechmus (about 375–325 B.C.), Euclid (about 330–275 B.C.), Archimedes (287–212 B.C.), and Apollonius (260–200 B.C.).

The only geometry known to the Egyptian priests was that of surfaces, together with a sketch of that of solids, a geometry consisting of the knowledge of the areas contained by some simple plane and solid figures, which they had obtained by actual trial. Thales introduced the ideal of establishing by exact reasoning the *relations* between the different parts of a figure, so that some of them could be found by means of others in a manner strictly rigorous. This was a phenomenon quite new in the world, and due, in fact, to the abstract spirit of the Greeks. In connection with the new impulse given to geometry, there arose with Thales, moreover, scientific astronomy, also an abstract science, and undoubtedly a Greek creation. The astronomy of the Greeks differs from that of the Orientals in this respect: the astronomy of the latter, which is altogether concrete and empirical, consisted merely in determining the duration of some periods or in indicating, by means of a mechanical process, the motions of the sun and planets; whilst the astronomy of the Greeks aimed at the discovery of the geometrical laws of the motions of the heavenly bodies.

Let us consider a simple case. The area of a right-angled field of length 80 yards and breadth 50 yards is 4000 square yards. Other fields which are not rectangular can be approximately measured by mentally dissecting them—a process which often requires great ingenuity and is a familiar problem to land-surveyors. Now, let us suppose that we have a circular field to measure. Imagine from the centre of the circle a large number of radii drawn, and let each radius make equal angles with the next ones on each side of it. By joining the points in succession where the radii meet the circumference of the circle, we get a large number of triangles of equal area, and the sum of the areas of all these triangles gives an approximation to the area of the circle. It is particularly instructive repeatedly to go over this and the following examples mentally, noticing how helpful the abstract ideas we call "straight line," "circle," "radius," "angle," and so on, are. We all of us know them, recognise them, and can easily feel that they are trustworthy and accurate ideas. We feel at home, so to speak, with the idea of a square, say, and can at once give details about it which are *exactly* true for it, and *very nearly* true for a field which we know is

There are two branches of mathematics which began to be cultivated by the Greeks, and which allow a connection to be formed between the spirits of ancient and modern mathematics.

The first is the method of geometrical analysis to which Plato seems to have directed attention. The analytical method of proof begins by assuming that the theorem or problem is solved, and thence deducing some result. If the result be false, the theorem is not true or the problem is incapable of solution: if the result be true, if the steps be reversible, we get (by reversing them) a synthetic proof; but if the steps be not reversible, no conclusion can be drawn. We notice that the leading thought in analysis is that which is fundamental in algebra, and which we have noticed in the case of Ahmes: the calculation or reasoning with an unknown entity, which is denoted by a conventional sign, as if it were known, and the deduction at last, of some relation which determines what the entity must be.

And this brings us to the second branch spoken of: algebra with the later Greeks. Diophantus of Alexandria, who probably lived in the early half of the fourth century after Christ, and probably was the original inventor of an algebra, used letters for unknown quantities in arithmetic and treated arithmetical problems analytically. Juxtaposition of symbols represented what we now write as "+," and "−" and "=" were also represented by symbols. All these symbols are mere abbreviations for words, and perhaps the most important advantage of symbolism—the power it gives of carrying out a complicated chain of reasoning almost mechanically—was not made much of by Diophantus. Here again we come across the economical value of symbolism: it prevents the wearisome expenditure of mental and bodily energy on those processes which can be carried out mechanically. We must remember that this economy both emphasises the unsubjugated—that is to say, unsystematised—problems of science, and has a charm—an æsthetic charm, it would seem—of its own.

Lastly, we must mention "incommensurables," "loci," and the beginnings of "trigonometry."

Pythagoras was, according to Eudemus and Proclus, the discoverer of "incommensurable quantities." Thus, he is said to have found that the diagonal and the side of a square are "incommensurable." Suppose, for example, that the side of the square is one unit in length; the diagonal is longer than this, but it is not two units in length. The excess of the length of the diagonal over one unit is not an integral submultiple of the unit. And, expressing the matter arithmetically, the remainder that is left over after each division of a remainder into the preceding divisor is not an integral submultiple of the remainder used as divisor. That is to say, the rule given in text-books on arithmetic and algebra for finding the greatest